Homework 11

2.6 If possible, assume the contrary, i.e., $\sim(n \not\parallel m)$ or $n \parallel m$. Equivalently $m \parallel n$. Since $l \parallel m$, we have $l \parallel n$, a contradiction to $l \not\parallel n$. Thus $n \not\parallel m$.

3.1 By the ruler axiom, there is a co-ordinate system g on $\overleftrightarrow{PQ}$. If $g(Q) < g(P)$, then set $h(A) = -g(A)$ $\forall A \in \overleftrightarrow{PQ}$. If $g(Q) > g(P)$, then define $h(A) = g(A)$ $\forall A \in \overleftrightarrow{PQ}$. In either cases, h defines a co-ordinate system on $\overleftrightarrow{PQ}$ having the property that $h(P) < h(Q)$. Now define $f : \overleftrightarrow{PQ} \to \mathbb{R}$ by:

$$f(A) = -h(P) + h(A) \quad \forall A \in \overleftrightarrow{PQ}.$$

f is a co-ordinate system & $f(P) = 0$, $f(Q) = h(Q) - h(P) > 0$.

3.4 Let $h(A) = -g(P) + g(A)$ $\forall A \in \overleftrightarrow{PQ}$. h is a co-ordinate system (as $|h(A)\ h(B)| = |-g(P)+g(A) - (-g(P)+g(B))| = |-g(B) + g(A)| = |AB|$) & $h(P) = 0$, $h(Q) = g(Q) - g(P) > 0$. Since f is a co-ordinate system with the same properties, by Thm 3.1, h (or f) is unique, i.e., $f(A) = h(A)$ $\forall A \in \overleftrightarrow{PQ}$. Thus $f(A) = -g(P) + g(A)$.

Let $g(A) = h(P) - h(A)$ $\forall A \in \overleftrightarrow{PQ}$. g is a co-ordinate system (argue as in the previous case), $g(P) = 0$ & $g(Q) = h(P) - h(Q) > 0$. Since f is also a co-ordinate system with the same properties, $f = g$. Hence $f(A) = h(P) - h(A)$ $\forall \in \overleftrightarrow{PQ}$.

3.5 $A-B-C$ means $\exists f : \overleftrightarrow{AB} \to \mathbb{R}$ s.t $f(A) < f(B) < f(C)$. Let $g = -f$. Then $g : \overleftrightarrow{AB} \to \mathbb{R}$ is a co-ordinate system & $g(A) > g(B) > g(C)$. Thus, $C-B-A$ holds.

Now, reversing the steps above proves $C-B-A \Rightarrow A-B-C$ and thus $A-B-C$ holds iff $C-B-A$ holds.

3.9 Choose a co-ordinate system $f: \overleftrightarrow{VA} \to \mathbb{R}$ s.t. $f(V)=0$ & $f(A)>0$. Then $P \in \overrightarrow{VA}$ iff $f(P) \geq 0$. Since $B \in \overrightarrow{VA}$, B, A & V lie on the line $\overleftrightarrow{VA}$ $(= \overleftrightarrow{VB})$. Let $f: \overleftrightarrow{VB} \to \mathbb{R}$ be the same co-ordinate system as above Then $Q \in \overrightarrow{VB}$ iff $f(Q) \geq 0$ (since $f(B) > 0$ as $B \in \overrightarrow{VA}$). Since elements of $\overrightarrow{VA}$ & $\overrightarrow{VB}$ are defined by the same condition on f, $\overrightarrow{VA} = \overrightarrow{VB}$.

3.14 Choose a co-ordinate system $f: \overleftrightarrow{AB} \to \mathbb{R}$ s.t $f(A)=0$ & $f(B)=|AB|$. Since f is a bijection, set $M \in \overleftrightarrow{AB}$ as $f^{-1}(|AB|/2)$, i.e, $f(M) = \frac{1}{2}|AB|$. Then $|AM| = f(M) = \frac{|AB|}{2}$ & $|MB| = |f(B) - f(M)| = |AB| - \frac{|AB|}{2} = \frac{|AB|}{2}$. Thus, $|AM| = \frac{1}{2}|AB| = |MB|$. Uniqueness of M follows by the injectivity of f, for if $\exists M_1, M_2 \in \overleftrightarrow{AB}$ s.t. $f(M_1) = \frac{1}{2}|AB| = f(M_2)$, then $M_1 = M_2$.

4.2 $\Rightarrow$ D lies in the interior of ΔABC implies D doesn't lie on the sides $\overleftrightarrow{AB}$ & $\overleftrightarrow{AC}$, i.e., $D \notin \overleftrightarrow{AB}$ & $D \notin \overleftrightarrow{AC}$. If $\overline{DB} \cap \overleftrightarrow{AC} \neq \emptyset$, then the intersection is a pt $P \in \overleftrightarrow{AC}$ and since $P \in \overline{BD} = \overline{DB}$, this implies D & B lie on different sides of $\overleftrightarrow{AC}$, a contradiction. Similarly, if $\overline{DC} \cap \overleftrightarrow{AB} = \{Q\}$, then D & C lie on different sides of $\overleftrightarrow{AB}$, a contradiction. Hence all four conditions are satisfied.

$\Leftarrow$ If $D \notin \overleftrightarrow{AB}$ & $D \notin \overleftrightarrow{AC}$ then D doesn't lie on the sides of the ΔBAC. Suppose D & B lie on different sides of $\overleftrightarrow{AC}$. Then $\overline{DB} \cap \overleftrightarrow{AC} = \{P\}$ since $\overline{DB} \times \overleftrightarrow{AC}$, a contradiction to the given hypothesis. Similarly, if D & C lie on different sides of $\overleftrightarrow{AB}$, then $\overline{DC} \cap \overleftrightarrow{AB} = \{Q\}$, contradicting the given hypothesis. Thus, D & B lie on the same side of $\overleftrightarrow{AC}$ & D & C lie on the same side of $\overleftrightarrow{AB}$. Hence D lies in the interior of ΔBAC.

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4.1 By the Protractor axiom, take the half-ray $\overrightarrow{AD}$ in H s.t. $m\angle BAD = \alpha$. Now choose a co-ordinate system on $\overleftrightarrow{AD}$ s.t

$$f : \overleftrightarrow{AD} \to \mathbb{R} \quad , \quad f(A) = 0 \; , \; f(D) > 0 .$$

For the given $r > 0$, set $C = f^{-1}(r)$. Then $|AC| = r$ & since $C \in \overrightarrow{AD}$, $\angle BAC$ has measure α.

4.6 $\Leftarrow$ If $\overrightarrow{AD}$ is inside $\angle BAC$, then by the Protractor axiom (4),

$$m\angle BAC = m\angle BAD + m\angle DAC > m\angle BAD .$$

$\Rightarrow$ If $m\angle BAD < m\angle BAC$, then look at the half plane H which contains C & is obtained by $\overleftrightarrow{AB}$ dividing the plane. Now, since C & D lie on the same side of $\overleftrightarrow{AB}$, $D \in H$. Now, there are two mutually exclusive & exhaustive cases.

(i) $\overrightarrow{AC}$ lies inside $\angle BAD$ (which is possible iff $m\angle BAD = m\angle BAC + m\angle CAD > m\angle BAC$; a contradiction!) OR

(ii) $\overrightarrow{AD}$ lies inside $\angle BAC$ (which means $m\angle BAC = m\angle BAD + m\angle CAD$).

Thus, $\overrightarrow{AD}$ lies inside $\angle BAC$.

4.7 Let $\overrightarrow{AD} \cap \overline{BC} = \{P\}$. Then $m\angle BAC = m\angle BAP + m\angle PAC$ & $m\angle BAP = m\angle BAD$ & $m\angle PAC = m\angle DAC$ as $P \in \overrightarrow{AD}$. If C & D are on the same side of $\overleftrightarrow{AB}$, then by Thm 4.2, we're done. If not, then let $\overline{CD} \cap \overleftrightarrow{AB} = \{Q\}$. Then $m\angle DAC = m\angle DAQ + m\angle QAC$. Since $Q \in \overleftrightarrow{AB}$, $\angle DAQ = \angle DAB$ & $\angle QAC = \angle BAC$. Thus,

$$m\angle DAC = m\angle DAB + m\angle BAC \qquad \cdots (1)$$

We also have:

$$m\angle BAC = m\angle BAD + m\angle DAC \qquad \cdots (2)$$

Adding (1) & (2) we have:

$$0 = 2\, m\angle BAD \quad \text{or} \quad m\angle BAD = 0 .$$

Thus, $D \in \overleftrightarrow{AB}$. Then $\overrightarrow{AD} \cap \overline{BC} = \{B\}$, a contradiction. This completes the proof.

4.8 Let $\angle BAC$ have measure less than π. Suppose $\overrightarrow{AD}$ lies inside $\angle BAC$. Then either

(i) $\overrightarrow{AD} \cap \overline{BC} \neq \{\emptyset$ OR

(ii) $\overrightarrow{AD} \cap \overline{BC} = \emptyset$.

In (ii) there are two possibilities; either $\overleftrightarrow{AD} \cap \overleftrightarrow{BC} \neq \emptyset$ or $\overleftrightarrow{AD}$ & $\overleftrightarrow{BC}$ are parallel. If $P \in \overleftrightarrow{BC}$, then P either lies on the same side of $\overleftrightarrow{AB}$ as D OR P lies on the same side of $\overleftrightarrow{AB}$ as C. Now, if $P \in \overleftrightarrow{AB}$, $P \in \overrightarrow{AD}$ means P lies in the same side of $\overleftrightarrow{AB}$ as D & P lies in the same side of $\overleftrightarrow{AC}$ as D also. If $P \in \overleftrightarrow{AD} \setminus \overrightarrow{AD}$, then P lies on the other side of $\overleftrightarrow{AB}$ as D & also on the other side of $\overleftrightarrow{AC}$ as D.

Thus, if $\overleftrightarrow{AD} \cap \overleftrightarrow{BC} = \{P\}$. Then since $P \in \overleftrightarrow{BC}$, P lies on the same side of $\overleftrightarrow{AB}$ as D & lies on the other side of $\overleftrightarrow{AC}$ as D OR P lies on the other side of $\overleftrightarrow{AB}$ as D & on the same side of $\overleftrightarrow{AC}$ as D. Also, since $P \in \overleftrightarrow{AD}$, P lies on the same side of $\overleftrightarrow{AB}$ as D & on the same side of $\overleftrightarrow{AC}$ as D OR different side(s) of $\overleftrightarrow{AB}$ & $\overleftrightarrow{AC}$ as D. This gives a contradiction to the existence of P. Hence $\overleftrightarrow{AD} \cap \overleftrightarrow{BC} = \emptyset$ & $\overleftrightarrow{AD} \parallel \overleftrightarrow{BC}$.

Proof of 4.3 We need only show that $\overleftrightarrow{AD} \parallel \overleftrightarrow{BC}$ can't happen. Since $\overrightarrow{AD}$ lies inside $\angle BAC$, $m\angle BAD + m\angle CAD = m\angle BAC > 0$. $\overleftrightarrow{AB} \times \overleftrightarrow{BC}$ & $\overleftrightarrow{AB} \times \overleftrightarrow{AD}$ and either $m\angle BAD = m\angle BAC + m\angle CAD$, whence

$2\, m\angle CAD = 0$, a contradiction OR

$m\angle BAD = \pi - m\angle BAC - m\angle CAD$, whence

$2\, m\angle BAC = \pi$, i.e., $m\angle BAC = \pi/2$.