MAT 141 FALL 2002
FINAL

!!! WRITE YOUR NAME, SUNY ID N. AND SECTION BELOW !!!

NAME :

SUNY ID N. :

SECTION :

THERE ARE 10 PROBLEMS. THEY DO NOT HAVE EQUAL VALUE.
SHOW YOUR WORK!!!

1	45	
2	40	
3	40	
4	35	
5	35	
6	40	
7	40	
8	40	
9	45	
10	40	
Total	400	

1. [**45 points**] Determine the following limits.

a) $\lim_{x\to\infty}\left(\sqrt{x^2+4x+1}-x\right)$

$$= \lim_{x\to\infty}\frac{\left(\sqrt{x^2+4x+1}-x\right)\left(\sqrt{x^2+4x+1}+x\right)}{\sqrt{x^2+4x+1}+x} =$$

$$\lim_{x\to\infty}\frac{x^2+4x+1-x^2}{\sqrt{x^2+4x+1}+x} = \lim_{x\to\infty}\frac{4x+1}{\sqrt{x^2+4x+1}+x} = \lim_{x\to\infty}\frac{4+\frac{1}{x}}{\sqrt{1+\frac{4}{x}+\frac{1}{x^2}}+1} =$$

$$\frac{4}{1+1} = 2$$

b) $\lim_{x\to 1}\frac{x-1}{\sqrt[3]{x}-1}$

(Let $t=\sqrt[3]{x}-1$. Then $x=(t+1)^3$ and $t\to 0$ as $x\to 1$)

$$= \lim_{t\to 0}\frac{(t+1)^3-1}{t} = \lim_{t\to 0}\frac{t^3+3t^2+3t}{t} = \lim_{t\to 0} t^2+3t+3 = 3$$

c) $\lim_{x\to 0}\frac{\sin(-4x)}{x}$

(Let $t=-4x$. Then $x=-\frac{1}{4}t$ and $t\to 0$ as $x\to 0$)

$$= \lim_{t\to 0}\frac{\sin t}{-\frac{1}{4}t} = -4\lim_{t\to 0}\frac{\sin t}{t} = -4$$

2. [**40 points**] $y = \frac{x^2+3}{x+1}$.

a) Find all asymptotes of this curve.
Setting the denominator zero, we have the vertical asymptotes :

$$x + 1 = 0 \Rightarrow x = -1$$

$y = \frac{x^2+3}{x+1} = \frac{x^2-1+4}{x+1} = x - 1 + \frac{4}{x+1}$
So, there is an oblique asymptote : $y = x - 1$

$$\lim_{x\to\pm\infty} \frac{x^2+3}{x+1} = \pm\infty$$

So, no horizontal asymptotes.

b) What are the left and right limit behaviors at -1?
(i.e. $\lim_{x\to-1^-} \frac{x^2+3}{x+1} =?$, $\lim_{x\to-1^+} \frac{x^2+3}{x+1} =?$)

(Let $t = x+1$. Then $x = t-1$, $t \to 0^- asx \to 0^-$ and $t \to 0^+ asx \to 0^+$)

$$\lim_{x\to-1^-} \frac{x^2+3}{x+1} = \lim_{t\to0^-} \frac{(t-1)^2+3}{t} = \lim_{t\to0^-} \frac{t^2-2t+4}{t} = \lim_{t\to0^-} t-2+4\frac{1}{t} = -\infty$$

$$\lim_{x\to-1^+} \frac{x^2+3}{x+1} = \lim_{t\to0^+} \frac{(t-1)^2+3}{t} = \lim_{t\to0^+} \frac{t^2-2t+4}{t} = \lim_{t\to0^+} t-2+4\frac{1}{t} = +\infty$$

Here, we have used the fact that $\lim_{t\to0^+} \frac{1}{x} = +\infty$ and $\lim_{t\to0^-} \frac{1}{x} = -\infty$

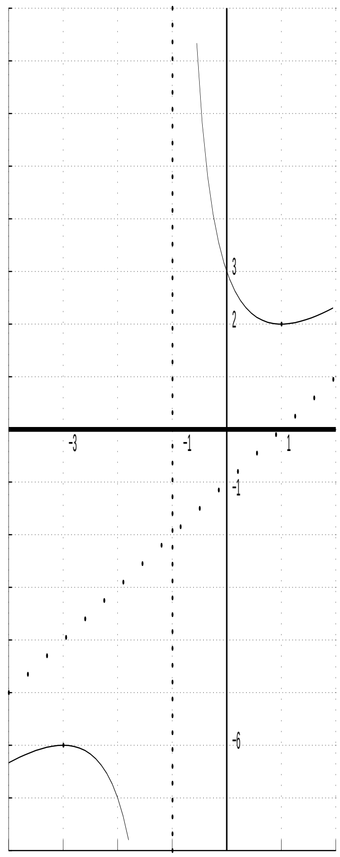
c) Find all critical points and inflection points of this curve.
Using $y = x - 1 + \frac{4}{x+1}$ from a) , we differentiate twice.
$y = x - 1 + 4(x+1)^{-1}$
$y' = 1 - 4(x+1)^{-2} = 1 - \frac{4}{x^2+2x+1} = \frac{x^2+2x+1-4}{x^2+2x+1} = \frac{x^2+2x-3}{x^2+2x+1}$
$y'' = 8\frac{1}{(x+1)^3}$
At critical points, $y' = 0$ and at inflection points $y'' = 0$. In order for these to be zero, the numerator needs to be zero.
$y' = 0 \Rightarrow x^2 + 2x - 3 = 0 \Rightarrow (x+3)(x-1) = 0 \Rightarrow x = -3\,, 2.$
At $x = -3$, we have $y = -6$, $y'' < 0$ and at $x = 1$ we have $y = 2$,$y'' > 0$.
So $(-3, -6)$ is a local maximum and $(1, 2)$ is a local minimum.

$y'' = 0 \Rightarrow 8 = 0$ Impossible. Hence there are no inflection points.

d) Sketch the graph of the curve including all the information from a) $\sim$ c). Also include the y intercept

Plug 0 into x to get y-intercept as 3.

	x<-3	-3<x<-1	-1<x<1	1<x
y'	$-$	$+$	$+$	$-$
y''	$+$	$+$	$-$	$-$
y	$\smile$		$\frown$	


3
2
-3
-1
1
-1
-6

3. [**40 points**] Let $f(x)$ and $g(x)$ be differentiable real valued functions defined on the interval $(-\infty, +\infty)$.

a) Write down the limit definition of $f'(x)$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

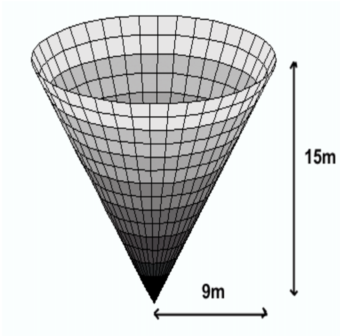
b) Prove the product rule: $(f(x) \cdot g(x))' = f'(x) \cdot g(x) + f(x) \cdot g'(x)$.

$$(f(x) \cdot g(x))' = \lim_{h \to 0} \frac{f(x+h) \cdot g(x+h) - f(x) \cdot g(x)}{h}$$

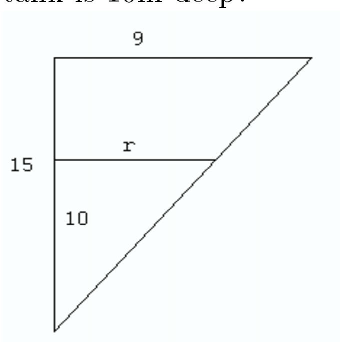
$$= \lim_{h \to 0} \frac{f(x+h) \cdot g(x+h) - f(x+h) \cdot g(x) + f(x+h) \cdot g(x) - f(x) \cdot g(x)}{h}$$

$$= \lim_{h \to 0} f(x+h) \frac{g(x+h) - g(x)}{h} + \frac{f(x+h) - f(x)}{h} g(x) = f(x) \cdot g'(x) + f'(x) \cdot g(x)$$

4. [**35 points**] Water is pumped into a conic tank at the rate of 6 m$^3/hr$. The cone is 15m high and the top of the cone is a circle of radius 9m.



a) What is the radius of the water's surface when the water in the tank is 10m deep?



$15 : 9 = 10 : r$

$r = \frac{90}{15} = 6$

b) At what rate is the depth increasing when the water is 10m deep?

$15 : 9 = h : r$

$r = \frac{9h}{15} = \frac{3h}{5}$

$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(\frac{3h}{5})^2 h = \frac{3}{25}\pi h^3$

$\frac{dV}{dt} = \frac{9}{25}\pi h^2 \frac{dh}{dt}$. When water is 10m deep, h=10 and since $\frac{dV}{dt} = 6m^3/hr$,

$$6 = \frac{9}{25}\pi 10^2 \frac{dh}{dt}$$
$$\frac{dh}{dt} = \frac{1}{6\pi} m/hr$$

5. [**35 points**] Differentiate $f(x) = \frac{\left(x^2+1\right)^{\frac{1}{2}}\left(x^2-x+1\right)^{\frac{1}{3}}}{(x^4+2)^{\frac{1}{4}}(x^6+3)^{\frac{1}{6}}}$ using logarithmic differentiation. (No partial credit will be given if logarithmic differentiation is not used.)

$$\ln f(x) = \ln\left(\frac{\left(x^2+1\right)^{\frac{1}{2}}\left(x^2-x+1\right)^{\frac{1}{3}}}{(x^4+2)^{\frac{1}{4}}(x^6+3)^{\frac{1}{6}}}\right)$$

$$\ln f(x) = \tfrac{1}{2}\ln\left(x^2+1\right) + \tfrac{1}{3}\ln\left(x^2-x+1\right) - \tfrac{1}{4}\ln\left(x^4+2\right) - \tfrac{1}{6}\ln\left(x^6+3\right)$$

Differentiating, we have

$$\frac{f'(x)}{f(x)} = \frac{x}{x^2+1} + \frac{\frac{1}{3}(2x-1)}{x^2-x+1} - \frac{x^3}{x^4+2} - \frac{x^5}{x^6+3}$$

$$f'(x) = f(x)\left(\frac{x}{x^2+1} + \frac{\frac{1}{3}(2x-1)}{x^2-x+1} - \frac{x^3}{x^4+2} - \frac{x^5}{x^6+3}\right)$$

$$f'(x) = \left(\frac{\left(x^2+1\right)^{\frac{1}{2}}\left(x^2-x+1\right)^{\frac{1}{3}}}{(x^4+2)^{\frac{1}{4}}(x^6+3)^{\frac{1}{6}}}\right)\left(\tfrac{1}{2}\ln\left(x^2+1\right) + \tfrac{1}{3}\ln\left(x^2-x+1\right) - \tfrac{1}{4}\ln\left(x^4+2\right) - \tfrac{1}{6}\ln\left(x^6+3\right)\right)$$

6. [**40 points**]

a) State the fundamental theorem of calculus part I and part II.

Part I. If f is continuous on $[a,b]$, then the function

$$F(x) = \int_a^x f(t)dt$$

has a derivative at every point x in $[a,b]$, and

$$\frac{dF}{dx} = \frac{d}{dx}\int_a^x f(t)dt = f(x).$$

Part II. If f is continuous at every point of $[a,b]$ and if F is any antiderivative of f on $[a,b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

b) Use the fundamental theorem of calculus Part I to prove the fundamental theorem of calculus part II.

Define a function $G(x)$ by

$$G(x) = \int_a^x f(t)\,dt$$

By the fundamental theorem of calculus Part I, $\frac{d}{dx}G(x) = f(x)$. Since $F(x)$ is also an antiderivative,

$$\frac{d}{dx}(F(x) - G(x)) = \frac{d}{dx}F(x) - \frac{d}{dx}G(x) = f(x) - f(x) = 0$$

. Hence $F(x) - G(x) = C$.

$$F(b)-F(a) = [G(b)+C]-[G(a)+b] = G(b)-G(a) = \int_a^b f(t)dt - \int_a^a f(t)\,dt = \int_a^b f(t)dt.$$

7. [**40 points**] Differentiate

$$f(x) = \int_{-e^x}^{\sin x} \ln\left(1+t^2\right) dt$$

$f(x) = \int_0^{\sin x} \ln(1+t^2)\, dt + \int_{-e^x}^{0} \ln(1+t^2)\, dt$
$= \int_0^{\sin x} \ln(1+t^2)\, dt - \int_0^{-e^x} \ln(1+t^2)\, dt$

By the Fundamental Theorem of Calculus I and the chain rule,
$f'(x) = \ln(1+\sin^2 x)\cdot\cos x - \ln(1+(-e^x)^2)(-e^x)$
$= \ln(1+\sin^2 x)\cdot\cos x + \ln(1+(-e^x)^2)e^x$
This can be simplified to
$= \ln(1+\sin^2 x)\cdot\cos x + \ln(1+e^{2x})e^x$

Solution 2)
Another way to solve this is using Fundamental Theorem of Calculus II.
Let $G(x)$ be an antiderivative of $\ln(1+t^2)$. Then by the Fundamental Theorem of Calculus II, we have

$$f(x) = \int_{-e^x}^{\sin x} \ln\left(1+t^2\right) dt = G(\sin x) - G(-e^x)$$

SInce $G(x)$ is an antiderivative of $\ln(1+t^2)$, $G'(x) = \ln(1+t^2)$. Combining this with the chain rule we get
$f'(x) = G'(\sin x)\cdot\cos x - G'(-e^x)(-e^x) = \ln(1+\sin^2 x)\cdot\cos x - \ln(1+(-e^x)^2)(-e^x)$
$= \ln(1+\sin^2 x)\cdot\cos x + \ln(1+(-e^x)^2)e^x$
This can be simplified to
$= \ln(1+\sin^2 x)\cdot\cos x + \ln(1+e^{2x})e^x$

8. [**40 points**] Compute the integral

$$\int_{\ln\frac{\pi}{4}}^{\ln\frac{\pi}{2}} -3e^t \sin e^t \, dt$$

$u = e^t$. Then $du = e^t dt$. Also when $t = \ln\frac{\pi}{2}$, $u = e^{\ln\frac{\pi}{2}} = \frac{\pi}{2}$ and when $t = \ln\frac{\pi}{4}$, $u = e^{\ln\frac{\pi}{4}} = \frac{\pi}{4}$.

$$\int_{\ln\frac{\pi}{4}}^{\ln\frac{\pi}{2}} -3e^t \sin e^t \, dt \; = \; \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} -3\sin u \, du = 3\cos u|_{\frac{\pi}{4}}^{\frac{\pi}{2}} \; = \; 3{\cdot}0{-}3{\cdot}\frac{\sqrt{2}}{2} = -3\frac{\sqrt{2}}{2}$$

9. **[45 points]**

a) Use Simpson's rule to estimate the integral

$$\int_1^2 -\frac{2}{x}\,dx$$

with n=4.

$h = \frac{2-1}{4} = 0.25$.

$S = \frac{0.25}{3}\left(f(1) + 4f(1.25) + 2f(1.5) + 4f(1.75) + f(2)\right) = -\frac{1747}{1260} \approx -1.39.$

b) Use the "error estimate for Simpson's rule" to give an upper bound for E_s. (No partial credit will be given for computing E_s directly.)

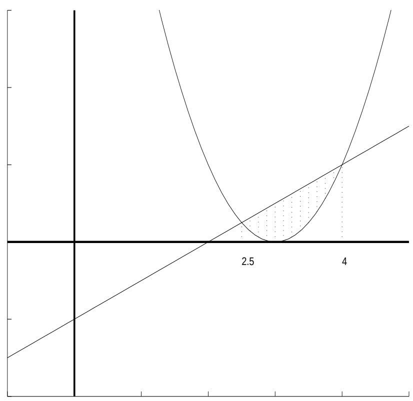
$|f^{(4)}(x)| = |-\frac{48}{x^5}|.$
In $[1, 2]$, $|f^{(4)}(x)| = \frac{48}{x^5}$ and since this is decreasing, maximum is when $x = 1$. So $M = 48$. Using the formula, we have

$$|E_s| \leq \frac{2-1}{180}(0.25)^4 \cdot 48 = \frac{1}{960}$$

10.[**40 points**] Use the "cylindrical shell" method to find the volume of the solid region obtained by rotating about the line $x = 1$ the region bounded by the line $y = \frac{1}{2}x - 1$ and the graph of the function $y = x^2 - 6x + 9$.

To see where the two graphs meet, we solve

$$\frac{1}{2}x - 1 = x^2 - 6x + 9$$
$$x = 2.5, 4$$



$$2\pi \int_{2.5}^{4} (x-1)(\frac{1}{2}x - 1 - x^2 + 6x - 9)\, dx \;=\; \frac{81}{32}\pi$$