Renormalization of three dimensional Hénon map

A Dissertation Presented

by

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to

The Graduate School

in Partial Fulfillment of the Requirements

for the Degree of

Doctor of Philosophy

in

Mathematics

Stony Brook University

December 2011

Stony Brook University

The Graduate School

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Variety of research and managerial roles
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This dissertation is accepted by the Graduate School.

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Abstract of the Dissertation

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Stony Brook University

2011

The three dimensional Hénon-like map

$$F(w) = (f(x) - \varepsilon(w),\, x,\, \delta(w))$$

on $\mathbb{R}^3$ is defined in three dimensional space. The geometric properties of the Cantor attractor, $\mathcal{O}_F$ is studied for the map, $F \in \mathcal{I}_B(\bar{\varepsilon})$ in the set of infinitely renormalizable maps. The n^{th} renormalized map, R^nF has universal asymptotic behavior. For example, Jacobian determinant of R^nF is as follows

$$\operatorname{Jac} R^nF = b^{2^n} a(x)(1 + O(\rho^n))$$

with the average Jacobian $b = b_F$.

Let $\mathcal{M}$ be set of model maps which satisfies $\varepsilon(x,y,z) \equiv \varepsilon(x,y)$. Then $\mathcal{M}$ is an invariant class under renormalization. Moreover, for the maps in $\mathcal{M}$ and a perturbation F with the small enough $\| \partial_z \varepsilon \|$, C^r invariant surfaces under $R^n F$ exist. By the C^r conjugation, the renormalization of two dimensional C^r Hénon-like maps is constructed. The geometric properties of Cantor attractor, for instance, non rigidity, typical unbounded geometry of $\mathcal{O}_F$ and discontinuity of invariant line field on $\mathcal{O}_F$ are involved with two dimensional C^r Hénon-like maps. Moreover, another subclass, $\mathcal{N}$ of $\mathcal{I}_B(\bar{\varepsilon})$ is considered, which is invariant under renormalization satisfying the following condition.

$$\partial_y \delta \circ F(w) + \partial_z \delta \circ F(w) \cdot \partial_x \delta(w) \equiv 0$$

In contrast with the map in $\mathcal{M}$, the two dimensional Hénon renormalization theory is not applied in the class $\mathcal{N}$, but the recursive formula of scaling maps is analyzed directly to study the geometry of the Cantor attractor. However, the same geometric properties of Cantor set, in particular, the non rigidity of $\mathcal{O}_F$ and typical unbounded geometry of $\mathcal{O}_F$ are also proved.

To my family

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Acknowledgements

I am very happy with the opportunity to express my gratitude to people who helped me to complete my thesis. First of all, I would like to thank my advisor, Marco Martens. This thesis could not be literally completed without his help. Then I would like to describe my impression to him.

During the work for the thesis, he gave me many invaluable suggestions and constantly encouraged me. From beginning to the advanced level of thesis topic, he answered the repeated questions or described the answers in a few different situations. I was deeply impressed with his endurance as well as his kindness when I discussed with him. Moreover, sometimes he spent the more time than that was expected in order to suggest ideas, calculate estimation for some quantities or read the manuscripts critically. Also he delighted the progressed work about thesis topic with me. Then his unselfish enthusiasm for the study revived that of me through the work with him.

The years with the professor Marco Martens was very happy. I deeply appreciate all his instruction and concern about my study.

I also would like to thank the committee members of my thesis defense, professor Misha Lyubich, Ranaan Schul and Charles Tresser. Misha Lyubich also suggested valuable comments about the further academic topics. I also thank the professor Scott Sutherland, who showed an interest about my thesis and allowed the chance for me to introduce it.

Finally I thank my family — my wife, Ran Jong Park, my son, Dennis Nam and my daughter, Dian Nam. Especially Ran Jong devoted with the family all the time and her daily support was my great comfort to complete my work. She also shared every delight and concern with me and always wished my success.

I also thank my mother, Yeong Ja Kim and younger brother, Young Taek Nam in Korea. Their support enabled me to study in United States. I am very grateful to them who always encouraged me and showed me the family love and belief.

Chapter 1

Introduction

The universality of one dimensional dynamical system was discovered by Feigenbaum and independently by Coullet and Tresser in the mid 1970's. Moreover, the universality of the higher dimensional maps is conjectured by Coullet and Tresser in [CT]. This topic which is about the transition from regular dynamics to the chaotic one has been especially the central theme of the one dimensional dynamics for last 30 years or even longer. The study of the universality and rigidity is essentially related to the study of the corresponding renormalization operator. The hyperbolicity at the fixed point of the renormalization operator is finally proved in the one dimensional holomorphic dynamical systems by Lyubich in [Lyu] using quadratic-like maps in the holomorphic germs. This hyperbolicity theorem is extended to the C^r renormalizable interval maps for $r \geq 3 + \alpha$ where α is close to one in [dFdMP]. The similar universality properties are expected in higher dimensional maps which are strongly dissipative and close to the one dimensional maps. In particular, renormalizable maps with *periodic doubling type* are interesting in two or higher dimensional maps. The universality of two dimensional strongly dissipative infinitely renormalizable Hénon-like maps is justified in [CLM] and the topological properties of the invariant attractors is explored in subsequent paper, [LM]. The Cantor attractor of two dimensional Hénon maps is the counterpart of that of one dimensional maps but it has different small scale geometric properties. The Cantor attractor of the two dimensional maps have non rigidity and typically unbounded geometry. These geometric properties of the two dimensional map is generalized in the highly dissipative three dimensional Hénon family.

1.1 Renormalization of unimodal maps

In the one dimensional map in the interval, the renormalizability is defined as follows in general.

Definition 1.1.1. Let f be the unimodal map on the interval I and c be the critical point of f. If there exists a proper subinterval J of I such that $c \in J$ and $f^n|_J \subset J$ for some $n \geq 2$ and $f^i(J) \cap J = \varnothing$ for all positive $i < n$, then we call f is *renormalizable.*

The periodic doubling renormalization operator was introduced to study the small scale geometry of the attractor of the family of unimodal maps with the single critical point c which is quadratic, that is, $f''(c) \neq 0$. For example, the family of quadratic map with parameter λ, $x \mapsto \lambda x(1-x)$ can be considered. Let us define the *periodic doubling* renormalization operator of the one dimensional map, f on the interval, I.

Definition 1.1.2 (Renormalization of periodic doubling type)**.** f is renormalizable if it has two disjoint subintervals which are exchanged by f.

Let the two smallest disjoint intervals which are exchanged by f be $\mathcal{C}_1 = \{I^1_0, I^1_1\}$ where I^1_0 contains the critical point c and I^1_1 contains the critical value v. The rescaled map of the first return map

$$f^2 : I^1_0 \to I^1_0$$

with affine conjugation defines the renormalization operator R_c. Similarly. the operator R_v is defined on I^1_1. If f is infinitely renormalizable, then the n^{th} renormalized map of f, $R^n f$ has the cycle of the pairwise disjoint intervals

$$\mathcal{C}_n = \{I^n_i \mid i = 0, 1, 2, \ldots, 2^n - 1\}$$

where $f(I^n_i) = I^n_{i+1}$ and

$$\bigcup \mathcal{C}_{n+1} \subset \bigcup \mathcal{C}_n.$$

The nested sequence of $\mathcal{C}_n$ implies the Cantor set is the attractor of f.

$$\mathcal{C} = \bigcap \bigcup \mathcal{C}_n$$

The topological properties of unimodal maps with Cantor attractor is deeply affected by the orbit of the critical point, which lead to the kneading sequence. If the given map f is infinitely renormalizable, then f acts on the dyadic adding machine on this attractor. For the introduction of dyadic adding machine and kneading sequence, see [BB]. The universality of renormalizable map says the small scale geometry of two maps has asymptotically same around the renormalization fixed point. The rigidity means that if two infinitely renormalizable maps, f and g are conjugated by a homeomorphism, h on the domain of two maps, that is,

$$h \circ f = g \circ h$$

then h is differentiable on the Cantor attractor. Moreover, de Melo and Pinto proved one dimensional infinitely renormalizable maps have the rigidity in [dMP].

The *topology* of the dynamical system implies the *geometry* of it.

1.2 Hénon maps and bifurcation of the homoclinic tangency

The Hénon map is a polynomial diffeomorphism from $\mathbb{R}^2$ to itself as follows.

$$H_{a,b}(x, y) = (1 - ax^2 + y,\ bx)$$

Hénon introduced this above map on 1974 and there were numerical experiments about it. A famous conjecture is there exists the *strange attractor* at the parameter $a = 1.4$ and $b = 0.3$. The first significant achievement about the Hénon map with parameter space (a, b) was done by Benedics and Carleson in [BC]. There exists the strange attractor for the positive measure of the parameter space, (a, b) such that $a_0 < a < 2$ and $b < b_0$ where a_0 is close to 2 and b_0 is small. Moreover, this parameter values which are considered in [BC] is a generalization of the one dimensional Misiuriewicz maps in [Jak]. Jakobson proved that the maps which have absolutely continuous invariant measure with respect to Lebegue measure has positive measure on the parameter space. So is in the Hénon family in [BC]. Young and Wang use the geometric condition to generalize Hénon family in [WY1]. Furthermore, it is generalized to arbitrary finite dimension of the rank one attractor, that is, attractor with the neutral or repulsive direction is one dimensional in [WY2]. The statistical properties, for instance, the existence of SRB measure on the invariant set, are important in the generic dynamics of the chaotic region.

The hyperbolic systems have been studied from 1960s. Moreover, those systems were expected to be generic in the whole dynamical systems. This generic hyperbolicity is the main conjecture of the rational maps on the Riemann sphere. In the quadratic polynomial case, it is known that MLC (Mandelbrot set is locally connected) conjecture is equivalent to the generic hyperbolicity. However, in the dynamical systems of two or higher dimensional maps Newhouse proved that the maps in the chaotic region contains an open set. The proof used a perturbation of homoclinic tangency with certain condition of the invariant Cantor set. After Newhouse proof, Palis suggested a new conjecture about the generic dynamical system at p.134 in [PT].

Conjecture 1.2.1 (Palis conjecture)**.** Every C^r diffeomorphism in Diff (M) for $r \geq 1$ can be approximated by a hyperbolic diffeomorphism or else by one exhibiting a homoclinic bifurcation involving a homoclinic tangency or a cycle of hyperbolic periodic saddles with different indices.

Let us consider a homoclinic tangency of the two dimensional maps. Then the dimension of both unstable and stable manifolds at the homoclinic point is one. After bifurcation of the homoclinic tangency, let us consider the case that stable and unstable manifold is (transversally) intersected around the homoclinic point. Let us choose the bounded region on which this bifurcation occurs and consider the first return map, H. Then after appropriate smooth coordinate change, the image of the horizontal lines in the bounded region is the vertical line. Then the simplest example of map of this form is the Hénon map. [1]

$$H_{a,\,b}(x, y) = (x^2 - a + by,\ x).$$

However, in general the first coordinate map of the first return map is not generally a polynomial but is a perturbation of a unimodal one dimensional map, say $f(x)$. Then we call the first return map which is of the following form

$$F(x, y) = (f(x) - \varepsilon(x, y),\ x)$$

the Hénon-like map where $f(x)$ is a unimodal map.

A perturbation of homoclinic tangency could occur in higher dimension, that is, in arbitrary (finite) dimension, the unstable or stable manifold at the homoclinic point may have its dimension greater than one. However, in order to make that the first return map has Hénon-like form in the first two coordinates, let us assume that dimension of unstable manifold at the homoclinic point is one in higher dimension. Then for example, in the three dimensional space we get

$$F(x, y, z) = (f(x) - \varepsilon(x, y, z),\ x,\ \bullet\,)$$

[1] The Hénon map in dynamical system means that the family of Hénon maps up to the linear conjugacy. The parameter space $\{(a, b)\}$ of each expression is also changed by the same linear conjugacy.

where $f(x)$ is a unimodal map. Then the first return map in higher dimension has first two coordinates similar to the Hénon-like map. If the unstable manifold of a fixed point is the attractor which is maximal backward invariant, then it is called *rank one attractor*. This viewpoint is reflected in the paper of Wang and Young, [WY2] in higher dimension for the maps on the chaotic region, maps with positive entropy. Hénon renormalization of two dimensional Hénon-like maps is defined on [CLM] for the maps on the regular region, namely, maps with the entropy zero.

1.3 Statement of results

Hénon renormalization of two dimensional map has common and different properties of the one dimensional renormalizable maps. Dynamical system of two dimensional Hénon-like map has *universality but non-rigidity*.

We expect that the three or higher dimensional system has the above properties. The extension of Hénon renormalization theory in three or higher dimension has the two goals in general.

- Finding the same or similar results of two dimensional theory in three dimension.
- Finding the new phenomena which appear only on the three or higher dimensional maps.

In this paper, we explore three dimensional Hénon-like maps for the first part of the general goals. In particular, it is shown that the small scale geometry of the Cantor attractor for three dimensional Hénon-like maps has the same properties for two dimensional Hénon-like maps.

The three dimensional Hénon-like map F from the cubic box B to $\mathbb{R}^3$ is defined as follows

$$F : (x, y, z) \mapsto (f(x) - \varepsilon(x, y, z),\ x,\ \delta(x, y, z))$$

where $f(x)$ is a unimodal map. Let us assume that $\|\,\varepsilon\|_{C^3}, \|\,\delta\|_{C^3} \leq \bar{\varepsilon}$ are sufficiently small $\bar{\varepsilon} > 0$. We would call *three dimensional Hénon-like maps* just Hénon-like maps unless the name could make confusion between two and three dimensional maps.

F has two hyperbolic fixed points, β_0 which has positive eigenvalues and β_1 which has both positive and negative eigenvalues. Since $\|\,\delta\|$ is sufficiently small, each fixed point has only one expanding direction and we may assume that product of two different eigenvalues is strictly less than one. The Hénon-like map is called *renormalizable* if $W^u(\beta_0)$ intersects $W^s(\beta_1)$ at the orbit of a single point. However, the renormalizable map (with periodic doubling type) has the invariant domain in B under F^2.

We need the non linear scaling map for universal limit of the renormalized map. For this, let us define the horizontal-like diffeomorphism H is defined as follows also.

$$H : (x, y, z) \mapsto (f(x) - \varepsilon(x, y, z),\, y,\, z - \delta(y, f^{-1}(y), 0))$$

The renormalized map RF of the three dimensional Hénon-like map F is defined as

$$RF = \Lambda \circ H \circ F^2 \circ H^{-1} \circ \Lambda^{-1}$$

where H is the horizontal-like diffeomorphism and Λ is linear scaling map.

Moreover, the n^{th} renormalization $R^n F$ is defined inductively. Assume that F is an infinitely renormalizable perturbed Hénon-like map. Then $R^n F$ converges to the degenerate map $F_* = (f_*(x),\, x,\, 0)$ where f_* is the fixed point of the renormalization operator of one dimensional unimodal maps. Furthermore, F_* is the hyperbolic fixed point of the renormalization operator, $R : F \mapsto RF$. Then we extend the renormalization theory of two dimensional Hénon-like maps to the three dimensional maps. On the remainder of this introduction we assume that F is three dimensional infinitely renormalizable analytic map.

Assume that F is renormalizable. Let the scaling map $\psi^1_v \equiv H^{-1} \circ \Lambda^{-1}$ and denote $\psi^1_c \equiv F \circ \psi^1_v$. Moreover, if F is twice renormalizable, then let ψ^2_v and ψ^2_c be the corresponding coordinate change maps for second renormalization. The composition of scaling maps are expressed as follows.

$$\Psi^2_{vc} = \psi^1_v \circ \psi^2_c, \quad \Psi^2_{cv} = \psi^1_v \circ \psi^2_c, \quad \Psi^2_{vv} = \psi^1_v \circ \psi^2_v$$

In general, we define the coordinate change map as the conjugation between F^{2^n} and $R^n F$ as follows

$$\Psi^n_{\mathbf{w}} = \psi^1_{w_1} \circ \psi^2_{w_2} \circ \cdots \circ \psi^n_{w_n}$$

where $\mathbf{w} = (w_1\, w_2\, \ldots w_n) \in \{v, c\}^n$ is a word of length n. Moreover, the set $B^n_{\mathbf{w}}$ is defined as $\Psi^n_{\mathbf{w}}(B)$.

The critical Cantor set is defined

$$\mathcal{O}_F = \bigcap_{n \geq 1}^{\infty} \bigcup_{\mathbf{w} \in W^n} B^n_{\mathbf{w}}$$

where $\mathbf{w} \in W^n$ is the word of the Cartesian product of $\{v, c\}$. The counter part of the critical value of one dimensional map is called the *tip*

$$\{\tau_F\} \equiv \bigcap_{n \geq 1}^{\infty} B^n_{\mathbf{v}}$$

where $\mathbf{v} = v^k$. Moreover, F acts as the dyadic adding machine on $\mathcal{O}_F$. The average Jacobian is defined on the critical Cantor set

$$b_F = \exp \int_{\mathcal{O}_F} \log \operatorname{Jac} F \, d\mu$$

where μ is the unique ergodic measure on $\mathcal{O}_F$.

With the above definitions, the Jacobian determinant of $R^n F$ has the universal limit $a(x)$ with exponential convergence.

Theorem 1.3.1 (Universality of $\operatorname{Jac} R^n F$)**.** *Let $F \in \mathcal{I}_B(\bar{\varepsilon})$ for sufficiently small $\bar{\varepsilon} > 0$.*

$$\operatorname{Jac} R^n F = b^{2^n} a(x)(1 + O(\rho^n))$$

where $b = b_F$ is the average Jacobian of F, $a(x)$ is the universal function and $\rho \in (0, 1)$.

The number $\log b_F$ is the sum of the Lyapunov exponents on the Cantor set, $\mathcal{O}_F$. The maximal exponent is zero. However, in contrast with two dimensional maps, $\log b_F$ is the sum of two exponents, that is, $\log b_F = \log b_1 + \log b_2$. Furthermore, the universality of the Jacobian determinant does not seem to imply the universality of the map $R^n F$ because the Jacobian determinant,

$$\operatorname{Jac} R^n F = \partial_y \varepsilon_n \cdot \partial_z \delta_n - \partial_z \varepsilon_n \cdot \partial_y \delta_n$$

has four different partial derivatives. In general, the asymptotic expression of all of these cannot be recovered using only the single number b_F and the universal function. Then instead of constructing the universal geometric theory of the invariant set of the three dimensional maps in $\mathcal{I}_B(\bar{\varepsilon})$, let us take subset of $\mathcal{I}_B(\bar{\varepsilon})$ as invariant classes under renormalization and construct the geometric properties of Cantor attractor.

Let Hénon-like maps with the condition $\partial_z \varepsilon \equiv 0$ be the *model maps* and denote it to be F_{mod}. Then the universality of the model map is re-constructed using the universality of two dimensional Hénon-like maps.

$$F_{\text{mod},\, n} \equiv R^n F_{\text{mod}} = (f_n(x) + b_1^{2^n} a(x)\, y\, (1 + O(\rho^n)),\; x,\; b_2^{2^n} z + \widetilde{\delta}_n(x, y))$$

where f_n is the unimodal map converging to f_* exponentially fast as $n \to \infty$, $b_1 b_2 = b_F$ and $\| \widetilde{\delta}_n \| = O(\bar{\varepsilon}^{2^n})$ with sufficiently small $\bar{\varepsilon} > 0$. In the class of model maps, b_1 is actually the average Jacobian of the two dimensional Hénon-like map and b_2 is the attracting rate which comes from the third coordinate direction.

Let us assume that $b_2 \ll b_1$ on the class of model map. Then there exists an invariant cone field on any given compact invariant set because of the universality theorem of two dimensional Hénon-like maps. Then there exists the continuous plane field on the global attractor

$$\mathcal{A}_F = \bigcap_{k \geq 0} F^k(B) \cap B.$$

The complementary invariant line field is the set of straight lines which are perpendicular to $xy-$plane. Furthermore, Hénon-like map, F which is close enough to model maps in the C^1 sense also has an invariant cone field under DF. Then The map F is called *a small perturbation of the model map* F_{mod} where $\varepsilon(x,y,z) = \varepsilon(x,y) + \widetilde{\varepsilon}(x,y,z)$ and $\|\partial_z \varepsilon\|$ is small enough.

With the existence of the invariant plane and line fields, the pseudo unstable manifold theorem says the existence of the local invariant C^r surfaces with $3 \leq r < \infty$ at the small neighborhood of $\mathcal{A}_F$. Furthermore, there exists a single invariant surface Q under F such that it contains $\mathcal{A}_{F^{2^n}}$ in $B^n_{v^n}$ for each sufficiently big $n \in \mathbb{N}$ (Lemma 10.2.1). Additionally if F is infinitely renormalizable, then there exists an invariant surface Q_n under $R^n F$ as the graph of C^r map, ξ_n from $xy-$plane to $z-$axis (Lemma 10.3.1).

Then two dimensional C^r Hénon-like map is defined as follows

$$F_{2d,\,\xi}(x,y) = (f(x) - \varepsilon(x,y,\xi),\, x)$$

where graph(ξ) is a C^r invariant surface of the three dimensional Hénon-like map $F \colon (x,y,z) \mapsto (f(x) - \varepsilon(x,y,z),\, x,\, \delta(x,y,z))$. The C^r diffeomorphism from the invariant surface to $xy-$plane, $\pi^{\xi_n}_{xy,\,n} \colon (x,y,\xi_n) \mapsto (x,y)$ on each level $n \in \mathbb{N}$ define the renormalization of C^r Hénon-like maps, $R^n F_{2d,\,\xi}$ on $xy-$plane which is same as the renormalization using the horizontal diffeomorphism and dilation

$$R^n F_{2d,\,\xi}(x,y) = (f_n(x) - \varepsilon_n(x,y,\xi_n),\, x). \tag{1.3.1}$$

Similarly, non linear scaling map between k^{th} and n^{th} renormalized Hénon-like maps is defined as follows

$${}_{2d}\Psi^n_{k,\,\xi} \equiv \pi^{\xi_k}_{xy,\,k} \circ \Psi^n_k \circ (\pi^{\xi_n}_{xy,\,n})^{-1}.$$

The properties of invariant surfaces under R^nF, the universality theorem of infinitely renormalizable C^r Hénon-like maps are obtained (Theorem 11.1.3).

Theorem 1.3.2 (Universality of C^r Hénon-like maps with C^r conjugation for $3 \leq r < \infty$)**.** *Let Hénon-like map $F_{2d,\,\xi}$ be the C^r map for some $3 \leq r < \infty$ which is defined in (1.3.1). Suppose that $F_{2d,\,\xi}$ is infinitely renormalizable. Then*

$$R^nF_{2d,\,\xi} = (f_n(x) - b_{1,\,2d}^{2^n}\,a(x)\,y\,(1 + O(\rho^n)),\ x)$$

where $f_n(x)$ is the unimodal map which converges to $f_*(x)$ exponentially fast as $n \to \infty$ for some $0 < \rho < 1$.

Moreover, the asymptotic expression of the scaling map ${}_{2d}\Psi_k^n$ has the similar expression of the analytic two dimensional Hénon-like maps (Theorem 11.1.4).

The dynamical properties on Cantor attractor of Hénon-like maps depend much on the asymptotic expression of the renormalized map and that of scaling maps. $R^nF_{2d,\xi}$ and ${}_{2d}\Psi_k^n$ for C^r Hénon-like maps has the asymptotic expressions similar to the analytic Hénon-like maps in [CLM]. The geometric properties of the Cantor attractor of C^r Hénon-like map is the same as that of Cantor attractor for analytic two dimensional Hénon-like maps.

For example, the Cantor attractor of C^r Hénon-like map also has the geometric properties, in particular, discontinuity of the invariant line field (Theorem 11.2.2), non-rigidity (Theorem 11.3.2) and typical unbounded geometry (Theorem 11.4.3). Moreover, all of these dynamical properties are transferred to the Cantor attractor of three dimensional Hénon-like map F through its invariant surfaces.

Let us see the Non rigidity theorem below.

Theorem 1.3.3. *Let $F, \widetilde{F} \in \mathcal{I}_B(\bar{\varepsilon})$ be small perturbation of model maps. Suppose that $b_2 \ll b_1$ and $\widetilde{b}_2 \ll \widetilde{b}_1$. Suppose also that each of F and $\widetilde{F}$ has invariant C^r surfaces which contains the global attracting set. Let $\mathcal{O}_F$ and $\mathcal{O}_{\widetilde{F}}$ be the critical Cantor set of F and $\widetilde{F}$ respectively. Let ϕ be a homeomorphism between $\mathcal{O}_F$ and $\mathcal{O}_{\widetilde{F}}$ with $\phi_{2d}(\tau_{\widetilde{F}}) = \tau_F$. Assume that $b_1 > \widetilde{b}_1$. Then the Hölder exponent α of ϕ_{2d} satisfies the following.*

$$\alpha \leq \frac{1}{2}\left(1 + \frac{\log b_1}{\log \widetilde{b}_1}\right)$$

There is another subspace of $\mathcal{I}_B(\bar{\varepsilon})$ invariant under renormalization. For the renormalizable maps, the recursive equation of each partial derivatives of δ and

δ_1 which are third coordinate maps of F and RF respectively. For example, let us consider the recursive formula of $\partial_z \delta_1$.

$$\begin{aligned}\partial_z \delta_1(w) = &\boxed{\left[\partial_y \delta \circ (F \circ H^{-1}(\sigma_0 w)) + \partial_z \delta \circ (F \circ H^{-1}(\sigma_0 w)) \cdot \partial_x \delta \circ H^{-1}(\sigma_0 w)\right]} \\ &\cdot \partial_z \phi^{-1}(\sigma_0 w) + \partial_z \delta \circ (F \circ H^{-1}(\sigma_0 w)) \cdot \partial_z \delta \circ H^{-1}(\sigma_0 w)\end{aligned}$$

where $\phi^{-1}(w)$ is the first coordinate map of $H^{-1}(w)$. The part in the box of the above equation also appears on the recursive equation for $\partial_y \delta_1$ and $\partial_x \delta_1$. Thus we can let this common part be the identically zero and consider the set of Hénon-like maps which satisfies the following equation.

$$\partial_y \delta \circ F(w) + \partial_z \delta \circ F(w) \cdot \partial_x \delta(w) \equiv 0 \tag{1.3.2}$$

where $w \in \psi_v^1(B) \cup \psi_c^1(B)$. Let the set of Hénon-like maps satisfying above equation be $\mathcal{N}$.

Theorem 1.3.4. *Let the set of Hénon-like maps which satisfies (1.3.2) be $\mathcal{N}$. Then the set $\mathcal{N} \cap \mathcal{I}_B(\bar{\varepsilon})$ is invariant under renormalization.*

Moreover, for the map, $R^n F \in \mathcal{N} \cap \mathcal{I}_B(\bar{\varepsilon})$, we obtain the universal expression of $\partial_z \delta_n$ by Proposition 13.2.1. By the chain rule, the recursive formula of $\partial_z \delta_n$ is as follows by (13.2.1) and induction.

$$\begin{aligned}\partial_z \delta_n(w) = &\ \partial_z \delta_{n-1} \circ (F_{n-1} \circ H_{n-1}^{-1}(\sigma_{n-1} w)) \cdot \partial_z \delta_{n-1} \circ H_{n-1}^{-1}(\sigma_{n-1} w) \\ = &\ \partial_z \delta_{n-1} \circ \psi_c^n(w) \cdot \partial_z \delta_{n-1} \circ \psi_v^n(w) \\ = &\ \prod_{\mathbf{w} \in W^n} \partial_z \delta \circ \Psi_{\mathbf{w}}^n(w)\end{aligned}$$

The logarithmic average of the right hand side converges a definite number as $n \to \infty$.

$$\frac{1}{2^n} \sum_{\mathbf{w} \in W^n} \log |\, \partial_z \delta \circ \Psi_{\mathbf{w}}^n(w)| \longrightarrow \int_{\mathcal{O}_F} \log |\partial_z \delta| \; d\mu$$

Define this limit as $\log b_2$. Then $\partial_z \delta_n = b_2^{2^n}(1 + O(\rho^n))$ and it means the contracting rate from the third coordinate map. Moreover, the ratio of b_2 and the average Jacobian b is defined b_1, [2]that is, $b = b_1 b_2$.

Then the coordinate change map Ψ_k^n of the three dimensional Hénon-like maps

[2]For the Hénon-like map $F \in \mathcal{I}_B(\bar{\varepsilon}) \cap \mathcal{N}$, it is not clear both $\log b_1$ and $\log b_2$ are Lyapunov exponents on the Cantor attractor.

is analyzed by asymptotic expression related to $\partial_y \delta_k$ and $\partial_y \varepsilon_k$. Then when the distance of image of two points under Ψ_k^n is measured, the distance of $z-$coordinate of points is incorporated to the two dimensional distance as the product $b_1^{2^k}$ and dilations. In the counter part of two dimensional Hénon-like map, the contracting of the $y-$distance is stronger than $x-$distance contraction with the factor b_1. Then if $F \in \mathcal{N} \cap \mathcal{I}_B(\bar{\varepsilon})$, then the method of measuring distances is essentially same as two dimensional maps. Furthermore, the affection b_2 is not visible even if b_2 is larger than b_1.

Then non-rigidity and the typical unbounded geometry are proved in the space $\mathcal{N} \cap \mathcal{I}_B(\bar{\varepsilon})$ but the method used in this space is an analysis of the recursive equations. Then this method is very different from that in a small perturbation of model maps. In this space the constructed two dimensional Hénon renormalization with invariant surfaces is applied to the three dimensional maps.

1.4 An open problem

We have seen that two or higher dimensional Hénon renormalizable maps have universality but non-rigidity. In the two dimensional Hénon renormalization theory, the different average Jacobians separate one smooth invariant class from another. Then the question about rigidity in the set of maps with same average Jacobian arises. It is suggested in [CLM] as an open problem. In similar way, the Hénon-like maps in three dimension has the question about smooth invariant class with the same two contracting rates, b_1 and b_2.

- If the two different three dimensional Hénon-like maps in $\mathcal{I}_B(\bar{\varepsilon})$ have same b_1 and b_2, then are these maps conjugated by C^1 or smoother map?

If the Hénon-like maps in $\mathcal{I}_B(\bar{\varepsilon})$ is of the form

$$(x, y, z) \mapsto (f(x) - \varepsilon(x, y),\, x,\, b_2 z),$$

then the above question is the same as rigidity question about two dimensional Hénon-like maps. This question for the the three dimensional Hénon-like maps in the whole class $\mathcal{I}_B(\bar{\varepsilon})$ may be difficult. Then we can restrict our attention to the space of a small perturbation of model maps or the space $\mathcal{N}$.

Chapter 2

Notations and conventions

For given map F, if the set A is related to F, then we denote it to be $A(F)$ or A_F and the F can be skipped if there is no confusion without F. The domain of the function F is denoted to $\text{Dom}(F)$ and the image of the set B under a function F is denoted by $F(B)$. If $F(B) \subset B$ then we call B is an (forward) invariant set under F. Similarly, if $F^{-1}(B) \subset B$, then we call B is an backward invariant set under F.

Let $\mathbb{N}$ be the set of the natural numbers, $\{1, 2, 3, \ldots\}$ and $\mathbb{N}_+ = \mathbb{N} \cup \{0\}$. Let the distance between two points p and q be on the metric space X be $\text{dist}_X(p, q)$. However, let us call the set distance $\text{dist}_{\min}(R, S)$ as the minimal distance between two sets, R and S as follows.

$$\text{dist}_{\min}(R, S) = \inf \{ \ \text{dist}(r, s) \text{ for all } r \in R \text{ and } s \in S \ \}$$

Let $f : X \to X$ is a continuous function on the metric space X. The stable manifold at some point p under f as follows.

$$W^s(p) = \{q \in X \mid \text{dist}(f^n(p), f^n(q)) \to 0 \text{ as } n \to \infty\}$$

The local stable manifold at p bounded by $\varepsilon' > 0$ is

$$W^s_{\varepsilon'}(p) = \{q \in X \mid \text{dist}(f^n(p), f^n(q)) \leq \varepsilon' \text{ for all } n \in \mathbb{N} \cup \{0\}\}$$

where dist is the distance along stable manifold. The (local) unstable manifold is defines as the set if the distance under f^{-n} is used instead of f^n. Without specified size of the local manifold, we denote the local stable manifold at the point p to be $W^s_{loc}(p)$ where p is on a certain bounded neighborhood which is connected on X.

If the unstable manifold is one dimensional, then we can express the curve connecting two points along the unstable manifold in the give space X is following.

$$[p, q]^u_w \subset W^u(w)$$

The square bracket means the given set $[p, q]^u_w$ is homeomorphic image of the closed interval $[-1, 1]$ under continuous map from $\mathbb{R}$ to X. The points p and q are the end points of the curve.

Denote the set of periodic points of F to be Per_F. The orbit of the point w under the map f is denoted to be $\mathrm{Orb}(w, f)$. We can express the (complete) orbit of w to be $\mathrm{Orb}(w)$ unless the map is emphasized or is ambiguous on the context in the related description. The omega limit set of a point x under the map F, $\omega(x)$ is the set of accumulation points of the forward orbit under F. Similarly the alpha limit set, $\alpha(x)$ of x under F is the set of accumulation points of the backward orbit under F. Thus

$$\omega(x) = \bigcap_{n\in\mathbb{N}} \overline{\{F^k(x) : k > n\}}, \qquad \alpha(x) = \bigcap_{n\in\mathbb{N}} \overline{\{F^{-k}(x) : k > n\}}$$

If there exists a neighborhood U of x and $N \geq 0$ such that

$$F^n(U) \cap U = \varnothing$$

for all $n \geq N$, then x is called a wandering point. If x is not a wandering point, then it is called nonwandering point and the set of nonwandering point, Ω_F is called *non wandering set*.

For three dimensional map, let us the projection from $\mathbb{R}^3$ to its $x-$axis, $y-$axis and $z-$axis be π_x, π_y and π_z respectively. Moreover, the projection from $\mathbb{R}^3$ to $xy-$plane be π_{xy} and so on. Furthermore, if there exists a surface which is embedded on $\mathbb{R}^3$ as the graph of the function ξ, for example $\{(x, y, \xi(x, y))\}$, then we define the projection from the surface to its domain, say π^ξ_{xy} as $(x, y, \xi(x, y)) \mapsto (x, y)$. Denote the partial derivatives of the function f over x, y and z to be $\partial_x f$, $\partial_y f$ and $\partial_z f$ respectively. The second partial derivatives are $\partial_{xx} f$, $\partial_{xy} f$ and so on. However, for a set S, ∂S without any subscript means the topological boundary of the set S.

$A = O(B)$ means that there exists a positive number C such that $A \leq CB$. Moreover, $A \asymp B$ means that there exists a positive number C which satisfies $\dfrac{1}{C}B \leq A \leq CB$.

Chapter 3

Preliminaries

Let us introduce two dimensional Hénon-like maps as a perturbation of one dimensional maps and define renormalization of two dimensional Hénon-like map. Many topological properties of two dimensional renormalizable Hénon-like map are well adapted to the three dimensional Hénon-like maps.

3.1 Hénon-like map as a perturbation of one dimensional map

Let $f : I \to I$ be a C^3 or smoother unimodal map with non-degenerate critical point $c \in I$ and f's Schwarzian derivative is negative on I. f is called (periodic doubling) *renormalizable* map if there exists the closed interval $c \in J \subset \mathrm{Int}I$ such that $J \cap f(J) = \varnothing$ and $f^2(J) \subset J$, that is, J is invariant under f^2. Then $f^2 : J \to J$ is also a unimodal map on J. We can choose the minimal intervals $J_c = [f^4(c), f^2(c)]$ and $J_v = [f^3(c), f(c)]$ which is invariant under f^2. Moreover, J_c and J_v are disjoint from each other. By the conjugation of the affine rescaling from J to I, we can define *renormalization* $R_c f$ at the critical point as $R_c f$ is defined as $sf^2(s^{-1}x)$ for some $s < -1$. The domain of the renormalizable map f, $I = [f^4(c), f(c)]$ contains the critical point, the critical value and one repelling fixed point whose eigenvalue is negative, say β_1. Without loss of generality we may assume that f can be extend on a sufficiently bigger symmetric interval at the origin which has another fixed point β_0 with positive eigenvalue such that the interval, $[-|f(c)|, |f(c)|\,]$ is compactly contained on this extended interval. Let us say this extended interval of I to be also I in order to save the notation. Then f has another repelling fixed point on the (extended) interval I whose eigenvalue is positive, say β_0.

Let us f be an infinitely renormalizable map. Then there is the unique fixed point f_* of the (periodic doubling) renormalization operator R_c with the universal scaling factor $\sigma = 0.73\ldots$. The scaling factor of the n^{th} renormalization converges to σ exponentially fast as $n \to \infty$.

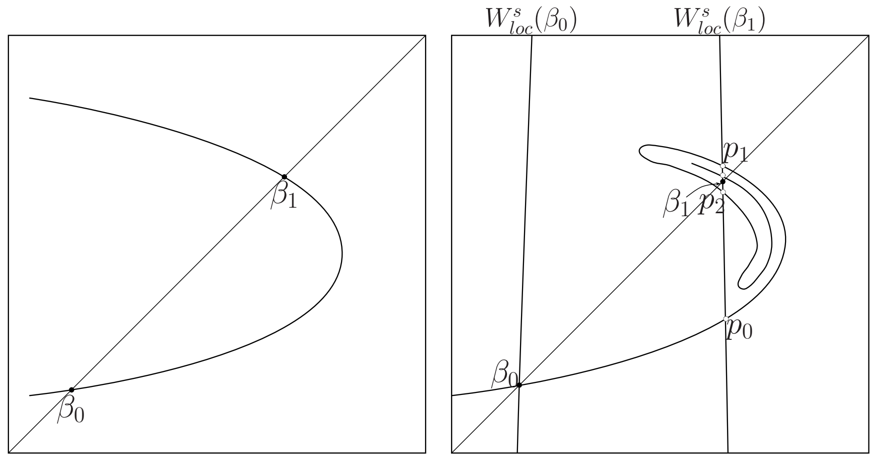
The graph of f is a parabolic-like curve. Since f is infinitely renormalizable by the assumption, the two dimensional degenerate map $F_\bullet : (x, y) \mapsto (f(x), x)$ on $I \times I$ contains the reflected image of the graph of f on the diagonal line going through the origin. Let us call two fixed points of $F_\bullet$ be β_0 and β_1 like the fixed points of the one dimensional map. Moreover, the parabolic-like curve $\{x \in I | (f(x), x)\}$ containing the fixed point β_0 is the unstable manifold of β_0, $W^u(\beta_0)$ under the degenerate map $F_\bullet$.

Let B be the square region whose center is the origin, that is $B = I^h \times I^v$ where I^h and I^v are the (appropriately extended) symmetric intervals at zero of the one-dimensional renormalizable map f. I^h and I^v mean that they are parallel to x-axis and y-axis respectively. The map $F : B \longrightarrow \mathbb{R}^2$ is called *Hénon-like* map if the image of the vertical line is a horizontal line and the image of the horizontal line is the parabolic-like curve. Then as a small perturbation of the one dimensional map f, the Hénon-like map F is of the following form.

$$F(x, y) = (f(x) - \varepsilon(x, y), x)$$

If the Jacobian determinant of F is non-zero at every point, F is called the *Hénon-like diffeomorphism*. On the followings, Hénon-like map always means Hénon-like diffeomorphism unless any other statements are specified. As a (small) perturbation of the one dimensional map, we assume that the Hénon-like map F has two saddle fixed points β_0 with positive eigenvalues — *flip saddle* — and β_1 with negative eigenvalues — *regular saddle* — .

Denote the local stable manifold at w, $W^s_{loc}(w)$ to be the component of the stable manifold $W^s(w)$ which contains the point w in B and keep the similar notation for the local unstable manifold. If $|f''(x)|$ is big enough then $W^s_{loc}(\beta_1)$ and $W^u(\beta_0)$ meets transversally at least two points. Let p_0 be the farthest point from β_1 along $W^s_{loc}(\beta_1)$ which is in the intersection of $W^u(\beta_0) \cap W^s_{loc}(\beta_1)$. Moreover, let us call the second and third farthest point from β_1 along $W^s_{loc}(\beta_1)$ in $W^u(\beta_0) \cap W^s_{loc}(\beta_1)$ be p_1 and p_2 respectively. p_2 is on the opposite side to p_1 from β_1 along $W^s_{loc}(\beta_1)$ because β_1 has negative eigenvalues. Then we can define p_n similarly for every $n \in \mathbb{N}$. Then $p_n \to \beta_1$ as $n \to +\infty$.



(a) A parabolic-like curve of the degenerate map as $W^u(\beta_0)$

(b) (Un)stable manifolds of Hénon-like map

Figure 3.1.1: Unstable manifolds of a degenarate map and a Hénon-like map

3.2 Topological properties of renormalizable two dimensional Hénon-like map

The renormalization of Hénon-like map was defined on [CLM] as the following. Let us call the orientation preserving Hénon-like map is *renormalizable* if the unstable manifold of β_0, $W^u(\beta_0)$ intersects the stable manifold of β_1, $W^s(\beta_1)$, on the single orbit of the points, say $\mathrm{Orb}_{\mathbb{Z}}(w)$ for some $w \in B$. Let $p_0 \in \mathrm{Orb}_{\mathbb{Z}}(w)$ be the point which is farthest point from β_1 along the local stable manifold of β_1, $W^s_{loc}(\beta_1)$. Denote $p_k = F^k(p_0)$ for each $k \in \mathbb{Z}$. Then the forward orbit of p_0, $\mathrm{Orb}_{n\geq 0}(p_0)$ is on $W^s_{loc}(\beta_1)$ and the local stable manifold of p_{-n}, $W^s_{loc}(p_{-n})$ where $n \leq 0$ is pairwise disjoint component of $W^s(\beta_1)$ and $W^s_{loc}(p_{-n})$ converges to $W^s(\beta_1)$ because p_{-n} converges to β_0 as $n \to +\infty$.

Denote $W^s_{loc}(p_{-n})$ to be M_{-n} for every $n \geq 0$. Then $W^s_{loc}(\beta_1)$ is denoted as M_0. Moreover, we can define M_1 as the component of $W^s(\beta_1)$ whose image under F is contained on M_1 and which does not have any point of $\mathrm{Orb}_{\mathbb{Z}}(w)$. It is on the opposite side of M_{-1} from M_0. We may assume that M_1 is a curve connecting the up and down sides of the square domain B inside. Then we can easily check the curves $[p_0, p_1]^u_{\beta_0}$ and $[p_1, p_2]^u_{\beta_0}$ does not intersect M_1 and M_{-1} respectively when F is renormalizable.

On the domain B, the dynamical region for renormalizable Hénon-like maps is the closure of the component of $B \setminus W^s(\beta_0)$ containing β_1, say B_0 because it is an (forward) invariant region under F. Let each region between M_{-n} and

M_{-n+1} be A_{-n} for every $n \geq 0$. Since $F(M_{-n}) \subset M_{-n+1}$ for each $n \geq 0$, we can see $F(A_{-n}) \subset A_{-n+1}$ for each $n \geq 0$. But the image of A_0 under F is contained on A_{-1}, that is, $F(A_0) \subset A_{-1}$. In other words, $W^s_{loc}(\beta_1)$ intersects $W^u(\beta_0)$ at p_1 transversally. Since M_0 is an invariant curve under F and $F(M_1)$ is a part of M_{-1}, if we take a curve γ connecting p_1 and a point in M_1, then $F(\gamma)$ is a curve connecting a point of M_{-1} and p_2 in A_{-1}.

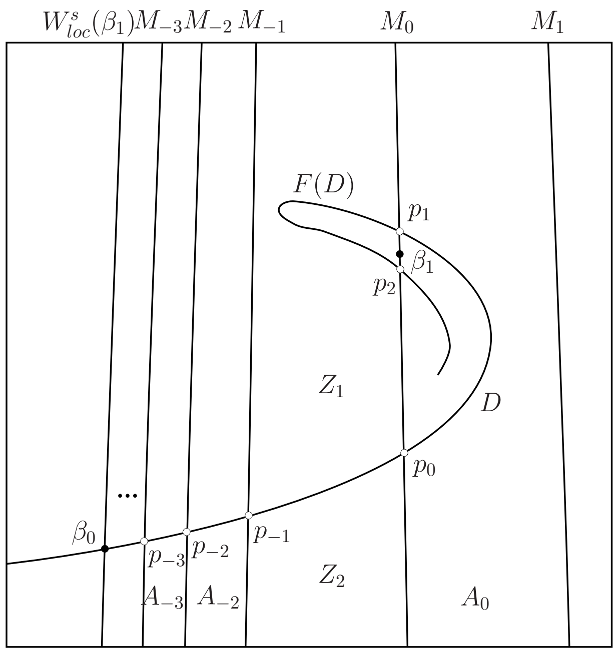


Figure 3.2.1: Regions between local stable manifolds

Let the region above the curve $[p_{-1}, p_0]^u_{\beta_0}$ in A_{-1} be Z_1 and the region below the same curve in A_{-1} be Z_2. Let the interior enclosed by two curves $[p_0, p_1]^u_{\beta_0}$ and $[p_0, p_1]^s_{\beta_1}$ be D.

Then for the renormalizable Hénon-like map, the local stable manifolds of p_{-n} and the regions A_{-n} between two successive local stable manifolds M_{-n} and M_{-n+1} have the following properties.

(1) M_0 is invariant under F.

(2) $F(M_{-n}) \subset M_{-n+1}$ for each $n \geq 0$.

(3) $F(M_1) \subset M_{-1}$.

(4) $F(A_{-n}) \subset A_{-n+1}$ for each $n \geq 1$. In particular, $F(A_{-1}) \subset A_0$.

(5) Let the region on the right side of M_1 be A_1. Then $F(A_1) \subset A_{-2}$.

(6) $F(A_0) \subset Z_1 \subset A_{-1}$.

(7) $F(Z_1) \subset D$.

(8) $W^u(\beta_0)$ intersects $W^s_{loc}(\beta_1)$ at p_0, p_1 and p_2 transversally.

(9) $F(D)$ is the interior enclosed by two curves $[p_1, p_2]^u_{\beta_0}$ and $[p_1, p_2]^s_{\beta_1}$ in A_{-1}.

Then the fact that $F(A_0) \subset Z_1 \subset A_{-1}$ and $F(Z_1) \subset D$ implies $F^2(A_0) \subset D$. Hence D is invariant under F^2 and furthermore any neighbourhood of D in A_0 is also invariant under F^2. Since $F(D)$ is also invariant under F^2, $D \cup F(D)$ is an invariant domain under F. The maximal invariant region under F is $\overline{B}_0$ — closure of the component of $B \setminus W^s(\beta_0)$ containing the fixed point β_1.

Lemma 3.2.1. *Let F be the renormalizable Hénon-like map. Then $\overline{B}_0$ is invariant under F and for every point $w \in B_0$, there exist $k \in \mathbb{N}$ such that $F^k(w) \in \overline{D}$.*

Proof. $W^s(\beta_0)$ is invariant under F and every M_{-n} for some $-n \leq -1$ are components of the stable manifold $W^s(\beta_0)$. Then we see that $F^n(M_{-n}) \subset M_0$ where $-n \leq -1$. Moreover, $F^2(M_1) \subset M_0$ because $F(M_1) \subset M_{-1} \cap \partial Z_1$. Since M_0 is the local stable manifold of the fixed point of β_1, we see that $F(M_0) \subset [p_0, p_1]^s_{\beta_1} \subset \partial D$. Then we can choose $k = n + 1$ where $-n \leq 0$ and $k = 3$ where $-n = 1$.

Now let us take a point $w \notin \bigcup_{n \leq 1} M_n$. Then it is sufficient to show that $F^k(w) \in \overline{D}$ for some $k \geq 0$. We may assume that w is contained in some region A_{-n} for some $-n \leq 1$ because each region A_{-n} is separated by M_{-n} and B_0 is the union of M_{-n} and A_{-n}. If $w \in A_{-n}$ where $-n \leq -1$, $F^{n-1}(w)$ is on A_{-1}. Let us say $w' = F^{n-1}(w)$. Then w' is contained in one of the following set — Z_1, $[p_{-1}, p_0]^u_{\beta_0}$ or Z_2. If $w' \in Z_2$, then by the property (4) of the regions between components of stable manifold of β_1, the image of w' under F^2 is in Z_1, that is, $F^2(w') \in Z_1$. However, $F(Z_1) \subset D$ and it implies $F^3(w') \in D$. Moreover, the fact that $[p_{-1}, p_0]^u_{\beta_0} \subset \partial Z_1$ implies that $F(w') \in \partial D$ for $w' \in [p_{-1}, p_0]^u_{\beta_0}$. For $n = 0$ case, we see that $F^2(A_0) \subset \overline{D}$. Hence, we can choose $k = n + 2$ for $n \geq 0$. For $n = 1$, we know that $F(A_1) \subset A_{-2}$. Then we can choose $k = 5$. □

Corollary 3.2.2. *Let F be the renormalizable Hénon-like map. Denote the region between two local stable manifolds M_0 and M_1 to be A_0. Then $F^2(A_0) \subset D$. In particular, any open neighbourhood of D in A_0 is invariant under F^2.*

3.3 Properties of renormalization operator of two dimensional Hénon-like maps

We have the invariant domain D under F^2 for periodic doubling renormalization from the previous subsection. However, F^2 is not Hénon-like map because the image of the vertical line, $\{x = const.\}$, under F^2 is not the horizontal line, $\{y = const.\}$. Then we need the non-linear coordinate change map to define renormalization of Hénon-like maps. We would call this non linear coordinate change map the horizontal diffeomorphism.

Define *horizontal diffeomorphism* H as the following.

$$H(x,y) = (f(x) - \varepsilon(x,y), y)$$

Then by the direct calculation (Lemma 3.4 in [CLM]), the map $H \circ F^2 \circ H^{-1}$ is also a Hénon-like map. It is called *pre-renormalization* of F and it is denoted to be PRF. There exists an interval V containing the critical point of f such that PRF is defined on the region $V \times I$ and it is invariant under PRF.[1] The square region with the center as the origin which is the restriction of $V \times I$ is the domain of PRF. The $\mathrm{Dom}(PRF)$ is extendible to the topological region A_1 if necessary. Moreover, the image of the $\mathrm{Dom}(PRF)$ under H^{-1} is the region whose boundaries are curves, $f(x) - \varepsilon(x,y) = const.$ and $y = const.$

Thus we define the domain of H as the region enclosed by curves $f(x) - \varepsilon(x,y) = const.$ and $y = const.$ and if this region is the minimal invariant region under F^2 then it is called B^1_v. Moreover, B^1_v is compactly contained in A_0. If the map $\varepsilon(x,y)$ is identically zero, then $H(B^1_v)$ is the square with the center origin. Furthermore, if the upper bounds of $|\varepsilon|$ are sufficiently small, then $H(B^1_v)$ is the rectangle on which the ratio of sides perpendicular to each other is $1 : 1+O(\bar{\varepsilon})$. Then the image of the slightly extended region of B^1_v under H is the square with center the origin. We would also say that this extended region to be B^1_v. Then $H(B^1_v)$ is invariant under PRF. Let us choose the expansion $\Lambda(x,y) \equiv (sx, sy)$ with some $s < -1$ such that the image of $H(B^1_v)$ under Λ is same as B. By the definition of Λ, we see that $H(B^1_v) = \Lambda^{-1}(B)$.

Define the region B^1_c to be $F(B^1_v)$. Then the map H^{-1} from $\Lambda^{-1}(B)$ to B^1_v is a horizontal map and the map $F \circ H^{-1}$ from $\Lambda^{-1}(B)$ to B^1_c is a vertical map.

[1] The closed interval V is the closure of the small neighborhood of J_c. If the rectangle $V \times I$ has the full height in B, then V contains every interval J_c of maps $x \mapsto f(x) - \varepsilon(x, y_0)$ for each $y_0 \in I^v$. If $\varepsilon \equiv 0$, then we can choose V to be J_c for $x \mapsto f(x)$. Furthermore, the rectangle $V \times I$ is contained in the region A_{-1}.

For simplicity let us denote the H and H^{-1} as the following.

$$\begin{aligned} H(x,y) &\equiv (f(x)-\varepsilon(x,y),\, y) = (\phi(x,y),\, y) \\ H^{-1}(x,y) &\equiv (\phi^{-1}(x,y),\, y) \end{aligned}$$

Then $\phi^{-1}(x,y)$ is a perturbation of the map $f^{-1}(x)$ in the two dimensional domain as if the map $\phi(x,y)$ is a perturbation of f. Moreover, $\phi^{-1} \circ H = x$. By the definition, H^{-1} is the horizontal map from $\Lambda^{-1}(B)$ to B^1_v. Similarly by the direct calculation $F \circ H^{-1}$ is a vertical map from $\Lambda^{-1}(B)$ to B^1_c. Then B^1_v is disjoint from B^1_c.

$$\begin{aligned} H^{-1} : \Lambda^{-1}(B) &\longrightarrow B^1_v, \qquad && (x,y) \mapsto (\phi^{-1}(x,y), y) \\ F \circ H^{-1} : \Lambda^{-1}(B) &\longrightarrow B^1_c, \qquad && (x,y) \mapsto (x, \phi^{-1}(x,y)) \end{aligned}$$

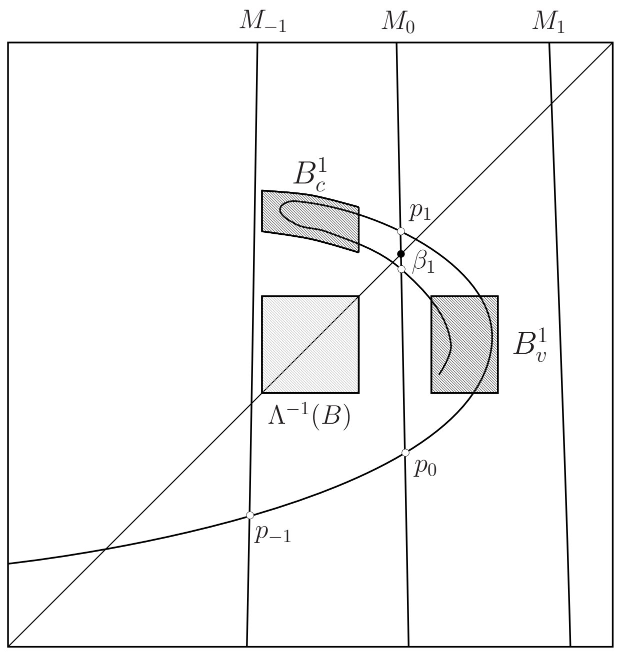


Figure 3.3.1: Restricted pieces for renormalization

Lemma 3.3.1 (Lemma 3.4 on [CLM]). *Assume that F is renormalizable and both f and ε are C^2 with the small norm of ε, $\|\varepsilon\| \leq \bar{\varepsilon}$, then*

$$H \circ F^2 \circ H^{-1} = (f_1(x) - \varepsilon_1(x,y), x)$$

for some unimodal map f_1 on V such that $\|f^2 - f_1\|_V \leq C\bar{\varepsilon}$ for some $C > 0$

and $\|\varepsilon_1\| = O(\bar{\varepsilon}^2)$.

Let us define the (first) *renormalization* of F with the appropriate scaling map, $\Lambda(x,y) = (sx, sy)$ with $s < -1$ as the following.

$$RF = \Lambda \circ H \circ F^2 \circ H^{-1} \circ \Lambda^{-1}$$

Moreover, if F is $\mathrm{n}+1$ times renormalizable, then the renormalized map $R^{n+1}F$ is defined recursively, that is, $R^{n+1}F = \Lambda_n \circ H_n \circ (R^nF)^2 \circ H_n^{-1} \circ \Lambda_n^{-1}$ where $n \geq 0$ and $R^0F \equiv F$. The map R^nF is also a Hénon-like map on the domain B.

Suppose the Hénon-like map F is an infinitely renormalizable map and let $R^nF(x,y) = (f_n(x) - \varepsilon_n(x,y), x)$. Then $\|\varepsilon_n\| = O(\bar{\varepsilon}^{2^n})$ by the above Lemma. Moreover, R^nF converges to the degenerate map $F_* = (f_*(x), x)$ exponentially fast as $n \to \infty$ where f_* is the fixed point of the renormalization operator of the one dimensional map. The hyperbolicity of the analytic unimodal map is proved in [Lyu]. The renormalization operator has the codimension one stable manifold and one dimensional unstable manifold at the fixed point f_*. The uniform norm of the analytic operator bounds all of C^r norm of the operator. Then the exponential convergence to the one dimensional fixed point $(f_*(x), 0)$ of R^nF and super-exponential decay of ε_n of the map R^nF implies the vanishing spectrum of DR, the derivative of renormalization operator. Hence, the unstable manifold at the fixed point of the Hénon renormalization operator is same as the unstable manifold of the renormalization operator of the unimodal maps. See the Section 4 on [CLM].

Chapter 4

Renormalization of the three dimensional Hénon-like maps

Three dimensional perturbed Hénon-like maps are introduced as a small perturbation of two dimensional Hénon-like map.

4.1 Hénon-like maps in three dimension

Let B_{2d} be the square region with the center origin and let this set tbe the domain of two dimensional Hénon-like map. Let B be the box domain which is a thickened domain of two dimensional Hénon-like map, that is, $B = B_{2d} \times [-c, c]$ for some $c > 0$. The length of the sides parallel to z axis is called the *thickness* or *height* of the domain B of the perturbed Hénon-like map in three dimension. Let us define the *perturbed Hénon-like map on three dimension* as the following with the cube B of which center is the origin. For simplicity, let us assume that the thickness of B is same as the length of the sides parallel to x or y axis.

$$F(x, y, z) = (f(x) - \varepsilon(x, y, z),\ x,\ \delta(x, y, z)) \tag{4.1.1}$$

where $f : I^x \longrightarrow I^x$ is a unimodal map.

Let us express the domain as $B = I^x \times \mathbf{I}^v$ where I^x is the line parallel to x-axis and $\mathbf{I}^v = I^y \times I^z$ where I^y and I^z are lines parallel to y-axis and z-axis respectively.

Remark 4.1.1. On the following section, some objects defined on the two dimensional space has the subscript $2d$. For example, B_{2d} is the square domain of the two dimensional Hénon-like map and F_{2d} is the two dimensional Hénon-like map defined on B_{2d}. However, same notation without any index indicates the three dimensional object. For instance, F and B are the perturbed Hénon-like map in three dimension and its box domain.

The image of the plane, $\{x = C\}$ parallel to yz-plane under F is contained in $\{y = C\}$ parallel to xz-plane.

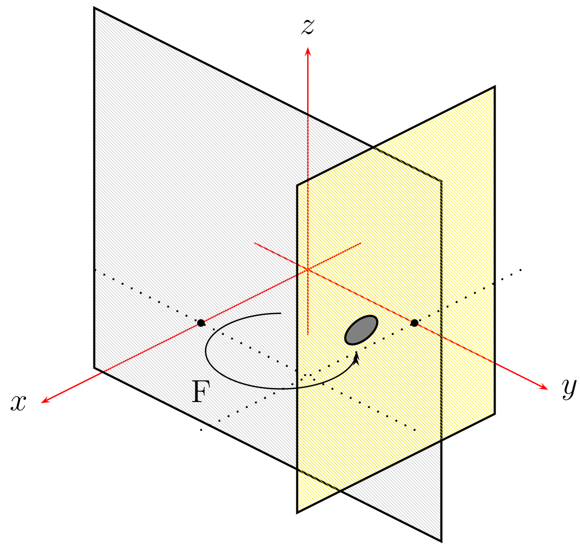


Figure 4.1.1: Image of $\{x = const.\}$ under the three dimensional Hénon-like map

Let us assume that $\|\varepsilon\|_{C^3} \leq \bar{\varepsilon}$ and $\|\delta\|_{C^3} \leq \bar{\delta}$ with sufficiently small positive numbers $\bar{\varepsilon}$ and $\bar{\delta}$. Assume that f is an infinitely renormalizable unimodal map. Since the norm of the third coordinate of F is sufficiently small, that is, $\|\delta\|_{C^3} \leq \bar{\delta} < 1$, F has only two fixed points like the two dimensional Hénon-like map by the contraction mapping theorem. Let these two saddle fixed points be β_0 and β_1 which is close to the regular and saddle fixed points of the two dimensional map $\pi_{xy} \circ F$ respectively. Moreover, β_0 and β_1 have stable manifolds of codimension one and one dimensional unstable manifolds. The orientation preserving perturbed Hénon-like map is called *renormalizable* if $W^u(\beta_0)$ and $W^s(\beta_1)$ intersects in a single orbit of a point.

On the local stable manifold of β_1, the distance of two points is defined as the distance along the shortest path connecting two points. This distance is close to the Euclidean distance on the domain B because of the Corollary 4.1.4 on the following. Let p_0 be the intersection point in $W^u(\beta_0) \cap W^s_{loc}(\beta_1)$ which is farthest from β_1 on $W^s_{loc}(\beta_1)$. Moreover, we define p_1 and p_2 to be the second and third farthest point from β_1 in $W^u(\beta_0) \cap W^s_{loc}(\beta_1)$ on the local stable manifold $W^s_{loc}(\beta_1)$ respectively. The points p_n are similarly defined for every $n \in \mathbb{N}$. If F is renormalizable, then p_n is $F^n(p_0)$ because $W^s_{loc}(\beta_1)$ is invariant under F and furthermore we can define p_k to be the forward or backward image of p_0 under F^k, that is, $p_k = F^k(p_0)$ for each $k \in \mathbb{Z}$. Then the intersection of the unstable manifold of β_0 and the stable manifold of β_1 is the (full) orbit of p_0, that is, $W^u(\beta_0) \cap W^s(\beta_1) = \mathrm{Orb}_{\mathbb{Z}}(p_0)$. In other words, every local stable manifolds of p_k, $W^s_{loc}(p_k)$ for all $k \in \mathbb{Z}$ are components of the stable manifolds of β_1, $W^s(\beta_1)$.

The topological properties of the renormalizable two dimensional Hénon-like maps are well extended to the renormalizable perturbed Hénon-like map in three dimension. Let B_0 be the component of $B \setminus W^s(\beta_0)$ containing β_1, which is invariant under F. Denote $W^s_{loc}(p_{-n})$ to be M_{-n} for $n \geq 0$ and define M_1 as the component of $W^s(\beta_1)$ in B_0 such that M_1 does not have any point of $\mathrm{Orb}_{\mathbb{Z}}(p_0)$ and its forward image under F is contained in M_{-1}, namely, $F(M_1) \subset M_{-1}$. Let each region in B between M_n and M_{-n+1} be A_n for $n \geq 0$ and let the region in B on the right side of M_1 be A_1. Then since $W^s_{loc}(\beta_1)$ is (forward) invariant under F and it is the common boundary of the regions A_{-1} and A_0, we can see that $F(A_{-1}) \subset A_0$ and $F(A_0) \subset A_{-1}$. In particular, A_0 is invariant under F^2 and $F^2(A_0)$ contains a small neighborhood of $[p_0, p_1]^u_{\beta_0}$ in A_0 and its boundary is disjoint from M_1 which is the component of $W^s(\beta_1)$ on the right hand side of $W^s_{loc}(\beta_1)$. Thus denote D to be the region $F(A_1)$, which is invariant under F^2 in A_0. Then the following properties are same as the two dimensional Hénon-like maps.

(1) M_0 is invariant under F.

(2) $F(M_{-n}) \subset M_{-n+1}$ for each $n \geq 0$.

(3) $F(M_1) \subset M_{-1}$.

(4) $F(A_{-n}) \subset A_{-n+1}$ for each $n \geq 0$. In particular, $F(A_{-1}) \subset A_0$.

(5) Let the region on the right side of M_1 be A_1. Then $F(A_1) \subset A_{-2}$.

(6) $W^u(\beta_0)$ intersects $W^s_{loc}(\beta_1)$ at p_0, p_1 and p_2 transversally.

Then the following lemma holds and the proof is similar to that of the two dimensional Hénon-like map case.

Lemma 4.1.1. *Let F be the renormalizable three dimensional Hénon-like map. Then $\overline{B}_0$ is the invariant under F and for every point $w \in B_0$, there exist $k \in \mathbb{N}$ such that $F^k(w) \in \overline{D}$.*

Proof. $W^s(\beta_0)$ is invariant under F and every M_{-n} for some $-n \leq -1$ are components of the stable manifold $W^s(\beta_0)$. Then we see that $F^{n-1}(M_{-n}) \subset M_{-1}$ where $-n \leq -2$. Moreover, $F(M_1) \subset M_{-1}$. Furthermore, by the definition of D, we see $F(M_{-1}) \subset \partial D$ and $F(M_0) \subset \partial D$. Then we can take $k = n$ where $-n \leq -1$, $k = 1$ where $n = 0$ and $k = 2$ where $n = 1$.

Now let us take a point $w \notin \bigcup_{n \leq 1} M_n$. Then it is sufficient to show that $F^k(w) \in \overline{D}$ for some $k \geq 0$. We may assume that w is contained in some region A_{-n} for some $-n \leq 1$ because each region A_{-n} is separated by M_{-n} and B_0 is the union of M_{-n} and A_{-n}. If $w \in A_{-n}$ where $-n \leq -1$, $F^{n-1}(w)$ is on A_{-1}. Let us say $w' = F^{n-1}(w)$. Then by the definition of D, $F(w') \in D$. Moreover, if $w \in A_0$ then by the invariance of D under F^2, $F^2(w) \in D$. If $w \in A_1$, then $F(w) \in A_{-2}$. Hence, we can choose $k = n$ where $-n \leq -1$, $k = 2$ where $n = 0$ and $k = 3$ where $n = 1$. $\square$

Corollary 4.1.2. *Let F be the renormalizable three dimensional Hénon-like map. Denote the region between two local stable manifolds M_0 and M_1 to be A_0. Then $F^2(A_0) \subset D$. In particular, any open neighbourhood of D in A_0 is invariant under F^2.*

Proof. Let us take any neighborhood of D in A_0, say D'. Then we get the following inclusion order

$$F^2(D) \subset F^2(D') \subset F^2(A_0) \subset F(A_{-1}) = D \subset D'$$

Hence, $F^2(D') \subset D'$. $\square$

As a result, any (thickened) domain D' is invariant under F^2. Then we can choose arbitrary region D' containing D as an invariant domain under F^2. Let us take an extended region as the domain such that $\pi_{xy}(D)$ compactly contains $D_{2d} \cap A_0$ in A_0 where the region D_{2d} is enclosed by curves, $[p_0, p_1]^s_{\beta_1}$ and $[p_0, p_1]^u_{\beta_0}$. Denote this extended region to be also D to save the notation on the following section.

Proposition 4.1.3. *Let $F(x, y, z) = (f(x) - \varepsilon(x, y, z), x, \delta(x, y, z))$ be a perturbed Hénon-like map with $\|\varepsilon\|_{C^1} \leq \bar{\varepsilon}$ and $\|\delta\|_{C^1} \leq \bar{\delta}$ where both $\bar{\varepsilon}$ and $\bar{\delta}$ are sufficiently small positive numbers. Suppose that there are intervals U and $U' \subset I^h$ such that f is injective on $V' \Supset U'$ with*

$$f(U') \supset \overline{U}$$

Then if there exists the map

$$\eta\colon \mathbf{I}^v \longrightarrow U$$

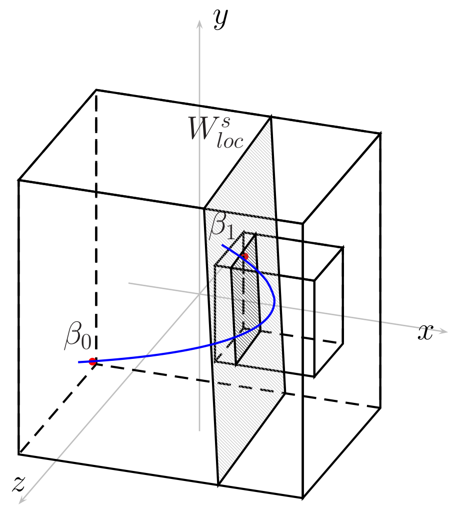


Figure 4.1.2: The local stable manifold of β_1, $W^s_{loc}(\beta_1)$ and the unstable manifold of β_0, $W^u(\beta_0)$

such that $\|D\eta\| \leq C_0(\bar{\varepsilon} + \bar{\delta})$ for some constant $C_0 > 0$, then the image of η under F^{-1} in B, namely, $F^{-1}(graph(\eta)) \cap (U' \times \mathbf{I}^v)$ is the graph of some function $\xi \colon \mathbf{I}^v \to U'$ with

$$\|D\xi\| \leq C(\bar{\varepsilon} + \bar{\delta})$$

for some constant $C > 0$.

Proof. Firstly we show that there exists the unique $x \in U'$ for each $(y', z') \in \mathbf{I}^v$ such that $F(x, y, z) = (\eta(y', z'), y', z') \in$ graph(η). Then

$$\phi_{y,z}(x) \equiv f(x) - \varepsilon(x, y, z) = \eta(x, \delta(x, y, z)) \tag{4.1.2}$$

The injectivity of f on U' with small enough $\bar{\varepsilon}$ implies that $f(x) - \varepsilon(x, y, z)$ has the inverse function for every point $(y, z) \in \mathbf{I}^v$. Moreover, η is the contraction with the small norm $\|\delta\|$. Then

$$\phi^{-1}_{y,z} \circ \eta(x, \delta(x, y, z)) \colon U' \to U'$$

is a well-defined contraction. Thus contraction mapping theorem implies unique existence of x for (4.1.2). Then $F^{-1}(\text{graph}(\eta)) \cap (U' \times \mathbf{I}^v)$ is the graph of some function, say ξ. Secondly, consider the image of the graph of ξ under F.

$$(\xi(y, z), y, z) \equiv (x, y, z) \mapsto (\eta(y', z'), y', z')$$

Then the formula of the perturbed Hénon-like map implies the following.

$$\eta(y', z') = \eta(x, \delta(x, y, z)) = f(x) - \varepsilon(x, y, z)$$

By the chain rule,

$$\begin{aligned} D\eta(y', z') &= Df \cdot D\xi(y, z) - \frac{\partial \varepsilon}{\partial x} D\xi(y, z) - D\varepsilon|_{\mathbf{I}^v}(y, z) \\ &= D\eta(\xi, \delta) \cdot \begin{pmatrix} 2D\xi \\ \frac{\partial \delta}{\partial x} D\xi + D\delta|_{\mathbf{I}^v} \end{pmatrix} \\ &= 2\frac{\partial \eta}{\partial y} D\xi(y, z) + \frac{\partial \eta}{\partial z}\left(\frac{\partial \delta}{\partial x} D\xi(y, z) + D\delta|_{\mathbf{I}^v}\right) \end{aligned}$$

Hence, when we solve the above equation in terms of $D\xi(y, z)$, we obtain that

$$D\xi(y, z) = \frac{D\varepsilon|_{\mathbf{I}^v}(y, z) + \frac{\partial \eta}{\partial z} D\delta|_{\mathbf{I}^v}(y, z)}{Df(x) - \frac{\partial \varepsilon}{\partial x} - 2\frac{\partial \eta}{\partial y} - \frac{\partial \eta}{\partial z}\frac{\partial \delta}{\partial x}}$$

Therefore, $\|D\xi\| \leq C(\bar{\varepsilon} + \bar{\delta})$. □

By the above proposition, the function from $\mathbf{I}^v$ to $U' \subset I^h$ with the small norm of derivative keeps its order under the (graph) transformation F^{-1}. Next we show that the local stable manifold $W^s_{loc}(\beta_1)$ can be the graph of some function from $\mathbf{I}^v$ to I^h by the standard graph transform technique.

Corollary 4.1.4. *$W^s_{loc}(\beta_1)$ is the graph of a function from $\mathbf{I}^v$ to I^h with the norm bounded by $C(\bar{\varepsilon} + \bar{\delta})$ for some constant $C > 0$.*

Proof. Since the β_1 is a fixed point of F, $\pi_x(\beta_1)$ is away from the critical point of f on I^h. Then we can take some neighbourhood $B_{2\rho}(\pi_x(\beta_1))$ of the $\pi_x(\beta_1)$ for some $\rho > 0$ such that $|Df(x)| \geq C > 1$ with a uniform constant C on $B_\delta(\pi_x(\beta_1))$. Denote that $U = B_\rho$ and $V = B_{2\rho}$. Thus let us consider the family of the functions as following.

$$\mathcal{G}_K = \{\eta \colon \mathbf{I}^v \to I^h \mid \eta\left(\pi_y(\beta_1), \pi_z(\beta_1)\right) = \pi_x(\beta_1),\ \|D\eta\| \leq K(\bar{\varepsilon} + \bar{\delta})\}$$

Moreover, we may assume that

$$\operatorname{diam}(\eta(\mathbf{I}^v)) \leq K(\bar{\varepsilon} + \bar{\delta}) \cdot \operatorname{diam}(I^h) < \rho_0$$

for some $0 < \rho_0 < 1$. Then for $\eta \in \mathcal{G}_K$, we have $\eta(\mathbf{I}^v) \subset U$. Applying the Proposition 4.1.3 with small enough $\bar{\varepsilon}$, the connected component of $F^{-1}(\text{graph}(\eta))$ containing β_1 in B is the graph of the some function η'. If we take $K > 0$ large enough, then we have $\eta' \in \mathcal{G}_K$. Then we can define the graph transformation $\mathcal{T} \colon \mathcal{G}_K \to \mathcal{G}_K$ with

$$\mathcal{T} \colon \eta \longrightarrow \eta'$$

This transformation is defined globally on the graph of η. Since the function f is expanding on U and $\bar{\varepsilon} + \bar{\delta}$ is small, this graph transformation contracts C^0 distance on $\mathcal{G}_K$. Hence the unique fixed point of $\mathcal{T}$, say η_0 is $W^s_{loc}(\beta_1) \in \mathcal{G}_K$ and it is the graph of the function in $\mathcal{G}_K$. $\square$

Let the function from $\mathbf{I}^v$ to I^h whose graph is the local stable manifold be ζ. Then by the Proposition 4.1.3 the norm $\|D\zeta\| \leq C(\bar{\varepsilon} + \bar{\delta})$ for some $C > 0$.

4.2 Hénon renormalization of maps in three dimension

In this section we construct the renormalization operator of three dimensional Hénon-like maps and the box domain of the conjugated map with the non linear coordinate change. The region D which contains $F(A_{-1})$ is an invariant domain under F^2. But F^2 is not Hénon-like map because the image of the plane, $\{x = C\}$ in B under F^2 is not part of the plane, $\{y = C\}$. Then for renormalization operator we need the non-linear coordinate change map. Let us call this map the *horizontal-like diffeomorphism* H and it is defined as the following.

$$H(x, y, z) = (f(x) - \varepsilon(x, y, z),\ y,\ z - \delta(y, f^{-1}(y), 0)) \tag{4.2.1}$$

Let the point in B be $w = (x, y, z)$. For simplicity, we express the map H and H^{-1} on the following.

$$\begin{aligned} H(x, y, z) &\equiv (f(x) - \varepsilon(w),\, y,\, z - \delta(y,\ f^{-1}(y),\, 0)) \\ H^{-1}(x, y, z) &\equiv (\phi^{-1}(w),\, y,\, z + \delta(y,\ f^{-1}(y),\, 0)) \end{aligned}$$

Then $\phi^{-1}(w) : B \to \mathbb{R}$ is an $\varepsilon-$ perturbation of the map $f^{-1}(x)$ in the three dimensional space as if the map $\phi(w)$ is an $\varepsilon-$ perturbation of f. Recall J_c the minimal invariant interval under f^2 containing the critical point of f. Let V be a closed interval which contains the small neighborhood of every J_c if the given unimodal maps are $f(x) - \varepsilon(x,\, y_0,\, z_0)$ for every $(y_0, z_0) \in \mathbf{I}^\mathbf{v}$. Then $H \circ F^2 \circ H^{-1}$ is a Hénon-like map on the domain V. Let $\mathcal{U}_U$ be the space of unimodal maps on the set U and $\mathcal{H}_U$ be the set of perturbed Hénon-like map on the set U.

Proposition 4.2.1. *Let H be the horizontal-like diffeomorphism defined on (4.2.1) and let $F = (f(x) - \varepsilon(w),\ x,\ \delta(w))$ is C^2 the Hénon-like map. Suppose that $\|\varepsilon\|_{C^2} \leq \bar{\varepsilon}$ and $\|\delta\|_{C^2} \leq \bar{\delta}$ with sufficiently small positive numbers $\bar{\varepsilon}$ and $\bar{\delta}$. Then there exists a unimodal map $f_1 \in \mathcal{U}_V$ such that $\|f_1 - f^2\|_V < C\bar{\varepsilon}$ and the map $H \circ F^2 \circ H^{-1}$ is a Hénon-like map $(x, y, z) \mapsto (f_1(x) - \varepsilon_1(x, y, z),\ x,\ \delta_1(x, y, z))$ of the class $\mathcal{H}_{V \times \mathbf{I}^v}$ with the norm, $\|\varepsilon_1\| = O(\bar{\varepsilon}^2 + \bar{\varepsilon}\bar{\delta})$ and $\|\delta_1\| = O(\bar{\varepsilon}\bar{\delta} + \bar{\delta}^2)$.*

Proof. Let us calculate $\phi^{-1}(w) - f^{-1}(x)$, $\varepsilon \circ F \circ H^{-1}$ and $\varepsilon \circ F^2 \circ H^{-1}$ for estimating $\|\varepsilon_1\|$ and $\|\delta_1\|$ later. The fact that $H \circ H^{-1}(w) = (x, y, z)$ implies that $f \circ \phi^{-1}(w) - \varepsilon \circ H^{-1}(w) = x$. Moreover,

$$\begin{aligned}\phi^{-1}(w) &= f^{-1}(x + \varepsilon \circ H^{-1}(w)) \\ &= f^{-1}(x) + (f^{-1})'(x) \cdot \varepsilon \circ H^{-1}(w) + \text{higher order terms}\end{aligned}$$

Then we get

$$\phi^{-1}(w) - f^{-1}(x) = (f^{-1})'(x) \cdot \varepsilon \circ H^{-1}(w) + \text{higher order terms} \tag{4.2.2}$$

Let $v(x) = \varepsilon(x, f^{-1}(x), 0)$. Then $v \circ f(x) = \varepsilon(f(x), x, 0)$.

$$\begin{aligned} &\varepsilon \circ F \circ H^{-1}(w) \\ &= \varepsilon(x, \phi^{-1}(w), \delta \circ H^{-1}(w)) \\ &= \varepsilon(x, f^{-1}(x), 0) + \partial_y \varepsilon(x, f^{-1}(x), 0) \cdot (\phi^{-1}(w) - f^{-1}(x)) \\ &\quad + \partial_z \varepsilon(x, f^{-1}(x), 0) \cdot \delta \circ H^{-1}(w) + h.o.t. \\ &= v(x) + \partial_y \varepsilon(x, f^{-1}(x), 0) \cdot (f^{-1})'(x) \cdot \varepsilon \circ H^{-1}(w) \\ &\quad + \partial_z \varepsilon(x, f^{-1}(x), 0) \cdot \delta \circ H^{-1}(w) + h.\,o.\,t. \end{aligned} \tag{4.2.3}$$

Similarly, we estimate $\varepsilon \circ F^2 \circ H^{-1}$.

$$\begin{aligned} &\varepsilon \circ F^2 \circ H^{-1}(w) \\ &= \varepsilon(f(x) - \varepsilon \circ F \circ H^{-1}(w),\ x,\ \delta \circ F \circ H^{-1}(w)) \\ &= \varepsilon(f(x),\ x,\ 0) + \partial_x \varepsilon(f(x),\ x,\ 0) \cdot \varepsilon \circ F \circ H^{-1}(w) \\ &\quad + \partial_z \varepsilon(f(x),\ x,\ 0) \cdot \delta \circ F \circ H^{-1}(w) + h.\,o.\,t. \\ &= v \circ f(x) + \partial_x \varepsilon(f(x),\ x,\ 0) \cdot \varepsilon \circ F \circ H^{-1}(w) \\ &\quad + \partial_z \varepsilon(f(x),\ x,\ 0) \cdot \delta \circ F \circ H^{-1}(w) + h.\,o.\,t. \end{aligned} \tag{4.2.4}$$

By the straightforward calculation, we obtain the coordinate functions of $H \circ$

$F^2 \circ H^{-1}$.

$$
\begin{aligned}
&(x, y, z) \\
\xrightarrow{H^{-1}}\ & (\phi^{-1}(w),\, y,\, z + \delta(y, f^{-1}(y), 0)) \\
\xrightarrow{F}\ & (x,\, \phi^{-1}(w),\, \delta \circ H^{-1}(w)) \\
\xrightarrow{F}\ & (f(x) - \varepsilon \circ F \circ H^{-1}(w),\, x,\, \delta \circ F \circ H^{-1}(w)) \\
\xrightarrow{H}\ & (f(f(x) - \varepsilon \circ F \circ H^{-1}(w)) - \varepsilon \circ F^2 \circ H^{-1}(w), \\
& \qquad\qquad x,\; \delta \circ F \circ H^{-1}(w) - \delta(x, f^{-1}(x), 0))
\end{aligned}
$$

Thus the first coordinate function of $H \circ F^2 \circ H^{-1}$ is

$$
f(f(x) - \varepsilon \circ F \circ H^{-1}(w)) - \varepsilon \circ F^2 \circ H^{-1}(w)
$$

By (4.2.2), (4.2.3) and (4.2.4), we get the following estimation.

$$
\begin{aligned}
& f(f(x)) - \varepsilon \circ F \circ H^{-1}(w)) - \varepsilon \circ F^2 \circ H^{-1}(w) \\
=\ & f^2(x) - f'(f(x)) \cdot \varepsilon \circ F \circ H^{-1}(w) - \big[\, \varepsilon(f(x),\, x,\, 0) \\
& + \partial_x \varepsilon(f(x),\, x,\, 0) \cdot \varepsilon \circ F \circ H^{-1}(w) + \partial_z \varepsilon(f(x),\, x,\, 0) \cdot \delta \circ F \circ H^{-1}(w) \,\big] \\
& + h.\,o.\,t. \\
=\ & f^2(x) - v \circ f(x) - [f'(f(x)) - \partial_x \varepsilon(f(x),\, x,\, 0)] v(x) \\
& - [f'(f(x)) - \partial_x \varepsilon(f(x),\, x,\, 0)] \cdot \big[\, \partial_y \varepsilon(x, f^{-1}(x), 0) \cdot (f^{-1})'(x) \cdot \varepsilon \circ H^{-1}(w) \\
& + \partial_z \varepsilon(x, f^{-1}(x), 0) \cdot \delta \circ H^{-1}(w) \,\big] - \partial_z \varepsilon(f(x),\, x,\, 0) \cdot \delta \circ F \circ H^{-1}(w) \\
& + h.\,o.\,t.
\end{aligned}
$$

Then the unimodal map, $f_1(x)$ of the first component of $H \circ F^2 \circ H^{-1}$ is the following.

$$
f^2(x) - v \circ f(x) - [f'(f(x)) - \partial_x \varepsilon(f(x),\, x,\, 0)] \cdot v(x)
$$

Thus $\|f_1(x) - f^2(x)\| = O(\|\varepsilon\|)$. Moreover, $\|\varepsilon_1(w)\| = O\big(\|\varepsilon\|^2 + \|\varepsilon\|\,\|\delta\|\,\big)$. Let us estimate the third coordinate of $H \circ F^2 \circ H^{-1}$. Recall $\delta_1(w) = \delta \circ F \circ$

$H^{-1}(w) - \delta(x,\ f^{-1}(x),\ 0)$.

$$\begin{aligned}
&\delta \circ F \circ H^{-1}(w) - \delta(x,\ f^{-1}(x),\ 0) \\
&= \delta(x, \phi^{-1}(w), \delta \circ H^{-1}(w)) - \delta(x,\ f^{-1}(x),\ 0) \\
&= \partial_y \delta(x,\ f^{-1}(x),\ 0) \cdot (\phi^{-1}(w) - f^{-1}(x)) + \partial_z \delta(x,\ f^{-1}(x),\ 0) \cdot \delta \circ H^{-1}(w) \\
&\quad + h.o.t. \\
&= \partial_y \delta(x,\ f^{-1}(x),\ 0) \cdot (f^{-1})'(x) \cdot \varepsilon \circ H^{-1}(w) + \partial_z \delta(x,\ f^{-1}(x),\ 0) \cdot \delta \circ H^{-1}(w) \\
&\quad + h.o.t.
\end{aligned}$$

Then $\|\delta_1\|$ is $O(\|\varepsilon\|\,\|\delta\| + \|\delta\|^2)$.

□

Define *pre-renormalization* of F as $H \circ F^2 \circ H^{-1}$ and denote it to be PRF. With the conjugation of the expanding map $\Lambda(x, y, z) = (sx, sy, sz)$ for some $s < -1$, we define the *renormalization* of the perturbed Hénon-like map F in three dimension and denote it to be RF.

The domain of the renormalized map is also B, the domain of F. To recover the domain of the renormalized map, $\operatorname{Dom}(PRF)$ must be the cube of which center is the origin and the box $\operatorname{Dom}(PRF)$ is invariant under PRF. Let us take a closed interval as the small neighborhood of each intervals J_c containing the critical point of the map $x \mapsto f(x) - \varepsilon(x, y_0, z_0)$ where $(y_0, z_0) \in \mathbf{I^v}$. Then this interval can be extended to the symmetric interval at the 0. Let this extended closed interval be V' and take the square in $\pi_{xy}(B)$, say ${}_{2d}B_0$ such that each sides are parallel to x and y axes and length of each side is same as that of V'. After that let us take $H^{-1}({}_{2d}B_0)$. Then this region is enclosed two lines parallel to the $x-$axis and two curves, $f(x) - \varepsilon(x, y, 0) = C_i$ where $i = 0, 1$. Furthermore, we can extend this region on the three dimensional domain with the full height, namely, $H^{-1}({}_{2d}B_0) \times I^z$ in B. Since the domain of H is necessary to be invariant under F^2, we modify the constants C_0 and C_1 such that $H^{-1}({}_{2d}B_0) \times I^z$ is the minimal invariant domain under F^2. Then the actual domain of PRF is the pillar with the rectangle base of which side's ratio is $1 : 1 + O(\bar{\varepsilon})$. Let us slightly extend the short sides of the rectangular base to be the square one with the origin as the center. Moreover, let us restrict the height of this cube to make the cube with the same sides with the origin as the center and call this cube $\operatorname{Dom}(PRF)$. Afterwards, we define the domain of the horizontal diffeomorphism H as $H^{-1}(\operatorname{Dom}(PRF))$ and denote this region to be B^1_v.

Let B^1_c be $F(B^1_v)$. By the construction of $\operatorname{Dom}(PRF)$, $\Lambda(\operatorname{Dom}(PRF))$ is the original box domain B where $\Lambda(x, y, z) = (sx, sy, sz)$ is a scaling map with

$s < -1$. Thus we express the domain $\mathrm{Dom}(PRF)$ to be $\Lambda^{-1}(B)$. Then the map H^{-1} from $\Lambda^{-1}(B)$ to B^1_v preserves the planes parallel to $xz-$plane and the map $F \circ H^{-1}$ from $\Lambda^{-1}(B)$ to B^1_c preserves the planes parallel to $yz-$plane.

$$
\begin{aligned}
H^{-1} : \Lambda^{-1}(B) \longrightarrow B^1_v, &\quad (x,y,z) \mapsto (\phi^{-1}(x,y,z),\ y,\ z + \delta(y, f^{-1}(y), 0)) \\
F \circ H^{-1} : \Lambda^{-1}(B) \longrightarrow B^1_c, &\quad (x,y,z) \mapsto (x,\ \phi^{-1}(x,y,z),\ \delta \circ H^{-1}(w))
\end{aligned}
$$

Since the H^{-1} is the horizontal map and $F \circ H^{-1}$ is the vertical map, $\pi_x(B^1_v)$ and $\pi_y(B^1_v)$ are disjoint from $\pi_x(B^1_c)$ and $\pi_y(B^1_c)$ respectively.

Definition 4.2.1 (Renormalization)**.** Let V be the (minimal) closed subinterval of I^x such that $V \times \mathbf{I^v}$ is invariant under $H \circ F^2 \circ H^{-1}$ and let $s \colon V \to I$ be the orientation reversing affine rescaling. With the rescaling map $\Lambda(x,y,z) = (sx, sy, sz)$, The *renormalization* of the three dimensional Hénon-like map is defined as $\Lambda \circ H \circ F^2 \circ H^{-1} \circ \Lambda^{-1}$ on the domain $B \equiv I^x \times \mathbf{I^v}$.

$$RF = \Lambda \circ H \circ F^2 \circ H^{-1} \circ \Lambda^{-1}$$

If RF is also renormalizable, we can define the second renormalization of F as the renormalization of RF. Then if F is n times renormalizable, then the n^{th} renormalization is defined successively.

$$R^nF = \Lambda_{n-1} \circ H_{n-1} \circ (R^{n-1}F)^2 \circ H^{-1}_{n-1} \circ \Lambda^{-1}_{n-1}$$

where $R^{n-1}F$ is the $(n-1)^{th}$th renormalization of F for $n \geq 1$.

Let $\mathcal{U}_J$ be the set of the unimodal maps on the interval J and $\mathcal{H}_B$ is the set of the perturbed Hénon-like maps on the domain B. Let us assume that the unimodal map on the interval $J \subset I^x$ can be extended on I^x. Then there exists a natural inclusion from $\mathcal{U}_J$ to $\mathcal{H}_B$.

$$
\begin{array}{rccc}
\imath : & \mathcal{U}_J & \hookrightarrow & \mathcal{H}_B \\
& f(x) & \mapsto & (f(x),\, x,\, 0)
\end{array}
\tag{4.2.5}
$$

Thus the degenerate maps can be treated as the one dimensional maps in the space of the perturbed Hénon-like maps. The renormalized map R_cf of the unimodal map f is defined as $s \circ f^2 \circ s^{-1}(x)$ for some $s < -1$. Let the perturbed Hénon-like map such that $\varepsilon \equiv 0$ and $\delta \equiv 0$ be the degenerate map $F_\bullet$, that is, $F_\bullet : (x,y,z) \mapsto (f(x), x, 0)$. The corresponding horizontal diffeomorphism is the map $H_\bullet : (x,y,z) \mapsto (f(x), y, z)$. Since the renormalization operator is defined on the non-diffeomorphic Hénon-like map, the renormalized map $RF_\bullet$

is the following by the direct calculation.

$$
\begin{aligned}
RF_{\bullet}(w) &= \Lambda \circ (f^2(x),\, x,\, 0) \circ \Lambda^{-1} \\
&= (s \circ f^2 \circ s^{-1}(x),\, x,\, 0)
\end{aligned}
\tag{4.2.6}
$$

Hence, the renormalization operator of the perturbed Hénon-like maps is an extension of the operator of the unimodal maps. Moreover, Proposition 4.2.1 implies that the unimodal map f_1 of $RF = (f_1 - \varepsilon_1,\, x,\, \delta_1)$ is an ε perturbation of $R_c f$, that is,

$$\|f_1 - R_c f\| \leq C\|\varepsilon\| \leq C\bar{\varepsilon}$$

for some $C > 0$.

Let the N^{th} renormalized map of F be $R^N F = (f_N - \varepsilon_N,\, x,\, \delta_N)$ for $N \geq 1$. Using the induction with the Proposition 4.2.1, we have

$$\|f_N - R_c f_{N-1}\| \leq C\|\varepsilon_{N-1}\| \leq C\bar{\varepsilon}^{2^{N-1}}$$

for some $C > 0$ depending on f and the domain B. The perturbation decreases super exponentially fast as $N \to \infty$. Then the renormalized Hénon-like map, $R^N F$ converges to the fixed point of the renormalization operator, $F_* = (f_*(x),\, x,\, 0)$ exponentially fast.

Lemma 4.2.2. *Let F is infinitely renormalizable Hénon-like map with sufficiently small $\|\varepsilon\| \leq \bar{\varepsilon}$ and $\|\delta\| \leq \bar{\delta}$. Then for all big enough $n \geq 1$, $R^n F$ converge to the degenerate map $F_* = (f_*(x), x, 0)$ exponentially fast as $n \to \infty$.*

Proof. Let the degenerate map be $F_{f_N} = (f_N,\, x,\, 0)$ where $R^N F = (f_N - \varepsilon_N,\, x,\, \delta_N)$ and let $F_{R_c^N f} = (R_c^N f,\, x,\, 0)$ where $R_c^N f$ is the N^{th} renormalized map of f for $N \geq 1$. Then for big enough N, we get the following estimation.

$$
\begin{aligned}
\|R^N F - F_*\| &\leq \|R^N F - F_{f_N}\| + \|F_{f_N} - F_{R_c^N f}\| + \|F_{R_c^N f} - F_*\| \\
&= \|(\varepsilon_N,\, 0,\, \delta_N)\| + \|f_N - R_c^N f\| + \|R_c^N f - f_*\| \\
&\leq C_2 \big(\bar{\varepsilon} + \bar{\delta}\big)^{2^N} + \|f_N - R_c^N f\| + C_0 \rho_0
\end{aligned}
$$

for some $C_0,\, C_2 > 0$ and $0 < \rho_0 < 1$. From the theory of the renormalization of the unimodal maps, $R_c^{Nk} f$ converges to f_* exponentilly fast as $k \to \infty$ for sufficiently large N. Using the adapted metric in [PS], we can take $N = 1$. Then for every $n \geq 1$, we obtain

$$\|R^n F - F_*\| \leq C_2 \big(\bar{\varepsilon} + \bar{\delta}\big)^{2^n} + \|f_n - R_c^n f\| + C_0 \rho_0^n$$

for some $C_0,\, C_1 > 0$ and $0 < \rho_0 < 1$.

Moreover,

$$\begin{aligned}\|f_n - R_c^n f\| \le \|f_n - R_c f_{n-1}\| + \|R_c f_{n-1} - R_c^2 f_{n-2}\| + \|R_c^2 f_{n-2} - R_c^3 f_{n-3}\| + \cdots \\ + \|R_c^{m-1} f_{n-m+1} - R_c^m f_{n-m}\| + \|R_c^m f_{n-m} - R_c^{m+1} f_{n-m-1}\| + \cdots \\ + \|R_c^{n-1} f_1 - R_c^n f\|\end{aligned}$$

For sufficiently large m and $n-m$, by Lemma 8 in [dMP] on the space of the quadratic-like maps we have C_0 distance contraction and by the Main Theorem on [AMdM] we obtain the C^r contractions, $r \ge 3$.[1]

$$\|R_c^m f_{n-m} - R_c^{m+1} f_{n-m-1}\| + \cdots + \|R_c^{n-1} f_1 - R_c^n f\| \le C_m \rho_m^{n-m} + \cdots + C_n \rho_n^n \tag{4.2.7}$$

for some $0 < C_i = O(\bar{\varepsilon}^{2^i})$ and $0 < \rho_i < 1$ where $i = m, m+1, \ldots, n$. The numbers C_is and ρ_i are independent of n. Thus the sum (4.2.7) is bounded above by $C_1 \rho_1^{n-m}$ for some $C_1 > 0$ and $0 < \rho_1 < 1$. Moreover, by the direct calculations of each terms we obtain

$$\begin{aligned}&\|f_n - R_c f_{n-1}\| + \|R_c f_{n-1} - R_c^2 f_{n-2}\| + \cdots + \|R_c^{m-1} f_{n-m+1} - R_c^m f_{n-m}\| \\ &\quad \le C_n \bar{\varepsilon}^{2^{n-1}} + C_{n-1}^2 \bar{\varepsilon}^{2^{n-2}} + \cdots + C_{n-m}^m \bar{\varepsilon}^{2^{n-m}}\end{aligned} \tag{4.2.8}$$

for some $0 < C_i$, $i = n-m, \ldots, n$. For sufficiently big $n-m$, the sum (4.2.8) is $O(\bar{\varepsilon}_0^{2^{n-m}})$ for $\bar{\varepsilon}_0 < \bar{\varepsilon}$. Then $\|f_n - R_c^n f\| \le C_1 \rho_1^{n-m} + O(\bar{\varepsilon}_0^{2^{n-m}})$. Hence,

$$\|R^n F - F_*\| \le C_2 \big(\bar{\varepsilon} + \bar{\delta}\big)^{2^n} + C_1 \rho_1^{n-m} + O(\bar{\varepsilon}_0^{2^{n-m}}) + C_0 \rho_0^n \le C \rho^n$$

for some $C > 0$ and $0 < \rho < 1$. Therefore, $R^n F$ converges to F_* exponentially fast. □

In the following sections, we suppress the bound of small norms of ε and δ to be $\bar{\varepsilon}$, that is, we denote $\bar{\varepsilon} = \max\{\bar{\varepsilon}, \bar{\delta}\}$.

4.3 Hyperbolicity of renormalization operator

The hyperbolicity of the renormalization operator at its fixed point was proved by M. Lyubich in [Lyu] using quadratic-like maps. This theory was for the one

[1] The theorems of on [dMP] and [AMdM] assumed that the maps are infinitely renormalizable with bounded combinotorics. On the [AMdM], the infinitely renormalized unimodal maps f and g has the same bounded type. We assume that every renormalizable functions has the type of periodic doubling on this article. This fixed and bounded single combinotorics is much simpler than the actual hypothesis on [dMP] or [AMdM].

dimensional complex analytic mappings. This hyperbolicity extended to the renormalization operator of C^r maps on the interval where $r = 3 + \alpha$ if α is close to one by de Faria, de Melo and Pinto in [dFdMP]. However, the renormalization operator of C^r maps is not differentiable. Thus the linear operator at the fixed point for hyperbolicity should be established. The contraction or repulsion along the stable and unstable manifold should be considered in the much bigger space than the space of the analytic maps.

We assume that every perturbed Hénon-like maps are analytic. In the [Lyu], Lyubich proved that the renormalization operator at the fixed point has the one dimensional unstable manifold and codimension one stable manifold on the complex sense. By Theorem 2.4 and Theorem 3.9 in [dFdMP], the renormalization of the real analytic map also has the one dimensional unstable manifold and codimension one stable manifold. The renormalization operator R of analytic maps has its derivative and the uniform norm bounds the norm of the derivative of the analytic operator.[2]

The renormalization operator of the degenerate maps is embedded under the natural inclusion from the renormalization operator of the one dimensional maps in the space of renormalizable Hénon-like map. Moreover, since this embedded operator is a closed subset of the renormalization of the Hénon-like maps, the quotient space $\mathcal{I}_B(\bar{\varepsilon})/\mathcal{I}_{I^x}$ is defined with the quotient norm where I^x is the interval as the invariant domain of the renormalizable unimodal maps. Then the super exponential convergence of $\| \varepsilon_n \|$ and $\| \delta_n \|$ to the zero as $n \to \infty$ by Proposition 4.2.1 implies the hyperbolicity of the renormalization operator of the perturbed Hénon-like maps.

Lemma 4.3.1. *The degenerate map F_* is the hyperbolic fixed point of the renormalization operator R of the perturbed Hénon-like maps. The derivative of the operator at F_*, $DR(F_*)$ acting on the quotient space $T\mathcal{I}_B(\bar{\varepsilon})/T\mathcal{I}_{I^x}$ has vanishing spectrum.*

Proof. Let $A = T\mathcal{I}_B(\bar{\varepsilon})/T\mathcal{I}_{I^x}$. The analytic operator $DR(F_*)$ has the norm, $\|DR(F_*)\| = O\big((\bar{\varepsilon} + \bar{\delta})^{2^n}\big)$ by Proposition 4.2.1. $\square$

Then the vanishing spectrum says that the stable manifold in $\mathcal{I}_B(\bar{\varepsilon})$ at the fixed point F_*, say $\mathcal{W}^s(F_*)$ is the extension of the stable manifold at f_*, $\mathcal{W}^s(f_*)$ of the unimodal renormalizable maps with the strong stable directions. The unstable manifold is not extended and it is a one dimensional analytic manifold on the space $\mathcal{I}_B(\bar{\varepsilon})$. Furthermore, the faster convergence than any exponential convergence of ε_n and δ_n keeps the hyperbolicity at the fixed point F_*.

[2] M. Lyubich pointed out the uniform norm bounds the norm of all derivatives of the analytic operator.

Corollary 4.3.2. *At the fixed point of the renormalization, F_*, it has one dimensional unstable manifold, $\mathcal{W}^u(F_*)$ which intersects transversally the stable manifold, $\mathcal{W}^s(F_*)$*

Chapter 5

Critical Cantor set

The minimal attracting set for two dimensional infinitely renormalizable Hénon-like maps is the Cantor set which is the dyadic adding machine. The topological construction of the invariant Cantor set of three dimensional Hénon-like map is exactly same as that for two-dimensional Hénon-like map (Corollary 5.2.3 below). Thus we use the same definition and notions of the two dimensional case in this section. The definitions and notions of the three dimensional Hénon-like maps are basically identical with the two-dimensional case. See [CLM].

5.1 Branches

Let $\Psi^1_v \equiv \psi^1_v := H^{-1} \circ \Lambda^{-1}$ be the coordinate change map which conjugates F^2 to RF on $\Psi^1_v(B)$ which is invariant under F^2, and let $\Psi^1_c \equiv \psi^1_c := F \circ \psi_v$. The subscript v and c are associated to the maps with the *critical value* and the *critical point* respectively. Similarly, let ψ^2_v and ψ^2_c be the coordinate change maps conjugating RF to R^2F. Let

$$\Psi^2_{vv} = \psi^1_v \circ \psi^2_v, \quad \Psi^2_{cv} = \psi^1_c \circ \psi^2_v, \quad \Psi^2_{vc} = \psi^1_v \circ \psi^2_c, \ldots$$

Moreover, let us define the coordinate change map of the n^{th} level for any $n \in \mathbb{N}$ as following.

$$\Psi^n_{\mathbf{w}} = \psi^1_{w_1} \circ \cdots \circ \psi^n_{w_n}, \quad \mathbf{w} = (w_1, \ldots, w_n) \in \{v, c\}^n$$

where $\mathbf{w} = (w_1, \ldots, w_n)$ is the word of length n such that the element v and c without any relation. Each element of $\mathbf{w}$ is either v or c and $W^n = \{v, c\}^n$ is the n-fold Cartesian product of $\{v, c\}$.

Lemma 5.1.1. *Let $F \in \mathcal{I}_B(\bar{\varepsilon})$ for $n \geq 1$. There exist $C > 0$ and a domain depending only on B and $\bar{\varepsilon}$, on which the derivative of the map $\Psi^n_{\mathbf{w}}$ is exponentially shrinking for $n \in \mathbb{N}$ with σ, that is, $\|D\Psi^n_{\mathbf{w}}\| \leq C\sigma^n$ for every words $\mathbf{w} \in W^n$.*

Proof. Recall $\phi^{-1}(x, y, z)$ is the first coordinate function of H^{-1}.[1]

$$
\begin{aligned}
H^{-1}(x, y, z) &= (\phi^{-1}(x, y, z),\ y,\ z + \delta(y, f^{-1}(y), 0)) \\
F \circ H^{-1} &= (x,\ \phi^{-1}(x, y, z),\ \delta \circ H^{-1}(w))
\end{aligned}
$$

Moreover, the equation, $H^{-1} \circ H = \mathrm{id}$ implies that $\phi^{-1} \circ H(x, y, z) = x$, that is,

$$
\phi^{-1}(f(x) - \varepsilon(x, y, z),\ y,\ z - \delta(y, f^{-1}(y), 0)) = x
$$

and then

$$
\begin{aligned}
&\frac{\partial \phi^{-1}}{\partial x} \cdot \left(f'(x) - \frac{\partial \varepsilon}{\partial x} \right) = 1 \\
&\frac{\partial \phi^{-1}}{\partial y} + \frac{\partial \phi^{-1}}{\partial x} \cdot \left(-\frac{\partial \varepsilon}{\partial y} \right) + \frac{\partial \phi^{-1}}{\partial z} \cdot \left(-\frac{d}{dy}\,\delta(y, f^{-1}(y), 0) \right) = 0 \\
&\frac{\partial \phi^{-1}}{\partial z} + \frac{\partial \phi^{-1}}{\partial x} \cdot \left(-\frac{\partial \varepsilon}{\partial z} \right) = 0
\end{aligned}
\tag{5.1.1}
$$

Moreover, for sufficiently small $\bar{\varepsilon}$ the perturbation of the one-dimensional map $\phi^{-1}_{y,z}(x)$ is a contraction on the neighbourhood on J in I. Then $\|\partial\phi^{-1}/\partial x\|$ is bounded away from 1 on the $J \times \mathbf{I}^v$ and $\|\partial\phi^{-1}/\partial y\|$ and $\|\partial\phi^{-1}/\partial z\|$ are comparable with $\|\partial\varepsilon/\partial y + \partial\varepsilon/\partial z \cdot d\delta/dy\|$ and $\|\partial\varepsilon/\partial z\|$ respectively. Since the partial derivatives ϕ^{-1} over both y and z are small, the coordinate change maps, $\psi^1_v = H^{-1} \circ \Lambda^{-1}$ and $\psi^1_c = F \circ H^{-1} \circ \Lambda^{-1}$, are contracting faster than or equal to Λ^{-1} by the factor $\sigma(1 + O(\mathrm{dist}(F, F_*)))$. Furthermore, the norm of the maps ψ^n_w is $\sigma(1 + O(\rho^n))$ for some $\rho \in (0, 1)$ for each $n \in \mathbb{N}$ because $R^n F$ converges to F_* to exponentially fast. Therefore, the composition $\Psi^n_{\mathbf{w}}$ of these maps are contracting by the number $O(\sigma^n)$. □

[1] The first coordinate map of $H^{-1}(w)$, $\phi^{-1}(x, y, z)$ is *not the inverse function* of the some function $\phi(w)$. However, $\phi^{-1}(w)$ is a perturbation of $f^{-1}(x)$. More precisely,

$$
f \circ \phi^{-1}(w) - \varepsilon \circ H^{-1}(w) = x
$$

5.2 Pieces

Define $B_v^1 \equiv B_v^1(F)$ as $\psi_v^1(B)$ and $B_c^1 \equiv B_c^1(F)$ as $F \circ \psi_v^1(B)$ like the definition on the Section 4.2. Then $F(B_c^1) \subset B_v^1$. If the Hénon-like map F is n times renormalizable, we can define $B_v^1(R^n F)$ and $B_c^1(R^n F)$ as $\psi_v^{n+1}(B)$ and $F_n \circ \psi_v^{n+1}(B)$ respectively for each $n \geq 1$. Furthermore, the piece $B_c^1(F_*)$ is a part of the parabolic-like curve of $x = f_*(y)$ and $B_v^1(F_*)$ is the rectangular box which contains rectangular domain of the two dimensional Hénon-like map on it's interior.

Let us call the set $B_{\mathbf{w}}^n \equiv B_{\mathbf{w}}^n(F) = \Psi_{\mathbf{w}}^n(B)$ the *pieces* of the n^{th} level or n^{th} generation where $\mathbf{w} \in W^n$. For each n, the number of pieces are 2^n. Moreover, W^n can be a additive group under the following correspondence from W to the numbers with base 2 of mod 2^n.

$$\mathbf{w} \mapsto \sum_{k=0}^{n-1} w_{k+1} 2^k \qquad (\operatorname{mod} 2^n)$$

where the symbols v and c are corresponding to 0 and 1 respectively. Let $P\colon W^n \to W^n$ be the operation of adding 1 in this group. The following lemma comes from Lemma 5.3 in [CLM].

Lemma 5.2.1. (1) *The pieces for the above maps are nested :*

$$B_{\mathbf{w}\nu}^n \subset B_{\mathbf{w}}^{n-1}, \quad \mathbf{w} \in W^{n-1},\ \nu \in W.$$

(2) *The pieces $B_{\mathbf{w}}^n$, $\mathbf{w} \in W$ are pairwise disjoint.*

(3) *Under F, the pieces are permuted as following. $F(B_{\mathbf{w}}^n) = B_{P(\mathbf{w})}^n$ unless $P(\mathbf{w}) = v^n$. If $P(\mathbf{w}) = v^n$, then $F(B_{\mathbf{w}}^n) \subset B_{v^n}^n$.*

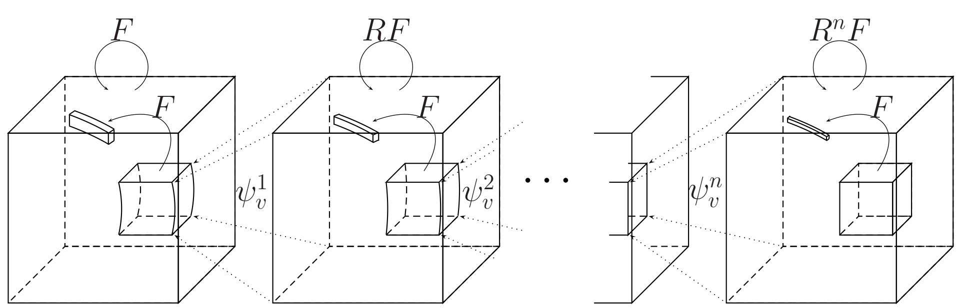


Figure 5.2.1: Coordinate change ψ_v^n around the tip at each level

Then the following diagram is commutative.

$$
\begin{tikzcd}
B \arrow[r, "R^n F"] \arrow[d, "\Psi^n_{\mathbf{w}}"'] & B \arrow[d, "\Psi^n_{\mathbf{w}}"] \\
\Psi^n_{\mathbf{w}}(B) \arrow[r, "F^{2^n}"] & \Psi^n_{\mathbf{w}}(B)
\end{tikzcd}
$$

Furthermore, Lemma 5.1.1 implies the following corollary.

Corollary 5.2.2. *The diameter of each piece shrinks exponentially fast for each $n \geq 1$, that is,* $\operatorname{diam}(B^n_{\mathbf{w}}) \leq C\sigma^n$ *for all* $\mathbf{w} \in W^n$ *where the constant* $C > 0$ *depend only on* B *and* $\bar{\varepsilon}$*.*

Define the invariant set of the infinitely renormalizable perturbed Hénon-like map F as follows.

$$\mathcal{O} \equiv \mathcal{O}_F = \bigcap_{n=1}^{\infty} \bigcup_{\mathbf{w} \in W^n} B^n_{\mathbf{w}}.$$

Then $\mathcal{O}$ is the invariant Cantor set under F. Since each Ψ of $B^n_{\mathbf{w}}$ is a diffeomorphism on its image, passing the limit with the result of Lemma 5.2.1 we can show that the constructed Cantor set is invariant under F.
Let us consider the inverse limit of W^n, $W^\infty = \varprojlim W^n$. The elements of this set are the infinite sequences $(w_1 w_2 \ldots)$ of symbols. This space is the set formal power series of numbers with base 2 when v and c corresponds to 0 and 1 respectively.

$$\mathbf{w} \mapsto \sum_{k=0}^{\infty} w_{k+1} 2^k$$

Then W^∞ is the *dyadic* group and it is also a Cantor set with the topology induced by the following metric.

$$\sum_{i=0}^{\infty} \frac{|\, v_i - w_i|}{2^i}$$

where $\mathbf{v} = (v_1 v_2 v_3 \ldots)$ and $\mathbf{w} = (w_1 w_2 w_3 \ldots)$. For detailed construction of the dyadic group as a Cantor set, see [BB].
The *adding machine* $P \colon W^\infty \to W^\infty$ is the operation of adding 1 in this group. The non negative integers with base 2 are embedded as the set of finite numbers in this dyadic group. Moreover, F acts on the critical Cantor set like the adding machine of the dyadic group.

Corollary 5.2.3. *The map $F|_{\mathcal{O}}$ is topologically conjugate to the adding machine $P\colon W^\infty \longrightarrow W^\infty$. The conjugacy is the following homeomorphism $h\colon W^\infty \longrightarrow \mathcal{O}$.*

$$h\colon \mathbf{w} = (w_1\, w_2 \ldots) \mapsto \bigcap_{n=1}^{\infty} B^n_{w_1 \ldots w_n}$$

Furthermore, there exists the unique invariant probability measure μ whose support is the Cantor set $\mathcal{O}$.

Proof. Consider the following diagram.

$$\begin{tikzcd}
W^\infty \arrow[r, "P"] \arrow[d, "h"'] & W^\infty \arrow[d, "h"] \\
\mathcal{O} \arrow[r, "F"] & \mathcal{O}
\end{tikzcd}$$

Take a word $\mathbf{w} \in W$. Let $\mathbf{w}_i = (w_1 w_2 w_3 \ldots w_i)$ be the first consecutive i concatenations of the word $\mathbf{w} = (w_1 w_2 w_3 \ldots)$. Then by the Lemma 5.2.1, $F(B^i_{\mathbf{w}_i}) = B^{i+1}_{\mathbf{w}_i+1}$ if $\mathbf{w}_i \neq v^n$. Otherwise, $F(B^i_{\mathbf{w}_i}) \subset B^{i+1}_{\mathbf{w}_i+1}$. Each domain $B^i_{\mathbf{w}_i}$ shrinks to a point of $\mathcal{O}_F$ when $i \to \infty$. Then passing the limit

$$F\left(\bigcap_{i=1}^{\infty} B^i_{\mathbf{w}_i}\right) = \bigcap_{i=1}^{\infty} B^{i+1}_{\mathbf{w}_i+1}$$

It means $F(h(\mathbf{w})) = h(\mathbf{w}+1)$. Then the above diagram is commutative. If two words $\mathbf{v}$ and $\mathbf{w}$ have the different i^{th} letter but not before, then $B^i_{\mathbf{v}_i}$ and $B^i_{\mathbf{w}_i}$ are disjoint from each other. Moreover, every point of $\mathcal{O}$ has its word and two different points of $\mathcal{O}$ have the different words by construction of the critical Cantor set. Hence, h is the bijection. The metric of the dyadic group implies the (uniform) continuity of h. Furthermore, the same topological structure and continuous bijection implies that h is a homeomorphism between two compact spaces. $\square$

Remark 5.2.1. The formal power series of the numbers with base 2 comes from the combinatorics of the renormalization. If the combinatorics of the renormalization is not period doubling but constant p-tupling, then we can construct the *p-adic* additive group of the numbers with base p using the same notions. Compare [Haz] for the p-tupling renormalization of the two dimensional Hénon-like map.

We will call the set $\mathcal{O}_F$ constructed above the *critical Cantor set* of F.

5.3 Periodic points and the critical Cantor set

There exists a one-to-one correspondence h between the critical Cantor set and the set of one sided infinite sequences of dyadic numbers by Corollary 5.2.3. Thus for every $w \in \mathcal{O}$, the unique sequence $\mathbf{w} \in W^{\infty}$ such that $h(\mathbf{w}) = w$. This corresponding word $\mathbf{w}$ to the point $w \in \mathcal{O}$ is called the *address* of w. The subscript $\mathbf{w}_n$ of the domain $B^n_{\mathbf{w}_n} \equiv \Psi^n_{\mathbf{w}_n}(B)$ is called the *address of the box domain* and the length of the address is called the *depth* of the box. Similarly, we can define the address $\mathbf{w}_n$ of the coordinate change map $\Psi^n_{\mathbf{w}_n}$. In this case, the length of the address is called the *level* of the coordinate change map.

Let us take a word, $\mathbf{w} = (w_1\, w_2\, w_3 \ldots w_n \ldots)$ as an address. The word of the first n concatenations, $\mathbf{w}_n = (w_1\, w_2\, w_3 \ldots w_n)$ is defined as the *subaddress* of the word $\mathbf{w}$.

Proposition 5.3.1. *Let F be the infinitely renormalizable Hénon-like map, namely $F \in \mathcal{I}_B(\bar{\varepsilon})$. Then the box domain $B^k_{\mathbf{w}_k}$ contains the two periodic points with the period 2^k for each $k \in \mathbb{N}$. Furthermore, $B^k_{\mathbf{w}_k}$ contains 2^n periodic points with the period 2^{n+k} for every $n \in \mathbb{N}$.*

Proof. The images of the fixed points on the box domain $B(R^kF)$ under $\Psi^k_{\mathbf{w}_k}$ for each $\mathbf{w}_k$ are the periodic points in the boxes $B^k_{\mathbf{w}_k} \Subset B$ which are mutually disjoint. Then $B^k_{\mathbf{w}_k}$ with the fixed address $\mathbf{w}_k$ contains two periodic points with the period 2^k. Similarly, each box domain $B^n_{\mathbf{w}_n}$ contains two periodic points with the period 2^n. However, the box of depth k and the depth n is defined as $B^k_{\mathbf{w}_k} \equiv \Psi^k_{\mathbf{w}_k}\big(B(R^kF)\big)$ and $B^n_{\mathbf{w}_n} \equiv \Psi^k_{\mathbf{w}_k}\big(\Psi^{n-k}_{\mathbf{w}_{n-k}}(B(R^kF))\big)$ for every $n > k$ and $k \in \mathbb{N}$. Hence, each $B^k_{\mathbf{w}_k}$ contains the mutually disjoint 2^{n-k} boxes $B^n_{\mathbf{w}_n}$ where the address $\mathbf{w}_k$ is the common subaddress of every addresses $\mathbf{w}_n$ and k is the maximal length of the all common subaddresses of $\mathbf{w}_n$. $\square$

Let the image of the fixed point $\beta_1(R^kF)$ under $\Psi^k_{v^k}$ be β_k, the periodic point under F of which period is 2^k. Then all periodic points with the period 2^k are contained in the orbit, $\operatorname{Orb}(\beta_k, F)$ for every $k \in \mathbb{N}$. Then we can let the address of β_n be v^{n+1} which is the sequence of 0s of length $n+1$. Recall v is defined to be act as 0 on the sequence of dyadic numbers on the proof of Corollary 5.2.2. $\Psi^k_{v^k}(\beta_i(R^kF))$ for $i = 0, 1$ are the periodic points with minimal period in the box domain $\Psi^k_{v^n}$. Moreover, the address of the periodic points of the minimal period, $\Psi^k_{v^k}(\beta_i(R^kF))$ for $i = 0, 1$ on each box domain defined as follows.

$$
\begin{aligned}
\mathbf{w}_n v = &\ (w_1\, w_2\, w_3 \ldots w_n\, v) \quad \text{where } i = 1 \\
\mathbf{w}_n c = &\ (w_1\, w_2\, w_3 \ldots w_n\, c) \quad \text{where } i = 0
\end{aligned}
$$

Then there is a bijection between every word with the finite length in $\bigcup_{k\geq 1} W^k$ and the set of periodic points, Per_F, that is, the set $\bigcup_{k\geq 1} W^k$ has the addresses of the every periodic points and each periodic points has the distinguishable address in $\bigcup_{k\geq 1} W^k$.

Lemma 5.3.2. *Let F be the infinitely renormalizable Hénon-like map, that is, $F \in \mathcal{I}_B(\bar{\varepsilon})$ with sufficiently small positive $\bar{\varepsilon}$. Then the set of accumulation points of periodic points of F is $\mathcal{O}_F$. In other words, $\overline{\mathrm{Per}}_F = \mathrm{Per}_F \cup \mathcal{O}_F$.*

Proof. F has 2^{k+1} periodic points with the period 2^k for every $k \in \mathbb{N}_+$. Thus if the sequence of the periodic points has the bounded maximal period, then it is a finite sequence. Since any point of the finite sequence is an isolated point, it has no accumulation point. Let us take any infinite sequence of periodic points whose period is unbounded. Every periodic points in the sequence has the address $\mathbf{w}_n v$ or $\mathbf{w}_n c$ for some $n \in N$. Select the single box domain on the depth one, B^1_v or B^1_c which contains the infinitely many periodic points. This selected box domain contains two box domains of the depth two. After repeating this process, we can find a sequence of the addresses $\{\mathbf{w}_{n_k} \,|\, k \in \mathbb{N}\}$ such that each address $\mathbf{w}_{n_i}$ is the subaddress of $\mathbf{w}_{n_{i+1}}$ where $n_k \to \infty$ as $k \to \infty$. Then the limit point w has the address of the word $\mathbf{w}$ whose length is infinity and $w \in \mathcal{O}_F$. Then $\mathcal{O}_F$ contains the set of accumulation points. For the reverse inclusion, it suffice to show that for each $w \in \mathcal{O}_F$ there exists a sequence of the periodic points converging to w. By the construction of $\mathcal{O}_F$, there exist the sequence of the box domain which converges to w. Each box domain $B^k_{\mathbf{w}_k}$ contains the periodic points such that the common subadress with maximal length of every periodic points is $\mathbf{w}_k$. Hence, the set of accumulation points of Per_F contains $\mathcal{O}_F$.

In the conclusion, we see

$$\mathrm{Per}_F \cup \mathcal{O}_F \subset \overline{\mathrm{Per}}_F$$

Furthermore, since every periodic points in $\overline{\mathrm{Per}}_F$ is isolated, we obtain

$$\mathrm{Per}_F \cup \mathcal{O}_F = \overline{\mathrm{Per}}_F$$

□

Chapter 6

Average Jacobian

Let us consider the average Jacobian of the infinitely renormalizable map F and show that the biggest Lyapunov exponent is 0 on Theorem 6.0.5.

Let the Jacobian determinant of F at w be $\operatorname{Jac} F(w)$.

$$\log \left| \frac{\operatorname{Jac} F(y)}{\operatorname{Jac} F(z)} \right| \leq C \quad \text{for any} \quad y, z \in B$$

by some constant C which is not depending on y or z. Moreover, Lemma 5.1.1 says the diameter of the domain $B^n_{\mathbf{w}}$ converges to zero exponentially fast. Then this implies the following lemma.

Lemma 6.0.3 (Distortion Lemma)**.** *There exist a constant C and the positive number $\rho < 1$ satisfying the following estimate.*

$$\log \left| \frac{\operatorname{Jac} F^k(y)}{\operatorname{Jac} F^k(z)} \right| \leq C\rho^n \quad \textit{for any} \quad y, z \in B^n_{\mathbf{w}}$$

where $k = 1, 2, 2^2, \ldots, 2^n$

Existence of the unique invariant probability measure, say μ, on $\mathcal{O}_F$ enable us to define the average Jacobian.

$$b_F \equiv b = \exp \int_{\mathcal{O}_F} \log \operatorname{Jac} F \, d\mu$$

On each level n, the measure μ on $\mathcal{O}_F$ satisfies that $\mu(B^n_{\mathbf{w}_n} \cap \mathcal{O}_F) = 1/2^n$ for every $\mathbf{w}_n$ where $\mathbf{w}_n$ is a word of length n.

Corollary 6.0.4. *For any piece of $B^n_{\mathbf{w}}$ on the level n and any point $w \in B^n_{\mathbf{w}}$,*

$$\operatorname{Jac} F^{2^n}(w) = b^{2^n}(1 + O(\rho^n))$$

where b is the average Jacobian of F for some positive $\rho < 1$.

Proof. Since

$$\int_{B^n_{\mathbf{w}}} \log \mathrm{Jac}\, F^{2^n} d\mu = \int_{\mathcal{O}} \log \mathrm{Jac}\, F \; d\mu = \log b,$$

there exists a point $\eta \in B^n_{\mathbf{w}}$ such that $\log \mathrm{Jac}\, F^{2^n}(\eta) = \dfrac{\log b}{\mu(B^n_{\mathbf{w}})} = 2^n \log b$
For any $w \in B^n_{\mathbf{w}}$, $\log\, \mathrm{Jac}\, F^{2^n}(z) \leq C\rho^n + \log\, \mathrm{Jac}\, F^{2^n}(\eta)$, and $O(\rho^n) = \log(1 + O(\rho^n))$ for a fixed constant ρ. Then

$$\begin{aligned} \log \mathrm{Jac}\, F^{2^n}(w) &= \log(1 + O(\rho^n)) + \log \mathrm{Jac}\, F^{2^n}(\eta) \\ &= \log(1 + O(\rho^n)) \cdot b^{2^n} \end{aligned}$$

Therefore $$\mathrm{Jac}\, F^{2^n}(w) = b^{2^n}(1 + O(\rho^n))$$

□

Three Lyapunov exponents χ_0, χ_1 and χ_2 exist for the three dimensional map. Let χ_0 be the maximal one. Since F is ergodic with respect to the invariant finite measure μ on the critical Cantor set, we get the following inequality.

$$|\mu|\, \chi(x) \leq \int_{\mathcal{O}_F} \log \|DF(x)\| \; d\mu(x)$$

where $|\mu|$ is the total mass of μ on $\mathcal{O}_F$.

Theorem 6.0.5. *The maximal Lyapunov exponent of F on $\mathcal{O}_F$ is 0.*

Proof. Let μ_n be $2^n\mu|_{B^n_w}$, an invariant measure under F^{2^n} and let ν_n be the (unique) invariant measure on $R^nF|_{\mathcal{O}_{R^nF}}$. Then

$$2^n\chi_0(F, \mu) = \chi_0(F^{2^n}|_{B^n_{v^n}}, \mu_n) = \chi_0(R^nF, \nu_n) \leq \int_{B^n_w} \log \|D(R^nF)\|\, d\nu_n \leq C$$

for every $n \in \mathbb{N}$, where C is a constant independent of n. The last inequality comes from the uniformly bounded C^1 norm of derivative of R^nF. Then the maximal Lyapunov exponent $\chi_0 \leq 0$. If $\chi_0 < 0$, then the support of μ contains some periodic cycles by Pesin's theory. But $\mathcal{O}_F$ does not contain any periodic cycle because F acts on $\mathcal{O}_F$ as a dyadic adding machine. Therefore, $\chi_0 = 0$ and the sum of the other exponents, $\chi_1 + \chi_2$, is $\log b$. □

Chapter 7

Universality around the tip

The universality of average Jacobian comes from the asymptotic behavior of the coordinate change Ψ^n between renormalized map $F_n \equiv R^n F$ and F^{2^n} for each $n \in \mathbb{N}$. Ψ_0^n conjugate F^{2^n} to F_n. Thus using the chain rule and Corollary 6.0.4, $\operatorname{Jac} F_n$ is the product of the average Jacobian of F^{2^n} and the ratio of the $\operatorname{Jac} \Psi_0^n$ at w and $F_n(w)$.

$$\begin{aligned}\operatorname{Jac} F_n(w) &= \operatorname{Jac} F^{2^n}(\Psi_0^n(w)) \frac{\operatorname{Jac} \Psi_0^n(w)}{\operatorname{Jac} \Psi_0^n(F_n(w))} \\ &= b^{2^n} \frac{\operatorname{Jac} \Psi_0^n(w)}{\operatorname{Jac} \Psi_0^n(F_n(w))} (1 + O(\rho^n)).\end{aligned} \tag{7.0.1}$$

where Ψ_0^n is $\Psi_{v^n}^n$.

Then on Theorem 7.5.1 below, we see that the universality of the Jacobian of the coordinate change map Ψ_0^n implies the universality of $\operatorname{Jac} F_n$. The asymptotic expression of non-linear part of Ψ_0^n is essential to the universality of $\operatorname{Jac} \Psi_0^n$.

7.1 Asymptotic of Ψ_k^n for fixed k^{th} level

For every infinitely renormalizable Hénon-like map F, we have a well defined *tip*:

$$\{\tau\} = \{\tau_F\} \equiv \bigcap_{n \geq 0} B_{v^n}^n \tag{7.1.1}$$

where the pieces B_w^n are defined in the previous sections. Let us denote the tip of the renormalizations, $\tau_k = \tau(R^k F)$ for each $k \in \mathbb{N}$. In order to simplify the notation, we would let the tip move to the origin as a fixed point of each

$\Psi^1_v(R^kF)$ for every $k \in \mathbb{N}$ by conjugation of the appropriate translations. Let us define Ψ^{k+1}_k.

$$\Psi_k \equiv \Psi^{k+1}_k = \Psi^1_v(R^kF)\,(w + \tau_{k+1}) - \tau_k \tag{7.1.2}$$

Let the derivative of the map defined Ψ_k on (7.1.2) at 0 be $D_k \equiv D^{k+1}_k$.

$$\begin{aligned} D^{k+1}_k \equiv D_k = D\Psi^{k+1}_k(0) &= D(\Psi^1_v(R^kF))(\tau_{k+1}) \\ &= D(T_k \circ \Psi^1_v(R^kF) \circ T^{-1}_{k+1})(0) \end{aligned}$$

where $T_k : w \mapsto w - \tau_k$ for each k. Then we can decompose D_k into the matrix of which diagonal entries are 1s and the diagonal matrix.

$$D_k = \begin{pmatrix} 1 & t_k & u_k \\ & 1 & \\ & d_k & 1 \end{pmatrix} \begin{pmatrix} \alpha_k & & \\ & \sigma_k & \\ & & \sigma_k \end{pmatrix} = \begin{pmatrix} \alpha_k & t_k\sigma_k & u_k\sigma_k \\ & \sigma_k & \\ & d_k\sigma_k & \sigma_k \end{pmatrix} \tag{7.1.3}$$

Moreover, we can express Ψ^{k+1}_k with the linear and non-linear parts.

$$\Psi^{k+1}_k \equiv \Psi_k(w) = D_k \circ (\mathrm{id} + \mathbf{s}_k)(w) \tag{7.1.4}$$

where $w = (x, y, z)$ and $\mathbf{s}_k(w) = (s_k(w),\ 0,\ r_k(y)) = O(|w|^2)$ near the origin.

Comparing the derivative of $H^{-1} \circ \Lambda^{-1}$ at 0 and D_k and (5.2.2), we obtain the following estimates

$$\begin{aligned} t_k &= \partial_y \phi^{-1}_k(\tau_{k+1}) = \partial_x \phi^{-1}_k(\tau_{k+1}) \cdot \partial_y \varepsilon_k(\tau_k) + \partial_z \phi^{-1}_k(\tau_{k+1}) \cdot d_k = O(\bar{\varepsilon}^{2^k}) \\ u_k &= \partial_z \phi^{-1}_k(\tau_{k+1}) = \partial_x \phi^{-1}_k(\tau_{k+1}) \cdot \partial_z \varepsilon_k(\tau_k) = O(\bar{\varepsilon}^{2^k}) \\ \text{and} \quad d_k &= \frac{d}{dy}\, \delta_k \left(\pi_y(\tau_{k+1}), f^{-1}_k(\pi_y(\tau_{k+1})), 0\right) = O(\bar{\varepsilon}^{2^k}) \end{aligned} \tag{7.1.5}$$

where $\phi^{-1}_k(w) = \pi_x \circ H^{-1}_k(w)$. Furthermore, $\sigma_k = -\sigma\left(1 + O(\rho^k)\right)$ and $\alpha_k = \sigma^2\left(1 + O(\rho^k)\right)$ for some $\rho \in (0,1)$ because $\partial_x \phi^{-1}_k$ exponentially converges to σ uniformly as $k \to \infty$.

Lemma 7.1.1. *Let s_k be the function defined on (7.1.4). For each $k \in \mathbb{N}$*

(1) $|\,\partial_x s_k| = O(1), \qquad |\,\partial_y s_k| = O(\bar{\varepsilon}^{2^k}), \qquad |\,\partial_z s_k| = O(\bar{\varepsilon}^{2^k})$

(2) $|\,\partial^2_{xx} s_k| = O(1)$, $\qquad |\,\partial^2_{xy} s_k| = O(\bar\varepsilon^{2^k})$, $\qquad |\,\partial^2_{yy} s_k| = O(\bar\varepsilon^{2^k})$

(3) $|\,\partial^2_{yz} s_k| = O(\bar\varepsilon^{2^k})$, $\qquad |\,\partial^2_{zx} s_k| = O(\bar\varepsilon^{2^k})$, $\qquad |\,\partial^2_{zz} s_k| = O(\bar\varepsilon^{2^k})$

(4) $|\,r_k(y)| = O(\bar\varepsilon^{2^k})$, $\qquad |\,r'_k(y)| = O(\bar\varepsilon^{2^k})$, $\qquad |r''_k(y)| = O(\bar\varepsilon^{2^k})$

Proof. Ψ_k has the two expressions, $D_k \circ (\mathrm{id} + \mathbf{s}_k)(w)$ and $T_k \circ H_k^{-1} \circ \Lambda_k \circ T_{k+1}^{-1}$. That is,

$$\begin{aligned}\Psi_k &= D_k \circ (\mathrm{id} + \mathbf{s}_k)(w) \\ &= T_k \circ H_k^{-1} \circ \Lambda_k \circ T_{k+1}^{-1} = H_k^{-1} \circ \Lambda_k(w + \tau_{k+1}) - \tau_k\end{aligned}$$

In order to obtain the asymptotic behavior of the non-linear part of Ψ_k, we need to compare the third and the first coordinates of these two expressions of Ψ_k. Let $\tau_k = (\tau_k^x, \tau_k^y, \tau_k^z)$ for each $k \geq 1$.
Let us compare the third coordinates of these two expression of Ψ_k.

$$\begin{aligned}\sigma_k(d_k y + z + r_k(y)) &= \pi_z\big(H_k^{-1} \circ \Lambda_k(w + \tau_{k+1}) - \tau_k\big) \\ &= \sigma_k(z + \tau_{k+1}^z) + \delta\big(\sigma_k(y + \tau_{k+1}^y),\ f^{-1}(\sigma_k(y + \tau_{k+1}^y)),\ 0\big) \\ &\quad - \tau_k^z\end{aligned}$$

Thus we have the following equation.

$$\sigma_k r_k(y) = -\sigma_k d_k y + \delta\big(\sigma_k(y + \tau_{k+1}^y),\ f^{-1}(\sigma_k(y + \tau_{k+1}^y)),\ 0\big) + \sigma_k \tau_{k+1}^z - \tau_k^z$$

Then $|\,r_k(y)| \leq C\big(|\,d_k y| + \|\delta\|_{C^0}\big)$ for some $C > 0$. The domain is bounded and $\|\delta\|$ is $O(\bar\varepsilon^{2^n})$. Hence, $|r_k(y)| = O(\bar\varepsilon^{2^k})$. Moreover,

$$r'_k(y) = -d_k + \frac{d}{dy}\,\delta\big(\sigma_k(y + \tau_{k+1}^y),\ f^{-1}(\sigma_k(y + \tau_{k+1}^y),\ 0\big)$$

Then $|\,r'_k(y)| = O(\bar\varepsilon^{2^k})$. The second derivative $|\,r'_k(y)|$ is also controlled by $\|\delta\|_{C^2}$. Then $|\,r''_k(y)| = O(\bar\varepsilon^{2^k})$.
Comparison of first coordinates implies the following.

$$\alpha_k x + \alpha_k s_k(w) + \sigma_k t_k y + \sigma_k(u_k z + r_k(y)) = \phi_k^{-1}(\sigma_k w + \sigma_k \tau_{k+1}) - \pi_x(\tau_k). \tag{7.1.6}$$

It implies the following equations.

$$
\begin{aligned}
\partial_x s_k &= \sigma_k \partial_x \phi_k^{-1} - \alpha_k \\
\partial_y s_k &= \sigma_k \partial_y \phi_k^{-1} - \sigma_k t_k - \sigma_k r_k'(y) \\
\partial_z s_k &= \sigma_k \partial_z \phi_k^{-1} - \sigma_k u_k
\end{aligned}
\tag{7.1.7}
$$

The norm of ε_k and δ_k is uniformly bounded above on the domain $B(F_k)$. Then by the equations (5.1.1), $|\,\partial_x \phi_k^{-1}| = O(1)$, $|\,\sigma_k\,\partial_y \phi_k^{-1}| = O\big(\bar{\varepsilon}^{2^k}\big)$ and $|\,\sigma_k\,\partial_z \phi_k^{-1}| = O\big(\bar{\varepsilon}^{2^k}\big)$. Moreover, by (7.1.5) t_k and u_k is $O\big(\bar{\varepsilon}^{2^k}\big)$. Hence, $|\,\partial_x s_k| = O(1)$, $|\,\partial_y s_k| = O\big(\bar{\varepsilon}^{2^k}\big)$ and $|\,\partial_z s_k| = O\big(\bar{\varepsilon}^{2^k}\big)$.
By the above equation (7.1.7), each second partial derivatives of s_k are comparable with the second partial derivatives of ϕ^{-1} over the same variables because $|r_k''(y)| = O\big(\bar{\varepsilon}^{2^k}\big)$. When calculating each partial derivatives, we obtain the bounds of each second partial derivatives of ϕ^{-1} is $O\big(\bar{\varepsilon}^{2^k}\big)$. For example, the second equation of (5.1.1)

$$
\phi_y^{-1} + \phi_x^{-1} \cdot (-\varepsilon_y) + \phi_z^{-1} \cdot \left(-\frac{d}{dy}\, \delta(y, f^{-1}(y), 0) \right) = 0
$$

implies that

$$
\begin{aligned}
\phi_{yy}^{-1} + \phi_{xy}^{-1} \cdot (-\varepsilon_y) + \phi_x^{-1} \cdot (-\varepsilon_{yy}) + \phi_{zy}^{-1} \cdot \left(-\frac{d}{dy}\, \delta(y, f^{-1}(y), 0) \right) \\
+ \phi_z^{-1} \cdot \left(-\frac{d^2}{dy^2}\, \delta(y, f^{-1}(y), 0) \right) = 0
\end{aligned}
$$

Then C^2 norms of ε and δ, each bounds of first and second partial derivatives of ϕ^{-1} except ϕ_{yy}^{-1} itself imply that the bounds of $|\phi_{yy}^{-1}|$ is $O\big(\bar{\varepsilon}^{2^k}\big)$. $\square$

7.2 The estimation of non linear part S_k^n from level k to the fixed level n

We consider the behavior of the coordinate change map from kth level to nth level. Let

$$
\Psi_k^n = \Psi_k \circ \cdots \circ \Psi_{n-1}, \quad B_k^n = \operatorname{Im} \Psi_k^n
$$

By Lemma 5.1.1,

$$
\operatorname{diam}(B_k^n) = O(\sigma^{n-k}) \qquad \text{for} \quad k < n
$$

Then combining Lemma 5.1.1 and Lemma 7.1.1, we have the following corollary.

Corollary 7.2.1. *For each $w \in B^n_k$ where $k < n$, we have*

$$|\,\partial_x s_k(w)| = O(\sigma^{n-k}) \qquad |\,\partial_y s_k(w)| = O\big(\bar{\varepsilon}^{2^k}\sigma^{n-k}\big) \qquad |\,\partial_z s_k(w)| = O\big(\bar{\varepsilon}^{2^k}\sigma^{n-k}\big)$$
$$|\,r_k(y)| = O\big(\bar{\varepsilon}^{2^k}\sigma^{n-k}\big) \qquad |\,r'_k(y)| = O\big(\bar{\varepsilon}^{2^k}\sigma^{n-k}\big)$$

Since the origin is the fixed point of each Ψ_j and D_j is $\Psi_j(0)$ for every $k \leq j \leq n$, we can let the derivative of Ψ^n_k at the origin be the composition of consecutive D_is for $k \leq i \leq n-1$.

$$D^n_k = D_k \circ D_{k+1} \circ \cdots \circ D_{n-1}$$

We can decompose D^n_k to two matrices, the matrix whose diagonal entries are ones and the diagonal matrix.

Lemma 7.2.2. *The derivative of Ψ^n_k at the origin, D^n_k is decomposed the sheer and scaling parts as follows.*

$$D^n_k = \begin{pmatrix} 1 & t_{n,\,k} & u_{n,\,k} \\ & 1 & \\ & d_{n,\,k} & 1 \end{pmatrix} \begin{pmatrix} \alpha_{n,\,k} & & \\ & \sigma_{n,\,k} & \\ & & \sigma_{n,\,k} \end{pmatrix}$$

Moreover, $\alpha_{n,\,k} = (\sigma^2)^{n-k}(1+O(\rho^k))$ and $\sigma_{n,k} = (-\sigma)^{n-k}(1+O(\rho^k))$ for some $\rho \in (0,1)$. Each $t_{n,\,k}$, $u_{n,\,k}$ and $d_{n,\,k}$ are comparable with the $t_{k+1,\,k}$, $u_{k+1,\,k}$ and $d_{k+1,\,k}$ respectively and converges to the numbers $t_{*,\,k}$, $u_{*,\,k}$ and $d_{*,\,k}$ super exponentially fast as $n \to \infty$.

Proof. Using the definition of each derivatives of Ψ_j on (7.1.3) at the fixed point zero, we obtain the following.

$$D^n_k = \prod_{j=k}^{n-1} D_j = \prod_{j=k}^{n-1} \begin{pmatrix} \alpha_j & t_j\,\sigma_j & u_j\,\sigma_j \\ & \sigma_j & \\ & d_j\,\sigma_j & \sigma_j \end{pmatrix}$$

By the straightforward calculation,

$$D_k^n = \begin{pmatrix} \prod_{j=k}^{n-1} \alpha_j & T_{n,k} & U_{n,k} \\ & \prod_{j=k}^{n-1} \sigma_j & \\ & \prod_{j=k}^{n-1} \sigma_j \sum_{j=k}^{n-1} d_j & \prod_{j=k}^{n-1} \sigma_j \end{pmatrix} \tag{7.2.1}$$

where

$$\begin{aligned} U_{n,k} &= \sigma_k \, \sigma_{k+1} \, \sigma_{k+2} \cdots \sigma_{n-2} \, \sigma_{n-1} \, u_k \\ &+ \alpha_k \, \sigma_{k+1} \, \sigma_{k+2} \cdots \sigma_{n-2} \, \sigma_{n-1} \, u_{k+1} \\ &+ \alpha_k \, \alpha_{k+1} \, \sigma_{k+2} \cdots \sigma_{n-2} \, \sigma_{n-1} \, u_{k+2} \\ &\quad \vdots \\ &+ \alpha_k \, \alpha_{k+1} \, \alpha_{k+2} \cdots \alpha_{n-2} \, \sigma_{n-1} \, u_{n-1} \end{aligned}$$

$$\begin{aligned} &T_{n,k} \\ = \;& \sigma_k \, \sigma_{k+1} \, \sigma_{k+2} \cdots \sigma_{n-2} \, \sigma_{n-1} \left[\; u_k \, (d_{k+1} + d_{k+2} + d_{k+3} + \cdots + d_{n-1}) + t_k \right] \\ + \;& \alpha_k \, \sigma_{k+1} \, \sigma_{k+2} \cdots \sigma_{n-2} \, \sigma_{n-1} \left[u_{k+1} \, (\qquad d_{k+2} + d_{k+3} + \cdots + d_{n-1}) + t_{k+1} \right] \\ + \;& \alpha_k \, \alpha_{k+1} \, \sigma_{k+2} \cdots \sigma_{n-2} \, \sigma_{n-1} \left[u_{k+2} \, (\qquad\qquad d_{k+3} + \cdots + d_{n-1}) + t_{k+2} \right] \\ & \quad \vdots \\ + \;& \alpha_k \, \alpha_{k+1} \, \alpha_{k+2} \cdots \alpha_{n-2} \, \sigma_{n-1} \left[u_{n-1} + t_{n-1} \right] \end{aligned}$$

Then we have the followings.

$$\begin{aligned} \sigma_{n,k} &= \prod_{j=k}^{n-1} \sigma_j = \prod_{j=k}^{n-1} (-\sigma)(1 + O(\rho^j)) = (-\sigma)^{n-k}(1 + O(\rho^k)) \\ \alpha_{n,k} &= \prod_{j=k}^{n-1} \alpha_j = \prod_{j=k}^{n-1} \sigma^2(1 + O(\rho^j)) = \sigma^{2(n-k)}(1 + O(\rho^k)) \end{aligned} \tag{7.2.2}$$

By the definition of $d_{n,k}$ and (7.2.2), each components of the sheer part and the scaling part are separated.

$$
\begin{aligned}
d_{n,\,k} &= \sum_{j=k}^{n-1} d_j \\
u_{n,\,k} &= \sum_{j=k}^{n-1} (-\sigma)^{j-k} u_j \,(1 + O(\rho^k)) \\
t_{n,\,k} &= \sum_{j=k}^{n-1} (-\sigma)^{j-k} \left[u_j \sum_{i=j}^{n-2} d_{i+1} + u_{n-1} + t_j \right] (1 + O(\rho^k))
\end{aligned}
\tag{7.2.3}
$$

Since $d_j = O\big(\bar{\varepsilon}^{2^j}\big)$, $u_j = O\big(\bar{\varepsilon}^{2^j}\big)$ and $t_j = O\big(\bar{\varepsilon}^{2^j}\big)$ for each $j \in \mathbb{N}$, each terms of the series in (7.2.3) shrinks super exponentially fast. Then the sum $d_{n,\,k}$, $u_{n,\,k}$ and $t_{n,\,k}$ are comparable with first terms of each series. Moreover, $d_{n,\,k}$, $u_{n,\,k}$ and $t_{n,\,k}$ converges to some numbers $d_{*,\,k}$, $u_{*,\,k}$ and $t_{*,\,k}$ as $n \to \infty$ super exponentially fast respectively. $\square$

After reshuffling of Ψ^n_k we can factor out D^n_k from the map Ψ^n_k. Then we have

$$
\Psi^n_k = D^n_k \circ (\mathrm{id} + \mathbf{S}^n_k) \tag{7.2.4}
$$

where $\mathbf{S}^n_k = (S^n_k(w),\ 0,\ R^n_k(y)) = O(|w|^2)$ near the origin. When we calculate directly the composition from $H^{-1}_k \circ \Lambda^{-1}_k$ to $H^{-1}_{n-1} \circ \Lambda^{-1}_{n-1}$, R^n_k depends only on y, the second coordinate of the point.

Proposition 7.2.3. *The third coordinate of $\mathbf{S}^n_k$, $R^n_k(y)$ has the following asymptotic.*

$$
|R^n_k| = O\big(\bar{\varepsilon}^{2^k}\big), \quad |(R^n_k)'| = O\big(\bar{\varepsilon}^{2^k} \sigma^{n-k}\big) \quad \textit{and} \quad |(R^n_k)''| = O\big(\bar{\varepsilon}^{2^k} (\sigma^2)^{n-k}\big)
$$

for all $k < n$.

Proof. The proof comes from the recursive formula between each partial derivatives of S^n_k and S^n_{k+1}. So before proving this lemma we need some intermediate calculations. For a point $w = (x, y, z) \in B$, let

$$
w^n_{k+1} = \begin{pmatrix} x^n_{k+1} \\ y^n_{k+1} \\ z^n_{k+1} \end{pmatrix} = \Psi^n_{k+1}(w) \in B^{n-k}_{v^{n-k}}(R^{n-k}F)
$$

By (7.2.4), we have

$$\begin{pmatrix} x_{k+1}^n \\ y_{k+1}^n \\ z_{k+1}^n \end{pmatrix} = \begin{pmatrix} \alpha_{n,\,k+1} & \sigma_{n,\,k+1} \cdot t_{n,\,k+1} & \sigma_{n,\,k+1} \cdot u_{n,\,k+1} \\ & \sigma_{n,\,k+1} & \\ & \sigma_{n,\,k+1} \cdot d_{n,\,k+1} & \sigma_{n,\,k+1} \end{pmatrix} \begin{pmatrix} x + S_{k+1}^n(w) \\ y \\ z + R_{k+1}^n(y) \end{pmatrix}$$

Then each coordinate of w_{k+1}^n are

$$\begin{aligned} x_{k+1}^n &= \alpha_{n,\,k+1}(x + S_{k+1}^n(w)) + \sigma_{n,\,k+1} t_{n,\,k+1} \cdot y + \sigma_{n,\,k+1} u_{n,\,k+1}(z + R_{k+1}^n(y)) \\ y_{k+1}^n &= \sigma_{n,\,k+1} \cdot y \\ z_{k+1}^n &= \sigma_{n,\,k+1}\, d_{n,\,k+1} \cdot y + \sigma_{n,\,k+1}(z + R_{k+1}^n(y)) \end{aligned} \tag{7.2.5}$$

Moreover, for any fixed $n > k$ the recursive formula for k is

$$\begin{aligned} D_k^n \circ (\mathrm{id} + \mathbf{S}_k^n) = \Psi_k^n = \Psi_k \circ \Psi_{k+1}^n &= D_k \circ (\mathrm{id} + \mathbf{s}_k) \circ \Psi_{k+1}^n \\ &= D_k^n \circ (\mathrm{id} + \mathbf{S}_{k+1}^n) + D_k \circ \mathbf{s}_k \circ \Psi_{k+1}^n \end{aligned} \tag{7.2.6}$$

$$\text{Thus} \qquad \Psi_k^n(w) = D_k^n \circ (\mathrm{id} + \mathbf{S}_{k+1}^n)(w) + D_k \circ \mathbf{s}_k(w_{k+1}^n)$$

and note that

$$D_k \circ \mathbf{s}_k(w_{k+1}^n) = \begin{pmatrix} \alpha_k & t_k \sigma_k & u_k \sigma_k \\ & \sigma_k & \\ & d_k \sigma_k & \sigma_k \end{pmatrix} \begin{pmatrix} s_k(w_{k+1}^n) \\ 0 \\ r_k(y_{k+1}^n) \end{pmatrix}$$

Moreover, the first partial derivatives of each coordinate are

$$\begin{aligned}
\frac{\partial x_{k+1}^n}{\partial x} &= \alpha_{n,\,k+1}\left(1+\frac{\partial S_{k+1}^n}{\partial x}(w)\right)\\
\frac{\partial x_{k+1}^n}{\partial y} &= \alpha_{n,\,k+1}\frac{\partial S_{k+1}^n}{\partial y}(w)+\sigma_{n,\,k+1}t_{n,\,k+1}+\sigma_{n,\,k+1}u_{n,k+1}(R_{k+1}^n)'(y)\\
\frac{\partial x_{k+1}^n}{\partial z} &= \alpha_{n,\,k+1}\frac{\partial S_{k+1}^n}{\partial z}(w)+\sigma_{n,\,k+1}u_{n,k+1}\\
\frac{\partial y_{k+1}^n}{\partial y} &= \frac{\partial z_{k+1}^n}{\partial z}=\sigma_{n,\,k+1}\\
\frac{\partial z_{k+1}^n}{\partial y} &= \sigma_{n,k+1}d_{n,\,k+1}+\sigma_{n,\,k+1}(R_{k+1}^n)'(y)\\
\frac{\partial y_{k+1}^n}{\partial x} &= \frac{\partial y_{k+1}^n}{\partial z}=\frac{\partial z_{k+1}^n}{\partial x}=0
\end{aligned} \tag{7.2.7}$$

In order to estimate of $R_k^n(y)$, compare the third coordinates of the functions in (7.2.6) (and recall $\sigma^{-1}=\lambda$). Then

$$\begin{aligned}
z_k^n &= \sigma_{n,\,k}\,d_{n,\,k}\cdot y+\sigma_{n,\,k}(z+R_k^n(y))\\
&= \sigma_{n,\,k}\,d_{n,\,k}\cdot y+\sigma_{n,\,k}(z+R_{k+1}^n(y))+\sigma_k\cdot r_k(y_{k+1}^n)\\
\text{Then}\quad R_k^n(y) &= R_{k+1}^n(y)+\sigma_{n,\,k}^{-1}\cdot\sigma_k\cdot r_k(y_{k+1}^n)
\end{aligned}$$

where $\sigma_{n,\,k}^{-1}\cdot\sigma_k$ is $(-\lambda)^{n-k-1}(1+O(\rho^k))$. By (7.2.7), the recursive relation between $R_k^n(y)$, $R_{k+1}^n(y)$ and the bounds of $r_k(y_{k+1}^n)$, we obtain the following formulas.

$$\begin{aligned}
R_k^n(y) &= R_{k+1}^n(y)+O\big((-\lambda)^{n-k-1}r_k(y_{k+1}^n)\big)\\
(R_k^n)'(y) &= (R_{k+1}^n)'(y)+O\big(r_k'(y_{k+1}^n)\big)\\
\text{and}\quad (R_k^n)''(y) &= (R_{k+1}^n)''(y)+O\big(\sigma^{n-k}\cdot r_k''(y_{k+1}^n)\big)
\end{aligned}$$

Hence, by the equation (7.2.5) and the chain rule

$$\begin{aligned}
|R_k^n| &\le |R_{k+1}^n|+K_0\bar{\varepsilon}^{2^k}\\
|(R_k^n)'| &\le |(R_{k+1}^n)'|+K_1\bar{\varepsilon}^{2^k}\sigma^{n-k}\\
|(R_k^n)''| &\le |(R_{k+1}^n)''|+K_2\bar{\varepsilon}^{2^k}(\sigma^2)^{n-k}
\end{aligned}$$

for all $k < n$. Then,

$$\begin{aligned}
|R_k^n| &= O(\bar{\varepsilon}^{2^k}), \\
|(R_k^n)'| &= O(\bar{\varepsilon}^{2^k}\sigma^{n-k}) \\
\text{and} \quad |(R_k^n)''| &= O(\bar{\varepsilon}^{2^k}(\sigma^2)^{n-k}) \quad \text{for all} \quad k < n
\end{aligned}$$

□

Lemma 7.2.4. *For $k < n$ we have*

(1) $|\partial_x S_k^n| = O(1)$, $\quad |\partial_y S_k^n| = O(\bar{\varepsilon}^{2^k})$, $\quad |\partial_z S_k^n| = O(\bar{\varepsilon}^{2^k})$

(2) $|\partial^2_{xy} S_k^n| = O(\bar{\varepsilon}^{2^k}\sigma^{n-k})$, $\quad |\partial^2_{xz} S_k^n| = O(\bar{\varepsilon}^{2^k}\sigma^{n-k})$

(3) $|\partial^2_{yz} S_k^n| = O(\bar{\varepsilon}^{2^k})$, $\quad |\partial^2_{zz} S_k^n| = O(\bar{\varepsilon}^{2^k})$

Proof. Compare the first coordinates of Ψ_k^n in (7.2.6). Thus

$$\begin{aligned}
x_k^n &= \alpha_{n,k}(x + S_k^n(w)) + \sigma_{n,k}\, t_{n,k} \cdot y + \sigma_{n,k}\, u_{n,k}\big(z + R_k^n(y)\big) \\
&= \alpha_{n,k}(x + S_{k+1}^n(w)) + \sigma_{n,k}\, t_{n,k} \cdot y + \sigma_{n,k}\, u_{n,k}\big(z + R_{k+1}^n(y)\big) \\
&\quad + \alpha_k \cdot s_k(w_{k+1}^n) + u_k \cdot r_k(y_{k+1}^n)
\end{aligned}$$

Then we obtain the recursive formula for S_k^n.

$$\begin{aligned}
S_k^n(w) = S_{k+1}^n(w) &+ \alpha_{n,k}^{-1}\alpha_k \cdot s_k(w_{k+1}^n) + \alpha_{n,k}^{-1}\sigma_{n,k}\, u_{n,k}\big(R_{k+1}^n(y) - R_k^n(y)\big) \\
&+ \alpha_{n,k}^{-1}\, u_k \cdot r_k(y_{k+1}^n)
\end{aligned}$$

Let us take the first partial derivatives of each side of above equation and use (7.2.7). Then we can have the recursive formulas of each first partial derivatives of $S_k^n(w)$.

$$
\begin{aligned}
\frac{\partial S_k^n}{\partial x} &= \frac{\partial S_{k+1}^n}{\partial x}\left(1+\frac{\partial s_k}{\partial x_{k+1}^n}\right)+\frac{\partial s_k}{\partial x_{k+1}^n} \\
\frac{\partial S_k^n}{\partial y} &= \left(1+\frac{\partial s_k}{\partial x_{k+1}^n}\right)\frac{\partial S_{k+1}^n}{\partial y}+K_1\lambda^{n-k-1}\left[\Big(t_{n,\,k+1}+u_{n,\,k+1}\cdot(R_{k+1}^n)'(y)\Big)\frac{\partial s_k}{\partial x_{k+1}^n}\right. \\
&\quad \left.+\frac{\partial s_k}{\partial y_{k+1}^n}+\Big(d_{n,\,k+1}+(R_{k+1}^n)'(y)\Big)\frac{\partial s_k}{\partial z_{k+1}^n}\ \right] \\
&\quad +K_2\lambda^{n-k}u_{n,\,k}\Big((R_{k+1}^n)'(y)-(R_k^n)'(y)\Big)+K_3\lambda^{n-k+1}u_k\cdot r_k'(y_n^{k+1}) \\
\frac{\partial S_k^n}{\partial z} &= \left(1+\frac{\partial s_k}{\partial x_{k+1}^n}\right)\frac{\partial S_{k+1}^n}{\partial z}+K_1\lambda^{n-k-1}\left[u_{n,\,k+1}\frac{\partial s_k}{\partial x_{k+1}^n}+\frac{\partial s_k}{\partial z_{k+1}^n}\ \right]
\end{aligned}
$$

where $\alpha_{n,\,k}^{-1}\cdot\alpha_k\cdot\sigma_{n,\,k+1}=K_1(-\lambda)^{n-k-1}$, $\alpha_{n,\,k}^{-1}\cdot\sigma_{n,\,k}=K_2(-\lambda)^{n-k}$ and $\alpha_{n,\,k}^{-1}\cdot\sigma_{n,\,k+1}=K_3(-\lambda)^{n-k+1}$.
By Corollary 7.2.1 and Proposition 7.2.3 , we have the following estimation

$$
\left|\frac{\partial s_k}{\partial x_{k+1}^n}\right|=O(\sigma^{n-k}),\qquad \left|\frac{\partial s_k}{\partial y_{k+1}^n}\right|=O(\bar\varepsilon^{2^k}\sigma^{n-k}),\qquad \left|\frac{\partial s_k}{z_{k+1}^n}\right|=O(\bar\varepsilon^{2^k}\sigma^{n-k})
$$

Moreover, $|\,t_{n,\,k}|$, $|\,u_{n,\,k}|$ and $|\,d_{n,\,k}|$ are $O(\bar\varepsilon^{2^k})$. With all these facts, the bounds of each partial derivatives of S_k^n are on the following.

$$
\begin{aligned}
\left|\frac{\partial S_k^n}{\partial x}\right| &\le (1+O(\rho^{n-k}))\left|\frac{\partial S_{k+1}^n}{\partial x}\right|+C\sigma^{n-k} \\
\left|\frac{\partial S_k^n}{\partial y}\right| &\le \big(1+O(\rho^{n-k})\big)\left|\frac{\partial S_{k+1}^n}{\partial y}\right|+C\bar\varepsilon^{2^k} \\
\left|\frac{\partial S_k^n}{\partial z}\right| &\le \big(1+O(\rho^{n-k})\big)\left|\frac{\partial S_{k+1}^n}{\partial z}\right|+C\bar\varepsilon^{2^k}
\end{aligned}
$$

for some constant $C>0$ and $\rho\in(0,1)$.
Hence, using above recursive formulas we have

$$
\left|\frac{\partial S_k^n}{\partial x}\right|=O(\sigma),\qquad \left|\frac{\partial S_k^n}{\partial y}\right|=O(\bar\varepsilon^{2^k})\quad\text{and}\quad \left|\frac{\partial S_k^n}{\partial z}\right|=O(\bar\varepsilon^{2^k})
$$

for all $k<n$.
For later use let us calculate the second partial derivatives of w_{k+1}^n using (7.2.7).

The second partial derivatives are

$$
\begin{aligned}
&\frac{\partial^2 x_{k+1}^n}{\partial x^2} = \alpha_{n,\,k+1}\frac{\partial^2 S_{k+1}^n}{\partial x^2}(w), \qquad \frac{\partial^2 x_{k+1}^n}{\partial xy} = \alpha_{n,\,k+1}\frac{\partial^2 S_{k+1}^n}{\partial xy}(w)\\
&\frac{\partial^2 x_{k+1}^n}{\partial xz} = \alpha_{n,\,k+1}\frac{\partial^2 S_{k+1}^n}{\partial xz}(w)\\
&\frac{\partial^2 x_{k+1}^n}{\partial y^2} = \alpha_{n,\,k+1}\frac{\partial^2 S_{k+1}^n}{\partial y^2}(w) + \sigma_{n,\,k+1}u_{n,\,k+1}(R_{k+1}^n)''(y)\\
&\frac{\partial^2 x_{k+1}^n}{\partial yz} = \alpha_{n,\,k+1}\frac{\partial^2 S_{k+1}^n}{\partial yz}(w), \qquad \frac{\partial^2 z_{k+1}^n}{\partial y^2} = \sigma_{n,\,k+1}(R_{k+1}^n)''(y)
\end{aligned}
\tag{7.2.8}
$$

and other second order partial derivatives are identically 0.

The second partial derivatives of S_k^n are the following.

$$
\begin{aligned}
\frac{\partial^2 S_k^n}{\partial xy} = &\left(1+\frac{\partial s_k}{\partial x_{k+1}^n}\right)\frac{\partial^2 S_{k+1}^n}{\partial xy} + \alpha_{n,\,k+1}\left(1+\frac{\partial S_{k+1}^n}{\partial x}\right)\frac{\partial^2 s_k}{\partial (x_{k+1}^n)^2}\frac{\partial S_{k+1}^n}{\partial y}\\
&+\sigma_{n,\,k+1}\left(1+\frac{\partial S_{k+1}^n}{\partial x}\right)\left[\left(t_{n,\,k+1}+u_{n,\,k+1}(R_{k+1}^n)'(y)\right)\frac{\partial^2 s_k}{\partial (x_{k+1}^n)^2}\right.\\
&\left.+\frac{\partial^2 s_k}{\partial x_{k+1}^n y_{k+1}^n} + \left(d_{n,\,k+1}+(R_{k+1}^n)'(y)\right)\frac{\partial^2 s_k}{\partial x_{k+1}^n z_{k+1}^n}\right]
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 S_k^n}{\partial xz} = &\left(1+\frac{\partial s_k}{\partial x_{k+1}^n}\right)\frac{\partial^2 S_{k+1}^n}{\partial xz} + \alpha_{n,\,k+1}\left(1+\frac{\partial S_{k+1}^n}{\partial x}\right)\frac{\partial S_{k+1}^n}{\partial z}\\
&+\sigma_{n,\,k+1}\left(1+\frac{\partial S_{k+1}^n}{\partial x}\right)\cdot\left[u_{n,\,k+1}\frac{\partial^2 s_k}{\partial (x_{k+1}^n)^2}+\frac{\partial^2 s_k}{\partial x_{k+1}^n z_{k+1}^n}\right]
\end{aligned}
$$

$$\frac{\partial^2 S^n_k}{\partial yz} = \left(1 + \frac{\partial s_k}{\partial x^n_{k+1}}\right) \frac{\partial^2 S^n_{k+1}}{\partial yz} + \Bigg[\alpha_{n,\,k+1} \frac{\partial S^n_{k+1}}{\partial z} \frac{\partial S^n_{k+1}}{\partial y} + \sigma_{n,\,k+1} u_{n,\,k+1} \frac{\partial S^n_{k+1}}{\partial y}$$

$$+ \sigma_{n,\,k+1}\Big(t_{n,\,k+1} + u_{n,\,k+1}(R^n_{k+1})'(y)\Big)\Big(\frac{\partial S^n_{k+1}}{\partial z} + K_1(-\lambda)^{n-k-1} u_{n,\,k+1}\Big)\Bigg]$$

$$\cdot \frac{\partial^2 s_k}{\partial (x^n_{k+1})^2} + \sigma_{n,\,k+1}\left(\frac{\partial S^n_{k+1}}{\partial z} + K_1(-\lambda)^{n-k-1} u_{n,\,k+1}\right) \frac{\partial^2 s_k}{\partial x^n_{k+1} y^n_{k+1}}$$

$$+ \sigma_{n,k+1}\Bigg[\frac{\partial S^n_{k+1}}{\partial y} + \big(d_{n,\,k+1} + (R^n_{k+1})'(y)\big)\frac{\partial S^n_{k+1}}{\partial z}$$

$$+ K_4\left(t_{n,\,k+1} + u_{n,\,k+1} d_{n,\,k+1} + 2u_{n,\,k+1}(R^n_{k+1})'(y)\right)\Bigg] \frac{\partial^2 s_k}{\partial x^n_{k+1} z^n_{k+1}}$$

$$+ K_4 \frac{\partial^2 s_k}{\partial y^n_{k+1} z^n_{k+1}} + K_4\Big(d_{n,\,k+1} + (R^n_{k+1})'(y)\Big)\frac{\partial^2 s_k}{\partial (z^n_{k+1})^2}$$

$$\frac{\partial^2 S^n_k}{\partial z^2} = \left(1 + \frac{\partial s_k}{\partial x^n_{k+1}}\right) \frac{\partial^2 S^n_{k+1}}{\partial z^2} + \left(\sigma_{n,\,k+1} u_{n,\,k+1} \frac{\partial S^n_{k+1}}{\partial z} + K_4 u^2_{n,k+1}\right) \frac{\partial^2 s_k}{\partial (x^n_{k+1})^2}$$

$$+ \left(\sigma_{n,\,k+1} \frac{\partial S^n_{k+1}}{\partial z} + 2K_4 u_{n,\,k+1}\right) \frac{\partial s_k}{\partial x^n_{k+1} z^n_{k+1}} + K_4 \frac{\partial^2 s_k}{\partial (z^n_{k+1})^2}$$

where $K_4 = \alpha^{-1}_{n,\,k} \alpha_k\, \sigma^2_{n,\,k+1} = O(1)$.

By Lemma 7.1.7, Corollary 7.2.1, and Proposition 7.2.3 , the bounds of each second derivatives of s_k is the following

$$\left|\frac{\partial^2 s_k}{\partial (x^n_{k+1})^2}\right| = O(\sigma^{n-k}), \quad \left|\frac{\partial^2 s_k}{\partial uv}\right| = O(\bar{\varepsilon}^{2^k} \sigma^{n-k})$$

where $u, v = x^n_{k+1}, y^n_{k+1}, z^n_{k+1}$ but both u and v are not x^n_{k+1} simultaneously. With the bounds of first partial derivatives of s_k, the estimation of $|\,t_{n,\,k}|$, $|\,u_{n,\,k}|$ and $|d_{n,\,k}|$ and the bounds of second derivatives of s_k, we have the bounds of second derivatives of S_k^n as follows.

$$\begin{aligned}
\left|\frac{\partial^2 S_k^n}{\partial xy}\right| &\leq \big(1+O(\rho^{n-k})\big)\left|\frac{\partial^2 S_{k+1}^n}{\partial xy}\right| + C\bar{\varepsilon}^{2^k}\sigma^{n-k} \\
\left|\frac{\partial^2 S_k^n}{\partial xz}\right| &\leq \big(1+O(\rho^{n-k})\big)\left|\frac{\partial^2 S_{k+1}^n}{\partial xz}\right| + C\bar{\varepsilon}^{2^k}\sigma^{n-k} \\
\left|\frac{\partial^2 S_k^n}{\partial yz}\right| &\leq \big(1+O(\rho^{n-k})\big)\left|\frac{\partial^2 S_{k+1}^n}{\partial yz}\right| + C\bar{\varepsilon}^{2^k} \\
\left|\frac{\partial^2 S_k^n}{\partial z^2}\right| &\leq \big(1+O(\rho^{n-k})\big)\left|\frac{\partial^2 S_{k+1}^n}{\partial z^2}\right| + C\bar{\varepsilon}^{2^k}
\end{aligned}$$

Hence, $|\,\partial^2_{xy} S_k^n| = O(\bar{\varepsilon}^{2^k}\sigma^{n-k})$, $|\,\partial^2_{xz} S_k^n| = O(\bar{\varepsilon}^{2^k}\sigma^{n-k})$, $|\,\partial^2_{yz} S_k^n| = O(\bar{\varepsilon}^{2^k})$, and $|\,\partial^2_{zz} S_k^n| = O(\bar{\varepsilon}^{2^k})$. □

7.3 Universal properties of the scaling map Ψ_k^n

On the following Lemma 7.3.3, we would show that the non-linear part of the coordinate change map $\mathrm{id} + S(x,y,z)$ is the small perturbation of the one-dimensional universal function. The content of this section is to rephrase some parts of the section 7 in [CLM].

Recall the one dimensional map $f_* \colon I \to I$ is the fixed point of the (periodic doubling) renormalization operator of the unimodal maps, namely, $Rf_* = f_*$. Let the critical point of f_* be c_* and $I = [-1, 1]$. Also assume that $f_*(c_*) = 1$ and $f_*^2(c_*) = -1$. Let us take the intervals $J_c^* = [-1, f_*^4(c_*)]$ and $J_v^* = f_*(J_c^*) = [f_*^3(c_*), 1]$. Then these intervals are the smallest renormalization invariant intervals under f_*^2 around the critical point and the critical value respectively. Observe that the critical point c_* is in J_c^* and $f_*(J_v^*) = J_c^*$.

Let the onto map $s \colon J_c^* \to I$ be the orientation reversing affine rescaling. Thus $s \circ f_* \colon J_v^* \to [-1, 1]$ is an expanding diffeomorphism. We can consider the inverse contraction

$$g_* \colon I \to J_v^*, \quad g_* = f_*^{-1} \circ s^{-1}$$

where f_*^{-1} is the branch of the inverse function which maps J_c^* onto J_v^*. The map g_* is called the *presentation function* and it has the unique fixed point at 1.

By the definition of g_* implies that

$$f_*^2|J_v^* = g_* \circ f_* \circ (g_*)^{-1}$$

Then by the appropriate rescaling of the presentation function, g_*, we can define the *renormalization at the critical value* v, $R_v^n f_*$. Inductively we can define g_*^n on the smallest interval $J_v^*(n)$ containing the critical value 1 with period 2^n. Let $G_*^n \colon I \to I$ be the diffeomorphism of the rescaled map of g_*^n.

Then the fact that g_* is the contraction implies the existence of the limit.

$$u_* = \lim_{n\to\infty} G_*^n \colon I \to I$$

and the convergence is exponentially fast in C^3 topology.

Moreover, we see the following lemmas in [CLM].

Lemma 7.3.1 (Lemma 7.1 in [CLM])**.** *For every* $n \geq 1$

(1) $J_v^*(n) = g_*^n(I)$

(2) $R_v^n f_* = G_*^n \circ f_* \circ (G_*^n)^{-1}$

(3) $u_* \circ f_* = f^* \circ u_*$

Lemma 7.3.2 (Lemma 7.3 in [CLM])**.** *Assume that there is the sequence of smooth functions* $g_k \colon I \to I, \quad k = 1, 2, \ldots, n$ *such that* $\|g_k - g_*\|_{C^3} \leq C\rho^k$ *where the* $g_* = \lim_{k\to\infty} g_k$ *for some constant* $C > 0$ *and* $\rho \in (0, 1)$*. Let* $g_k^n = g_k \circ \cdots \circ g_n$ *and let* $G_k^n = a_k^n \circ g_k^n \colon I \to I$*, where* a_k^n *is the affine rescaling of* $\operatorname{Im} g_k^n$ *to* I*. Then* $\|G_k^n - G_*^{n-k}\|_{C^1} \leq C_1\rho^{n-k}$*, where* C_1 *depends only on* ρ *and* C*.*

Let us normalize the functions u_* and g_* which have the fixed point at the origin and the derivatives at the origin is 1. Let

$$v_*(x) = \frac{u_*(x+1) - 1}{u_*'(1)}$$

Abusing notation, we denote the normalized function of $g_*(x)$ to be also the $g_*(x)$ in the following lemma.

Lemma 7.3.3. *There exists the positive constant* $\rho < 1$ *such that for all* $k < n$ *and for every* $y \in I^y$ *and* $z \in I^z$

$$\begin{aligned} &|\operatorname{id} + S_k^n(\,\cdot\,, y, z) - v_*(\,\cdot\,)| = O(\bar{\varepsilon}^{2^k} y + \bar{\varepsilon}^{2^k} z + \rho^{n-k}) \\ \textit{and} \quad &|1 + \partial_x S_k^n(\,\cdot\,, y, z) - v_*'(\,\cdot\,)| = O(\rho^{n-k}) \end{aligned}$$

Proof. The map $\mathrm{id} + S^n_k(\,\cdot\,, y, z)$ is the normalized function of Ψ^n_k such that the derivative at the origin is the identity map, id, and $v_*(\,\cdot\,)$ is also the normalized map of u_*, which is the conjugation of the renormalization fixed point at the critical point and the critical value in Lemma 7.3.1. Thus the normalized map, $\mathrm{id} + S^n_k(\,\cdot\,, 0, 0)$ and the one dimensional map, G^n_* converge to the same function $v_*(\,\cdot\,)$ as $n \to \infty$ because the critical value of f and the tip of F moved to the origin as the fixed point of each function g^n_* by the appropriate affine conjugation.

By Lemma 4.2.2

$$\mathrm{dist}_{C^3}(\,\mathrm{id} + s_k(\,\cdot\,, 0, 0),\ g_*(\,\cdot\,)) = O(\rho^k)$$

and by Lemma 7.3.2, we obtain

$$\mathrm{dist}_{C^1}(\,\mathrm{id} + S^n_k(\,\cdot\,, 0, 0),\ G^{n-k}_*(\,\cdot\,)) = O(\rho^{n-k}) \tag{7.3.1}$$

Since the $G^n_* \to v_*$ exponentially fast, we have the exponential convergence of the function $\mathrm{id} + S^n_k(\,\cdot, 0, 0)$ to $v_*(\,\cdot\,)$. Moreover, by Lemma 7.2.4 we have

$$|\,\partial_y S^n_k| \;= O(\bar{\varepsilon}^{2^k}), \qquad |\,\partial_z S^n_k| \;= O(\bar{\varepsilon}^{2^k})$$

Hence, the above asymptotic and the exponential convergence at the origin prove the first part of the lemma. Furthermore, C^1 convergence of (7.3.1) implies that

$$|\,1 + \partial_x S^n_k(\,\cdot\,, 0, 0) - v'_*(\,\cdot\,)| = O(\rho^{n-k})$$

where $\rho \in (0, 1)$. ∎

7.4 The estimation of the quadratic part of S^n_k for n

We estimate the asymptotic of S^n_k using the estimation of the partial derivatives and recursive formulas. Then it implies that the estimation of the asymptotic of the non-linear part of Ψ^n_k as $n \to \infty$. In order to simplify notations, we would treat the case $k = 0$ and consider the behaviour of S^n_0 instead of S^n_k.

Lemma 7.4.1. *The following asymptotic is true*

$$|\,[\,x + S^n_0(x, y, z)] - [\,v_*(x) + a_{F,1}\,y^2 + a_{F,2}\,yz + a_{F,3}\,z^2]| = O(\rho^n)$$

where constants $|a_{F,1}|, |a_{F,2}|\ |a_{F,3}|$ *are* $O(\bar{\varepsilon})$ *for some* $\rho \in (0, 1)$.

Remark 7.4.1. The notations $t_{n+1,n}, u_{n+1,n}$ and $d_{n+1,n}$ are simplified as t_n, u_n and d_n, which is $O(\bar{\varepsilon}^{2^n})$ like the notations used in (7.1.3) . Moreover, factors of dilation parts, $\alpha_{n+1,n}, \sigma_{n+1,n}$ are abbreviated as α_n, σ_n respectively. Thus $\alpha_n = \sigma^2(1+O(\rho^n))$ and $\sigma_n = -\sigma(1+O(\rho^n))$. Using the similar abbreviation, D_n denote D_n^{n+1} and s_n is the s_n^{n+1}.

Proof. For any fixed $k \geq 0$, the recursive formula for $n > k$ comes from the $\Psi_k^{n+1} = \Psi_k^n \circ \Psi_n^{n+1}$. Thus

$$\mathbf{S}_k^{n+1}(w) = \mathbf{s}_n(w) + D_n^{-1} \circ \mathbf{S}_k^n \circ D_n \circ (\mathrm{id} + \mathbf{s}_n)(w) \tag{7.4.1}$$

Let $k=0$ for simplicity, and compare each coordinates of the both sides. Then

$$\begin{aligned}
&(S_0^{n+1}(w), 0, R_0^{n+1}(y)) \\
&= (s_n(w), 0, r_n(y)) \ + \begin{pmatrix} \alpha_n^{-1} & \alpha_n^{-1}(-t_n + d_n u_n) & -\alpha_n^{-1} u_n \\ & \sigma_n^{-1} & \\ & -\sigma_n^{-1} d_n & \sigma_n^{-1} \end{pmatrix} \begin{pmatrix} S_0^n(w) \\ 0 \\ R_0^n(y) \end{pmatrix} \\
&\qquad \circ \begin{pmatrix} \alpha_n & \sigma_n t_n & \sigma_n u_n \\ & \sigma_n & \\ & \sigma_n d_n & \sigma_n \end{pmatrix} \begin{pmatrix} x + s_n(w) \\ y \\ z + r_n(y) \end{pmatrix}
\end{aligned}$$

By the direct calculation,

$$\begin{aligned}
&(S_0^{n+1}(w),\ 0,\ R_0^{n+1}(y)) \\
&= (s_n(w),\ 0,\ r_n(y)) + \left(\frac{1}{\alpha_n} S_0^n(w) - \frac{1}{\alpha_n} u_n R_0^n(y),\ 0,\ \frac{1}{\sigma_n} R_0^n(y)\right) \circ \\
&\qquad \Big(\alpha_n(x + s_n(w)) + \sigma_n t_n y + \sigma_n u_n (z + r_n(y)),\ \sigma_n y,\ \sigma_n d_n y + \sigma_n (z + r_n(y))\Big) \\
&= (s_n(w),\ 0,\ r_n(y)) \\
&\quad + \Bigg(\frac{1}{\alpha_n} S_0^n\Big(\alpha_n(x + s_n(w)) + \sigma_n t_n y + \sigma_n u_n (z + r_n(y)),\ \sigma_n y, \\
&\qquad \sigma_n d_n y + \sigma_n (z + r_n(y))\Big) - \frac{1}{\alpha_n} u_n R_0^n(\sigma_n y),\ 0,\ \frac{1}{\sigma_n} R_0^n(\sigma_n y)\Bigg)
\end{aligned}$$

Firstly, let us compare the third coordinates of each side of the above equation.

Using the Taylor's expansion and Lemma 7.1.1, we obtain

$$\begin{aligned} R_0^{n+1}(y) &= r_n(y) + \frac{1}{\sigma_n} R_0^n(\sigma_n y) \\ &= \frac{1}{\sigma_n}\, R_0^n(\sigma_n y) + c_n y^2 + O(\bar{\varepsilon}^{2^n} y^3) \quad \text{where} \quad c_n = O(\bar{\varepsilon}^{2^n}) \end{aligned}$$

Then we have the following form of $R_0^n(y)$.

$$\begin{aligned} R_0^n(y) &= a_n y^2 + A_n(y) y^3 \\ \text{Thus} \quad R_0^{n+1}(y) &= \frac{1}{\sigma_n}\Big(a_n(\sigma_n y)^2 + A_n(\sigma_n y)\cdot(\sigma_n y)^3\Big) + c_n y^2 + O(\bar{\varepsilon}^{2^n} y^3) \end{aligned}$$

Thus $a_{n+1} = \sigma_n a_n + c_n$ and $\|A_{n+1}\| \leq \|\,\sigma_n\|^2 \|A_n\| + O(\bar{\varepsilon}^{2^n})$.

Hence, $A_n \to 0$ and $a_n \to 0$ exponentially fast as $n \to \infty$. The image of the vertical plane $(y,z) \to (0,y,z)$ under the map $\mathrm{id} + \mathbf{S}_0^n$ is the graph of the function $\xi_n \colon \mathbf{I}^v \to \mathbb{R}$ defined as

$$\xi_n(y,z) = (S_0^n(0,y,z), 0, R_0^n(y))$$

Since $R_0^n(y)$ is vanished exponentially fast, $|\,\xi_n(y,z)| = |\,S_0^n(0,y,z)| + O(\rho^n)$. Moreover, the second part of Lemma 7.3.3 implies the following equation.

$$\big|\,[\,x + S_0^n(x,y,z)] - [\,v_*(x) + S_0^n(0,y,z)]\big| = O(\rho^n) \tag{7.4.2}$$

Secondly, compare the first coordinates of (7.4.1) at $(0,y,z)$.

$$\begin{aligned} &S_0^{n+1}(0,y,z) \\ &= s_n(0,y,z) \\ &\quad + \frac{1}{\alpha_n}\, S_0^n\Big(\alpha_n(x + s_n(0,y,z)) + \sigma_n t_n y + \sigma_n u_n(z + r_n(y)),\ \sigma_n y, \\ &\quad \sigma_n d_n y + \sigma_n(z + r_n(y))\Big) - \frac{1}{\alpha_n}\, u_n R_0^n(\sigma_n y) \end{aligned}$$

The estimation of $|\partial^2_{xy} S_k^n|$, $|\partial^2_{xz} S_k^n|$ and $|\partial^2_{yz} S_k^n|$, $|\,\partial^2_{zz} S_k^n|$ in Lemma 7.2.4 implies that

$$\frac{\partial S_0^n}{\partial x}(0,y,z) = O(\sigma^n y + \sigma^n z) \quad \text{and} \quad \frac{\partial S_0^n}{\partial z}(0,y,z) = O(y + z)$$

respectively. The order of the t_n, u_n, r_n and Taylor's expansion of S_0^n at

$(0, \sigma_n y, \sigma_n z)$ implies that

$$
\begin{aligned}
& S_0^{n+1}(0, y, z) \\
&= s_n(0, y, z) + \frac{1}{\alpha_n}\,[S_0^n(0,\, \sigma_n y,\, \sigma_n z) \\
&\quad + \frac{\partial S_0^n}{\partial x}(0,\, \sigma_n y,\, \sigma_n z) \cdot \Big(\alpha_n s_n(0, y, z)) + \sigma_n t_n y + \sigma_n u_n (z + r_n(y))\Big) \\
&\quad + \frac{\partial S_0^n}{\partial z}(0,\, \sigma_n y,\, \sigma_n z) \cdot \Big(\sigma_n d_n y + \sigma_n r_n(y)\Big)\Big] - \frac{1}{\alpha_n}\, u_n R_0^n(\sigma_n y) \\
&\quad + O\left(\bar{\varepsilon}^{2^n} \sum_{j=0}^{3} y^{3-j} z^j\right) \\
&= \frac{1}{\alpha_n}\, S_0^n(0,\, \sigma_n y,\, \sigma_n z) + \sum_{i=0}^{2} e_{n,\,i}\, y^{2-i} z^i + O\left(\bar{\varepsilon}^{2^n} \sum_{j=0}^{3} y^{3-j} z^j\right)
\end{aligned}
$$

where $e_{n,\,i} = O(\bar{\varepsilon}^{2^n})$ for $i = 0, 1, 2$.

Then we can express $S_0^n(0, y, z)$ as the quadratic and higher order terms,

$$
S_0^n(0, y, z) = a_{n,\,1}\, y^2 + a_{n,\,2}\, yz + a_{n,\,3}\, z^2 + A_n(y, z) \left(\sum_{j=0}^{3} c_j\, y^{3-j} z^j\right)
$$

The recursive formula for $S_0^n(0, y, z)$ implies that

$$
\begin{aligned}
& S_0^{n+1}(0, y, z) \\
&= \frac{1}{\alpha_n}\left[a_{n,\,1}(\sigma_n y)^2 + a_{n,\,2}(\sigma_n y\; \sigma_n z) + a_{n,\,3}(\sigma_n z)^2 \right. \\
&\quad \left. + A_n(\sigma_n y, \sigma_n z) \left(\sum_{j=0}^{3} c_j\, (\sigma_n y)^{3-j} (\sigma_n z)^j\right) \right] + \sum_{i=0}^{2} e_{n,\,i}\, y^{2-i} z^i \\
&\quad + O\left(\bar{\varepsilon}^{2^n} \sum_{j=0}^{3} y^{3-j} z^j\right)
\end{aligned}
$$

Hence, $a_{n+1,\,i} = \dfrac{\sigma^2}{\alpha_n}\, a_{n,\,i} + \displaystyle\sum_{j=0}^{2} e_{n,j}$ for $i = 0, 1, 2$ and moreover, $\|A_{n+1}\| \leq \|A_n\| \cdot \dfrac{|\,\sigma_n|^3}{|\,\alpha_n|} + O(\bar{\varepsilon}^{2^n})$. It implies that $a_{n,\,i} \to a_{F,\,i}$ for $i = 0, 1, 2$ and $\|A_n\| \to 0$ exponentially fast as $n \to \infty$. The exponential convergence of

$S_0^n(0,y,z)$ to the quadratic function of y and z and (7.4.2) proves this global asymptotic behaviour of $S_0^n(x,y,z)$. $\square$

Remark 7.4.2. The above Lemma can be generalized for S_k^n as follows.

$$\big|\,[\,x + S_k^n(x,y,z)] - [\,v_*(x) + a_{F,1}\,y^2 + a_{F,2}\,yz + a_{F,3}\,z^2]\big| = O(\rho^{n-k})$$

The constants $|a_{F,i}|$ for $i = 1,2,3$ of S_k^n are $O(\bar{\varepsilon}^{2^k})$.

7.5 Universality of the Jacobian determinant, $\operatorname{Jac} R^nF$

Let the n^{th} renormalized map of F be $R^nF \equiv F_n = (f_n - \varepsilon_n, x, \delta_n)$ and let $\Psi^n_{\text{tip}} \equiv \Psi^n_{v^n}$ from n^{th} level to 0^{th} level. Recall that the tip $\tau_F \in B^n_{v^n}$ for each n. Then Ψ^n_{tip} is the original coordinate change rather than the normalized function Ψ^n_0 conjugated by translations T_n.
Recall (7.0.1) again.

$$\begin{aligned}
\operatorname{Jac} F_n(w) &= \operatorname{Jac} F^{2^n}(\Psi^n_{\text{tip}}(w))\frac{\operatorname{Jac}\Psi^n_{\text{tip}}(w)}{\operatorname{Jac}\Psi^n_{\text{tip}}(F_nw)}\\
&= b^{2^n}\frac{\operatorname{Jac}\Psi^n_{\text{tip}}(w)}{\operatorname{Jac}\Psi^n_{\text{tip}}(F_nw)}(1+O(\rho^n)).
\end{aligned}$$

Theorem 7.5.1 (Universal limit of Jacobian determinant)**.** *For the function $F \in \mathcal{I}_B(\bar{\varepsilon})$ for sufficiently small $\bar{\varepsilon} > 0$, we obtain that*

$$\operatorname{Jac} F_n = b^{2^n} a(x)\,(1 + O(\rho^n))$$

where b is the average Jacobian of F, $\rho \in (0,1)$, and $a(x)$ is the universal function which is positive.

Proof. Let us consider the affine maps

$$T\colon w \mapsto w - \tau, \qquad T_n\colon w \mapsto w - \tau_n$$

where τ_n is the tip of R^nF. Then we can consider the map

$$L^n\colon w \mapsto (D^n_0)^{-1}(w - \tau)$$

as the local chart of B^n. On these local charts, we write maps with the boldface if the maps are conjugated by its local charts in this proof.

$$\mathbf{F}_n = T_n \circ F_n \circ T_n^{-1}, \qquad \mathrm{id} + \mathbf{S}_0^n = L^n \circ \Psi_{\mathrm{tip}}^n \circ T_n^{-1}$$

By the definition of the coordinate change map Ψ_{tip}^n and the normalized map Ψ_0^n, we can obtain the following commutative diagram.

$$\begin{tikzcd}
T_n(B) \arrow[d, "\Psi_0^n"'] & B \arrow[l, "T_n"'] \arrow[r, "F_n"] \arrow[d, "\Psi_{\mathrm{tip}}^n"'] & F_n(B) \arrow[r, "T_n"] \arrow[d, "\Psi_{\mathrm{tip}}^n"] & (T_n \circ F_n)(B) \arrow[d, "\Psi_0^n"] \\
T(B_n) & B^n \arrow[l, "T"'] \arrow[r, "F^{2^n}"] & F^{2^n}(B^n) \arrow[r, "T"] & (T \circ F^{2^n})(B^n)
\end{tikzcd}$$

Since any translation does not affect Jacobian determinant, the ratio of Jacobian determinant of the coordinate change map is following.

$$\frac{\mathrm{Jac}\,\Psi_{\mathrm{tip}}^n(w)}{\mathrm{Jac}\,\Psi_{\mathrm{tip}}^n(F_n w)} = \frac{\mathrm{Jac}\,\Psi_0^n(\mathbf{w}_n)}{\mathrm{Jac}\,\Psi_0^n(\mathbf{F}_n\mathbf{w}_n)} = \frac{1 + \partial_x S_0^n(\mathbf{w}_n)}{1 + \partial_x S_0^n(\mathbf{F}_n\mathbf{w}_n)} \tag{7.5.1}$$

where $\mathbf{w}_n = T_n(w)$. By Theorem 4.2.2, the tip τ_n converges to $\tau_\infty = (1, c_*, 0)$ exponentially fast where c_* is the critical point of $f_*(x)$. It implies the following limits

$$\begin{aligned}
T_n &\to T_\infty \colon w \mapsto w - \tau_\infty \\
\mathbf{w}_n = T_n(w) &\to T_\infty(w) \\
\mathbf{F}_n\mathbf{w}_n &\to \mathbf{F}_* \circ T_\infty(w) = T_\infty \circ F_*(w) = (f_*(x) - 1, x - c_*, 0)
\end{aligned}$$

and each convergence is exponentially fast.
Hence, Lemma 7.4.1 implies that the following convergence

$$1 + \partial_x S_0^n \to v_*' \tag{7.5.2}$$

is exponentially fast.
Combining (7.5.1), (7.5.2) and convergence of $\mathbf{F}_n\mathbf{w}_n$ to the $\mathbf{F}_* \circ T_\infty$, we have

$$\frac{\mathrm{Jac}\,\Psi_{\mathrm{tip}}^n(w)}{\mathrm{Jac}\,\Psi_{\mathrm{tip}}^n(F_n w)} \longrightarrow \frac{v_*'(x-1)}{v_*'(f_*(x)-1)} \equiv a(x) \tag{7.5.3}$$

where $w = (x, y, z)$.

Moreover, this convergence is exponentially fast. The positivity of $a(x)$ comes from two facts. Firstly, the Jacobian determinant of the orientation preserving diffeomorphism is non-negative at every point and we assumed that each infinitely renormalizable map, $F \in \mathcal{I}(\bar{\varepsilon})$, is orientation preserving on each level. Secondly, the renormalization theory of the one dimensional map at the critical value implies the non vanishing property of v'_* with the sufficiently small perturbation. $\square$

Remark 7.5.1. The universality of the Jacobian does not imply the universality of the renormalized map F_n because the Jacobian determinant, $\partial_y \varepsilon_n \cdot \partial_z \delta_n - \partial_z \varepsilon_n \cdot \partial_y \delta_n$ cannot make the universal expression of each element of the Jacobian matrix, DF_n.

Chapter 8

The trapping regions and the global attracting set

The critical Cantor set is defined as the limit of the union of the boxes $B^n_{\mathbf{w}_n}$. However, it can be constructed by the topological invariant sets which is called the trapping regions.

Recall the M_{-n} where $n \geq -1$ to be the component of the stable manifold at β_1 and especially $M_0 \equiv W^s_{\mathrm{loc}}(\beta_1)$ to be the component of the stable manifold containing β_1. Recall the definition of the regions A_{-n}, the region between M_{-n} and M_{-n+1}. Thus let us denote the region $D_0 \equiv D_0(F)$ to be $F(A_{-1}) \subset A_0$. Then D_0 is invariant under F^2 and it is the $\bar{\varepsilon}$ neighborhood of the curve $[p_0, p_1]^u_{\beta_0} \subset W^u(\beta_0)$ in A_0. Moreover, one component of $\partial D_0 \cap W^s_{loc}(\beta_1)$ contains the point p_0 and the other components contains β_1 and all of p_k where $k \geq 1$.

Definition 8.0.1. Let F be the Hénon-like map with sufficiently small $\bar{\varepsilon} > 0$. The invariant domain as the $\bar{\varepsilon}$ neighborhood of the curve $[p_0, p_1]^u_{\beta_0} \subset W^u(\beta_0)$ is called $D_0(F)$ and it is defined as follows.

$$D_0 \equiv D_0(F) = F(A_{-1})$$

If F is infinitely renormalizable, the invariant region under F^{2^n} is defined successively.

$$D_n \equiv D_n(F) = \Psi^n_{v^n}\big(D_0(R^nF)\big)$$

where $D_0(R^nF) = R^nF\big(A_{-1}(R^nF)\big)$. The n^{th} *trapping region* of F for $n \geq 1$ is defined as follows.

$$\mathrm{Trap}_n \equiv \bigcup_{k \geq 0} F^k(D_n) \tag{8.0.1}$$

Remark 8.0.2. The n^{th} trapping region has 2^{n+1} components in $B(F)$ because $D_0(R^nF)$ is invariant under $(R^nF)^2$ and by the conjugation $\Psi^n_{v^n}$, D_n is invariant under $F^{2^{n+1}}$.

Proposition 8.0.2. *Let $F \in \mathcal{I}_B(\bar{\varepsilon})$ an infinitely renormalizable Hénon-like map. Then the critical Cantor set $\mathcal{O}_F$ is the intersection of the trapping regions.*

Proof. Let us show that $\operatorname{Trap}_{n+1} \Subset \operatorname{Trap}_n$ for every $n \geq 1$. Recall the following commutative diagram.

$$\begin{tikzcd} B \arrow[r, "R^nF"] \arrow[d, "\Psi^n_{\mathbf{w}}"'] & B \arrow[d, "\Psi^n_{\mathbf{w}}"] \\ \Psi^n_{\mathbf{w}}(B) \arrow[r, "F^{2^n}"] & \Psi^n_{\mathbf{w}}(B) \end{tikzcd}$$

The case that $n = 1$, by the definition of D_1, we see the fact $D_1 \Subset D_0$ from the following set inclusion.

$$\begin{aligned} D_1 \equiv \Psi^1_v\big(D_0(RF)\big) &= \Psi^1_v \circ RF\big(A_{-1}(RF)\big) \\ &\subseteq \Psi^1_v \circ RF\big(B(RF)\big) \\ &= F^2 \circ \Psi^1_v(B(RF)) = F^2(B^1_v) \Subset F^2(A_0) \subset F(A_{-1}) = D_0. \end{aligned} \tag{8.0.2}$$

Similarly, the commutative diagram is valid between the map R^nF and $R^{n+1}F$ with some coordinate change map. Then by induction the set relation $D_{n+1} \Subset D_n$ is true and furthermore, $\operatorname{Trap}_{n+1} \Subset \operatorname{Trap}_n$ for every $n \geq 1$ because F is a diffeomorphism between the domain of F and its image.[1]

The fixed point β_1 is contained in ∂D_0 and similarly each fixed point $\beta_1(R^nF)$ is in $\partial D_0(R^nF)$. By the definition of the point, $\beta_{n+1} \equiv \Psi^n_{v^n}(\beta_1)$, β_{n+1} is contained in D_n. Moreover, the fact that $\operatorname{Trap}_{n+1} \Subset \operatorname{Trap}_n$, each trapping region Trap_n contains all periodic points with period 2^{k+1} where[2] $k \geq n$ but dose not have any periodic points with period less than 2^{n+1}. Since, every

[1] If we define the 0^{th} trapping region as the union of D_0 and its image under F, that is, $\operatorname{Trap}_0 \equiv D_0 \cup F(D_0)$, then $\operatorname{Trap}_1 \Subset \operatorname{Trap}_0$. Actually the closure of Trap_0 covers the maximal compact set in $A_{-1} \cup A_0$ which is invariant under F^2.

[2] Each D_n contains two periodic points β_n and $\Psi^n_{v^n}\big(\beta_0(R^nF)\big)$. Moreover, the orbit of these two periodic points under F covers the all periodic points with the same period because R^nF has two fixed points for every $n \in \mathbb{N}$. In other words, every periodic points with fixed period has at most two different cycles.

point in the Cantor set $\mathcal{O}_F$ is the accumulation point of Per_F by Lemma 5.3.2, every trapping region contains the critical Cantor set.
Using the induction with the relation (8.0.2), we see that

$$D_n \subset B^n_{v^n}$$

The above relation and Lemma 5.2.1 implies that

$$\mathrm{Trap}_n \subset \bigcup_{\mathbf{w}_n \in W^n} B^n_{\mathbf{w}_n}$$

for every $n \in \mathbb{N}$. Hence, by the definition of the critical Cantor set and trapping region, the following set relation holds.

$$\mathcal{O}_F \subset \bigcap_{n\geq 1} \mathrm{Trap}_n \subset \bigcap_{n=1}^{\infty} \bigcup_{\mathbf{w}_n \in W^n} B^n_{\mathbf{w}_n} \equiv \mathcal{O}_F$$

Therefore,

$$\mathcal{O}_F = \bigcap_{n\geq 1} \mathrm{Trap}_n$$

□

Proposition 8.0.3. *Let F be the renormalizable Hénon-like map. Then the image of $\beta_0(RF)$ under $\Psi^1_v = H^{-1} \circ \Lambda^{-1}$ is the fixed point of F, β_1.*

Proof. Observe that F has only two fixed points β_0 and β_1 with the sufficiently small norm, $\|\delta\|_{C^1}$. The x and $y-$coordinates of β_0 and β_1 are negative and positive respectively. Let $\beta_1 = (\beta^x,\, \beta^x,\, z_0)$. Thus $\beta^x > 0$. Recall $H(w) = (f(x) - \varepsilon(w),\, y,\, z - \delta(y, f^{-1}(y), 0))$. Since β_1 is the fixed point under F^2, $H(\beta_1) = (\beta^x,\, \beta^x,\, z_0 - \delta(\beta^x,\, f^{-1}(\beta^x), 0)$ is a fixed point of the pre-renormalization, $PRF = H \circ F^2 \circ H^{-1}$. Then $\Lambda(H(\beta_1))$ is a fixed point of RF and $\beta_0(RF) = \Lambda \circ H(\beta_1)$ because $\Lambda(x,\, y,\, z) = (sx,\, sy,\, sz)$ with $s < -1$ and $\beta_0(RF)$ is the unique fixed point such that both x and $y-$coordinates are negative. Hence, $\beta_1 = H^{-1} \circ \Lambda^{-1}\big(\beta_0(RF)\big)$. □

Corollary 8.0.4. *Let F be the infinitely renormalizable Hénon-like map. Then every periodic points with period 2^n are contained in $\mathrm{Orb}(\beta_{n+1})$ for every $n \in \mathbb{N}$.*

Proof. Let $\beta_0(R^{n+1}F)$ be the regular fixed point under the map $R^{n+1}F$. Then by Proposition 8.0.3, $\Psi^n_{n-1}\big(\beta_0(R^nF)\big)$ is $\beta_1(R^{n-1}F)$. Then R^nF has only one periodic point with period 2. Then the image of these two points $\beta_0(R^nF)$ and $\beta_1(R^nF)$ under the conjugation map $\Psi^n_{v^n}$ are β_n and β_{n+1} respectively. Since the set $F^n(B^n_{v^n})$ contains β_n and β_{n+1} and the number of periodic points with period 2^n is at most 2^{n+1}, the following set

$$\bigcup_{\mathbf{w}_n \in W^n} B^n_{\mathbf{w}_n} = \bigcup_{n=0}^{2^n-1} F^n(B^n_{v^n})$$

has all periodic points with period 2^n. Then $\operatorname{Orb}(\beta_n) \cup \operatorname{Orb}(\beta_{n+1})$ contains all periodic points with period 2^n. However, β_n is the periodic point with the period 2^{n-1}. Hence, the orbit of β_{n+1} contains all periodic points with the period 2^n. ∎

Recall the region $B_\bullet$ is the component of $B \setminus W^s_{loc}(\beta_0)$ which contains β_1. The fact that the region $\Psi^1_v(B) = B^1_v$ contains β_1 implies that $\Psi^1_v(B_\bullet)$ is $B^1_v \cap A_0$. Since the region $B_\bullet$ is invariant under F^2 and the image of $\partial B_\bullet \setminus W^s_{loc}(\beta_0)$ under F^2 is in $B_\bullet$, $B^1_v \cap A_0$ contains the $\bar{\varepsilon}$ neighborhood $\mathcal{N}$ of the curve $[p_0, p_1]^u_{\beta_0}$ in $B^1_v \cap A_0$ and $\partial\mathcal{N} \cap B^1_v \cap A_0 \subset W^s_{loc}(\beta_1)$. We may also assume that $\mathcal{N}$ is $F(A_{-1}) \equiv D$ when we relax the condition that the box B^1_v is the image of the minimal cubic box for renormalization and then allow the $\bar{\varepsilon}$ neighborhood of B^1_v to be B^1_v.

Topological properties of the unstable manifolds of two dimensional Hénon-like maps and the three dimensional Hénon-like maps are similar. The following lemma is the three dimensional version of the topological properties of the invariant compact sets under Hénon-like maps. See the Theorem 4.1 in [LM].

Theorem 8.0.5. *Let F be the Hénon-like map in $\mathcal{I}_B(\bar{\varepsilon})$. Then the nonwandering set Ω_F is $\operatorname{Per}_F \cup \mathcal{O}_F$.*

Proof. Let x be the point on the domain $B_\bullet$ which does not converge to the any orbit of the periodic points. Then by Lemma 4.1.1, there exists a constant $k_0 \in \mathbb{N}$ depending on x such that $F^{k_0}(x) \in D_0$. Similarly, for $x_1 \in B_\bullet(RF)$, there exists a constant $k_1 \in \mathbb{N}$ such that $(RF)^{k_1}(x_1) \in D_0(RF)$. Observe that

$$D_0 \subset B^1_v \cap A_0 = \operatorname{Im}(\Psi^1_v)$$

Then

$$\operatorname{Orb}\big((RF)^{k_1}(x_1)\big) \subset \operatorname{Orb}\big(D_0(RF)\big)$$

So that

$$F^{k_0+2k_1}(x) \in D_1 \subset \operatorname{Trap}_1$$

Inductively, for every $x \in B_{\bullet}$ there exists $k \in \mathbb{N}$ such that $F^k(x) \in \mathrm{Trap}_n$ for every $n \in \mathbb{N}$. Hence, the omega limit set of x, $\omega(x)$ is $\mathcal{O}_F$.

Clearly, $\mathrm{Per}_F \cup \mathcal{O}_F \subset \Omega_F$. Let us take a point $x \in B_{\bullet} \setminus (\mathrm{Per}_F \cup \mathcal{O}_F)$ which is not convergent to any periodic orbit. Since $\mathcal{O}_F$ is compact and $\mathrm{Trap}_n \to \mathcal{O}_F$ as $n \to \infty$, there exists a neighborhood U of x disjoint from Trap_{n_0} for some $n_0 \in \mathbb{N}$. Moreover, by the above argument there exist $k \in \mathbb{N}$ such that $F^k(U) \subset \mathrm{Trap}_{n_0}$ for each fixed $n_0 \geq 1$. However, the fact that $x \notin \mathcal{O}_F$ implies that $x \notin \mathrm{Trap}_N$ for all sufficiently large N. Hence, x is wandering. Let us consider the non-periodic points which converges to the periodic orbit

$$x \in \bigcup_{w \in \, \mathrm{Per}_F} W^s(w)$$

Let us take a non-periodic point $x \in W^s(\beta_1) \setminus W^s_{loc}(\beta_1)$. Observe that the set $\overline{D_0 \cup F(D_0)} \equiv \overline{\mathrm{Trap}_0}$ is forward invariant under F and $\mathrm{Trap}_0 \cap W^s(\beta_1) \subset W^s_{loc}(\beta_1)$. By Lemma 4.1.1 , for each x, $F^k(x) \in \overline{\mathrm{Trap}_0}$ for some $k \in \mathbb{N}$. Moreover, $\overline{\mathrm{Trap}_0}$ is a topological handle body. So that we can choose the neighborhood U of x which is contained in $\overline{\mathrm{Trap}_0}$. Thus if $x \in W^s(\beta_1) \setminus \overline{\mathrm{Trap}_0}$, then x is wandering.

Let the component of $\partial D_0 \cap W^s_{loc}(\beta_1)$ which contains p_0 and β_1 be U_0 and V_0 respectively. Then the set V_0 contains $\{p_i \mid i \in \mathbb{N}\} \cup \{\beta_1\}$. Recall F is renormalizable and then $p_n = F^n(p_0)$ for each $n \in \mathbb{Z}$. Furthermore, we can define U_n and V_n as the component of $\partial\big(F^n(D_0)\big) \cap W^s_{loc}(\beta_1)$ which contains p_n and β_1 respectively for every $n \in \mathbb{N}$. Then the following is true.

(1) Each $F^n(D_0)$ is the handle body of which boundary in $W^s_{loc}(\beta_1)$ is U_n and V_n for every $n \in \mathbb{N}_+$.

(2) U_n is disjoint from V_n for every $n \in \mathbb{N}_+$.

(3) $\displaystyle\bigcup_{i \geq k+1} \Big(U_i \cup V_i\Big) \Subset V_k$ for every $k \in \mathbb{N}_+$.

(4) $F^{2k}(D_0) \subset A_0$ and $F^{2k+1}(D_0) \subset A_{-1}$ for every $k \in \mathbb{N}_+$.

(5) $\overline{F^{n+3}(D_0) \cup F^{n+2}(D_0)} \Subset \overline{F^{n+1}(D_0) \cup F^n(D_0)}$ for every $n \in \mathbb{N}$.

Let us take a point $x \in W^s_{loc}(\beta_1) \cap \overline{\mathrm{Trap}_0}$. If $x \in U_0$, then x is wandering. Since $\{p_n\} \to \beta_1$ as $n \to \infty$ exponentially fast, the diameter of V_n shrinks to zero as $n \to \infty$ also. If $x \in V_0$, then $x \in F^k\big(\overline{\mathrm{Trap}_0}\big) \setminus F^{k+2}\big(\overline{\mathrm{Trap}_0}\big)$ for some

$k \in \mathbb{N}_+$. Then by the above property (5), x is wandering. Since F is infinitely renormalizable, the same fact is true for the map R^nF on $B_\bullet(R^nF)$ for every $n \in \mathbb{N}$. Moreover, since the disjointness is preserved under the conjugation map, every non-periodic points in $\bigcup_{w \in \ \mathrm{Per}_F} W^s(w)$ is wandering. Hence, every points in $B_\bullet \setminus \big(\mathrm{Per}_F \cup \mathcal{O}_F\big)$ are wandering. Therefore,

$$\Omega_F = \mathrm{Per}_F \cup \mathcal{O}_F$$

□

Let Γ_j for $j \geq 1$ and Γ be smooth curves. Let us say that the Γ_j converge to Γ as $j \to \infty$ if the curve Γ_j converges in the C^1-topology and the each Γ_j and Γ has the smooth parametrization.

The following Lemma 8.0.6 and Theorem 8.0.7 and their proofs are same as Lemma 4.4 and Theorem 4.1 on [LM] respectively except that the map F is three dimensional Hénon-like map.

Lemma 8.0.6. *Let $F \in \mathcal{I}_B(\bar{\varepsilon})$ with sufficiently small $\bar{\varepsilon} > 0$. Let Γ is a curve contained in $W^u(\beta_n)$ for $n \geq 1$. Then there are curves $\Gamma_j \subset W^u(\beta_0)$ such that $F^{t_j}(\Gamma_j) \to \Gamma$ as $t_j \to \infty$.*

Proof. If the periodic point β_n is in the interior of the curve $\Gamma \subset W^u(\beta_n)$, then $\bigcup_{i \geq 0} F^i(\Gamma) = W^u(\beta_n)$. Then we may assume that the curve in $W^u(\beta_n)$ contains the fixed point β_n. Let us use induction for the proof. For $n = 1$, $W^u(\beta_0)$ and $W^s_{loc}(\beta_1)$ meets transversally at all of p_i for $i = 0, 1, 2, \ldots$. Moreover, $p_n \to \beta_1$ as $n \to \infty$. Then by the inclination Lemma (for example, see [Rob] Theorem 11.1 and its references) there exist arcs Γ_j for $j \geq 1$ such that $p_j \in \Gamma_j$ with the uniformly positive length and the time $t_j \to \infty$ such that

$$F^{t_j}(\Gamma_j) \to \Gamma \subset W^u(\beta_1)$$

Take an arc $\Gamma^n \subset W^u(\beta_n)$ and then we may assume that $\Gamma^n = \Psi^n_0(\hat{\Gamma})$ with $\hat{\Gamma} \subset W^u(\beta_1(R^nF))$. Since $R^nF \in \mathcal{I}_B(\bar{\varepsilon}^{2^n})$ with sufficiently small $\bar{\varepsilon}$, $W^u(\beta_0(R^nF))$ and $W^s_{loc}(\beta_1(R^nF))$ meets transversally at all of $p_i(R^nF)$ for $i = 0, 1, 2, \ldots$ and moreover, $p_n(R^nF) \to \beta_1(R^nF)$ as $n \to \infty$. Then the inclination Lemma implies that

$$\big(R^nF\big)^{\hat{t}_j}(\hat{\Gamma}_j) \to \hat{\Gamma} \subset W^u(\beta_1(R^nF)) \tag{8.0.3}$$

for every $n \in \mathbb{N}_+$.
Suppose that

$$F^{\,t_{j,\,k}}(\Gamma_j^k) \xrightarrow[j\,\to\,\infty]{} \Gamma^k \subset W^u(\beta_k)$$

for $k = 1, 2, 3, \ldots, n$. Since Ψ_0^n is a diffeomorphism and β_{n+1} is defined as $\Psi_0^n(\beta_1(R^nF))$, the convergence (8.0.3) is equivalent to the following.

$$F^{\,2^n\hat{t}_j}(\Psi_0^n(\hat{\Gamma}_j)) \xrightarrow[j\,\to\,\infty]{} \Psi_0^n(\hat{\Gamma}) \subset W^u(\beta_{n+1}) \tag{8.0.4}$$

Hence, the arc $\Gamma^n \subset W^u(\beta_{n+1})$ can be approximated by some arcs in $W^u(\beta_n)$ and arcs in $W^u(\beta_n)$ can be approximated by some arcs in $W^u(\beta_0)$, that is,

$$F^{\,t_i}(\Gamma_i) \xrightarrow[i\,\to\,\infty]{} \Gamma^n \subset W^u(\beta_n)$$

for each $n \in \mathbb{N}$. Therefore, every curve of $W^u(\beta_n)$ can be approximated by some curves in $W^u(\beta_0)$. $\square$

For the map $F : B \to \mathbb{R}^3$, the set $\bigcap_{k\geq 0} F^k(B)$ is called the *global attracting set*. Then it is the maximal backward invariant subset of B. Let us show the maximality of $\bigcap_{k\geq 0} F^k(B)$. Let the set Z be a backward invariant set in B. Thus $F^k(Z) \subset F^k(B)$ for each $k \in \mathbb{N}_+$. Since $F^k(Z) \subset F^{k+1}(Z)$ for every k, passing the limit $Z \subset \bigcap_{k\geq 0} F^k(B)$. For the infinitely renormalizable perturbed Hénon-like map $F \in \mathcal{I}_B(\bar{\varepsilon})$, take the following set

$$\mathcal{A}_F = \mathcal{O}_F \cup \bigcup_{w\in\,\mathrm{Per}_F} W^u(w) \cap B \tag{8.0.5}$$

Then $\mathcal{A}_F$ is backward invariant and $\mathcal{A}_F \cap B_\bullet$ is completely invariant under F. The image of $W^u(\beta_0)$ under F is extended outside of B.

Theorem 8.0.7. *Let $F \in \mathcal{I}_B(\bar{\varepsilon})$ the infinitely renormalizable Hénon-like map with sufficiently small $\bar{\varepsilon} > 0$. Then*

$$\mathcal{A}_F = \bigcap_{k\geq 0} F^k(B) \cap B = \overline{W^u(\beta_0)} \cap B$$

Proof. The fact that $\mathcal{A}_F$ is backward invariant under F implies that

$$\mathcal{A}_F \subset \bigcap_{k\geq 0} F^k(B) \cap B$$

For the opposite inclusion, take a point $x \in \bigcap_{k\geq 0} F^k(B) \cap B$. If $x \in \mathcal{O}_F$ then $x \in \mathcal{A}_F$. Let us assume that $x \notin \mathcal{O}_F$. Since the global attracting set is backward invariant, $F^{-k}(x) \in B$ for all $k \in \mathbb{N}_+$. The alpha limit set of x, $\alpha(x)$ is the set of accumulation of the backward image of x under F. Then it is completely invariant closed set in B and moreover, $\alpha(x) \subset \Omega_F$ for every $x \in \bigcap_{k\geq 0} F^k(B) \cap B$. Since $x \notin \mathcal{O}_F$, $x \notin \operatorname{Trap}_n$ for all sufficiently large $n \in \mathbb{N}$. The fact that $F(\operatorname{Trap}_n) \subset \operatorname{Trap}_n$ implies that $F^{-i}(x) \notin \operatorname{Trap}_n$ for all $i \in \mathbb{N}$. Then $\alpha(x) \cap \mathcal{O}_F = \varnothing$. Hence, $\alpha(x) \subset \operatorname{Per}_F$. It means that

$$x \in \bigcup_{w \in \operatorname{Per}_F} W^u(w) \cap B.$$

Then

$$\mathcal{A}_F = \bigcap_{k\geq 0} F^k(B) \cap B \tag{8.0.6}$$

The set $\overline{W^u(\beta_0)} \cap B$ is backward invariant. Then

$$\overline{W^u(\beta_0)} \cap B \subset \bigcap_{k\geq 0} F^k(B) \cap B = \mathcal{A}_F.$$

For the opposite inclusion, recall Lemma 5.3.2, the critical Cantor set $\mathcal{O}_F$ is the set of accumulation points of Per_F. Then $\mathcal{O}_F \subset \overline{\cup_{n\geq 0} \operatorname{Orb}(\beta_n)}$. Moreover, by Lemma 8.0.6, every unstable manifold of periodic points are contained in $\overline{W^u(\beta_0)}$. Hence, Definition 8.0.5 implies that

$$\overline{W^u(\beta_0)} \cap B = \mathcal{A}_F.$$

□

Corollary 8.0.8. *Let $F \in \mathcal{I}_B(\bar{\varepsilon})$ the infinitely renormalizable Hénon-like map with sufficiently small $\bar{\varepsilon} > 0$. Then $\overline{W^u(\beta_n)} \cap B^n_{v^n}$ is the invariant set under F^{2^n}, which is maximal backward invariant set under F^{2^n} for each $n \in \mathbb{N}$.*

Proof. Theorem 8.0.7 implies that if $F \in \mathcal{I}_B(\bar{\varepsilon})$, then $\overline{W^u(\beta_0(R^nF))}$ is the maximal backward invariant set on $B(R^nF)$ under R^nF for each $n \in \mathbb{N}$. Since $F^{2^n}(w) = \Psi^n_{v^n} \circ R^nF \circ (\Psi^n_{v^n})^{-1}(w)$ on $B^n_{v^n}$ and $B^n_{v^n}$ is $\Psi^n_{v^n}(B(R^nF))$, the backward maximality on $B(R^nF)$ under R^nF is inherited to F^{2^n} on the $B^n_{v^n}$ by the conjugation for each $n \in \mathbb{N}$. In particular, the stable (respectively unstable) manifold at a point moves to that of the corresponding point by the conjugation $\Psi^n_{v^n}$. Hence, $\Psi^n_{v^n}\big(\overline{W^u(\beta_0(R^nF))}\big) = \overline{W^u(\beta_n)}$ and $\overline{W^u(\beta_n)} \cap B^n_{v^n}$ is the invariant set under F^{2^n} which is maximal of the backward invariant under F^{2^n}. □

The above Corollary says that there exists the global attracting set under F^{2^n}, $\mathcal{A}_{F^{2^n}}$ on each domain $B^n_{v^n}$ for each $n \in \mathbb{N}$. Then this fact suggests the term *locally global attracting set*, which seems to be almost self contradictory.

Chapter 9

Small perturbation of model maps

The university of Jacobian does not imply asymptotic formula of ε or δ in general. However, with particular assumptions the dynamics of the critical Cantor set is well-controlled using Lyapunov exponents on it. The maximal exponent of the three dimensional Hénon-like map is 0 and it has two other exponents $\log b_1$ and $\log b_2$ which are strictly less than zero. In particular, if $\partial_z \varepsilon \equiv 0$, then one of these exponents comes from the two dimensional Hénon-like map and the other one represents how much $z-$directions are attracted. See Proposition 9.1.1. If the attraction of the third coordinate is sufficiently stronger than the other attractions uniformly on the compact invariant set, then there exists the dominated splitting on the given set under sufficiently high iterates of DF^{-1} by Lemma 9.2.3.

9.1 Renormalizable model maps

Let us take a two dimensional Hénon-like map $F_{2d}(x,y) = (f(x) - \varepsilon(x,y), x)$ which is infinitely renormalizable. We can consider the three dimensional perturbed Hénon-like map $F(x,y,z) = (f(x) - \varepsilon(x,y),\, x,\, \delta(x,y,z))$ as a perturbation of two dimensional Hénon-like map. See the condition $\partial_z \varepsilon \equiv 0$. Let these maps be the model maps and denote F_{mod}. In contrast with the general three dimensional Hénon-like map, F_{mod} has the special $\varepsilon(w)$ such that ε depends only on the first two variables x and y. The map ε_n in $F_{\text{mod},\,n}$ of on each level n depends only on x and y using the direct calculation of $H \circ F^2 \circ H^{-1}$ and the induction. See Proposition 4.2.1.

Proposition 9.1.1. *Let F_{mod} be the three dimensional Hénon-like model map in $\mathcal{I}_B(\bar{\varepsilon})$ with sufficiently small $\bar{\varepsilon} > 0$. Then $\varepsilon_n(w)$ in $R^n F_{\text{mod}}$ depends only* *on x and y as follows.*

$$F_{\mathrm{mod},\,n}(x,y,z) = (f_n(x) - \varepsilon_n(x,y), x, \delta_n(x,y,z))$$

Furthermore, ε_n and δ_n has following form.

$$\begin{aligned}\varepsilon_n(x,y) &= b_1^{2^n} a(x)\; y\; (1+O(\rho^n))\\ \delta_n(x,y,z) &= (b_2^{2^n} z + \widetilde{\delta}_n(x,y))(1+O(\rho^n))\end{aligned}$$

where $a(x)$ is the non-vanishing diffeomorphism on the I^x on B, $\|\widetilde{\delta}_n(x,y)\| = O\big(\bar{\varepsilon}^{2^n}\big)$ for some $\rho \in (0,1)$.

Proof. The direct calculation $\Lambda \circ H \circ F^2 \circ H^{-1}\Lambda^{-1}(w)$ with induction implies that $\varepsilon_n(w) = \varepsilon_n(x,y)$. In other words, The two dimensional map $F_{2d,\,n}$ is the composition of the projection on $xy-$plane and the model map, that is, $F_{2d,\,n} = \pi_{xy} \circ F_{\mathrm{mod},\,n}$ for every $n \in \mathbb{N}$ where $F_{\mathrm{mod},\,n} \in \mathcal{I}_B\big(\bar{\varepsilon}\big)$. Then $\varepsilon_n(x,y)$ has the universal expression for each $n \in \mathbb{N}$ because of universality theorem of the two dimensional Hénon-like map. See Theorem 7.9 in [CLM]. The universal function, $\partial_y \varepsilon_n(x,y) = b_1^{2^n} a_{2d}(x)\,(1+O(\rho^n))$. Moreover, the fact that $\partial_z \varepsilon \equiv 0$ implies that $\mathrm{Jac}\,F_{\mathrm{mod},\,n} = \partial_y \varepsilon_n\, \partial_z \delta_n - \partial_z \varepsilon_n\, \partial_y \delta_n = \partial_y \varepsilon_n \partial_z \delta_n$. Then by Theorem 7.5.1, the following holds.

$$\mathrm{Jac}\,F_{\mathrm{mod},\,n} = b^{2^n} a(x)\,(1+O(\rho^n)) = b_1^{2^n} a_{2d}(x)\,(1+O(\rho^n))\,\partial_z \delta_n$$

Then it is sufficient to show that the $a(x)$ is same as $a_{2d}(x)$, the universal function of two dimensional map $\pi_{xy} \circ F_{\mathrm{mod},\,n}$.

Recall (7.5.3). $a_{2d}(x)$ is the two dimensional version of the following limit.

$$a(x) = \lim_{n\to\infty} \frac{\mathrm{Jac}\,\Psi^n_{\mathrm{tip}}(w)}{\mathrm{Jac}\,\Psi^n_{\mathrm{tip}}(F_n w)}$$

Lemma 7.4.1 implies that the map $x + S_0^n(w)$ is asymptotically the sum of the universal function $v(x)$ of the variable x and the quadratic homogeneous polynomials of the other variables. Then both two and three dimensional Jacobian determinant of Ψ^n_{tip}, namely, $\mathrm{Jac}\,\Psi^n_{\mathrm{tip}} = 1 + \partial_x S_0^n$ converges to the universal one-dimensional map $v'(x-1)$. Moreover, the ratio of $\Psi^n_{\mathrm{tip}}(w)$ and $\Psi^n_{\mathrm{tip}}(F_n w)$ of both two and three dimensional Hénon-like maps has the same universal limit $a(x)$ by (7.5.3).

Hence, the fact that $b = b_1 b_2$ and exponential convergence of $\mathrm{Jac}\,F_n$ implies that $\partial_z \delta_n = b_2^{2^n}(1+O(\rho^n))$. $\square$

9.2 Invariant splitting of tangent bundle on invariant compact sets

The infinitely renormalizable model map, F_{mod} has the invariant constant sections, $E^{ss} = \{(0\ 0\ 1)\} \cdot \mathbb{R}$. If the contraction of DF_{mod} along this subbundle is strongest, then there exists an invariant cone field by Lemma 9.2.3.

For the given $w_0 = (x,\, y,\, z)$, let us denote $w_i = (x_i,\, y_i,\, z_i) = F^i(x,\, y,\, z)$. The DF_{mod} is the following matrix.

$$DF_{\text{mod}} = \left(\begin{array}{cc|c} \multicolumn{2}{c|}{\multirow{2}{*}{DF_{2d}}} & 0 \\ & & 0 \\ \hline \partial_x\delta & \partial_y\delta & \partial_z\delta \end{array}\right)$$

For simplicity, let us express the above matrix as the block matrix

$$DF_{\text{mod}}(x,y,z) = \begin{pmatrix} A_1 & \mathbf{0} \\ C_1 & D_1 \end{pmatrix} \equiv \begin{pmatrix} A_1(w) & \mathbf{0} \\ C_1(w) & D_1(w) \end{pmatrix}$$

where $A_1 = DF_{2d}(x,y)$, $\mathbf{0} = \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)$, $C_1 = (\partial_x\delta(w)\ \ \partial_y\delta(w))$ and $D_1 = \partial_z\delta(w)$.

Let each component of the derivative of $F^N_{\text{mod}}(w)$ at w_0 be A_N, $\mathbf{0}$, C_N and D_N as follows.

$$DF^N_{\text{mod}}(x_0,y_0,z_0) = \begin{pmatrix} A_N & \mathbf{0} \\ C_N & D_N \end{pmatrix} \equiv \begin{pmatrix} A_N(w_0) & \mathbf{0} \\ C_N(w_0) & D_N(w_0) \end{pmatrix} \tag{9.2.1}$$

Then for each $N \geq 1$,

$$A_N = \prod_{i=0}^{N-1} DF_{2d}(x_{N-1-i}, y_{N-1-i}), \quad D_N = \prod_{i=0}^{N-1} \partial_z\delta(w_{N-1-i})$$

Moreover,

$$\begin{pmatrix} A_N & \mathbf{0} \\ C_N & D_N \end{pmatrix} = \begin{pmatrix} A(w_{N-1}) & \mathbf{0} \\ C(w_{N-1}) & D(w_{N-1}) \end{pmatrix} \cdot \begin{pmatrix} A_{N-1} & \mathbf{0} \\ C_{N-1} & D_{N-1} \end{pmatrix} \tag{9.2.2}$$

Let A_0 and $D_0 \equiv 1$ for notational compatibility. Then we have the expression of C_N as follows.

$$\begin{aligned} C_N &= \sum_{i=0}^{N-1} D_i(w_{N-1-i})\, C(w_{N-1-i})\, A_{N-1-i} \\ &= \sum_{i=0}^{N-1} \left[\prod_{j=0}^{i-1} \partial_z \delta(w_{N-1-j}) \right] \cdot \big(\partial_x \delta(w_{N-1-i}) \ \ \partial_y \delta(w_{N-1-i}) \big) \\ &\quad \cdot \left[\prod_{i=0}^{N-i-1} DF_{2d}(x_i, y_i) \right] \end{aligned}$$

In order to prove the existence of the invariant splitting of the tangent bundle under the map DF^N_{mod}, it suffice to show that there exists the invariant cone field under DF^N_{mod} with uniform expansion or contraction. Denote the cone field with the vertical direction $(0\ 0\ 1)$ with some positive number γ at w to be

$$\mathcal{C}(\gamma)_w = \{ u + v \mid (u, v) \in \mathbb{R}^2 \times \mathbb{R} \ \text{ and } \ \|u\| < \gamma\, \|v\| \,\} \tag{9.2.3}$$

The cone field on a given compact invariant set Γ is the union of the cones at every points in Γ.

$$\mathcal{C}(\gamma) = \bigcup_{w \in \Gamma} \mathcal{C}(\gamma)_w \tag{9.2.4}$$

In order to construct the invariant splitting, we need to show that

$$DF^{-N}_{\text{mod}} \left(\mathcal{C}(\gamma) \right)_{w_N} \subset \mathcal{C} \left(\frac{1}{2} \gamma \right)_{w_0}$$

at every point w_0 of the invariant set under F^{-N}_{mod} for $N \in \mathbb{N}$. Let $\|DF\|$ the operator norm of DF. The minimum expansion rate (or the strongest contraction rate) of DF is defined by the equation, $\|DF^{-1}\| = \dfrac{1}{m(DF)}$.

The Jacobian determinant of F_{2d} is $\partial_y \varepsilon(x, y)$. Since F_{2d} is an orientation preserving diffeomorphism from $B_{2d} \equiv \text{Dom}(F_{2d})$ to its image, $\partial_y \varepsilon(x, y)$ has the positive infimum. Denote this infimum to be m_{2d}, that is,

$$m_{2d} = \inf_{(x,\, y) \in B(F_{2d})} \{ \partial_y \varepsilon(x, y) \} \,.$$

Similarly, define $m_{2d,\,n} = \inf_{(x,\,y) \in B(R^n F_{2d})} \{ \partial_y \varepsilon_n(x,y) \}$ for the n^{th} renormalized model map, $F_{2d,\,n} \equiv R^n F_{2d}$. Denote the two dimensional coordinate change map from level n to k to be ${}_{2d}\Psi^n_k$. Thus the derivative of Ψ^n_0 at the tip as follows.

$$D_{2d}\Psi^n_0 = \begin{pmatrix} \alpha_{n,\,0}\,(1+\partial_x S^n_0) & \alpha_{n,\,0}\,\partial_y S^n_0 + \sigma_{n,\,0}\, t_{n,\,0} \\ 0 & \sigma_{n,\,0} \end{pmatrix} \tag{9.2.5}$$

Proposition 9.2.1. *Let $F_{\mathrm{mod}} \in \mathcal{I}_B(\bar{\varepsilon})$ with sufficiently small $\bar{\varepsilon} > 0$. Then the infimum of the derivative of the two dimensional map, $m(DF^{2^n}_{2d}) \asymp \sigma^n b_1^{2^n}$ in $B^n_{v^n}$ for every $n \in \mathbb{N}$.*

Proof. Firstly, let us show that $m(DF^{2^n}_{2d}) \lesssim \sigma^n b_1^{2^n}$.

$$\frac{1}{m(DF_{2d})} = \|DF^{-1}_{2d}\| \geq \left\| \frac{1}{\partial_y \varepsilon} \begin{pmatrix} 0 & \partial_y \varepsilon \\ -1 & f'(x) - \partial_x \varepsilon \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\| \geq \frac{1}{\partial_y \varepsilon(x,y)}$$

Then $m(DF_{2d}) \leq \partial_y \varepsilon(x,y) = \operatorname{Jac} F_{2d}$ for every point (x,y). Similarly, since F_{2d} is infinitely renormalizable, $m(DF_{2d,\,n}) \leq \partial_y \varepsilon_n(x,y) = \operatorname{Jac} F_{2d,\,n}$ for every $n \in \mathbb{N}$. Then $m(DF_{2d,\,n}) \leq m_{2d,\,n}$. Let us estimate upper bound of the norm of $DF^{-2^n}_{2d}$.

$$\begin{aligned} \|DF^{-2^n}_{2d}\| &\geq \left\| D\Psi^n_0 \cdot DF^{-1}_{2d,\,n} \cdot D(\Psi^n_0)^{-1} \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \right\| \\ &= \left\| D\Psi^n_0 \cdot DF^{-1}_{2d,\,n} \cdot \frac{1}{\alpha_{n,\,0}\,(1+\partial_x S^n_0)} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\| \\ &= \left\| D\Psi^n_0 \cdot \frac{-1}{\partial_y \varepsilon_n \cdot \alpha_{n,\,0}\,(1+\partial_x S^n_0)} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\| \\ &= \left\| \frac{-1}{\partial_y \varepsilon_n \cdot \alpha_{n,\,0}\,(1+\partial_x S^n_0)} \cdot \begin{pmatrix} \alpha_{n,\,0}\,\partial_y S^n_0 + \sigma_{n,\,0}\, t_{n,\,0} \\ \sigma_{n,\,0} \end{pmatrix} \right\| \end{aligned}$$

$$
\begin{aligned}
&\geq \left\| \frac{-1}{\sup\limits_{w\in B(R^nF)} \{\partial_y \varepsilon_n\} \cdot \sup\limits_{w \in B(F)} \{1 + \partial_x S_0^n)\}} \cdot \frac{1}{\alpha_{n,\,0}} \begin{pmatrix} \alpha_{n,\,0}\, \partial_y S_0^n + \sigma_{n,\,0}\, t_{n,\,0} \\ \sigma_{n,\,0} \end{pmatrix} \right\| \\
(*) \quad &\geq \frac{1}{C_0 b_1^{2^n}} \left\| \frac{\sigma_{n,\,0}}{\alpha_{n,\,0}} \begin{pmatrix} \frac{\alpha_{n,\,0}}{\sigma_{n,\,0}}\, \partial_y S_0^n + t_{n,\,0} \\ 1 \end{pmatrix} \right\| \\
(**) \quad &\geq \frac{1}{C_0 C_1 b_1^{2^n} \sigma^n} (1 - C_2 \bar{\varepsilon}) \\
&\geq \frac{1}{C \sigma^n b_1^{2^n}}
\end{aligned}
$$

where C and C_i for $i = 0, 1, 2$ are some positive numbers. By the universality of the two dimensional Hénon-like maps, $1 + \partial_x S_0^n(w) = v_*'(x) + O(\rho^n)$ where $v_*(x)$ is a diffeomorphism on its domain and $\partial_y \varepsilon_n \asymp b_1^{2^n}$. Moreover, $|\,\sigma_{n,0}| \asymp \sigma^n$, $\alpha_{n,\,0} \asymp \sigma^{2n}$, $|\,t_{n,0}| = O(\bar{\varepsilon})$, and $\partial_y S_0^n = a_F y + O(\rho^n)$ for $a_F = O(\bar{\varepsilon})$. Then the inequality $(*)$ and $(**)$ holds. For the detailed proof about the above asymptotic of the two dimensional Hénon-like maps, see the Section 7 in [CLM]. Hence,

$$
\frac{1}{m(DF_{2d}^{2^n})} \geq \frac{1}{C \sigma^n b_1^{2^n}} \quad \text{for each } n \in \mathbb{N}
$$

Secondly, let us show that $m(DF_{2d}^{2^n}) \gtrsim \sigma^n b_1^{2^n}$. Let us observe the following fact which is used later. For the vector $\left(\begin{smallmatrix} v_1 \\ v_2 \end{smallmatrix}\right)$ whose length is 1, that is, $v_1^2 + v_2^2 = 1$, the following inequality holds by the Cauchy-Schwarz inequality.

$$
\left\| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right\| \leq \sqrt{a^2 + b^2 + c^2 + d^2} \tag{9.2.6}
$$

Moreover, if $ad - bc \neq 0$, then

$$
\left\| \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right\| \leq \frac{1}{(ad - bc)^2} \sqrt{a^2 + b^2 + c^2 + d^2} \tag{9.2.7}
$$

$$
F_{2d}^{-2^n} = \Psi_0^n \circ F_{2d,\,n}^{-1} \circ (\Psi_0^n)^{-1}
$$

Then

$$
\| DF^{-2^n} \| \leq \| D\Psi_0^n \| \cdot \| DF_{2d,\,n}^{-1} \| \cdot \| D(\Psi_0^n)^{-1} \|
$$

By the (9.2.6) and (9.2.7), the upper bounds of norms are following.

$$
\begin{aligned}
\|D\Psi_0^n\|^2 &= \left\| \begin{pmatrix} \alpha_{n,\,0}\,(1 + \partial_x S_0^n) & \alpha_{n,\,0}\,\partial_y S_0^n + \sigma_{n,\,0}\,t_{n,\,0} \\ 0 & \sigma_{n,\,0} \end{pmatrix} \right\|^2 \\
&\leq \sup_{w \in B(R^nF)} \{\alpha_{n,\,0}^2 (1 + \partial_x S_0^n)^2 + (\alpha_{n,\,0}\,\partial_y S_0^n + \sigma_{n,\,0}\,t_{n,\,0})^2 + \sigma_n^2\}
\end{aligned}
$$

$$
\begin{aligned}
\|DF_{2d,\,n}^{-1}\|^2 &= \left\| \frac{1}{\partial_y \varepsilon_n} \begin{pmatrix} 0 & \partial_y \varepsilon_n \\ -1 & f_n'(x) - \partial_x \varepsilon_n \end{pmatrix} \right\|^2 \\
&\leq \sup_{w \in B(R^nF)} \left\{ \frac{1}{(\partial_y \varepsilon_n)^2} \big((\partial_y \varepsilon_n)^2 + 1 + (f_n'(x) - \partial_x \varepsilon_n)^2 \big) \right\}
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\|DF_{2d}^{-2^n}\|^2 \\
\leq\ & \sup_{w \in B(R^nF)} \{\alpha_{n,\,0}^2 (1 + \partial_x S_0^n)^2 + (\alpha_{n,\,0}\,\partial_y S_0^n + \sigma_{n,\,0}\,t_{n,\,0})^2 + \sigma_n^2\} \\
&\cdot \sup_{w \in B(R^nF)} \left\{ \frac{1}{(\partial_y \varepsilon_n)^2} \big((\partial_y \varepsilon_n)^2 + 1 + (f_n'(x) - \partial_x \varepsilon_n)^2 \big) \right\} \\
&\cdot \sup_{w \in B(F)} \left\{ \frac{1}{\alpha_{n,\,0}^2 (1 + \partial_x S_0^n)^2 \cdot \sigma_{n,\,0}^2} \right. \\
&\qquad \left. \cdot \Big[\alpha_{n,\,0}^2 (1 + \partial_x S_0^n)^2 + (\alpha_{n,\,0}\,\partial_y S_0^n + \sigma_{n,\,0}\,t_{n,\,0})^2 + \sigma_n^2 \Big] \right\} \\
\leq\ & \frac{C_0}{b_1^{2n+1} \sigma^{2n}}
\end{aligned}
$$

for some $C_0 > 0$. Then

$$
\frac{1}{m(DF_{2d}^{2^n})} \leq \frac{C}{b_1^{2^n} \sigma^n} \quad \text{for } n \in \mathbb{N}
$$

□

Lemma 9.2.2. *Let the A_N, $\mathbf{0}$, C_N and D_N be components of DF_{mod}^N defined on (9.2.1) and suppose that $\|D\| \ll m_{2d}$ on B. Then $\|C_N\| \leq \kappa \|A_N\|$ for some $\kappa > 0$ which is independent of N. Moreover, $\|C_N \cdot A_N^{-1}\|$ is bounded above by the some $\kappa_0 > 0$ independent of N.*

Proof. Let us calculate the upper bounds of $\|D_i\|$ and the lower bounds of

$m(A_{i+1})$.

$$\|D_i\| \leq \prod_{j=0}^{i-1} \|\partial_z \delta(w_{N-1-i})\| \leq (C_1 b_2)^i \tag{9.2.8}$$

for some $C_1 > 0$. Recall

$$A_{i+1} = \prod_{j=0}^{i} DF_{2d}(x_{i-j}, y_{i-j})$$

and $m(A) = \|DF_{2d}^{-1}\|^{-1}$. Then by Proposition 9.2.1, we obtain that

$$\frac{1}{m(A_{i+1})} \leq \prod_{j=0}^{i} \|DF_{2d}^{-1}(x_j, y_j)\| \leq \frac{1}{m(A)^{i+1}} \leq \left(\frac{K}{\|\partial_y \varepsilon\|}\right)^{i+1} \leq \left(\frac{C_2}{b_1}\right)^{i+1} \tag{9.2.9}$$

for some $C_2 > 0$. Let us assume that $b_2 < C_1 C_2 b_1$. Then

$$\begin{aligned}
\| C_{N+1}\| = & \left\| \sum_{i=0}^{N} D_i(w_{N-i})\, C(w_{N-i})\, A_{N-i} \right\| \leq \sum_{i=0}^{N} \|D_i\| \|C\| \frac{\|A_N\|}{m(A_{i+1})} \\
\leq & \ \|C\| \|A_N\| \sum_{i=0}^{N} \frac{\|D_i\|}{m(A_{i+1})} \leq C_0 \bar{\varepsilon} \|A_N\| \sum_{i=0}^{N} \frac{\|D_i\|}{m(A_{i+1})} \\
\leq & \ C_0 \bar{\varepsilon} \|A_N\| \sum_{i=0}^{N} (C_1\, b_2)^i \left(\frac{C_2}{b_1}\right)^{i+1} \\
\leq & \ C_0 \bar{\varepsilon} \|A_N\| \frac{C_2}{b_1} \sum_{i=0}^{\infty} \left(\frac{b_2}{b_1} \cdot C_1 C_2\right)^i \\
= & \ \frac{C_0 C_2}{b_1 - C_1 C_2 b_2} \|A_N\| = \kappa_0 \|A_N\|
\end{aligned} \tag{9.2.10}$$

where $\kappa_0 = \dfrac{C_0 C_2}{b_1 - C_1 C_2 b_2}$. By the recursive relation (9.2.2), we get the following estimation.

$$D(w_N)\, C_N = C_{N+1} - C(w_N)\, A_N$$

Then by the above estimation (9.2.10), we have

$$\begin{aligned}\|D(w_N)\, C_N\| \leq\ & \|C_{N+1}\| + \|C(w_N)\|\|A_N\| \\ C_3 b_2 \|C_N\| \leq\ & \kappa_0 \|A_N\| + C_0 \bar{\varepsilon}\, \|A_N\|\end{aligned}$$

for some $K > 0$. Hence,

$$\|C_N\| \leq \frac{\kappa_0 + C_0\bar{\varepsilon}}{C_3 b_2}\, \|A_N\| = \kappa \|A_N\|$$

where $\kappa = \dfrac{\kappa_0 + C_0\bar{\varepsilon}}{C_3 b_2}$.

Let us calculate $A_{N-1-i} \cdot A_N^{-1}(w_0)$.

$$\begin{aligned}A_N(w_0) &= \prod_{j=0}^{N-1} A(w_{N-1-j}) = \prod_{j=0}^{i} A(w_{N-1-j}) \cdot \prod_{j=i+1}^{N-1} A(w_{N-1-j}) \\ &= \prod_{j=0}^{i} A(w_{N-1-j}) \cdot \prod_{k=0}^{N-2-i} A(w_{N-2-i-k}) \\ &= A_{i+1}(w_{N-1-i}) \cdot A_{N-1-i}(w_0)\end{aligned}$$

Then

$$\begin{aligned}A_{N-1-i}(w_0) \cdot A_N^{-1}(w_0) &= A_{N-1-i}(w_0) \cdot \big[A(w_N)\big]^{-1} \\ &= A_{N-1-i}(w_0) \cdot \big[A_{N-1-i}(w_0)\big]^{-1} \cdot \big[A_{i+1}(w_{N-1-i})\big]^{-1} \\ &= \big[A_{i+1}(w_{N-1-i})\big]^{-1}\end{aligned} \tag{9.2.11}$$

Then by the similar calculation of (9.2.10),

$$
\begin{aligned}
&\| C_N(w_0) A_N^{-1}(w_N)\| \\
= \;& \Big\| \sum_{i=0}^{N-1} D_i(w_{N-i})\, C(w_{N-i})\, A_{N-i} \cdot A_N^{-1}(w_N) \Big\| \le \; \sum_{i=0}^{N-1} \|D_i\| \|C\| \|A_{i+1}^{-1}\| \\
\le \;& \|C\| \sum_{i=0}^{N-1} \frac{\|D_i\|}{m(A_{i+1})} \le \; C_0\bar{\varepsilon} \sum_{i=0}^{N-1} \frac{\|D_i\|}{m(A_{i+1})} \\
\le \;& C_0\bar{\varepsilon} \sum_{i=0}^{N-1} \big(C_1\, b_2\big)^i \left(\frac{C_2}{b_1}\right)^{i+1} \\
\le \;& \kappa_0
\end{aligned} \tag{9.2.12}
$$

where κ_0 is defined above. $\square$

Lemma 9.2.3. *Let F_{mod} be the model map in $\mathcal{I}_B(\bar{\varepsilon})$ for sufficiently small $\bar{\varepsilon} > 0$. Suppose that $b_2 \ll b_1$. Then the cone field $\mathcal{C}(\gamma)$ is invariant under DF_{mod}^{-1} for all sufficiently small $\gamma > 0$. More precisely, $\mathcal{C}(\gamma)_w \subset DF_{\mathrm{mod}}^{-1}\big(\mathcal{C}(\frac{1}{2}\gamma)\big)_{F_{\mathrm{mod}}(w)}$ on every point of the any given compact set, Γ which is (completely) invariant under F.*

Proof. Let us take any vector $(u\ v) \in \mathbb{R}^2 \times \mathbb{R}$ in the cone field $\mathcal{C}(\gamma)$ such that $\|u\| < \gamma_0 \|v\|$ where $\gamma_0 < \gamma$. we may assume that $v = 1$.

$$
DF_{\mathrm{mod}}^{-N} \begin{pmatrix} u \\ 1 \end{pmatrix} = \begin{pmatrix} A_N^{-1} & \mathbf{0} \\ -D_N^{-1} C_N A_N^{-1} & D_N^{-1} \end{pmatrix} \begin{pmatrix} u \\ 1 \end{pmatrix} = \begin{pmatrix} A_N^{-1} \cdot u \\ -D_N^{-1} C_N A_N^{-1} \cdot u + D_N^{-1} \end{pmatrix}
$$

For the invariance of the cone field, it suffices to show that

$$
\frac{\|A_N^{-1} \cdot u\|}{\| - D_N^{-1} C_N A_N^{-1} \cdot u + D_N^{-1}\|} \le \frac{1}{2}\, \gamma_0.
$$

Observe that

$$
-D_N^{-1} C_N A_N^{-1} \cdot u + D_N^{-1} = D_N^{-1}(-C_N A_N^{-1} \cdot u + 1)
$$

Let us take small enough γ such that $\kappa_0\, \|u\| \le \frac{1}{2}$. Then by (9.2.12), we see that $\| -C_N A_N^{-1} \cdot u + 1\| \geq \frac{1}{2}$. Then

$$\frac{\|A_N^{-1} \cdot u\|}{\| -D_N^{-1} C_N A_N^{-1} \cdot u + D_N^{-1}\|} \leq \frac{\|A_N^{-1}\| \|u\|}{\frac{1}{2}\|D_N^{-1}\|} \leq \frac{2m(D_N)}{m(A_N)} \|u\| \leq 2\left(\frac{K b_2}{b_1}\right)^N \cdot \gamma_0$$

for some $K > 0$ and for all sufficiently small $\gamma_0 < \gamma$. Hence, the cone field $\mathcal{C}(\gamma_0)$ contracts with the uniform rate under DF_{mod}^{-N} for all $N \in \mathbb{N}$ if $b_2 \ll b_1$. $\square$

Remark 9.2.1. Whenever we assume that $b_2 < b_1$, there exists a big enough n_0 such that $b_2^{2^{n_0}} \ll \sigma^{n_0} b_1^{2^{n_0}}$. Then we may assume that $b_2 \ll b_1$ instead of taking $R^{n_0}F$ in order to separate two exponents.

9.3 small perturbation of the model maps with invariant cone field

The invariance of the cone field of the DF_{mod} holds under the sufficiently small continuous perturbation of entries in DF_{mod}. Let us express the perturbation of the given model map $F_{\text{mod}}(w) = (f(x) - \varepsilon(x,y),\, x,\, \delta(w))$.

$$F(w) = (f(x) - \varepsilon(x,y) - \widetilde{\varepsilon}(w),\, x,\, \delta(w)) \tag{9.3.1}$$

Recall the definition of the cone field (9.2.3) and (9.2.4).

$$\mathcal{C}(\gamma) = \bigcup_{w \in \Gamma} \mathcal{C}(\gamma_w)_w = \bigcup_{w \in \Gamma} \{u + v \mid (u,v) \in \mathbb{R}^2 \times \mathbb{R} \;\text{ and }\; \|u\| < \gamma_w \|v\| \}$$

for every γ_w is positive and $\gamma = \sup_{w \in \Gamma}\{\gamma_w\}$ where Γ is an invariant compact set under F. If $\gamma = \sup_{w \in \Gamma}\{\gamma_w\}$ is bounded, we call γ the *width* of the cone field with the direction $(0\ 0\ 1)$. Let us consider two cones, $\mathcal{C}_1$ and $\mathcal{C}_2$ at the same point w. If $\overline{\mathcal{C}}_1 \subset \mathcal{C}_2 \cup \{w\}$, then we say that the cone $\mathcal{C}_1$ is *properly contained* in the cone $\mathcal{C}_2$.

Let us denote the derivative of F as the matrix.

$$DF = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) = \left(\begin{array}{cc|c} \multicolumn{2}{c|}{\multirow{2}{*}{DF_{2d}}} & \partial_z \varepsilon(w) \\ & & 0 \\ \hline \partial_x \delta & \partial_y \delta & \partial_z \delta \end{array}\right) \tag{9.3.2}$$

where $DF_{2d} = \begin{pmatrix} f'(x) - \partial_x \varepsilon(w) & -\partial_y \varepsilon(w) \\ 1 & 0 \end{pmatrix}$.

By the universality of the two dimensional Hénon-like maps, $\partial_y \varepsilon(x, y, z_0) \asymp b_1$ for each $z_0 \in I^z$. If $\|\partial_z \varepsilon\|$ is sufficiently small, especially smaller than b_1, then by the Taylor's theorem, $\partial_y \varepsilon(x, y, z) \asymp b_1$. Moreover, by Proposition 9.2.1, $\partial_y \varepsilon(x, y, z) \asymp m(A)$.

If F is sufficiently close to F_{mod} in the C^1 sense, then DF invariant cone field which is same as the cone field invariant under DF_{mod}. On the following lemma, we quantify the perturbation of $\|\,\partial_z \varepsilon\|$ to obtain the invariant cone field under DF, the derivative of the perturbation of F_{mod}.

Lemma 9.3.1. *Let F_{mod} and F be infinitely renormalizable maps, that is, $F \in \mathcal{I}_B(\bar{\varepsilon})$. Let F be a perturbation of the model map F_{mod} defined on (9.3.1) with $b_2 \ll b_1$. Suppose that $\dfrac{\|\,\partial_z \varepsilon\|}{m(A) \cdot m(\partial_z \delta)} \leq \rho\,\gamma$ for some positive $\rho < 1$ where γ is the width of the invariant cone field of F_{mod}. Suppose also that $\|\partial_z \varepsilon\|$ is sufficiently small such that $\dfrac{m(A)}{m(A - BD^{-1}C)} \leq 1 + \varepsilon_0$ for any given number $\varepsilon_0 > 0$. Then F has the invariant cone field $\mathcal{C}(\gamma)$ such that $\mathcal{C}(\gamma)_w$ properly contained in $DF^{-1}\big(\mathcal{C}(\tfrac{1}{2}\gamma)\big)_{F(w)}$ on every point of the any given compact invariant set Γ under F.*

Proof. Let us denote DF to be

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

similar to (9.3.2). Then $D = \partial_z \delta$. By the direct calculation, DF^{-1} is

$$\begin{pmatrix} A^{-1} + \zeta_{11} & \zeta_{12} \\ -D^{-1}C(A^{-1} + \zeta_{11}) & D^{-1}\zeta_{22} \end{pmatrix}$$

where $\zeta_{12} = -(A - BD^{-1}C)^{-1}BD^{-1}$, $\zeta_{11} = -\zeta_{12}CA^{-1}$ and $\zeta_{22} = 1 - C\zeta_{12}$.

Since $\|\zeta_{12}\| \leq \dfrac{\|B\|}{m(A - BD^{-1}C) \cdot m(D)}$ and $B = \begin{pmatrix} \partial_z \varepsilon(w) \\ 0 \end{pmatrix}$, the small enough $\|\partial_z \varepsilon\|$ implies that $\|\zeta_{12}\| \leq \bar{\rho}\,\gamma$ for some $\bar{\rho} \leq (1 + \varepsilon_0)\rho$. Moreover, if γ is small enough then $\|\zeta_{11}\|$ has the same upper bound of $\|\zeta_{12}\|$ up to the uniform constant multiple because $\|CA^{-1}\|$ is uniformly bounded by Lemma 9.2.2. However, ζ_{22} is close to 1. Take the vector $(u\ v) \in \mathbb{R}^2 \times \mathbb{R}$ such that $\|u\| < \gamma\|v\|$.

Then we can normalize the vector v letting it be 1. Then $\|u\| < \gamma$.

$$
\begin{aligned}
DF^{-1}\begin{pmatrix} u \\ 1 \end{pmatrix} &= \begin{pmatrix} A^{-1}+\zeta_{11} & \zeta_{12} \\ -D^{-1}C(A^{-1}+\zeta_{11}) & D^{-1}\zeta_{22} \end{pmatrix}\begin{pmatrix} u \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} (A^{-1}+\zeta_{11})\cdot u+\zeta_{12} \\ -D^{-1}C(A^{-1}+\zeta_{11})\cdot u+D^{-1}\zeta_{12} \end{pmatrix}
\end{aligned}
$$

Let us calculate $\dfrac{\|(A^{-1}+\zeta_{11})\cdot u+\zeta_{12}\|}{\|-D^{-1}C(A^{-1}+\zeta_{11})\cdot u+D^{-1}\zeta_{12}\|}$ in order to obtain the invariance of the cone field. Observe that

$$
\begin{aligned}
(A^{-1}+\zeta_{11})\cdot u+\zeta_{12} = {}& A^{-1}u+\zeta_{12}(-CA^{-1}u+1) \\
D^{-1}C(A^{-1}+\zeta_{11})\cdot u+D^{-1}\zeta_{12} = {}& D^{-1}\big[\,C(A^{-1}+\zeta_{11})\cdot u+1-C\zeta_{12}\big]
\end{aligned}
$$

Then with the sufficiently small γ, $\|(A^{-1}+\zeta_{11})\cdot u+\zeta_{12}\| \leq c_0\|A^{-1}u\|$ for some $c_0>0$ and $\|-D^{-1}C(A^{-1}+\zeta_{11})\cdot u+D^{-1}\zeta_{12}\| \geq \frac{1}{2}\|D^{-1}\|$. Hence, by the similar proof of Lemma 9.2.3

$$
\frac{\|(A^{-1}+\zeta_{11})\cdot u+\zeta_{12}\|}{\|-D^{-1}C(A^{-1}+\zeta_{11})\cdot u+D^{-1}\zeta_{12}\|} \leq \frac{c_0\|A^{-1}u\|}{\frac{1}{2}\|D^{-1}\|} \leq \frac{2c_0\cdot m(A)\|u\|}{m(D)} \leq 2c_0\frac{Kb_2}{b_1}\gamma
$$

Then the cone field $\mathcal{C}(\gamma)$ is properly contained in $DF^{-1}(\mathcal{C}(\gamma))$. $\square$

Definition 9.3.1. Let $F_{\mathrm{mod}} \in \mathcal{I}_B(\bar{\varepsilon})$ be the model map defined as follows.

$$
F_{\mathrm{mod}}(x,y,z) = (f(x)-\varepsilon(x,y),\, x,\, \delta(x,y,z))
$$

Suppose that $\varepsilon(x,y) \asymp b_1$ where b_1 is the average Jacobian of the two dimensional map $\pi_{xy}\circ F_{\mathrm{mod}}$. Let the set $\mathcal{C}(\gamma) = \bigcup_{w\in\Gamma}\mathcal{C}(\gamma_w)_w$ be the invariant cone field under DF^{-1}_{mod} for sufficiently $\gamma>0$ such that $\mathcal{C}(\gamma)$ is properly contained in $DF^{-1}_{\mathrm{mod}}(\mathcal{C}(\gamma))$ on the given compact invariant set Γ under F. Let us define the *small perturbation* of the model map F_{mod} if the Hénon-like map F satisfies the following conditions.

(1) F is of the form in (9.3.1) and infinitely renormalizable.

(2) $\|D\widetilde{\varepsilon}\|$ is sufficiently small such that $\|\,\partial_y(\varepsilon(x,y)+\widetilde{\varepsilon}(w))\| \asymp b_1$.

(3) $\|\partial_z \widetilde{\varepsilon}\|$ sufficiently small implies that the cone field $\mathcal{C}^F = \bigcup_{w \in \Gamma} \mathcal{C}^F_w$ exists such that

(a) Every cone $\mathcal{C}^F_w$ is contained properly in $DF^{-1}(\mathcal{C}(\gamma)_{F(w)})$ at every point $w \in \Gamma$.

(b) Every cone $\mathcal{C}^F_w$ for all $w \in \Gamma$ contains the constant direction $(0\ 0\ 1)$.

Chapter 10

Invariant surfaces under the small perturbation of model maps

The existence of the invariant cone field on the invariant compact set implies the existence invariant splitting of the tangent bundle, in particular, the invariant plane field and the line field. The invariant plane field implies the existence of the local invariant surfaces by the pseudo-(un)stable manifold theorem. For instance, there exist surfaces invariant under F^{2^p} on the sufficiently small neighborhood of the periodic points with period 2^p. The periodic point β_N with the sufficiently close to the tip has the pseudo unstable manifold which contains all periodic points with period greater than N by Lemma 10.2.1. Since each point of the critical Cantor set is the accumulation point of periodic points, every pseudo-unstable manifolds at β_N as invariant surfaces also contains the Cantor set in the small neighborhood of β_N. Moreover, using the scoping map, $\Psi^n_{v^n}$ as the smooth conjugation between F^{2^n} and R^nF, it is shown that there exist global invariant surfaces as the graph from $I^x \times I^y$ to I^z under R^nF for every sufficiently large $n \in \mathbb{N}$ by Lemma 10.3.1.

10.1 Pseudo-unstable manifold on the compact invariant set

If there is the splitting of the contraction or expanding ratio is sufficiently large on the compact invariant set, there exists C^r pseudo (un)stable manifolds at every points on this set by Lemma 10.1.1.

Definition 10.1.1. Let $T : E \to E$ be a continuous linear map of the Banach space E. T is $\rho-$pseudo hyperbolic if there is a T invariant splitting of $E = E^1 \oplus E^2$ and there exist constants $0 < \lambda < \rho < \mu$ such that

(1) Let the restriction of T on E^1 be T_1. It is an isomorphism and
$\| T_1^n(v)\| \geq C_1\, \mu^n \|v\|$ for all $n \in \mathbb{N}$ and $v \in T_1$ and for some $C_1 > 0$.

(2) Let the restriction of T on E^1 be T_1.
$\| T_2^n(v)\| \leq C_2\, \lambda^n \|v\|$ for all $n \in \mathbb{N}$ and $v \in T_2$ and for some $C_2 > 0$.

If there exists a compact invariant set under a map (for example, diffeomorphism) f and there is an invariant splitting with the pseudo-hyperbolicity, then there exist the strong stable and pseudo-unstable (or pseudo-stable and strong unstable) manifold which are locally invariant under f. Then we can use the *strong stable manifold theorem* and *pseudo-unstable manifold theorem*.

Definition 10.1.2. Let $f : B \longrightarrow B$ be a differentiable map and ρ is a positive number and $d(w, q)$ be the distance between two points w and q in B. The pseudo-stable and pseudo-unstable set at a point $w \in B$ is the followings.

$$W^{ps}(w) = \left\{ q \in B \,\middle|\, \frac{d(f^n(w), f^n(q))}{\rho^n} \to 0 \quad as \quad n \to \infty \right\}$$
$$W^{pu}(w) = \left\{ q \in B \,\middle|\, \frac{d(f^{-n}(w), f^{-n}(q))}{\rho^{-n}} \to 0 \quad as \quad n \to \infty \right\}$$

The local pseudo-stable and unstable set is defined as following.

$$W^{ps}_{\varepsilon}(w) = \left\{ q \in B \,\middle|\, \frac{d(f^n(w), f^n(q))}{\rho^n} \leq \varepsilon \quad \text{for all} \quad n \in \mathbb{N}_+ \right\}$$
$$W^{pu}_{\varepsilon}(w) = \left\{ q \in B \,\middle|\, \frac{d(f^{-n}(w), f^{-n}(q))}{\rho^{-n}} \leq \varepsilon \quad \text{for all} \quad n \in \mathbb{N}_+ \right\}$$

If $\rho = 1$, then the above definitions are same as the usual (local) stable and unstable set. More generally, let us introduce the definition of the dominated splitting.

Definition 10.1.3. Let $F : M \to M$ be a C^1 map and Γ be a compact completely invariant set under F, that is, $F(\Gamma) = \Gamma$. The compact invariant set Γ has the *dominated splitting if*

(1) *The tangent bundle over* Γ *has an invariant subbundles —* $T_\Gamma M = E^1 \oplus E^2$

(2) $\|DF^n|_{E^1(x)}\|\|DF^n|_{E^2(F^n(x))}\| \leq C\lambda^n$, *for all* $x \in \Gamma$ *and* $n \geq 0$.

Moreover, the dominated splitting implies that invariant sections $w \mapsto E^1(w)$ and $w \mapsto E^2(w)$ are continuous by Theorem 1.2 in [New]. Furthermore, any dominated splitting has the adapted metric if every tangent spaces in the invariant tangent subbundle has the same constant dimension, that is, $\dim E^i(w)$ for $i = 1, 2$ is independent of w but dependent of each subbundle E^i. For the proof of the existence of adapted metric, see [Gou].

The following lemma and its complete proof is the Theorem IV.1 and the proof of it in [Shub].

Lemma 10.1.1 (Pseudo-unstable manifold theorem)**.** *Let* Γ *be an (compact) invariant set for the* C^r *diffeomorphism of* M *(which is a finite dimensional manifold). Suppose that the restricted tangent bundle* $T_\Gamma M$ *has a continuous* Df *invariant splitting*

$$T_\Gamma M = E^1 \oplus E^2$$

and there are constants $0 < \lambda < \rho < \mu$ *and* $0 < \lambda < 1$ *such that*

$$\begin{aligned} &\|Df(x)\,v\| \leq \lambda\|v\| \quad \textit{for all} \quad x \in \Lambda \quad \textit{and} \quad v \neq 0 \textit{ in } E^2 \\ \textit{and} \quad &\|Df(x)\,v\| \geq \mu\|v\| \quad \textit{for all} \quad x \in \Lambda \quad \textit{and} \quad v \neq 0 \textit{ in } E^1 \end{aligned}$$

with the adapted metric in $T_\Gamma M$*. Then there exist a positive number* ε *and for every point* $x \in \Gamma$ *there exist two embedded discs* $W^{ss}_\varepsilon(x)$*, local strong stable manifold and* $W^{pu}_\varepsilon(x)$*, local pseudo-unstable manifold which are tangent at* x *to* $E^2(x)$ *and* $E^1(x)$ *respectively. The* $W^{pu}_\varepsilon(x)$ *satisfies the following.*

(1) *If* $\lambda\rho^{-j} < 1$ *for* $1 \leq j \leq r$*, then* $W^{pu}_\varepsilon(x)$ *is* C^r*.*

(2) *The map* $x \mapsto W^{pu}_\varepsilon(x)$ *and* $x \mapsto W^{ss}_\varepsilon(x)$ *are continuous on* Γ*.*

(3) $f(W^{pu}_\varepsilon(x)) \cap B_\varepsilon(x) \subseteq W^{pu}_\varepsilon(f(x))$*, where*

$$B_\varepsilon(x) = \{\, y \in M \mid \ d(x, y) < \varepsilon \,\}$$

(4) *The* $W^{pu}_\varepsilon(x)$ *varies continuously as* C^r *embedded discs, that is, if* $\dim E^2 = k$*, then there is a neighborhood* U *of* x *and the continuous map* $\Theta : U \to Emb^r(D^n, M)$ *such that*

$$\Theta(x)(0) = x \quad \textit{and} \quad \Theta(x)(D^n) = W^{pu}_\varepsilon(x)$$

where $Emb^r(D^n, M)$ *is the set of* C^r *embedding from* D^n *to* M*.*

Let us denote E^1 on Lemma 10.1.1 to be E^{pu} and call it the invariant tangent subbundle with the pseudo unstable direction. Similarly, let us denote E^2 to be E^{ss} and call it the invariant tangent subbundle with the strong stable direction. If we take the invariant splitting at any point w which satisfies the assumption of (1) in the Lemma 10.1.1, then $W^{pu}_{\varepsilon}(w)$ is the graph of a C^r map from $E^{pu}(w)$ to $E^{ss}(w)$. Moreover, it is tangent to $E^{pu}(w)$ at each $w \in \Gamma$. The lower bounds of the size of the invariant manifolds is uniformly away from 0 at all points in Γ. Thus the splitting $T_wM = E^{pu}(w) \oplus E^{ss}(w)$ at $w \in \Gamma$ implies that we have the heteroclinic transversal intersection on every sufficiently small neighborhood of each point.

Remark 10.1.1. The existence of C^r invariant surfaces which are tangent to the invariant planes under DF comes from the pseudo-unstable manifold theorem. The proof of this theorem is similar to the proof of the unstable manifold theorem on the hyperbolic compact set. However, it requires to use the smooth cut-off function as an extension of some specific map. Moreover, the smoothness of the pseudo-unstable manifold is based on how much strong the splitting of the invariant directions under DF is on the compact invariant set. Then the pseudo-unstable manifold can be just finitely many differentiable although it can be any number depending on the splitting.

Proposition 10.1.2. *Let Γ be a compact invariant set under $f : B \to B$ where B is the compact manifold. Suppose that the tangent bundle on Γ has an invariant splitting under Df and this splitting is $\rho-$pseudo hyperbolic. Then there exists $\eta > 0$ for any $\varepsilon' > 0$ such that if $d(x, y) < \eta$ for any two points x, y in Γ, then the local pseudo-stable and pseudo-unstable manifolds meet transversally each other at a single point, say q, that is, $W^{pu}_{\varepsilon'}(x) \pitchfork W^{ps}_{\varepsilon'}(y) = \{q\}$.*

Proof. The invariant splitting of the tangent bundle with the invariant cone fields implies that the angle of two subspaces of the tangent space at each point $w \in \Gamma$, $\measuredangle(E^{pu}(w), E^{ps}(w))$ is positive (and has the uniform positive minimal angle). Moreover, since two locally invariant manifolds at w are tangent of $E^{pu}(w)$ and $E^{ps}(w)$ respectively. Furthermore, by the splitting of the tangent space T_wB, we have the equation $\dim(T_wB) = \dim(E^{pu}(w)) + \dim(E^{ps}(w))$. Then the dimension of the intersection of two manifolds is zero, because $\dim(T_wB) - \dim(E^{pu}(w)) - \dim(E^{ps}(w)) = 0$ and the dimension of submanifold is same as the dimension of the tangent subspace. Thus for sufficiently small ε', the intersection is connected and then it should be a single point. The compactness of the invariant set implies the existence of η independent of $w \in \Gamma$. $\square$

Remark 10.1.2. The assumption of Proposition 10.1.2 does not exclude the possibility that the compact manifold or Γ has boundaries. However, in order to keep the constant dimension of the intersection set we assume that the boundary of B and Γ is disjoint if these boundaries exist. However, the maximal global invariant set $\mathcal{A}_F$ has its image under F outside B. Then in order to apply Proposition 10.1.2, we need to take the smaller invariant set than $\mathcal{A}_F$. We can choose the set as an invariant compact set, for example, one of the sets $\overline{W^u(\beta_0)} \cap B_{\bullet}$ or $\overline{W^u(\beta_1)}$ and so on.

Let $\mathcal{O}_{\mathbf{w}_n} = \mathcal{O}_F \cap B^n_{\mathbf{w}_n}$ and let $\mathrm{Per}_{\mathbf{w}_n} = \mathrm{Per} \cap B^n_{\mathbf{w}_n}$ where $\mathbf{w}_n = (w_1 w_2 \ldots w_n)$ is the word of length n. Thus Per_{v^k} contains the periodic points β_n for all $n \geq k$ and its iterated images under F^{2^k} by Corollary 8.0.8.

10.2 Pseudo unstable manifolds as the C^r invariant surfaces under F

Recall the periodic point β_n on the domain of F as follows.

$$\beta_{n+1} \equiv \Psi^n_{v^n}\big(\beta_1(R^n F)\big)$$

for $n \geq 1$. Then β_{n+1} is a periodic point with period of 2^n. Furthermore, the sequence $\{\beta_n\}$ converges to the tip of F, τ_F as $n \to \infty$.

Let us take a convex neighborhood $\mathcal{N}$ of β_N in B for sufficiently big $N \in \mathbb{N}$ such that

(1) $\tau_F \in \mathcal{N}$

(2) $\partial\big(W^{pu}_{\varepsilon'}(\beta_N) \cap \mathcal{N}\big) \subset \partial \mathcal{N}$ and $\partial\big(W^{pu}_{\varepsilon'}(\beta_N) \cap B^{N-1}_{v^{N-1}}\big) \subset \partial B^{N-1}_{v^{N-1}}$

(3) $B^{N-1}_{v^{N-1}} \subset \mathcal{N}$.

(4) $W^{pu}_{\varepsilon'}(\beta_N) \cap \mathcal{N}$ is connected and simply connected.

(5) $W^{pu}_{\varepsilon'}(\beta_N) \pitchfork W^{ss}_{loc}(q)$ for every periodic points q in $\mathcal{N}$ and the intersection point is unique for each point q.

(6) $W^{pu}_{\varepsilon'}(\beta_N)$ is the graph of a C^r function η from $E^{pu}(\beta_N)$ to $E^{ss}(\beta_N)$ with $\|D\eta\| \leq C$ for some positive C independent of N.

Let $Q_{\beta_N} = W^{pu}_{\varepsilon'}(\beta_N) \cap \mathcal{N}$. Then Q_{β_N} is connected and it is the embedded C^r disc in $B^{N-1}_{v^{N-1}}$.
By the pseudo unstable manifold theorem, the local embedded disc $W^{pu}_{\varepsilon'}(w)$ at each point w is the graph of C^r function from $E^{pu}(w)$ to $E^{ss}(w)$. If $F \in \mathcal{I}_B(\bar{\varepsilon})$ is the small perturbation of the model maps, then $W^{pu}_{\varepsilon'}(w)$ is the graph of C^r map, ξ from $I^x \times I^y$ to I^z after C^1 coordinate change. Furthermore, C^1 norm of ξ is bounded because invariant cone field with $z-$direction contains $(0\ 0\ 1)$ and the angles between the invariant plane field and the line field has uniformly positive infinmum.

Lemma 10.2.1. *Let $F \in \mathcal{I}_B(\bar{\varepsilon})$ be a small perturbation of the model map for sufficiently small $\bar{\varepsilon} > 0$. Suppose that $b_2 \ll (b_1)^r$ for some $r \in \mathbb{N}$. Then for sufficiently large $N \in \mathbb{N}$, $W^{pu}_{\varepsilon'}(\beta_N)$ contains $\operatorname{Per}_{v^N} \cup \mathcal{O}_{v^N}$. Moreover, $W^{pu}_{\varepsilon'}(\beta_N) \cap B^{N-1}_{v^{N-1}}$ is the graph of C^r map from $I^x \times I^y$ to I^z.*

Proof. For sufficiently large N, β_N can be arbitrarily close the tip τ_F. Then there exists a pseudo unstable manifold of β_N in $\mathcal{N}$ which satisfies the above conditions. Let $Q_{\beta_N} = W^{pu}_{\varepsilon'}(\beta_N) \cap \mathcal{N}$. Since $b_2 \ll b_1 \ll 1$, $W^{u}_{\varepsilon'}(\beta_N) \cap \mathcal{N} \subset W^{pu}_{\varepsilon'}(\beta_N) \cap \mathcal{N}$ by the definition of the pseudo unstable manifold. However, the neighborhood the periodic point β_N contains the invariant domain under F^{2^N} around the tip, that is, $B^{N-1}_{v^{N-1}} \subset \mathcal{N}$ by the condition of $\mathcal{N}$. Then $W^u(\beta_N) \subset W^{pu}(\beta_N)$ in $B^{N-1}_{v^{N-1}}$. Moreover, since $W^u(\beta_n) \subset \overline{W^u(\beta_N)}$ for every $n \geq N$ by Theorem 8.0.7, Q_{β_N} contains the unstable manifolds of every periodic points with period 2^n for $n \geq N$.
Suppose that $q \in Q_{\beta_N} \cap W^{ss}(\beta_n)$ is different from β_n. Then $F^{2^n}(q) \to \beta_n$ as $n \to \infty$, that is, β_n is the accumulation point of the sequence $\{F^{2^n}(q)\}$. However, the fact that Q_{β_N} is invariant under F^{2^N} and $Q_{\beta_N} \pitchfork W^{ss}(\beta_n)$ implies that Q_{β_N} accumulate itself at β_n. It contradicts that Q_{β_N} is the embedded disc in $\mathcal{N}$. Then $q = \beta_n$. Hence, the single surface Q_{β_N} in $B^{N-1}_{v^{N-1}}$ which contains every periodic points β_n for $n \geq N$ is the only piece of $W^{pu}(\beta_N) \cap B^{N-1}_{v^{N-1}}$. The surface Q_{β_N} is the graph from $E^{pu}(\beta_N)$ to $E^{ss}(\beta_N)$ by the transversal intersection between $E^{pu}(\beta_N)$ and $E^{ss}(\beta_N)$.The strong stable manifolds $W^{ss}(\beta_n)$ for all $n \in \mathbb{N}$ are transversal to $I^x \times I^y$. Then by the C^1 coordinate change, $W^{pu}(\beta_N) \cap B^{N-1}_{v^{N-1}}$ is the graph of C^r map from $I^x \times I^y$ to I^z. $\square$

10.3 Invariant surfaces on each levels

There exists an invariant surface Q under F on $\pi_{xy}(B^n_{v^n})$ as the graph of the C^r function ξ with $\|D\xi\| \leq C\bar{\varepsilon}$ only if $b_2 \ll b_1$ and $\|\partial_z \varepsilon\| \ll b_2$. Then the image of Q under the smooth conjugation $\big(\Psi^n_{v^n}\big)^{-1}$ is also an invariant surface under R^nF with the same property. It means that there exists the C^r semi-conjugation between R^nF and a certain two dimensional Hénon-like maps $R^nF_{2d,\,\xi}$ at every deep level n.

Lemma 10.3.1. *Let Q be an invariant surface under $F \in \mathcal{I}_B(\bar\varepsilon)$ as the graph of a C^r function ξ on $\pi_{xy}(B^n_{v^n})$ such that $\|D\xi\| \leq C_0\bar\varepsilon$ for some $C_0 > 0$ and $\pi_{xy}(Q) = \pi_{xy}(B^n_{v^n})$. Then $Q_n \equiv \big(\Psi^n_{v^n}\big)^{-1}(Q)$ is an invariant surface under R^nF as the graph of a C^r function ξ_n on $\pi_{xy}\big(B(R^nF)\big)$ such that $\|D\xi_n\| \leq C\bar\varepsilon$ for some $C > 0$. In particular, $\|\partial_x\xi_n\| \leq C_1\bar\varepsilon\sigma^n$ for some $C_1 > 0$.*

Proof. Let us denote the graph(ξ) and the image of graph(ξ) under $\big(\Psi^n_{v^n}\big)^{-1}$ as follows.

$$\begin{aligned} \mathrm{graph}(\xi) = (x,\, y,\, \xi(x,y)) &= (x,\, y,\, z) \\ \big(\Psi^n_{v^n}\big)^{-1}\big(\mathrm{graph}(\xi)\big) &\equiv \{(x',\, y',\, z')\} \equiv Q_n \end{aligned}$$

By Lemma 7.2.2, we observe the following.

$$x = \alpha_{n,0}(x' + S^n_0(w')) + t_{n,0}\,\sigma_{n,0}\cdot y' + u_{n,0}\,\sigma_{n,0}\,(z' + R^n_0(y')) \tag{10.3.1}$$
$$y = \sigma_{n,0}\cdot y' \tag{10.3.2}$$
$$z = d_{n,0}\,\sigma_{n,0}\cdot y' + \sigma_{n,0}\,(z' + R^n_0(y')) \tag{10.3.3}$$

where $w' = (x',\, y',\, z')$. Firstly, let us show that $Q_n = \{(x',\, y',\, z')\}$ is the graph of a function ξ_n, that is, $z' = \xi_n(x',\, y')$. By the equations (10.3.2) and (10.3.3), we see that

$$y' = \frac{y}{\sigma_{n,0}} \tag{10.3.4}$$
$$z' = \frac{z - d_{n,0}\cdot y}{\sigma_{n,0}} - R^n_0\left(\frac{y}{\sigma_{n,0}}\right) \tag{10.3.5}$$

Thus

$$z' = \frac{\xi(x,\, \sigma_{n,0}\, y')}{\sigma_{n,0}} - d_{n,0}\cdot y' - R^n_0\,(y')$$

Then if x is well defined by x' and y', then z' is also a well defined function in terms of x' and y'. The invariant surface Q intersects only faces of $B^n_{v^n}$ which satisfies $\{(x,y,z)\,|\,\pi_x\circ\Psi^n_0(x,y,z) = \mathit{const.}\ \}$ and $\{y = \mathit{const.}\}$. The map Ψ^n_0 is a diffeomorphism between $B(R^nF)$ and $B^n_{v^n}$ and furthermore, the image of each face of $B(R^nF)$ is also corresponding faces of $B^n_{v^n}$. Since $B(R^nF)$ is the box domain and it's four faces are $\{x = const.\}$ and $\{y = const.\}$, the projected image of the surface and the box to the $xy-$plane are same as each other, that is, $\pi_{xy}(Q_n) = \pi_{xy}(B(R^nF))$. Then for each $z' \in \pi_z(Q_n)$ there exists x' and y' such that $(x', y', z') \in Q_n$. Moreover,

$$\pi_x \circ \Psi_0^n(x', y', z') = x.$$

Furthermore, there exist x' and y' which determines x because Ψ_0^n is a diffeomorphism for each $x \in \pi_{xy}(B^n_{v^n})$. Let us show that if x is defined, then it is uniquely determined in terms of x' and y'. On the equations between (x, y, z) and (x', y', z'), the variable x is only contained in (10.3.1). Let us define the map $G(x,\, y,\, x')$ as follows.

$$\begin{aligned} G(x,\, x',\, y') =&\ -x + \alpha_{n,\,0}\,(x' + S_0^n(w')) + t_{n,\,0}\,\sigma_{n,\,0} \cdot y' + u_{n,\,0}\,\sigma_{n,\,0}\,(z' + R_0^n(y')) \\ =&\ -x + \alpha_{n,\,0}\,\big(x' + S_0^n\big(x', y', z'\big)\big) + t_{n,\,0}\,\sigma_{n,\,0} \cdot y' \\ &\quad + u_{n,\,0}\,\big(\,\xi(x, y) - d_{n,\,0}\,\sigma_{n,\,0} \cdot y'\big) \end{aligned}$$

Then

$$\begin{aligned} \partial_x G(x,\, x',\, y') =&\ -1 + \alpha_{n,\,0} \cdot \partial_z S_0^n \cdot \partial_x z' + u_{n,\,0}\,\partial_x \xi \\ =&\ -1 + \frac{\alpha_{n,\,0}}{\sigma_{n,\,0}}\ \partial_z S_0^n \cdot \partial_x \xi + u_{n,\,0}\,\partial_x \xi \end{aligned}$$

Recall that $\alpha_{n,\,0} \asymp \sigma^{2n}$ and $\sigma_{n,\,0} \asymp (-\sigma)^n$. Lemma 7.2.4 implies that $\partial_z S_0^n = O(\bar{\varepsilon})$ and Lemma 7.2.2 implies that $|\, u_{n,0}| \leq C\bar{\varepsilon}$ for some $C > 0$. Then the partial derivative of G over x is away from zero.

$$|\,\partial_x G(x,\, x',\, y')| \geq 1 - C_0\,\bar{\varepsilon}^2\sigma^n - C_1\,\bar{\varepsilon}^2 \geq c > 0 \tag{10.3.6}$$

where the positive numbers C_0 and C_1 are uniform constants for all $n \in \mathbb{N}$ with sufficiently small $\bar{\varepsilon}$. The implicit function theorem implies that the x is a well defined function in terms of x' and y' on some neighborhood of the point (x', y'). Furthermore, it is globally well defined as the C^r function of x' and y' by the continuation using the neighborhoods of every points because of (10.3.6). Hence, z' is well defined as the function of variables x' and y' on the $\pi_{xy}\big(\mathrm{Dom}(R^nF)\big)$ and denote z' to be $\xi_n(x',\, y')$. Observe that $S_0^n(w') = S_0^n(x',\, y',\, \xi_n(x', y'))$.

Let us calculate the norm $\|D\xi_n(x',\, y')\|$. By the chain rule, the following holds.

$$\frac{\partial \xi_n}{\partial x'} = \frac{\partial \xi_n}{\partial x} \cdot \frac{\partial x}{\partial x'}$$
$$\frac{\partial \xi_n}{\partial y'} = \frac{\partial \xi_n}{\partial x} \cdot \frac{\partial x}{\partial y'} + \frac{\partial \xi_n}{\partial y} \cdot \frac{\partial y}{\partial y'}$$

By the (10.3.5) and Lemma 7.4.1, we see that

$$\begin{aligned}
\frac{\partial \xi_n}{\partial x'} &= \frac{1}{\sigma_{n,0}} \cdot \partial_x \xi \left[\alpha_{n,0} \big(1 + \partial_{x'} S_0^n(w')\big) + u_{n,0}\, \sigma_{n,0} \cdot \frac{\partial \xi_n}{\partial x'} \right] \\
\frac{\partial \xi_n}{\partial y'} &= \frac{1}{\sigma_{n,0}} \cdot \partial_x \xi \left[\alpha_{n,0}\, \partial_{y'} S_0^n(w') + t_{n,0}\, \sigma_{n,0} + u_{n,0}\, \sigma_{n,0} \Big(\frac{\partial \xi_n}{\partial y'} + (R_0^n)'(y') \Big) \right] \\
&\quad + \partial_y \xi - d_{n,0} - (R_0^n)' \left(\frac{y}{\sigma_{n,0}} \right)
\end{aligned} \tag{10.3.7}$$

Since $\sigma_{n,0} \asymp (-\sigma)^n$, $\alpha_{n,0} \asymp \sigma^{2n}$ for each $n \in N$ and $\|\partial_{y'} S_0^n(w')\| \leq C_3\, \bar{\varepsilon}$ for some $C_3 > 0$ by Proposition 7.2.3, the estimation of each partial derivatives of ξ_n is the following.

$$\begin{aligned}
\left\| \frac{\partial \xi_n}{\partial x'} \right\| &\leq \|\partial_x \xi\|\, C_0 \sigma^n \leq C \bar{\varepsilon}\, \sigma^n \\
\left\| \frac{\partial \xi_n}{\partial y'} \right\| &\leq \|\partial_x \xi\|\, C_1 \bar{\varepsilon}\, \sigma^n + \|\, \partial_x \xi \cdot t_{n,0} + \partial_y \xi - d_{n,0}\| + C_2 \bar{\varepsilon}\, \sigma^n \leq C \bar{\varepsilon}
\end{aligned}$$

for some $C > 0$. Therefore, $\|D\xi_n\| \leq C\bar{\varepsilon}$. □

Remark 10.3.1. In general, the renormalized map of a small perturbation of the model is not a small perturbation of model map on the deeper level. Moreover, the model map itself does not give any information about $\partial_y \delta_n$ on each level n. Then in order to obtain invariant surfaces on the successive levels, we used the scope map $(\Psi^n_{v^n})^{-1}$ instead of constructing universal expression of small perturbation of model maps on each level.

Chapter 11

Applications of two dimensional theory to the invariant surface

As F is a sufficiently small perturbation of the model maps with $b_2 \ll b_1$, we have obtained a C^r invariant surface Q_n of R^nF for every sufficiently deep level. Moreover, the invariant surface is the graph of a C^r map, ξ_n from $I^x \times I^y$ to I^z. Then using the graph map $(x, y, \xi_n) \mapsto (x, y)$, C^r Hénon-like maps is defined on the $\pi_{xy}(B)$ on each level. Moreover, we can define the coordinate change map in the similar way. The C^r Hénon-like maps on each level is actually the renormalized map defined by the conjugation with the horizontal diffeomorphism by Lemma 11.1.1. The dynamical and geometric properties of C^r Hénon-like maps are valid in the invariant surfaces Q_n, for instance, non existence of the continuous invariant line field, non-rigidity on the Cantor set and unbounded geometry of Cantor set. These negative results on the invariant surface is also valid on the three dimensional analytic Hénon-like map in no time.

11.1 Universality of C^r two dimensional Hénon-like map from invariant surfaces

Let $F \in \mathcal{I}_B(\bar{\varepsilon})$ be a small perturbation of the given model map $F_{\text{mod}} \in \mathcal{I}_B(\bar{\varepsilon})$ with the sufficiently small $\bar{\varepsilon} > 0$. Let Q_n and Q_k be invariant surfaces under R^nF and R^kF respectively and assume that $k < n$. Then by Lemma 10.3.1, we may assume that Ψ^n_k is the coordinate change map from level n to k such that $\Psi^n_k(Q_n) \subset Q_k$. Let us define the C^r two dimensional Hénon-like map ${}_{2d}F_{n,\xi}$ on level n as follows.

$$ {}_{2d}F_{n,\,\xi} \equiv \pi_{xy}^{\xi_n} \circ R^nF|_{Q_n} \circ (\pi_{xy}^{\xi_n})^{-1} \tag{11.1.1} $$

where the map $(\pi_{xy}^{\xi_n})^{-1} : (x,y) \mapsto (x,y,\xi_n(x,y))$ is a C^r diffeomorphism on the domain of two dimensional map, $\pi_{xy}(B)$. In particular, the map $F_{2d,\,\xi}$ is defined as follows

$$ F_{2d,\,\xi}(x,y) = (f(x) - \varepsilon(x,y,\xi),\, x) \tag{11.1.2} $$

where graph(ξ) is a C^r invariant surface of the three dimensional map F: $(x,y,z) \mapsto (f(x) - \varepsilon(x,y,z),\, x,\, \delta(x,y,z))$.

Let us assume that $3 \le r < \infty$. By Lemma 10.3.1, the invariant surfaces, Q_n and Q_k are the graph of C^r maps $\xi_n(x,y)$ and $\xi_k(x,y)$ respectively.

The coordinate change map ${}_{2d}\Psi^n_{k,\,\xi}$ is defined as the map which satisfies the following commutative diagram.

$$\begin{tikzcd}
(Q_n, \tau_n) \arrow[r, "\Psi^n_k"] \arrow[d, "\pi^{\xi_n}_{xy,\,n}"'] & (Q_k, \tau_k) \arrow[d, "\pi^{\xi_k}_{xy,\,k}"] \\
({}_{2d}B_n, \tau_{2d,\,n}) \arrow[r, "{}_{2d}\Psi^n_{k,\,\xi}"] & ({}_{2d}B_k, \tau_{2d,\,k})
\end{tikzcd}$$

where Q_n and Q_k are invariant C^r surfaces with $3 \le r < \infty$ of R^nF and R^kF respectively and $\pi^{\xi_n}_{xy,\,n}$ and $\pi^{\xi_k}_{xy,\,k}$ are the inverse of the graph maps, $(x,y) \mapsto (x,y,\xi_n)$ and $(x,y) \mapsto (x,y,\xi_k)$ respectively.

Using translations $T_k : w \mapsto w - \tau_k$ and $T_n : w \mapsto w - \tau_n$, we can let the tip move to the origin as the fixed point of the new coordinate change map, $\Psi^n_k := T_k \circ \Psi^n_k \circ T_n^{-1}$, which is defined on Section 7.1. Thus due to the above commutative diagram, the corresponding tips in ${}_{2d}B_j$ for $j = k, n$ is changed to the origin. Let $\pi_{xy} \circ T_j$ be $T_{2d,\,j}$ for $j = k, n$. This origin is also the fixed point of the map ${}_{2d}\Psi^n_{k,\,\xi} := T_{2d,\,k} \circ {}_{2d}\Psi^n_{k,\,\xi} \circ T^{-1}_{2d,\,n}$ where $T_{2d,\,j} = \pi_{xy,\,j} \circ T_j$ with $j = k, n$.

By the direct calculation, we obtain the expression of the map ${}_{2d}\Psi^n_{k,\,\xi}$ as follows.

$$
\begin{aligned}
{}_{2d}\Psi^n_{k,\,\xi} &= \pi^{\xi_k}_{xy,\,k} \circ \Psi^n_k(x, y, \xi_n) \\
&= \pi^{\xi_k}_{xy,\,k} \circ \begin{pmatrix} \alpha_{n,k} & t_{n,\,k}\,\sigma_{n,\,k} & u_{n,\,k}\,\sigma_{n,\,k} \\ & \sigma_{n,\,k} & \\ & d_{n,\,k}\,\sigma_{n,\,k} & \sigma_{n,\,k} \end{pmatrix} \begin{pmatrix} x + S^n_{k,\,\xi} \\ y \\ \xi_n + R^n_k(y) \end{pmatrix} \\
&= \big(\alpha_{n,\,k}(x + S^n_{k,\,\xi}) + t_{n,\,k}\,\sigma_{n,\,k}\; y + u_{n,\,k}\,\sigma_{n,\,k}(\xi_n + R^n_k(y)),\; \sigma_{n,\,k}\, y\big)
\end{aligned} \tag{11.1.3}
$$

where $S^n_{k,\,\xi} = S^n_k(x, y, \xi_n(x, y))$. Then

$$
\begin{aligned}
\operatorname{Jac}{}_{2d}\Psi^n_{k,\,\xi} &= \det \begin{pmatrix} \alpha_{n,\,k}(1 + \partial_x S^n_{k,\,\xi} + \partial_z S^n_{k,\,\xi} \cdot \partial_x \xi_n) + u_{n,\,k}\,\sigma_{n,\,k}\,\partial_x \xi_n & \bullet \\ 0 & \sigma_{n,\,k} \end{pmatrix} \\
&= \sigma_{n,\,k}\left(\alpha_{n,\,k}(1 + \partial_x S^n_{k,\,\xi} + \partial_z S^n_{k,\,\xi} \cdot \partial_x \xi_n) + u_{n,\,k}\,\sigma_{n,\,k}\,\partial_x \xi_n\right)
\end{aligned} \tag{11.1.4}
$$

If $F \in \mathcal{I}_B(\bar{\varepsilon})$ has the invariant surfaces as the graph from $I^x \times I^y$ to I^z on every level, then ${}_{2d}\Psi^{k+1}_{k,\xi}$ is the conjugation between $({}_{2d}F_{k,\,\xi})^2$ and ${}_{2d}F_{k+1,\,\xi}$ for each $k \in \mathbb{N}$. Then the two dimensional map $F_{2d,\,\xi}$ is called the *formally* infinitely renormalizable map with C^r conjugation. Moreover, the map defined on the equation (11.1.3) with $n = k + 1$, ${}_{2d}\Psi^{k+1}_{k,\,\xi}$ preserves the horizontal line and is the inverse of the horizontal map

$$
(x, y) \mapsto (f_k(x) - \varepsilon_k(x, y, \xi_k),\, y) \circ (\sigma_k x, \sigma_k y)
$$

by Lemma 11.1.1 as follows.

Lemma 11.1.1. *Let the coordinate change map between $({}_{2d}F_{k,\,\xi})^2$ and ${}_{2d}F_{k+1,\,\xi}$ be ${}_{2d}\Psi^{k+1}_{k,\xi}$ which is defined on (11.1.3) as the conjugation. Then*

$$
{}_{2d}\Psi^{k+1}_{k,\,\xi} = H^{-1}_{k,\,\xi} \circ \Lambda^{-1}_k
$$

for every $k \in \mathbb{N}$ where $H_{k,\,\xi}(x, y) = (f_k(x) - \varepsilon_k(x, y, \xi_k),\, y)$ and $\Lambda^{-1}_k(x, y) = (\sigma_k x, \sigma_k y)$.

Proof. Recall the definitions of the horizontal-like diffeomorphism H_k and it's inverse, H^{-1}_k as follows.

$$
\begin{aligned}
H_k(w) &= \; (f_k(x) - \varepsilon_k(w),\, y,\, z - \delta_k(y, f^{-1}_k(y), 0)) \\
H^{-1}_k(w) &= \; (\phi^{-1}_k(w),\, y,\, z + \delta_k(y, f^{-1}_k(y), 0))
\end{aligned}
$$

Observe that $H_k \circ H_k^{-1} = \mathrm{id}$ and $f_k \circ \phi_k^{-1}(w) - \varepsilon_k \circ H_k^{-1}(w) = x$ for all points $w \in B$. Then if we choose the set $\sigma_k \cdot \mathrm{graph}(\xi_{k+1}) \subset B$, then the similar identical equation holds.

By the definition of the map ${}_{2d}\Psi^n_{k,\,\xi}$, the following equations hold.

$$\begin{aligned}
{}_{2d}\Psi^{k+1}_{k,\,\xi}(x,y) &= \ \pi^{\xi_k}_{xy} \circ \Psi^{k+1}_k \circ (\pi^{\xi_{k+1}}_{xy})^{-1}(x,y) \\
&= \ \pi^{\xi_k}_{xy} \circ \Psi^{k+1}_k(x,y,\xi_{k+1}) \\
&= \ \pi^{\xi_k}_{xy} \circ H_k^{-1} \circ \Lambda_k^{-1}(x,y,\xi_{k+1}) \\
&= \ \pi^{\xi_k}_{xy} \circ H_k^{-1}(\sigma_k x,\, \sigma_k y,\, \sigma_k \xi_{k+1}) \\
(*) \qquad &= \ \pi^{\xi_k}_{xy} \big(\phi_k^{-1}(\sigma_k x,\, \sigma_k y,\, \sigma_k \xi_{k+1}),\, \sigma_k y,\, \xi_k(\phi_k^{-1}, \sigma_k y)\big) \\
&= \ (\, \phi_k^{-1}(\sigma_k x,\, \sigma_k y,\, \sigma_k \xi_{k+1}),\, \sigma_k y\,)
\end{aligned} \tag{11.1.5}$$

In the above equation, $(*)$ comes from the fact that $H_k^{-1} \circ \Lambda_k^{-1}(\,\mathrm{graph}(\xi_{k+1})) \subset \mathrm{graph}(\xi_k)$.

Let us calculate $H_{k,\xi} \circ {}_{2d}\Psi^{k+1}_{k,\,\xi}(x,y)$. The second coordinate function of it is just $\sigma_k y$. The first coordinate function is following.

$$\begin{aligned}
& f_k \circ \phi_k^{-1}(\sigma_k x,\, \sigma_k y,\, \sigma_k \xi_{k+1}) \\
& \qquad - \varepsilon_k\big(\phi_k^{-1}(\sigma_k x,\, \sigma_k y,\, \sigma_k \xi_{k+1}),\, \sigma_k y,\, \xi_k(\phi_k^{-1}, \sigma_k y)\big) \\
(*) \qquad &= \ f_k \circ \phi_k^{-1}(\sigma_k x,\, \sigma_k y,\, \sigma_k \xi_{k+1}) - \varepsilon_k \circ H_k^{-1}(\sigma_k x,\, \sigma_k y,\, \sigma_k \xi_{k+1}) \\
&= \ \sigma_k x
\end{aligned}$$

Hence, $H_{k,\,\xi} \circ {}_{2d}\Psi^{k+1}_{k,\,\xi}(x,y) = (\sigma_k x,\, \sigma_k y)$. However, $H_{k,\,\xi} \circ \big(H^{-1}_{k,\,\xi}(x,y) \circ \Lambda_k^{-1}(x,y)\big) = (\sigma_k x,\, \sigma_k y)$. Therefore, by the uniqueness of the inverse map of $H_{k,\,\xi}(x,y)$,

$${}_{2d}\Psi^{k+1}_{k,\,\xi} = H^{-1}_{k,\,\xi} \circ \Lambda_k^{-1}.$$

□

Recall the topological definition of the renormalizability, that is, $W^u(\beta_0) \cap W^s(\beta_1)$ is the single orbit of intersection point under F. It does not involve the analyticity of Hénon-like maps. Then this definition can be applied to the C^r Hénon-like maps. Moreover, Lemma 11.1.1 enable us to define the renormalization of the two dimensional C^r Hénon-like maps as the extension of renormalization of the analytic Hénon-like maps. Let $f : I \to I$ is the unimodal map and $J \subset \mathrm{int}(I)$ which contains the critical point such that $J \cap f(J) = \varnothing$ and $f^2(J) \subset J$.

Definition 11.1.1. Let $F : (x, y) \mapsto (f(x) - \varepsilon(x, y), x)$ be a C^r Hénon-like map with $r \geq 3$. If F is renormalizable, then RF, the *renormalization* of F is defined as follows.

$$RF = (\Lambda \circ H) \circ F^2 \circ (H^{-1} \circ \Lambda^{-1})$$

where $H(x, y) = (f(x) - \varepsilon(x, y), y)$. Define the linear scaling map $\Lambda(x, y) = (sx, sy)$ if $s : J \to I$ is the orientation reversing affine scaling and J is minimal such that $J \times I$ is invariant under $H \circ F^2 \circ H^{-1}$.

If F is renormalizable n times, then the above definition can be applied to $R^k F$ for $1 \leq k \leq n$ successively. The two dimensional map ${}_{2d}F_{n,\,\xi}$ with the C^r function ξ_n is same as $R^n F_{2d,\,\xi}$ by Lemma 11.1.1 and the above definition. Thus if the maps ${}_{2d}F_{n,\,\xi}$ are defined on every $n \in \mathbb{N}$, then the map ${}_{2d}F_{n,\,\xi}$ is denoted to be $R^n F_{2d,\,\xi}$ and it is called the n^{th} *renormalization* of $F_{2d,\,\xi}$.

Recall that every invariant surfaces as the pseudo unstable manifold in Lemma 10.2.1 contains the global attracting set, in particular, the critical Cantor set $\mathcal{O}_F$. Then the ergodic measure on the critical Cantor set restricted to the surface Q, say $\mu|_Q$, is same as the measure μ without restriction. Moreover, the critical Cantor set, $\mathcal{O}_{F_{2d,\,\xi}}$ is the image of $\mathcal{O}_F$ under π^{ξ}_{xy} and it is independent of invariant surfaces because all invariant surfaces contains the global attracting set. Then we suppress ξ in the notation of the Cantor set, that is, $\mathcal{O}_{F_{2d,\,\xi}} \equiv \mathcal{O}_{F_{2d}}$.
The ergodic measure on $\mathcal{O}_{F_{2d}}$ is defined as the push forward measure μ on $\mathcal{O}_F$ by the map π^{ξ}_{xy}. That is to say, $\mathcal{O}_{F_{2d}} \equiv \pi^{\xi}_{xy}(\mathcal{O}_F)$ and the ergodic measure on $\mathcal{O}_{F_{2d}}$ is defined as $(\pi^{\xi}_{xy})_*(\mu) \equiv \mu_{2d,\,\xi}$ where μ is the ergodic probability measure on $\mathcal{O}_F$. In particular, it is defined as follows.

$$\mu_{2d,\,\xi}\big(\pi^{\xi}_{xy}(\mathcal{O}_F \cap B^n_{\mathbf{w}})\big) = \mu_{2d,\,\xi}\big(\pi^{\xi}_{xy}(\mathcal{O}_F) \cap \pi^{\xi}_{xy}(B^n_{\mathbf{w}})\big) = \frac{1}{2^n}$$

The fact that $\pi^{\xi}_{xy}(\mathcal{O}_F)$ is independent of ξ implies that $\mu_{2d,\,\xi}$ is independent of ξ. Then we denote this measure to be μ_{2d}.
Let us define the *average Jacobian* of $F_{2d,\,\xi}$.

$$b_{1,\,2d} = \exp \int_{\mathcal{O}_{F_{2d}}} \log \operatorname{Jac} F_{2d,\,\xi} \; d\mu_{2d}$$

This average Jacobian is independent of the surface map ξ because every invariant surfaces has the same invariant tangent bundle under DF on the global attracting set $\mathcal{A}_F$ which contains $\mathcal{O}_{F_{2d}}$.

Lemma 11.1.2. *Let $F \in \mathcal{I}_B(\bar{\varepsilon})$ for sufficiently small $\bar{\varepsilon} > 0$ with $b_1 \gg b_2$. Suppose that there exist invariant C^r surface with $3 \leq r < \infty$ under R^nF for every $n \in \mathbb{N}_+$ and each surface contains the global attracting set of each R^nF such that graph(ξ_n) is the invariant surface where ξ_n is C^r map from $I^x \times I^y$ to I^z. Let $R^nF_{2d,\,\xi}$ be $\pi_{xy}^{\xi_n} \circ F_n|_{Q_n} \circ (\pi_{xy}^{\xi_n})^{-1}$ for each $n \geq 1$. Then*

$$\operatorname{Jac} R^nF_{2d,\,\xi} = b_{1,\,2d}^{2^n}\, a(x)(1 + O(\rho^n))$$

where $b_{1,\,2d}$ is the average Jacobian of $F_{2d,\,\xi}$, $\rho \in (0,1)$ and $a(x)$ is the universal function of x.

Proof. By the distortion Lemma 6.0.3 and Corollary 6.0.4, we obtain

$$\operatorname{Jac} F_{2d,\,\xi}^{2^n} = b_{1,\,2d}^{2^n}(1 + O(\rho^n))$$

Moreover, the chain rule implies that

$$\operatorname{Jac} R^nF_{2d,\,\xi} = b_{1,\,2d}^{2^n}\, \frac{\operatorname{Jac}_{\,2d}\Psi_{0,\,\xi}^n(w)}{\operatorname{Jac}_{\,2d}\Psi_{0,\,\xi}^n(R^nF_{2d,\,\xi}(w))}(1 + O(\rho^n))$$

where $w = (x, y)$. After letting the tip on every level move to the origin by the appropriate linear map, the equation (11.1.4) implies the Jacobian of $\Psi_{0,\,\xi}^n(w)$.

$$\operatorname{Jac}_{\,2d}\Psi_{0,\,\xi}^n = \sigma_{n,\,0}\left(\alpha_{n,\,0} \cdot \partial_x\big(x + S_0^n(x, y,\, \xi_n)\big) + u_{n,\,0}\,\sigma_{n,\,0} \cdot \partial_x \xi_n\right) \qquad (11.1.6)$$

Then in order to have the universal limit of the Jacobian, we need the asymptotic expression of the following.

(1) $\partial_x\big(x + S_0^n(x, y,\, \xi_n)\big)$

(2) $\dfrac{\sigma_{n,\,0}}{\alpha_{n,\,0}}\, \partial_x \xi_n$

By Lemma 7.4.2,

$$x + S_0^n(x, y,\, \xi_n) = v_*(x) + a_{F,\,1}\, y^2 + a_{F,\,2}\, y \cdot \xi_n + a_{F,\,3}\,(\xi_n)^2 + O(\rho^n)$$

with C^1 convergence. Then

$$\partial_x\big(x + S_0^n(x, y,\, \xi_n)\big) = v_*'(x) + a_{F,\,2}\, y \cdot \partial_x \xi_n + 2\, a_{F,\,3} \cdot \xi_n \cdot \partial_x \xi_n + O(\rho^n)$$

By the equation (10.3.7) on Lemma 10.3.1, we see $\|\partial_x \xi_n\| \leq C\bar{\varepsilon}\,\sigma^n$. Then

$$\partial_x\big(x + S_0^n(x,y,\,\xi_n)\big) = v_*'(x) + O(\rho^n) \tag{11.1.7}$$

By the equation (10.3.7) on Lemma 10.3.1,

$$\frac{\sigma_{n,\,0}}{\alpha_{n,\,0}}\,\frac{\partial \xi_n}{\partial x} = \partial_x \xi(\bar{x},\bar{y})\left[1 + \partial_x S_0^n(x,y,\,\xi_n) + \frac{\sigma_{n,\,0}}{\alpha_{n,\,0}}\,u_{n,\,0}\frac{\partial \xi_n}{\partial x}\right]$$

$$\frac{\sigma_{n,\,0}}{\alpha_{n,\,0}}\,\frac{\partial \xi_n}{\partial x} = \frac{\partial_x \xi(\bar{x},\bar{y})}{1 - u_{n,\,0}\,\partial_x \xi(\bar{x},\bar{y})}\big[1 + \partial_x S_0^n(x,y,\,\xi_n)\big]$$

where $(\bar{x},\bar{y}) \in B(F_{2d,\,\xi})$. Thus $(\bar{x},\bar{y})$ converges to the origin $(0,0)$ as $n \to \infty$ exponentially fast by Corollary 5.2.2.

$$\operatorname{diam}({}_{2d}\Psi^n_{0,\,\xi}) \le \operatorname{diam}(\Psi^n_0) \le C\sigma^n$$

for some $C > 0$. In addition to the exponential convergence of $\partial_x \xi(\bar{x},\bar{y})$ to $\partial_x \xi(0,0)$, $u_{n,\,0}$ converges to $u_{*,\,0}$ super exponentially fast. Then,

$$\frac{\sigma_{n,\,0}}{\alpha_{n,\,0}}\,\frac{\partial \xi_n}{\partial x} = \frac{\partial_x \xi(0,0)}{1 - u_{*,\,0}\,\partial_x \xi(0,0)}\;v_*'(x) + O(\rho^n) \tag{11.1.8}$$

Let $(x',y') = w' = {}_{2d}F_{n,\,\xi}(w)$. Then

$$\frac{\operatorname{Jac}{}_{2d}\Psi^n_{0,\,\xi}(w)}{\operatorname{Jac}{}_{2d}\Psi^n_{0,\,\xi}(w')} = \frac{1 + \partial_x(S^n_{0,\,\xi}(w)) + \dfrac{\sigma_{n,\,0}}{\alpha_{n,\,0}}\;u_{n,\,0}\,\partial_x \xi_n(x,y)}{1 + \partial_x(S^n_{0,\,\xi}(w')) + \dfrac{\sigma_{n,\,0}}{\alpha_{n,\,0}}\;u_{n,\,0}\,\partial_x \xi_n(x',y')} \tag{11.1.9}$$

where $S_0^n(x,y,\,\xi_n) = S^n_{0,\,\xi}(x,y)$. The translation does not affect Jacobian determinant and each translation from tip to the origin converges to the map $w \mapsto \tau_\infty$ exponentially fast. Then by the similar calculation in Theorem 7.5.1, the equation (11.1.9) converges to the following universal function exponentially fast.

$$
\begin{aligned}
\lim_{n\to\infty} \frac{\mathrm{Jac}_{\,2d}\Psi^n_{0,\,\xi}(w)}{\mathrm{Jac}_{\,2d}\Psi^n_{0,\,\xi}(w')} &= \frac{v'_*(x-1) + \dfrac{u_{*,\,0}\,\partial_x\xi(\pi_{xy}(\tau_F))}{1-u_{*,\,0}\,\partial_x\xi(\pi_{xy}(\tau_F))}\;v'_*(x-1)}{v'_*(f_*(x-1)) + \dfrac{u_{*,\,0}\,\partial_x\xi(\pi_{xy}(\tau_F))}{1-u_{*,\,0}\,\partial_x\xi(\pi_{xy}(\tau_F))}\;v'_*(f_*(x-1))} \\
&= \frac{v'_*(x-1)\Big(1+\dfrac{u_{*,\,0}\,\partial_x\xi(\pi_{xy}(\tau_F))}{1-u_{*,\,0}\,\partial_x\xi(\pi_{xy}(\tau_F))}\Big)}{v'_*(f_*(x-1))\Big(1+\dfrac{u_{*,\,0}\,\partial_x\xi(\pi_{xy}(\tau_F))}{1-u_{*,\,0}\,\partial_x\xi(\pi_{xy}(\tau_F))}\Big)} \\
&= \frac{v'_*(x-1)}{v'_*(f_*(x-1))} \equiv a(x)
\end{aligned}
\tag{11.1.10}
$$

□

Theorem 11.1.3 (Universality of C^r Hénon-like maps with C^r conjugation for $3 \leq r < \infty$)**.** *Let Hénon-like map $F_{2d,\,\xi}$ be the C^r map with $3 \leq r < \infty$ which is defined on (11.1.2). Suppose that $F_{2d,\,\xi}$ is infinitely renormalizable. Then*

$$
R^n F_{2d,\,\xi} = (f_n(x) - b_{1,\,2d}^{2^n}\,a(x)\,y\,(1+O(\rho^n)),\,x) \tag{11.1.11}
$$

where $f_n(x)$ is the unimodal map which converges to $f_*(x)$ exponentially fast as $n \to \infty$ for some $0 < \rho < 1$.

Proof. By the smooth conjugation of two dimensional map and $F_n|_{Q_n}$, we see that

$$
R^n F_{2d,\,\xi} = (f_n(x) - \varepsilon_n(x, y, \xi_n),\,x)
$$

Denote $\varepsilon_n(x, y, \xi_n)$ to be $\varepsilon_{n,\,\xi_n}(x, y)$. By Lemma 11.1.2, we have the universal expression of Jacobian determinant of two dimensional map, $\partial_y \varepsilon_{n,\,\xi_n}(x, y) = b_{1,\,2d}^{2^n}\,a(x)(1+O(\rho^n))$. Then

$$
\varepsilon_{n,\,\xi_n}(x, y) = b_{1,\,2d}^{2^n}\,a(x)\,y\,(1+O(\rho^n)) + U_n(x).
$$

The map $U_n(x)$ which depends only on the x variable can be incorporated to $f_n(x)$. □

Theorem 11.1.4. *Let $R^kF_{2d,\,\xi}$ be the C^r Hénon-like map defined as $\pi_{xy}^{\xi_k} \circ R^kF|_{Q_k} \circ (\pi_{xy}^{\xi_k})^{-1}$ for all sufficiently big $k \in \mathbb{N}$ where $R^kF \in \mathcal{I}_B(\bar{\varepsilon}^{2^k})$ with invariant surfaces $Q_k \equiv graph(\xi_k)$ under R^kF. Let the conjugation between $R^nF_{2d,\,\xi}$ and $\big(R^kF_{2d,\,\xi}\big)^{2^{n-k}}$ be ${}_{2d}\Psi^n_{k,\,\xi}$. Then the map ${}_{2d}\Psi^n_{k,\,\xi}$ is expressed as follows.*

$$\begin{aligned}
&\begin{pmatrix} 1 & {}_{2d}t_{n,\,k} \\ & 1 \end{pmatrix} \begin{pmatrix} \alpha_{n,\,k} & \\ & \sigma_{n,\,k} \end{pmatrix} \begin{pmatrix} x + {}_{2d}S^n_k(w) \\ y \end{pmatrix} \\
&= \big(\, \alpha_{n,\,k}\, (\, x + {}_{2d}S^n_k(w)) + \sigma_{n,\,k} \cdot {}_{2d}t_{n,\,k} \cdot y,\ \sigma_{n,\,k}\, y\, \big)
\end{aligned} \tag{11.1.12}$$

where

$$D_{2d}\Psi^n_{k,\,\xi}(0) = \begin{pmatrix} 1 & {}_{2d}t_{n,\,k} \\ & 1 \end{pmatrix} \begin{pmatrix} \alpha_{n,\,k} & \\ & \sigma_{n,\,k} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \tag{11.1.13}$$

$\sigma_{n,\,k} = (-\sigma)^{n-k}(1 + O(\rho^k))$ and $\alpha_{n,\,k} = \sigma^{2(n-k)}(1 + O(\rho^k))$. Moreover, $x + {}_{2d}S^n_k(w)$ has the asymptotic

$$x + {}_{2d}S^n_k(w) = v_*(x) + a_{F,\,k}\, y^2 + O(\rho^{n-k})$$

where $|\, a_{F,\,k}| = O(\varepsilon^{2^k})$.

Proof. By Lemma 11.1.1, the coordinate change map, ${}_{2d}\Psi^n_{k,\,\xi}$ is the composition of the inverse of horizontal diffeomorphisms with linear scaling as follows.

$$H^{-1}_{k,\,\xi} \circ \Lambda^{-1}_k \circ H^{-1}_{k+1,\,\xi} \circ \Lambda^{-1}_{k+1} \circ \cdots \circ H^{-1}_{n,\,\xi} \circ \Lambda^{-1}_n$$

Then after reshuffling non-linear and linear parts separately by the direct calculations and letting the tip move to the origin by the appropriate translations on each levels, the coordinate change map is of the form (11.1.12). In order to estimate ${}_{2d}S^n_k(w)$, the recursive formulas of the first and the second partial derivatives of ${}_{2d}S^n_k(w)$ are required. However, analyticity does not affect the calculation of any recursive formulas of derivatives. C^r map with $r \geq 3$ is sufficient. Then the exactly same calculation in Section 7.2 in [CLM] can be used. Since the recursive formulas with same estimations are applied to ${}_{2d}S^n_k(w)$, we just observe the following estimation

$$x + {}_{2d}S^n_k(w) = v_*(x) + a_{F,\,k}\, y^2 + O(\rho^{n-k})$$

where $|\, a_{F,\,k}| = O(\varepsilon^{2^k})$. □

Let us denote ${}_{2d}\, t_{k+1,\,k}$ to be ${}_{2d}\, t_k$ for simplicity. Compare the derivative of $H^{-1}_{k,\,\xi} \circ \Lambda^{-1}_k$ and the form (11.1.13) with $n = k+1$. Then $b^{2^k}_{1,\,2d} \asymp {}_{2d}\, t_k$ for each $k \in \mathbb{N}$. Hence, the C^r infinitely renormalizable Hénon-like maps (which are defined by the C^r conjugation from an invariant surface of the three dimensional map R^nF) has the Universality theorem and the asymptotic of the coordinate change maps which is similar to the analytic maps.

11.2 Non existence of the continuous invariant line field on Q_n

The C^r conjugation $(x, y) \mapsto (x, y, \xi_n(x, y))$ between $R^nF_{2d,\,\xi}$ and $F|_{Q_n}$ is as smooth as the invariant surface Q_n on each level n. Since every invariant surfaces contain the global attracting set which has periodic points and the critical Cantor set, any differentiable invariant properties on $\mathcal{O}_F$ is same as the properties on $\mathcal{O}_{F_{2d}}$ by the C^r conjugation.

Lemma 11.2.1. *Let $F_{2d,\,\xi}$ be a C^r Hénon-like map for $3 \leq r < \infty$. Suppose that there exists the C^r conjugation between three dimensional map $F \in \mathcal{I}_B(\bar{\varepsilon})$ restricted to its invariant surface, $F|_Q$ and $F_{2d,\,\xi}$. If $F_{2d,\,\xi}$ is the infinitely renormalizable map defined on Definition 11.1.1, then $F_{2d,\,\xi}$ has no continuous invariant line field on the critical Cantor set. Especially, every invariant line fields is discontinuous at the tip.*

Proof. C^r infinitely renormalizable Hénon-like map for $3 \leq r < \infty$ has the Universality theorem (Theorem 11.1.3) and estimation of scaling map Ψ^n_k by Lemma 11.1.4 similar to the analytic maps. Then actual proof of this theorem is essentially same as the proof of the analytic case. See Theorem 9.7 in [CLM] or Theorem 4.2.2 on [Haz]. □

Theorem 11.2.2. *Let $F \in \mathcal{I}_B(\bar{\varepsilon})$ be a small perturbation of the model maps with $b_2 \ll b_1$. Let Q be an invariant surface under F which contains the global attracting set. Then $F|_Q$ has no continuous invariant line fields on the critical Cantor set, $\mathcal{O}_F$. Especially if there exists invariant line field on $\mathcal{O}_F$, then it is discontinuous at the tip.*

Proof. We may assume that the map $F \in \mathcal{I}_B(\bar{\varepsilon})$ which is a small perturbation of model map with $b_2 \ll b_1$ has an invariant surface Q which is the graph of C^r map ξ, namely, $Q = \mathrm{graph}(\xi)$ from P to I^z. Let P the domain of the two dimensional Hénon-like map, $F_{2d,\,\xi}$ in particular, the square domain with the center origin on the $xy-$plane. For the notational simplicity, we suppress ξ in the notation of two dimensional map in this proof. For example, $F_{2d,\,\xi} = F_{2d}$. Let $T_{\mathcal{O}_{F_{2d}}}P$ be the tangent bundle on the critical Cantor set of DF_{2d}. For each point $w \in \mathcal{O}_{F_{2d}}$, let us assume that $T_{\mathcal{O}_{F_{2d}}}P$ is decomposed to the invariant subspaces $E^1_{2d} \oplus E^2_{2d}$ under DF_{2d}. In order to simplify the notation let the graph map $(x, y) \mapsto (x, y, \xi)$ be just ξ.

Then $T_{\mathcal{O}_F}Q$ has the splitting with invariant subspaces, $E^1 \oplus E^2$ under $DF|_Q$. Since Q is a C^r invariant surface under F and it contains the critical Cantor set $\mathcal{O}_F$, the following diagram is commutative.

$$\begin{tikzcd} T_{\mathcal{O}_{F_{2d}}}P \arrow[r, "(D\xi, \xi)"] \arrow[d, "\pi"'] & T_{\mathcal{O}_F}Q \arrow[d, "\pi'"] \\ \mathcal{O}_{F_{2d}} \arrow[r, "\xi"] & \mathcal{O}_F \end{tikzcd}$$

where the tangent map is defined as $(D\xi, \xi)(v, w) = (D\xi(w) \cdot v,\ \xi(w))$ for each $(v, w) \in T_{\mathcal{O}_{F_\xi}}P$ and both π and π' are the projections from the bundle to the base space, that is, for each $(v, w) \in$ bundle, $\pi(v, w) = w$ and $\pi'(v, w) = w$ respectively.

Furthermore, the image of any invariant tangent subbundle of $T_{\mathcal{O}_{F_{2d}}}P$ is an invariant subbundle of $T_{\mathcal{O}_F}Q$. Then without loss of generality, we may assume that $(D\xi, \xi)(E^1_{2d}) = E^1$. Let γ and γ' be the invariant sections under F_{2d} and $F|_Q$ respectively.

$$\begin{tikzcd} E^1_{2d} \arrow[r, "(D\xi, \xi)"] & E^1 \\ \mathcal{O}_{F_{2d}} \arrow[r, "\xi"] \arrow[u, "\gamma"] & \mathcal{O}_F \arrow[u, "\gamma'"'] \end{tikzcd}$$

Since ξ is C^r function, the tangent map $(D\xi, \xi)$ is continuous at $(v, w) \in E^1_{2d}$. Hence, the section γ is continuous if and only if γ' is continuous because ξ is a diffeomorphism. However, any invariant line field under DF_{2d} on the Cantor set $\mathcal{O}_{F_{2d}}$ is not continuous at the tip, $\tau_{F_{2d}}$ by Lemma 11.2.1. Hence, there is no continuous invariant line field under $DF|_Q$ on any C^r invariant surface Q under F. $\square$

11.3 Non rigidity of Hénon-like maps on the invariant surfaces

If two dimensional analytic Hénon-like maps F_{2d} and $\widetilde{F_{2d}}$ in $\mathcal{I}_B(\bar{\varepsilon})$ have different average Jacobians, b_1 and $\widetilde{b}_1$, then any conjugation ϕ_{2d} between $\mathcal{O}_{F_{2d}}$ and $\widetilde{\mathcal{O}_{F_{2d}}}$ is at most Hölder continuous at corresponding tips by Theorem 10.1 in [CLM]. This theorem relies on Universality theorem and the estimation of the tilt, t_k depending only on the average Jacobian b rather than analyticity of the Hénon-like map. However, C^r infinitely renormalizable Hénon-like maps defined by the invariant surfaces has Universality theorem, Theorem 11.1.3 and the universal estimation of the scaling maps, Theorem 11.1.4. These theorems are similar to the corresponding theorems of the analytic maps. Then if we follow the proof of the non rigidity theorem in [CLM] with C^r setting, then the same conclusion would appear.

Theorem 11.3.1. *Let $F_{2d}, \widetilde{F_{2d}} \in \mathcal{I}_B(\bar{\varepsilon})$ be the two dimensional C^r Hénon-like maps. Let $\mathcal{O}_{F_{2d}}$ and $\mathcal{O}_{\widetilde{F_{2d}}}$ be the critical Cantor set of F_{2d} and $\widetilde{F_{2d}}$ respectively. Let b_1 and $\widetilde{b}_1$ are average Jacobians of F_{2d} and $\widetilde{F_{2d}}$ respectively. Let ϕ_{2d} be a homeomorphism between $\mathcal{O}_{F_{2d}}$ and $\mathcal{O}_{\widetilde{F_{2d}}}$ with $\phi_{2d}(\tau_{\widetilde{F_{2d}}}) = \tau_{F_{2d}}$. Assume that $b_1 > \widetilde{b}_1$. Then the Hölder exponent α of ϕ_{2d} satisfies the following.*

$$\alpha \leq \frac{1}{2}\left(1 + \frac{\log b_1}{\log \widetilde{b}_1}\right)$$

Theorem 11.3.2. *Let $F, \widetilde{F} \in \mathcal{I}_B(\bar{\varepsilon})$ be the small perturbation of model maps with C^r invariant surfaces Q and $\widetilde{Q}$ respectively. Suppose that the average Jacobian of $F|_Q$ and $\widetilde{F|_Q}$ is b_1 and $\widetilde{b}_1$ respectively and also suppose that $b_1 > \widetilde{b}_1$. Let the homeomorphism ϕ between two critical Cantor sets, $\mathcal{O}_F$ and $\mathcal{O}_{\widetilde{F}}$ with $\phi(\tau_{\widetilde{F}}) = \tau_F$ be the map defined on Theorem 11.3.1. Then the Hölder exponent of ϕ is same as ϕ_{2d} if the distance on the critical Cantor sets is induced by the Riemannian metric on each invariant surfaces Q and $\widetilde{Q}$.*

Proof. Let F and $\widetilde{F} \in \mathcal{I}_B(\bar{\varepsilon})$ be small perturbations of model maps. We may assume that there exist invariant surfaces under R^nF and $R^n\widetilde{F}$ respectively for every $n \in \mathbb{N}$. Then the map ϕ between two critical Cantor sets of the three dimensional maps F and $\widetilde{F}$ is defined as follows.

$$\phi := \pi_{xy}^{-1} \circ \phi_{2d} \circ \widetilde{\pi_{xy}} \tag{11.3.1}$$

Every C^r map $F_{2d,\,\xi}$ on the critical Cantor set is independent of the invariant surface graph(ξ) because all invariant surfaces contains $\mathcal{O}_F$. Then we suppress the notation ξ in the two dimensional map F_{2d}. The following diagram is commutative. Then ϕ is the conjugation between F and $\widetilde{F}$ on each critical Cantor set. Moreover, π_{xy}, $\widetilde{\pi_{xy}}$ and ϕ_{2d} maps the tip of each domain to the tip of its image. Then we may assume that $\phi(\tau_{\widetilde{F}}) = \tau_F$.

$$
\begin{tikzcd}
\mathcal{O}_F \arrow[r, "\pi_{xy}"] \arrow[d, "F"'] & \mathcal{O}_{F_{2d}} \arrow[d, "F_{2d}"'] & \mathcal{O}_{\widetilde{F_{2d}}} \arrow[l, "\phi_{2d}"'] \arrow[d, "\widetilde{F_{2d}}"] & \mathcal{O}_{\widetilde{F}} \arrow[l, "\widetilde{\pi_{xy}}"'] \arrow[d, "\widetilde{F}"] \arrow[lll, bend right, "\phi"'] \\
\mathcal{O}_F \arrow[r, "\pi_{xy}"] & \mathcal{O}_{F_{2d}} & \mathcal{O}_{\widetilde{F_{2d}}} \arrow[l, "\phi_{2d}"'] & \mathcal{O}_{\widetilde{F}} \arrow[l, "\widetilde{\pi_{xy}}"'] \arrow[lll, bend left, "\phi"]
\end{tikzcd}
$$

The above commutative diagram implies the following equation.

$$
\begin{aligned}
\phi \circ \widetilde{F} &= \pi_{xy}^{-1} \circ \phi_{2d} \circ (\widetilde{\pi_{xy}} \circ \widetilde{F}) = \pi_{xy}^{-1} \circ (\phi_{2d} \circ \widetilde{F_{2d}}) \circ \widetilde{\pi_{xy}} \\
&= (\pi_{xy}^{-1} \circ F_{2d}) \circ \phi_{2d} \circ \widetilde{\pi_{xy}} = F \circ (\pi_{xy}^{-1} \circ \phi_{2d} \circ \widetilde{\pi_{xy}}) \\
&= F \circ \phi
\end{aligned}
$$

Since both π_{xy} and $\widetilde{\pi_{xy}}$ is differentiable, it is locally Lipschitz map near the tips of each Cantor set. If ϕ_{2d} is Hölder continuous with the Hölder exponent α, then $\phi := \pi_{xy}^{-1} \circ \phi_{2d} \circ \widetilde{\pi_{xy}}$ is also Hölder continuous map with the same exponent of ϕ_{2d}. Then the non rigidity with Hölder conjugation between the two critical Cantor sets with different Lyapunov exponent is same as that of two dimensional Hénon-like maps.

The Riemannian distance d_R between two points is the minimal distance along the path which connects two points on the surface. Since the invariant surfaces is the graph of C^r function ξ or $\widetilde{\xi}$ and we may assume that C^2 norm of each surfaces is uniformly bounded, $d_R(\xi(w_1), \xi(w_2)) \leq C\,\mathrm{dist}(w_1, w_2)$ for every w_1, w_2 on the small neighborhood of the tip where C depends only on $\|\xi\|_{C^2}$. Then π_{xy}, $\widetilde{\pi_{xy}}$ and inverses of these maps are locally Lipschitz function between the invariance surface and $xy-$plane. The composition of Hölder map and Lipschitz maps does not change the exponent of Hölder map. Then ϕ and ϕ_{2d} has the same Hölder exponent. $\square$

11.4 Unbounded geometry on the Cantor set

Recall the *pieces* $B^n_{\mathbf{w}} \equiv B^n_{\mathbf{w}}(F) = \Psi^n_{\mathbf{w}_n}(B)$ on the n^{th} level or n^{th} generation which is defined on the Chapter **??** where the word, $\mathbf{w}_n = (w_1 \dots w_n) \in W^n := \{v, c\}^n$ has length n. Recall that W^n is the additive group of numbers with base 2 (mod 2^n) and

$$\mathbf{w}_n = (w_1 \dots w_n) \mapsto \sum_{k=0}^{n-1} w_{k+1} 2^k$$

is the one to one correspondence between words of length n and the additive group. Denote the subset of the critical Cantor set on each pieces to $\mathcal{O}_{\mathbf{w}} \equiv B^n_{\mathbf{w}} \cap \mathcal{O}$. Then by the definition of $\mathcal{O}_{\mathbf{w}}$, Lemma 5.2.1 and Corollary 5.1.1, we have the following facts.

(1)
$$\mathcal{O}_F = \bigcup_{\mathbf{w} \in W^n} \mathcal{O}_{\mathbf{w}}$$

(2) $F(B^n_{\mathbf{w}}) \subset B^n_{\mathbf{w}+1}$ for every $\mathbf{w} = (w_1 \dots w_n) \in W^n$.

(3) $\operatorname{diam}(B^n_{\mathbf{w}}) \leq C\sigma^n$ for some $C > 0$ depending only on B and $\bar{\varepsilon}$.

Then we can define boxing of the Cantor set of n^{th} generation.

Definition 11.4.1. Let $F \in \mathcal{I}(\bar{\varepsilon})$. A collection of the simply connected sets with non-empty interior $\mathbf{B}^n = \{B^n_{\mathbf{w}} \Subset \operatorname{Dom}(F) \,|\, \mathbf{w} \in W^n\}$ is called *boxing* of $\mathcal{O}_F$ if

(1) $\mathcal{O}_{\mathbf{w}} \Subset B^n_{\mathbf{w}}$ for each $\mathbf{w} \in W^n$.

(2) $B^n_{\mathbf{w}}$ and $B^n_{\mathbf{w}'}$ has disjoint closure if $\mathbf{w} \neq \mathbf{w}'$.

(3) $F(B^n_{\mathbf{w}}) \subset B^n_{\mathbf{w}+1}$ for every $\mathbf{w} \in W^n$.

(4) Each element of $\mathbf{B}^n$ is nested for each n, that is,

$$B^{n+1}_{\mathbf{w}\nu} \subset B^n_{\mathbf{w}}, \;\; \mathbf{w} \in W^n, \quad \nu \in \{v, c\}$$

On the above definition, the elements of boxing are just topological boxes. However, the geometry of the boxing can depend on not only F but also the boxing itself. Then we define *canonical boxing*, $\mathbf{B}^n_{can}$ which is the set of pieces $B^n_{\mathbf{w}} \equiv \Psi^n_{\mathbf{w}}(B)$. Let us say that $\mathrm{Dom}(F_{2d}) := B_{2d}$ in order to distinguish the domain of three dimensional Hénon-like map from that of two dimensional map.

Let the minimal distance between two boxes B_1, B_2 be the infimum of the distance between all points of each boxes and call this distance $\mathrm{dist}_{\min}(B_1, B_2)$.

Definition 11.4.2. The boxing $\mathbf{B}^n$ defined on the Definition 11.4.1 has *bounded geometry* if

$$\begin{aligned} \mathrm{dist}_{\min}(B^{n+1}_{\mathbf{w}v}, B^{n+1}_{\mathbf{w}c}) &\asymp \mathrm{diam}(B^{n+1}_{\mathbf{w}\nu}) \quad \text{for } \nu \in \{v, c\} \\ \mathrm{diam}(B^n_{\mathbf{w}}) &\asymp \mathrm{diam}(B^{n+1}_{\mathbf{w}\nu}) \quad \text{for } \nu \in \{v, c\} \end{aligned}$$

for all $\mathbf{w} \in W^n$ and for all $n \geq 0$.

Moreover, if the boxing has bounded geometry, then we just call $\mathcal{O}_F$ has bounded geometry. If the given boxing does not have bounded geometry, then we call $\mathcal{O}_F$ has *unbounded geometry*.

The proof of the (un)bounded geometry of the Cantor set requires to compare the diameter of boxes and the minimal distance of two adjacent boxes in the boxing. In order to compare these quantities, we would use the maps, $\Psi^n_k(w)$, $F_k(w)$ and $\Psi^k_0(w)$ with the two points $w_1 = (x_1, y_1, z_1)$ and $w_2 = (x_2, y_2, z_2)$ in the domain of $F_n(w)$, namely, $\mathrm{Dom}(R^nF)$. Let us each successive image of w_j under $\Psi^n_k(w)$, $F_k(w)$ and $\Psi^k_0(w)$ be $\dot{w}_j$, $\ddot{w}_j$ and $\dddot{w}_j$ for $j = 1, 2$.

$$w_j \xmapsto{\Psi^n_k} \dot{w}_j \xmapsto{F_k} \ddot{w}_j \xmapsto{\Psi^k_0} \dddot{w}_j$$

Denote $\dot{w}_j = (\dot{x}_j, \dot{y}_j, \dot{z}_j)$ and the points $\ddot{w}_j$ and $\dddot{w}_j$ have the similar coordinate expressions. Let S_1 and S_2 be the (path) connected set on $\mathbb{R}^3$. If $\pi_x(\overline{S}_1) \cap \pi_x(\overline{S}_2)$ contains at least two points, then this intersection is called the $x-$*axis overlap* or *horizontal overlap* of S_1 and S_2. Moreover, we say S_1 *overlaps* S_2 on the $x-$axis or horizontally.

Let F_{2d} be an infinitely renormalizable two dimensional Hénon-like map and b_1 be the average Jacobian of F_{2d}. Then the unbounded geometry of the Cantor set depends on Universality theorem and the asymptotic of the tilt, $-t_k \asymp b_1^{2^k}$ but it does not depend on the analyticity of the map. The section 5.3 in [Haz] contains the proof of unbounded geometry Cantor set under the assumption $\dot{x}_1 - \dot{x}_2 = 0$. The infinitely renormalizable C^r Hénon-like maps defined by the invariant surfaces has Universality theorem and the asymptotic of the tilt $-{}_{2d}t_k \asymp b_1^{2^k}$.

Lemma 11.4.1. *Let F be an infinitely renormalizable C^r Hénon-like maps defined by the invariant surfaces. Let us choose two points $w_1 = (x_1, y_1)$ and $w_2 = (x_2, y_2)$ in ${}_{2d}B^1_v(R^nF) \cap \mathcal{O}_{R^nF}$ and ${}_{2d}B^1_c(R^nF) \cap \mathcal{O}_{R^nF}$ respectively. Suppose that ${}_{2d}B^{n-k}_{\mathbf{w}v}(R^kF)$ overlaps ${}_{2d}B^{n-k}_{\mathbf{w}c}(R^kF)$ on the $x-$axis for some $\mathbf{w} \in W^{n-k}$. Then for all sufficiently large k and n with $k < n$, we have the following estimate.*

$$\operatorname{dist}_{\min}({}_{2d}B^n_{\mathbf{w}v},\ {}_{2d}B^n_{\mathbf{w}c}) \leq C_0\, b_1^{2^k} \sigma^{2k} \sigma^{n-k}$$

for some $C_0 > 0$. Moreover,

$$\operatorname{diam}({}_{2d}B^n_{\mathbf{w}v}) \geq C_1 \sigma^{2(n-k)} \sigma^k$$

for some $C_1 > 0$.

Proof. The proof is the analytic case because it depends on the universality theorem and asymptotic of the tilt $-{}_{2d}t_k \asymp b_1^{2^k}$. Every C^r infinitely renormalizable maps defined by the invariant surfaces, $R^nF_{2d,\,\xi}$ has the universal limit $F_*(w) = (f_*(x), x, 0)$ as $n \to \infty$. This limit is same as the limit of analytic two dimensional Hénon-like maps, R^nF_{2d} in $\mathcal{I}_B(\bar{\varepsilon})$. Then we can adapt the proof of the analytic case with the analytic fixed point of renormalization, F_* and universal convergence of the renormalized maps. See Proposition 5.3.4 and Proposition 5.3.6 in [Haz] with the periodic doubling combinatorics of renormalization operator. $\square$

The unbounded geometry on the critical Cantor set holds if we choose $n > k$ such that $b_1^{2^k} \asymp \sigma^{n-k}$ for every sufficiently large $k \in \mathbb{N}$. The fact that $b_1^{2^k} \asymp \sigma^{n-k}$ for $n > k$ is the necessary and sufficient condition two adjacent boxes ${}_{2d}B^{n-k}_{v^{n-k}v}(R^kF)$ and ${}_{2d}B^{n-k}_{v^{n-k}c}(R^kF)$ has the $x-$axis (or horizontal) overlap.

Remark 11.4.1. The $x-$axis overlapping with the parameter $b_1 \in [0, 1]$ is the G_δ dense subset with full Lebesgue measure in $[0, 1]$ by Theorem 5.5.1 in [Haz] also.

Proposition 11.4.2. *Let $F \in \mathcal{I}(\bar{\varepsilon})$ be three dimensional Hénon-like map with $b_2 \ll b_1$. Suppose that there exists an invariant surface under R^kF, $Q_k :=$ graph(ξ_k) from $I^x \times I^y$ to I^z. Then Euclidean distance of any two points q_1, q_2 in $Q_k \subset \operatorname{Dom}(R^kF)$ is comparable with the two dimensional distance, $\operatorname{dist}(\pi_{xy}(q_1), \pi_{xy}(q_2))$.*

Proof. The invariance of the surface under means that $R^kF(Q_k) \subset Q_k$, that is,

$$R^kF(x,y,\xi) = (f_k(x) - \varepsilon_k(x,y,\xi), x, \delta_k(x,y,\xi)) \in \operatorname{graph}(\xi_k)$$

Thus

$$\xi_k\,(f_k(x) - \varepsilon_k(x,y,\xi_k),\, x) = \delta_k(x,y,\xi_k)$$

Then $\|D\xi_k\| \leq \|\,\xi_k\|_{C^1} \leq \|\delta_k\|_{C^1} \leq C\bar{\varepsilon}^{2^k}$ on $\pi_{xy}(Q_k \cap R^kF(B))$.
The mean value theorem implies that

$$\operatorname{dist}(\pi_z(q_1), \pi_z(q_2)) \leq \|D\xi_k\| \operatorname{dist}(\pi_{xy}(q_1), \pi_{xy}(q_2))$$

for any points q_1 and q_2 on S_k. Then

$$\begin{aligned}
\operatorname{dist}(\pi_{xy}(q_1), \pi_{xy}(q_2)) &\leq \operatorname{dist}(q_1, q_2) \\
&\leq \operatorname{dist}(\pi_{xy}(q_1), \pi_{xy}(q_2)) + \operatorname{dist}(\pi_z(q_1), \pi_z(q_2)) \\
&\leq \operatorname{dist}(\pi_{xy}(q_1), \pi_{xy}(q_2)) + C\bar{\varepsilon}^{2^k} \operatorname{dist}(\pi_{xy}(q_1), \pi_{xy}(q_2)) \\
&= (1 + C\bar{\varepsilon}^{2^k}) \operatorname{dist}(\pi_{xy}(q_1), \pi_{xy}(q_2))
\end{aligned}$$

□

Theorem 11.4.3. *Let the three dimensional Hénon-like map, $F \in \mathcal{I}(\bar{\varepsilon})$ be a small perturbation of the model map with $b_2 \ll b_1$. Suppose that F has an invariant surface Q as the graph of C^r map ξ from $I^x \times I^y$ to I^z for some $3 \leq r < \infty$. Suppose also that $B^{n-k}_{\mathbf{v}v}(R^kF)$ overlaps $B^{n-k}_{\mathbf{v}c}(R^kF)$ on the $x-$axis for $\mathbf{v} = v^{n-k} \in W^{n-k}$. Then the critical Cantor set $\mathcal{O}_F$ has unbounded geometry.*

Proof. The box on the invariant surface, Q is defined as the image of the box, ${}_{2d}B^n_{\mathbf{w}}$ of two dimensional Hénon-like map under the graph map $(x,y) \mapsto (x,y,\xi)$ for every $n \in \mathbb{N}$. For the minimal distance between two boxes, it is sufficient to know that each box on the invariant surface Q is contained in the three dimensional box with the same word. By Proposition 11.4.2, the minimal distance between two boxes on the surface and $xy-$plane with same words is comparable. Then the upper bound of the minimal distance of two dimensional box is also a upper bound of the three dimensional box up to the uniform constant independent of n. By Lemma 11.4.1, we have

$$\operatorname{dist}_{\min}(B^n_{\mathbf{w}v},\ B^n_{\mathbf{w}c}) \leq C_0\, b_1^{2^k} \sigma^{2k} \sigma^{n-k}$$

for the word $\mathbf{w} = v^{n-k-1}cv^k \in W^n$.

By Proposition 11.4.2, diameters of the two dimensional box, ${}_{2d}B^n_{\mathbf{w}}$ on the $xy-$plane and box on the surface Q, namely, $Q \cap B^n_{\mathbf{w}}$ is comparable for all sufficiently large $n \in \mathbb{N}$. The box of the three dimensional map F, $B^n_{\mathbf{w}}$ contains the box, $Q \cap B^n_{\mathbf{w}}$ with same word $\mathbf{w}$. Then the diameter of $B^n_{\mathbf{w}}$ is greater than that of $Q \cap B^n_{\mathbf{w}}$. However, if the word $\mathbf{w}$ is fixed the lower bound of $\operatorname{diam}_{2d} B^n_{\mathbf{w}}$ is also a lower bound of $\operatorname{diam} B^n_{\mathbf{w}}$ up to the uniform constant independent of n. By Lemma 11.4.1, we have

$$\operatorname{diam}(B^n_{\mathbf{w}v}) \geq C_1 \sigma^{2(n-k)} \sigma^k$$

for the word $\mathbf{w} = v^{n-k-1} c v^k$ on the above inequality for the minimal distance. The condition of $x-$axis overlapping of the adjacent two boxes in three dimension is same as the condition of the two dimension because of the existence of the invariant surface as the graph from the plane to $z-$axis. Then we may assume that

$$b_1^{2^k} \asymp \sigma^{n-k}$$

for all sufficiently large k. Hence, $\operatorname{dist}_{\min}(B^n_{\mathbf{w}v},\ B^n_{\mathbf{w}c}) \leq C\sigma^k \operatorname{diam}(B^n_{\mathbf{w}v})$ for some $C > 0$. Therefore, the critical Cantor set has unbounded geometry. $\square$

Chapter 12

Another invariant space under renormalization

12.1 Definition of the invariant subspace from recursive formulas about δ

Let F be a renormalizable three dimensional Hénon-like map. Recall *pre-renormalization* of F, PRF is defined as follows.

$$PRF = H \circ F^2 \circ H^{-1}$$

where $H(w) = (f(x) - \varepsilon(w),\, y,\, z - \delta(y, f^{-1}(y), 0))$. Recall the renormalized map RF is defined as $\Lambda \circ PRF \circ \Lambda^{-1}$ where $\Lambda(w) = (sx, sy, sz)$ for the appropriate number $s < -1$ from the renormalized one dimensional map, $f(x)$. Denote $\sigma_0 = 1/s$

Let the first coordinate map of $H^{-1}(w)$ be $\phi^{-1}(w)$. Then

$$H^{-1}(w) = (\phi^{-1}(w),\, y,\, z + \delta(y, f^{-1}(y), 0))$$

By the direct calculation PRF is as follows.

$$\begin{aligned} PRF(w) = (&f(f(x) - \varepsilon \circ F \circ H^{-1}(w)) - \varepsilon \circ F^2 \circ H^{-1}(w), \\ &x,\ \delta \circ F \circ H^{-1}(w) - \delta(x, f^{-1}(x), 0)) \end{aligned}$$

Let the perturbed part of the first coordinate map of PRF be $\operatorname{Pre} \varepsilon_1(w)$. Let the third coordinate map of PRF be $\operatorname{Pre} \delta_1(w)$. Moreover, $\operatorname{Pre} \varepsilon_1(w)$ and $\operatorname{Pre} \delta_1(w)$ is defined as the corresponding parts of PR^kF for each $k \in \mathbb{N}$.

Denote partial derivatives of the composition as follows.

$$\partial_x\{P\circ Q(w)\}\equiv\partial_x P(Q(w))\qquad \partial_x P \text{ at } Q(w) \text{ is } \partial_x P\circ Q(w)$$

The similar notation is defined for partial derivatives over any other variables also.
Then the relation between Pre $\varepsilon_k(w)$ and $\varepsilon_k(w)$ (and between Pre $\delta_k(w)$ and $\delta_k(w)$ respectively).

$$\text{Pre } \varepsilon_k(w)=\sigma_{k-1}\cdot\varepsilon_k\circ\left(\frac{1}{\sigma_{k-1}\cdot w}\right)\quad \text{and}\quad \text{Pre } \delta_k(w)=\sigma_{k-1}\cdot\delta_k\circ\left(\frac{1}{\sigma_{k-1}\cdot w}\right)$$

Thus each partial derivatives of ε_k (and δ_k) at a point w are the partial derivatives of Pre $\varepsilon_k(w)$ (and Pre $\delta_k(w)$ respectively) over the same variables at the point with the linear scaling, $\sigma_{k-1}\,w$ for every $k\in\mathbb{N}$. For example,

$$\partial_y\varepsilon_k(w)=\partial_y\big(\text{Pre }\varepsilon_k\big)\circ(\sigma_{k-1}\,w)$$

Let us calculate the recursive formula of each partial derivatives of Pre $\delta_1(w)$. By the definition of the pre-renormalization and the recursive formula (B.0.4), ∂_xPre δ_1 is the following.

$$\begin{aligned}
&\partial_x\big(\text{Pre }\delta_1\big)(w)\\
&=\partial_x(\delta\circ F\circ H^{-1}(w)-\delta(x,f^{-1}(x),0))\\
&=\boxed{\big[\partial_y\delta\circ(F\circ H^{-1}(w))+\partial_z\delta\circ(F\circ H^{-1}(w))\cdot\partial_x\delta\circ H^{-1}(w)\big]}\cdot\partial_x\phi^{-1}(w)\\
&\qquad+\partial_x\delta\circ(F\circ H^{-1}(w))-\frac{d}{dx}\,\delta(x,f^{-1}(x),0)
\end{aligned}$$

Similarly, by the recursive formula (B.0.5) ∂_yPre δ_1 is the following.

$$\begin{aligned}
&\partial_y\big(\text{Pre }\delta_1\big)(w)\\
&=\boxed{\big[\partial_y\delta\circ(F\circ H^{-1}(w))+\partial_z\delta\circ(F\circ H^{-1}(w))\cdot\partial_x\delta\circ H^{-1}(w)\big]}\cdot\partial_y\phi^{-1}(w)\\
&\qquad+\partial_z\delta\circ(F\circ H^{-1}(w))\cdot\Big[\partial_y\delta\circ H^{-1}(w)+\partial_z\delta\circ H^{-1}(w)\cdot\frac{d}{dy}\,\delta(y,f^{-1}(y),0)\Big]
\end{aligned}$$

The equation (B.0.6) implies that ∂_zPre δ_1 is expressed in terms of sum or products of the partial derivatives of the maps on the previous level.

$$
\begin{aligned}
&\partial_z\big(\mathrm{Pre}\ \delta_1\big)(w) \\
= &\boxed{\big[\partial_y\delta\circ(F\circ H^{-1}(w)) + \partial_z\delta\circ(F\circ H^{-1}(w))\cdot\partial_x\delta\circ H^{-1}(w)\big]}\cdot\partial_z\phi^{-1}(w) \\
&\quad + \partial_z\delta\circ(F\circ H^{-1}(w))\cdot\partial_z\delta\circ H^{-1}(w)
\end{aligned}
$$

Definition 12.1.1. In the space of the infinitely renormalizable maps, let us denote the set of three dimensional Hénon-like maps to be $\mathcal{N}$ if the following equations are satisfied

$$
\begin{aligned}
\partial_y\delta\circ(F\circ H^{-1}(\sigma_0 w)) + \partial_z\delta\circ(F\circ H^{-1}(\sigma_0 w))\cdot\partial_x\delta\circ H^{-1}(\sigma_0 w) &\equiv 0 \\
\partial_y\delta\circ(F^2\circ H^{-1}(\sigma_0 w)) + \partial_z\delta\circ(F^2\circ H^{-1}(\sigma_0 w))\cdot\partial_x\delta\circ(F\circ H^{-1}(\sigma_0 w)) &\equiv 0
\end{aligned}
\tag{12.1.1}
$$

for all $w \in B$[1].

12.2 Invariance of the space $\mathcal{N}$ under renormalization

Recall the following definitions.

$$
\Lambda_n^{-1}(w) = \sigma_n\cdot w, \quad \psi_v^{n+1} = H_n^{-1}(\sigma_n w), \quad \psi_c^{n+1} = F_n\circ H_n^{-1}(\sigma_n w)
$$

Proposition 12.2.1. *Let F be an infinitely renormalizable Hénon-like map. Denote R^kF to be F_k and let H_k is the horizontal-like diffeomorphism of F_k. Let B be the cubic box which is the domain of F_k for all $k\in\mathbb{N}$. Then the following is true*

$$
\begin{aligned}
\psi_v^k\circ\psi_c^{k+1}(w) = &\ F_{k-1}^2\circ H_{k-1}^{-1}(\sigma_{k-1}w') \in F_{k-1}^2\circ H_{k-1}(\sigma_{k-1}B) \\
\psi_c^k\circ\psi_v^{k+1}(w) = &\ F_{k-1}\circ H_{k-1}^{-1}(\sigma_{k-1}w') \in F_{k-1}\circ H_{k-1}(\sigma_{k-1}B) \\
\psi_c^k\circ\psi_c^{k+1}(w) = &\ F_{k-1}^3\circ H_{k-1}^{-1}(\sigma_{k-1}w') \in F_{k-1}\circ H_{k-1}(\sigma_{k-1}B)
\end{aligned}
$$

where $w' = H_k^{-1}(\sigma_k w)$ for every $k\in\mathbb{N}$.

[1]The domain of each renormalized map, R^nF is denoted to be $B(R^nF)$. However, all of $B(R^nF)$ are the same sized cubic with the center origin and each sides are parallel to the each axes in the rectangular coordinate. Then we condensed the notation $B(R^nF)$ to B.

Proof. Let us prove that the image of $\sigma_{k-1}w'$ under each function $F_{k-1}^{i} \circ H_{k-1}^{-1}$ is contained in the set $F_{k-1}^{j} \circ H_{k-1}(\sigma_{k-1}B)$ for $i = 1, 2, 3$ and $j = i \pmod 2$.

$$w' = H_k^{-1}(\sigma_k w) = H_k^{-1} \circ \Lambda_k^{-1}(w)$$

Since $H_k^{-1} \circ \Lambda_k^{-1}(B) \subset B$, $\sigma_{k-1}w'$ is contained in $\sigma_{k-1}B$ only if $w \in B$. Moreover, $F_{k-1} \circ H_{k-1}(\sigma_{k-1}w) = \psi_c^k(w)$ by the definition of ψ_c^k. However, the set $\psi_c^k(B)$ is invariant under F_{k-1}^2. Then we observe that following.

$$F_{k-1}^3 \circ H_{k-1}^{-1}(\sigma_{k-1}w') \in F_{k-1}^3 \circ H_{k-1}(\sigma_{k-1}B) \subset F_{k-1} \circ H_{k-1}(\sigma_{k-1}B) \quad (12.2.1)$$

Next let us prove the equality part of the Proposition. Recall the definition of the renormalization, $F_k = \Lambda_{k-1} \circ H_{k-1} \circ F_{k-1}^2 \circ H_{k-1}^{-1} \circ \Lambda_{k-1}^{-1}$.

$$\begin{aligned}
& \psi_v^k \circ \psi_c^{k+1}(w) \\
= \;& H_{k-1}^{-1}\big(\,\sigma_{k-1}\,\psi_c^{k+1}(w)\big) \\
= \;& H_{k-1}^{-1}\big(\,\sigma_{k-1}\,F_k \circ H_k^{-1}(\sigma_k w)\big) \\
= \;& H_{k-1}^{-1}\big(\,\Lambda_{k-1}^{-1} \circ F_k \circ H_k^{-1}(\sigma_k w)\big) \\
= \;& \big(H_{k-1}^{-1} \circ \Lambda_{k-1}^{-1}\big) \circ \big(\,\Lambda_{k-1} \circ H_{k-1} \circ F_{k-1}^2 \circ H_{k-1}^{-1} \circ \Lambda_{k-1}^{-1} \circ H_k^{-1}\,\big)(\sigma_k w) \\
= \;& F_{k-1}^2 \circ H_{k-1}^{-1} \circ \big(\,\Lambda_{k-1}^{-1} \circ H_k^{-1}\,\big)(\sigma_k w) \\
= \;& F_{k-1}^2 \circ H_{k-1}^{-1}(\sigma_{k-1}w')
\end{aligned} \quad (12.2.2)$$

By the definitions of ψ_c^k and ψ_v^{k+1}, we obtain the following equation.

$$\begin{aligned}
\psi_c^k \circ \psi_v^{k+1}(w) = \;& F_{k-1} \circ H_{k-1}^{-1} \circ \big(\,\sigma_{k-1} \circ H_k^{-1}\,\big)(\sigma_k w) \\
= \;& F_{k-1} \circ H_{k-1}^{-1}(\sigma_{k-1}w')
\end{aligned}$$

Recall the equation $\psi_c^k = F_{k-1} \circ \psi_v^k$. Then by the similar calculation of (12.2.2), we obtain the following.

$$\begin{aligned}
\psi_c^k \circ \psi_c^{k+1}(w) = \;& F_{k-1}^3 \circ H_{k-1}^{-1} \circ \big(\,\Lambda_{k-1}^{-1} \circ H_k^{-1}\,\big)(\sigma_k w) \\
= \;& F_{k-1}^3 \circ H_{k-1}^{-1}(\sigma_{k-1}w')
\end{aligned}$$

Hence, the second part of the proposition, $\psi_c^k \circ \psi_c^{k+1}(w) \in F_{k-1} \circ H_{k-1}(\sigma_{k-1}B)$ holds. $\square$

In the rest of this paper, we use the notation $q(y)$ and $q_k(y)$ as follows.

$$q(y) = \frac{d}{dy}\,\delta(y, f^{-1}(y), 0), \quad q_k(y) = \frac{d}{dy}\,\delta_k(y, f_k^{-1}(y), 0) \tag{12.2.3}$$

for each $k \in \mathbb{N}$. Similarly, we can define $q(x)$ or $q_k(x)$. Moreover, the value of q_k at different point, for instance at $\sigma_k y$ is expressed as $q_k \circ (\sigma_k y)$ and so on.

Theorem 12.2.2. *Let the set of Hénon-like maps defined on* (12.1.1) *be* $\mathcal{N}$*. The space* $\mathcal{N}$ *in the space of infinitely renormalizable maps,* $\mathcal{I}_B(\bar{\varepsilon})$ *is invariant under renormalization, that is, if* $F \in \mathcal{I}_B(\bar{\varepsilon}) \cap \mathcal{N}$*, then* $RF \in \mathcal{I}_B(\bar{\varepsilon}) \cap \mathcal{N}$*.*

Proof. Suppose the following equation holds for $F \in \mathcal{I}_B(\bar{\varepsilon}) \cap \mathcal{N}$

$$\begin{aligned}
&\partial_y\delta_n \circ (F_n \circ H_n^{-1}(\sigma_n w)) + \partial_z\delta_n \circ (F_n \circ H_n^{-1}(\sigma_n w)) \qquad \cdot\partial_x\delta_n \circ H_n^{-1}(\sigma_n w) \equiv 0 \\
&\partial_y\delta_n \circ (F_n^2 \circ H_n^{-1}(\sigma_n w)) + \partial_z\delta_n \circ (F_n^2 \circ H_n^{-1}(\sigma_n w)) \quad \cdot\partial_x\delta_n \circ (F_n \circ H_n^{-1}(\sigma_n w)) \\
&\equiv 0
\end{aligned} \tag{12.2.4}$$

for $n = 0, 1, 2, \ldots, k-1$ and for every $w \in B$. Then it suffice show that the above equation holds for $n = k$ by induction. Recall $\psi_v^{k+1} = H_k^{-1}(\sigma_k w)$ and $\psi_c^{k+1} = F_k \circ H_k^{-1}(\sigma_k w)$ for $k \in \mathbb{N} \cup \{0\}$.
Let us express each partial derivatives of δ_k in terms of $\partial_x\delta_{k-1}$, $\partial_y\delta_{k-1}$ and $\partial_z\delta_{k-1}$.

$$\begin{aligned}
&\partial_y\delta_k \circ \big(F_k \circ H_k^{-1}(\sigma_k w)\big) \\
={}& \partial_y\delta_k \circ \psi_c^{k+1}(w) \\
={}& \partial_z\delta_{k-1} \circ \big(F_{k-1} \circ H_{k-1}^{-1}\big(\sigma_{k-1}F_k \circ H_k^{-1}(\sigma_k w)\big)\big) \\
&\cdot \Big[\, \partial_y\delta_{k-1} \circ H_{k-1}^{-1}\big(\sigma_{k-1}F_k \circ H_k^{-1}(\sigma_k w)\big) \\
&\qquad + \partial_z\delta_{k-1} \circ H_{k-1}^{-1}\big(\sigma_{k-1}F_k \circ H_k^{-1}(\sigma_k w)\big) \cdot q_{k-1} \circ (\sigma_{k-1}\phi_k^{-1}(\sigma_k w)) \,\Big] \\
={}& \partial_z\delta_{k-1} \circ \big(\psi_c^k \circ \psi_c^{k+1}(w)\big) \cdot \Big[\, \partial_y\delta_{k-1} \circ \big(\psi_v^k \circ \psi_c^{k+1}(w)\big) \\
&+ \partial_z\delta_{k-1} \circ \big(\psi_v^k \circ \psi_c^{k+1}(w)\big) \cdot q_{k-1} \circ (\sigma_{k-1}\phi_k^{-1}(\sigma_k w)) \,\Big]
\end{aligned}$$

$$
\begin{aligned}
&\partial_z\delta_k \circ \big(F_k \circ H_k^{-1}(\sigma_k w)\big) \\
= \ & \partial_z\delta_k \circ \psi_c^{k+1}(w) \\
= \ & \partial_z\delta_{k-1} \circ \big(F_{k-1} \circ H_{k-1}^{-1}\big(\sigma_{k-1}F_k \circ H_k^{-1}(\sigma_k w)\big)\big) \\
& \cdot \partial_z\delta_{k-1} \circ H_{k-1}^{-1}\big(\sigma_{k-1}F_k \circ H_k^{-1}(\sigma_k w)\big) \\
= \ & \partial_z\delta_{k-1} \circ \big(\psi_c^k \circ \psi_c^{k+1}(w)\big) \cdot \partial_z\delta_{k-1} \circ \big(\psi_v^k \circ \psi_c^{k+1}(w)\big)
\end{aligned} \tag{12.2.5}
$$

$$
\begin{aligned}
&\partial_x\delta_k \circ \big(H_k^{-1}(\sigma_k w)\big) = \partial_x\delta_k \circ \psi_v^{k+1}(w) \\
= \ & \partial_x\delta_{k-1} \circ \big(F_{k-1} \circ H_{k-1}^{-1}\big(\sigma_{k-1}H_k^{-1}(\sigma_k w)\big)\big) - q_{k-1} \circ (\sigma_{k-1}\phi_k^{-1}(\sigma_k w)) \\
= \ & \partial_x\delta_{k-1} \circ \big(\psi_c^k \circ \psi_v^{k+1}(w)\big) - q_{k-1} \circ (\sigma_{k-1}\phi_k^{-1}(\sigma_k w))
\end{aligned} \tag{12.2.6}
$$

Then

$$
\begin{aligned}
&\partial_y\delta_k \circ \big(F_k \circ H_k^{-1}(\sigma_k w)\big) + \partial_z\delta_k \circ \big(F_k \circ H_k^{-1}(\sigma_k w)\big) \cdot \partial_x\delta_k \circ \big(H_k^{-1}(\sigma_k w)\big) \\
= \ & \partial_y\delta_k \circ \psi_c^{k+1}(w) + \partial_z\delta_k \circ \psi_c^{k+1}(w) \cdot \partial_x\delta_k \circ \psi_v^{k+1}(w) \\
= \ & \partial_z\delta_{k-1} \circ \big(\psi_c^k \circ \psi_c^{k+1}(w)\big) \cdot \Big[\partial_y\delta_{k-1} \circ \big(\psi_v^k \circ \psi_c^{k+1}(w)\big) \\
& \quad + \partial_z\delta_{k-1} \circ \big(\psi_v^k \circ \psi_c^{k+1}(w)\big) \cdot q_{k-1} \circ (\sigma_{k-1}\phi_k^{-1}(\sigma_k w))\Big] \\
& + \partial_z\delta_{k-1} \circ \big(\psi_c^k \circ \psi_c^{k+1}(w)\big) \cdot \partial_z\delta_{k-1} \circ \big(\psi_v^k \circ \psi_c^{k+1}(w)\big) \\
& \quad \cdot \Big[\partial_x\delta_{k-1} \circ \big(\psi_c^k \circ \psi_v^{k+1}(w)\big) - q_{k-1} \circ (\sigma_{k-1}\phi_k^{-1}(\sigma_k w))\Big] \\
= \ & \partial_z\delta_{k-1} \circ \big(\psi_c^k \circ \psi_c^{k+1}(w)\big) \cdot \Big[\partial_y\delta_{k-1} \circ \big(\psi_v^k \circ \psi_c^{k+1}(w)\big) \\
& \quad + \partial_z\delta_{k-1} \circ \big(\psi_v^k \circ \psi_c^{k+1}(w)\big) \cdot \partial_x\delta_{k-1} \circ \big(\psi_c^k \circ \psi_v^{k+1}(w)\big)\Big]
\end{aligned} \tag{12.2.7}
$$

By Proposition 12.2.1 , observe the following relations

$$
\psi_v^k \circ \psi_c^{k+1}(w) \in F_{k-1}^2 \circ H_{k-1}(\sigma_{k-1}B), \quad \psi_c^k \circ \psi_v^{k+1}(w) \in F_{k-1} \circ H_{k-1}(\sigma_{k-1}B).
$$

Moreover, by the same proposition we see

$$
F_{k-1} \circ \psi_c^k \circ \psi_v^{k+1}(w) = \psi_v^k \circ \psi_c^{k+1}(w) \tag{12.2.8}
$$

Hence, the first part of the equation holds by induction.
Recall that $F_k = \Lambda_{k-1} \circ H_{k-1} \circ F_{k-1}^2 \circ H_{k-1}^{-1} \circ \Lambda_{k-1}^{-1}$. Let us calculate the following equation for later use.

$$
\begin{aligned}
& H_{k-1}^{-1} \circ \big(\sigma_{k-1} \cdot F_k^2 \circ H_k^{-1}(\sigma_k w)\big) \\
= \; & H_{k-1}^{-1} \circ \big(\sigma_{k-1} \cdot \big(\Lambda_{k-1} \circ H_{k-1} \circ F_{k-1}^2 \circ H_{k-1}^{-1} \circ \Lambda_{k-1}^{-1}\big) \circ F_k \circ H_k^{-1}(\sigma_k w)\big) \\
= \; & F_{k-1}^2 \circ H_{k-1}^{-1} \circ \big(\sigma_{k-1} \cdot F_k \circ H_k^{-1}(\sigma_k w)\big) \\
= \; & F_{k-1}^2 \circ H_{k-1}^{-1} \circ \sigma_{k-1} \cdot \psi_c^{k+1}(w) \\
= \; & F_{k-1}^2 \circ \psi_v^k \circ \psi_c^{k+1}(w)
\end{aligned}
\tag{12.2.9}
$$

Let us express each partial derivatives of δ_k in terms of $\partial_x \delta_{k-1}$, $\partial_y \delta_{k-1}$ and $\partial_z \delta_{k-1}$.

$$
\begin{aligned}
& \partial_y \delta_k \circ \big(F_k^2 \circ H_k^{-1}(\sigma_k w)\big) \\
= \; & \partial_y \delta_k \circ \big(F_k \circ \psi_c^{k+1}(w)\big) \\
= \; & \partial_z \delta_{k-1} \circ \big(F_{k-1} \circ H_{k-1}^{-1}\big(\sigma_{k-1} F_k^2 \circ H_k^{-1}(\sigma_k w)\big)\big) \\
& \quad \cdot \Big[\partial_y \delta_{k-1} \circ H_{k-1}^{-1}\big(\sigma_{k-1} F_k^2 \circ H_k^{-1}(\sigma_k w)\big) \\
& \qquad + \partial_z \delta_{k-1} \circ H_{k-1}^{-1}\big(\sigma_{k-1} F_k^2 \circ H_k^{-1}(\sigma_k w)\big) \cdot q_{k-1} \circ (\sigma_{k-1} \cdot \sigma_k x)) \Big] \\
= \; & \partial_z \delta_{k-1} \circ \big(F_{k-1}^2 \circ \psi_c^k \circ \psi_c^{k+1}(w)\big) \cdot \Big[\partial_y \delta_{k-1} \circ \big(F_{k-1}^2 \circ \psi_v^k \circ \psi_c^{k+1}(w)\big) \\
& \quad + \partial_z \delta_{k-1} \circ \big(F_{k-1}^2 \circ \psi_v^k \circ \psi_c^{k+1}(w)\big) \cdot q_{k-1} \circ (\sigma_{k-1} \cdot \sigma_k x)) \Big]
\end{aligned}
\tag{12.2.10}
$$

$$
\begin{aligned}
& \partial_z \delta_k \circ \big(F_k^2 \circ H_k^{-1}(\sigma_k w)\big) \\
= \; & \partial_z \delta_k \circ \big(F_k \circ \psi_c^{k+1}(w)\big) \\
= \; & \partial_z \delta_{k-1} \circ \big(F_{k-1} \circ H_{k-1}^{-1}\big(\sigma_{k-1} F_k^2 \circ H_k^{-1}(\sigma_k w)\big)\big) \\
& \quad \cdot \partial_z \delta_{k-1} \circ H_{k-1}^{-1}\big(\sigma_{k-1} F_k^2 \circ H_k^{-1}(\sigma_k w)\big) \\
= \; & \partial_z \delta_{k-1} \circ \big(F_{k-1}^2 \circ \psi_c^k \circ \psi_c^{k+1}(w)\big) \cdot \partial_z \delta_{k-1} \circ \big(F_{k-1}^2 \circ \psi_v^k \circ \psi_c^{k+1}(w)\big)
\end{aligned}
\tag{12.2.11}
$$

$$
\begin{aligned}
&\partial_x\delta_k \circ \big(F_k \circ H_k^{-1}(\sigma_k w)\big) = \partial_x\delta_k \circ \psi_c^{k+1}(w) \\
=\ &\partial_x\delta_{k-1} \circ \big(F_{k-1} \circ H_{k-1}^{-1}\big(\sigma_{k-1}F_k \circ H_k^{-1}(\sigma_k w)\big)\big) - q_{k-1} \circ (\sigma_{k-1} \cdot \sigma_k x)) \\
=\ &\partial_x\delta_{k-1} \circ \big(F_{k-1} \circ \psi_v^k \circ \psi_c^{k+1}(w)\big) - q_{k-1} \circ (\sigma_{k-1} \cdot \sigma_k x))
\end{aligned}
\tag{12.2.12}
$$

Then

$$
\begin{aligned}
&\partial_y\delta_k \circ \big(F_k^2 \circ H_k^{-1}(\sigma_k w)\big) \\
&\quad + \partial_z\delta_k \circ \big(F_k^2 \circ H_k^{-1}(\sigma_k w)\big) \cdot \partial_x\delta_k \circ \big(F_k \circ H_k^{-1}(\sigma_k w)\big) \\
=\ &\partial_y\delta_k \circ \big(F_k \circ \psi_c^{k+1}(w)\big) + \partial_z\delta_k \circ \big(F_k \circ \psi_c^{k+1}(w)\big) \cdot \partial_x\delta_k \circ \big(F_k \circ \psi_v^{k+1}(w)\big) \\
=\ &\partial_z\delta_{k-1} \circ \big(F_{k-1}^2 \circ \psi_c^k \circ \psi_c^{k+1}(w)\big) \cdot \Big[\partial_y\delta_{k-1} \circ \big(F_{k-1}^2 \circ \psi_v^k \circ \psi_c^{k+1}(w)\big) \\
&\qquad + \partial_z\delta_{k-1} \circ \big(F_{k-1}^2 \circ \psi_v^k \circ \psi_c^{k+1}(w)\big) \cdot q_{k-1} \circ (\sigma_{k-1} \cdot \sigma_k x))\Big] \\
&\quad + \partial_z\delta_{k-1} \circ \big(F_{k-1}^2 \circ \psi_c^k \circ \psi_c^{k+1}(w)\big) \cdot \partial_z\delta_{k-1} \circ \big(F_{k-1}^2 \circ \psi_v^k \circ \psi_c^{k+1}(w)\big) \\
&\quad \cdot \Big[\partial_x\delta_{k-1} \circ \big(F_{k-1} \circ \psi_v^k \circ \psi_c^{k+1}(w)\big) - q_{k-1} \circ (\sigma_{k-1} \cdot \sigma_k x))\Big] \\
=\ &\partial_z\delta_{k-1} \circ \big(F_{k-1}^2 \circ \psi_c^k \circ \psi_c^{k+1}(w)\big) \cdot \Big[\partial_y\delta_{k-1} \circ \big(F_{k-1}^2 \circ \psi_v^k \circ \psi_c^{k+1}(w)\big) \\
&\quad + \partial_z\delta_{k-1} \circ \big(F_{k-1}^2 \circ \psi_v^k \circ \psi_c^{k+1}(w)\big) \cdot \partial_x\delta_{k-1} \circ \big(F_{k-1} \circ \psi_v^k \circ \psi_c^{k+1}(w)\big)\Big]
\end{aligned}
\tag{12.2.13}
$$

By Proposition 12.2.1 , $\psi_v^k \circ \psi_c^{k+1}(w) \in F_{k-1}^2 \circ H_{k-1}(\sigma_{k-1}B)$, that is,

$$
\psi_v^k \circ \psi_c^{k+1}(B) \subset F_{k-1}^2 \circ H_{k-1}(\sigma_{k-1}B)
$$

for all $w \in B$. Furthermore, since the region $H_{k-1}(\sigma_{k-1}B)$ is invariant under F^2,

$$
F_{k-1}^2 \circ \psi_v^k \circ \psi_c^{k+1}(w) \subset F_{k-1}^2 \circ H_{k-1}(\sigma_{k-1}B).
$$

Hence, the second part of the condition (12.2.4) for $n = k$ holds. Therefore, the space, $\mathcal{N} \cap \mathcal{I}_B(\bar{\varepsilon})$ is invariant under renormalization. $\square$

Then if $F \in \mathcal{I}_B(\bar{\varepsilon}) \cap \mathcal{N}$, then the recursive formula of the partial derivatives of $\delta_{k+1}(w)$ in terms of the partial derivatives of $\delta_k(w)$ is following.

$$
\begin{aligned}
\partial_x \delta_{k+1}(w) = {} & \partial_x \delta_k \circ (F_k \circ H_k^{-1}(\sigma_k w)) - \frac{d}{dx}\,\delta_k(\sigma_k x, f_k^{-1}(\sigma_k x), 0) \\
\partial_y \delta_{k+1}(w) = {} & \partial_z \delta_k \circ (F_k \circ H_k^{-1}(\sigma_k w)) \\
& \cdot \Big[\partial_y \delta_k \circ H_k^{-1}(\sigma_k w) + \partial_z \delta_k \circ H_k^{-1}(\sigma_k w) \cdot \frac{d}{dy}\,\delta_k(\sigma_k y, f_k^{-1}(\sigma_k y), 0)\Big] \\
\partial_z \delta_{k+1}(w) = {} & \partial_z \delta_k \circ (F_k \circ H_k^{-1}(\sigma_k w)) \cdot \partial_z \delta_k \circ H_k^{-1}(\sigma_k w)
\end{aligned}
\tag{12.2.14}
$$

Corollary 12.2.3. *Let $F \in \mathcal{N} \cap \mathcal{I}_B(\bar{\varepsilon})$ and the third coordinate function of F be $\delta(w)$. Then*

$$\partial_y \delta \circ \Psi^{n+1}_{v^n c}(w) + \partial_z \delta \circ \Psi^{n+1}_{v^n c}(w) \cdot \partial_x \delta \circ \Psi^{n+1}_{c^n v}(w) \equiv 0 \tag{12.2.15}$$

for every $n \in \mathbb{N}$.

Proof. Firstly recall the fact that $\Psi^n_{\mathbf{w}}(B)$ is invariant under F^{2^n} for every $\mathbf{w} \in W^n$. Moreover, recall also the fact that $F_n(w) = \Lambda_{n-1} \circ H_{n-1} \circ F^2_{n-1} \circ H_{n-1} \circ \Lambda_{n-1}(w)$ for every $n \in \mathbb{N}$. Then

$$
\begin{aligned}
& \Psi^{n+1}_{v^n c}(w) = \Psi^n_{v^n} \circ F_n \circ H_n^{-1}(\sigma_n w) \\
= {} & H^{-1} \circ \Lambda^{-1} \circ H_1^{-1} \circ \Lambda_1^{-1} \circ \cdots \circ H_{n-1}^{-1} \circ \Lambda_{n-1}^{-1} \circ F_n \circ H_n^{-1}(\sigma_n w) \\
= {} & H^{-1} \circ \Lambda^{-1} \circ H_1^{-1} \circ \Lambda_1^{-1} \circ \cdots \circ F^2_{n-1} \circ H_{n-1}^{-1} \circ \Lambda_{n-1} \circ H_n^{-1}(\sigma_n w) \\
& \quad \vdots \\
= {} & F^{2^n} \circ H^{-1} \circ \Lambda^{-1} \circ H_1^{-1} \circ \Lambda_1^{-1} \circ \cdots \circ H_{n-1}^{-1} \circ \Lambda_{n-1} \circ H_n^{-1} \circ \Lambda_n^{-1}(w) \\
= {} & F^{2^n} \circ \Psi^{n+1}_{v^{n+1}}(w)
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
F^{2^n} \circ \Psi^{n+1}_{v^{n+1}}(w) = {} & F^{2^{n-1}} \circ F^2 \circ H^{-1}\big(\sigma_0 \cdot \psi^2_v \circ \psi^3_v \circ \cdots \circ \psi^{n+1}_v(w)\big) \\
\in {} & F^{2^{n-1}} \circ F^2 \circ H^{-1}\big(\sigma_0 \cdot \psi^2_v \circ \psi^3_v \circ \cdots \circ \psi^{n+1}_v(B)\big) \\
\subset {} & F^2 \circ H^{-1}\big(\sigma_0 \cdot \psi^2_v \circ \psi^3_v \circ \cdots \circ \psi^{n+1}_v(B)\big) \\
\subset {} & F^2 \circ H^{-1}(B)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
& F \circ \Psi^{n+1}_{c^n v}(w) = F \circ \Psi^{n}_{c^n} \circ H_n^{-1}(\sigma_n w) \\
= \;& F \circ F \circ H^{-1} \circ \Lambda^{-1} F_1 \circ H_1^{-1} \circ \Lambda_1^{-1} \circ \cdots \circ F_{n-1} \circ H_{n-1}^{-1} \circ \Lambda_{n-1}^{-1} \circ H_n^{-1}(\sigma_n w) \\
& \quad \vdots \\
= \;& F \circ F \circ F^2 \circ F^{2^2} \circ \cdots \circ F^{2^{n-1}} \\
& \circ H^{-1} \circ \Lambda^{-1} \circ H_1^{-1} \circ \Lambda_1^{-1} \circ \cdots \circ H_{n-1}^{-1} \circ \Lambda_{n-1} \circ H_n^{-1} \circ \Lambda_n^{-1}(w) \\
= \;& F^{2^n} \circ \Psi^{n+1}_{v^{n+1}}(w)
\end{aligned}
$$

Hence, the proof is complete. $\square$

Chapter 13

Asymptotic of each partial derivatives of δ_n and related formula of $\partial_y \varepsilon_n$

13.1 Critical point and recursive formula of $\partial_x \delta_n$

Let us define the *critical point* of $F \in \mathcal{I}_B(\bar{\varepsilon})$ as the inverse image of the tip, τ_F under F and denote this point to be c_F. Recall the definition of the tip.

$$\{\tau_F\} = \bigcap_{n \geq 1} \Psi^n_{v^n}(B)$$

The above intersection is nested and each $\Psi^n_{v^n}(B)$ is connected. Then the tip is just the limit of the sequence of $\Psi^n_{v^n}(B)$ as follows.

$$\{\tau_F\} = \bigcap_{n \geq 1} \Psi^n_{v^n}(B) = \lim_{n \to \infty} \Psi^n_{v^n}(B) \tag{13.1.1}$$

Observe that the following fact

$$\begin{aligned} \Psi^n_{v^n} \circ F_n \circ H_n^{-1}(\sigma_n w) &\in \Psi^n_{v^n}(B) \\ \Psi^n_{c^n} \circ H_n^{-1}(\sigma_n w) &\in \Psi^n_{c^n}(B) \end{aligned}$$

for each $n \in \mathbb{N}$. Since $\operatorname{diam}(\Psi^n_{\mathbf{w}}) \leq C\sigma^n$ for some $C > 0$, the limit of $\Psi^n_{w^n}(B)$ as $n \to \infty$ is a single point and furthermore, it is same as the limit of the point set which is included in $\Psi^n_{w^n}(B)$ where $w = v$ or $c \in W$. By Corollary 12.2.3, the following equation holds

$$F \circ \Psi^n_{c^n} \circ H_n^{-1}(\sigma_n w) = \Psi^n_{v^n} \circ F_n \circ H_n^{-1}(\sigma_n w)$$

for every $n \in \mathbb{N}$. Passing the limit the following equation holds

$$\begin{aligned} F \circ \lim_{n\to\infty} \Psi^n_{c^n}(B) &= \lim_{n\to\infty} F \circ \Psi^n_{c^n}(B) \\ &= \lim_{n\to\infty} F \circ \Psi^n_{c^n} \circ H_n^{-1}(\{\sigma_n w\}) = \lim_{n\to\infty} \Psi^n_{v^n} \circ F_n \circ H_n^{-1}(\{\sigma_n w\}) \\ &= \lim_{n\to\infty} \Psi^n_{v^n}(B) = \{\tau_F\} \end{aligned} \tag{13.1.2}$$

where B is the domain of F_n for all $n \in \mathbb{N}$. Then the critical point of F, $\{c_F\}$ is $\lim_{n\to\infty} \Psi^n_{c^n}(B)$.

Definition 13.1.1. Let us express the notation of the composition of $\psi^k_w \circ \cdots \circ \psi^n_w$ where $w = v$ or $c \in W$ as follows.[1]

$$\begin{aligned} \psi^k_v \circ \psi^{k+1}_v \circ \cdots \circ \psi^n_v = \Psi^n_{k,\, v^{n-k}} \equiv \Psi^n_{k,\, \mathbf{v}} \\ \psi^k_c \circ \psi^{k+1}_c \circ \cdots \circ \psi^n_c = \Psi^n_{k,\, c^{n-k}} \equiv \Psi^n_{k,\, \mathbf{c}} \end{aligned}$$

Moreover, let us take the following notations

$$\Psi^n_{k,\, \mathbf{v}} \circ \psi^{n+1}_c \equiv \Psi^{n+1}_{k,\, \mathbf{v}c}, \quad \Psi^n_{k,\, \mathbf{c}} \circ \psi^{n+1}_v \equiv \Psi^{n+1}_{k,\, \mathbf{c}v}$$

for each $n \in \mathbb{N}$. Furthermore, the notation $\Psi^{n+2}_{k,\, \mathbf{v}cv}$ or $\Psi^{n+2}_{k,\, \mathbf{v}c^2}$ and any similar notations are allowed.

Proposition 13.1.1. *Let the Hénon-like map F is in the space $\mathcal{N} \cap \mathcal{I}_B(\bar{\varepsilon})$. Let $\delta_n(w)$ be he third coordinate map of F_n for each $n \in \mathbb{N}$. Then the following equation is true*

$$\partial_x \delta_n(w) = \partial_x \delta \circ \Psi^n_{c^n}(w) - \sum_{i=0}^{n-1} q_i \circ \big(\pi_x \circ \Psi^n_{i,\, \mathbf{c}}(w)\big)$$

[1] By the above definition 13.1.1, $\Psi^n_{v^n}$ and $\Psi^n_{c^n}$ can be also expressed as follows

$$\Psi^n_{v^n} = \Psi^n_{0,\, \mathbf{v}}, \quad \Psi^n_{c^n} = \Psi^n_{0,\, \mathbf{c}}$$

for each $n \in \mathbb{N}$. Moreover, passing the limit the following equation holds

$$\partial_x \delta(c_F) = \lim_{n\to\infty} \sum_{i=0}^{n-1} q_i \circ \big(\pi_x \circ \Psi^n_{i,\mathbf{c}}(w)\big) = \lim_{n\to\infty} \sum_{i=0}^{n-1} q_i(\pi_x(c_{F_i}))$$

where c_F is the critical point of F.

Proof. By the equation (12.2.14), we see

$$\partial_x \delta_n(w) = \ \partial_x \delta_{n-1} \circ (F_{n-1} \circ H^{-1}_{n-1}(\sigma_{n-1} w)) - \frac{d}{dx}\, \delta_{n-1}(\sigma_{n-1} x, f^{-1}_{n-1}(\sigma_{n-1} x), 0)$$

Recall the definition of $q_k(x)$ in the equation (12.2.3). Then

$$\begin{aligned}
\partial_x \delta_n(w) = &\ \partial_x \delta_{n-1} \circ \psi^n_c(w) - q_{n-1}(\pi_x \circ \psi^n_c(w)) \\
= &\ \partial_x \delta_{n-2} \circ (\psi^{n-1}_c \circ \psi^n_c(w)) \\
&\quad - q_{n-2} \circ (\pi_x \circ \psi^{n-1}_c \circ \psi^n_c(w)) - q_{n-1} \circ (\pi_x \circ \psi^n_c(w)) \\
&\quad \vdots \\
= &\ \partial_x \delta \circ \Psi^n_{c^n}(w) - \sum_{i=0}^{n-1} \ q_i \circ \big(\pi_x \circ \Psi^n_{i,\mathbf{c}}(w)\big)
\end{aligned}$$

Moreover, we observe that following limit in (13.1.2)

$$\lim_{n\to\infty} \Psi^n_{i,\mathbf{c}}(B) = \{c_{F_i}\} \tag{13.1.3}$$

for each fixed $i \in \mathbb{N}$. Since $\|\,\partial_x \delta_n\| \le C\bar{\varepsilon}^{2^n}$ for some $C > 0$, passing the limit we obtain

$$\partial_x \delta(c_F) = \lim_{n\to\infty} \sum_{i=0}^{n-1} q_i \circ \big(\pi_x \circ \Psi^n_{i,\mathbf{c}}(w)\big)$$

Furthermore, since the above limit is constant and the critical points of each level, c_{F_i} are in $\Psi^n_{i,\mathbf{c}}(B)$ for all $n \in \mathbb{N}$. Then

$$\partial_x \delta(c_F) = \lim_{n\to\infty} \sum_{i=0}^{n-1} q_i(\pi_x(c_{F_i}))$$

□

13.2 Universal number b_2 and the asymptotic of $\partial_z\delta_n$ and $\partial_y\delta_n$

Proposition 13.2.1. *Let the Hénon-like map F is in the space $\mathcal{N}\cap\mathcal{I}_B(\bar{\varepsilon})$. Denote the n^{th} renormalized map of F to be $R^nF \equiv F_n = (f_n(x)-\varepsilon_n(w), x, \delta_n(w))$. Assume that F is a diffeormorphism. Then*

$$\partial_z\delta_n = b_2^{2^n}(1+O(\rho^n))$$

where b_2 is a positive number for each $n\in\mathbb{N}$ and $0<\rho<1$.

Proof. Recall the equation (12.2.14) for $\partial_z\delta_n$. Then

$$\begin{aligned}
\partial_z\delta_n(w) &= \partial_z\delta_{n-1}\circ(F_{n-1}\circ H_{n-1}^{-1}(\sigma_{n-1}w))\cdot\partial_z\delta_{n-1}\circ H_{n-1}^{-1}(\sigma_{n-1}w)\\
&= \partial_z\delta_{n-1}\circ\psi_c^n(w)\cdot\partial_z\delta_{n-1}\circ\psi_v^n(w)\\
&= \partial_z\delta_{n-2}\circ(\psi_c^{n-1}\circ\psi_c^n(w))\cdot\partial_z\delta_{n-2}\circ(\psi_v^{n-1}\circ\psi_c^n(w))\\
&\quad\cdot\partial_z\delta_{n-2}\circ(\psi_c^{n-1}\circ\psi_v^n(w))\cdot\partial_z\delta_{n-2}\circ(\psi_v^{n-1}\circ\psi_v^n(w)) \qquad (13.2.1)\\
&\quad\vdots\\
&= \prod_{\mathbf{w}\in W^n}\partial_z\delta\circ\Psi_{\mathbf{w}}^n(w)
\end{aligned}$$

The number of word $\mathbf{w}\in W^n$ is 2^n. Let us take the logarithmic average of $|\partial_z\delta_n|$ on the regions $\Psi_{\mathbf{w}}^n(B)$ and let this map be $l_n(w)$ for each $n\in\mathbb{N}$.

$$l_n(w) = \frac{1}{2^n}\sum_{\mathbf{w}\in W^n}\log|\,\partial_z\delta\circ\Psi_{\mathbf{w}}^n(w)| \qquad (13.2.2)$$

If $\partial_z\delta(w)=0$ for some $w\in B$, then $\partial_y\delta(w)=0$ at the same point because $F\in\mathcal{N}$. Thus $\operatorname{Jac}F(w)=0$, that is, F cannot be a dffeomorphism. Moreover, $\partial_z\delta$ is defined on some compact set which contains the set $\bigcup_{\mathbf{w}\in W^n}\Psi_{\mathbf{w}}^n(B)$. Then we may assume that $\partial_z\delta(w)$ has the positive lower bounds (or negative upper bounds) on the given compact set.

$$l_n(w)\longrightarrow\int_{\mathcal{O}_F}\log|\partial_z\delta|\,d\mu$$

as $n\to\infty$ where μ is the unique ergodic probability measure on the Cantor set $\mathcal{O}_F$.

The limit of $l_n(w)$ as $n\to\infty$ is a function defined on the critical Cantor set, $\mathcal{O}_F$. However, the values of the limit function at all points of $\mathcal{O}_F$ are same as each other. Then the limit is a constant function. Let this limit be $\log b_2$ for some $b_2 > 0$. Moreover, since $\operatorname{diam}(\Psi^n_{\mathbf{w}}(B)) \leq C\sigma^n$ for all $\mathbf{w} \in W^n$ for some $C > 0$, the convergence of the above equation (13.2.2) is exponentially fast. In other words,

$$\frac{1}{2^n} \log |\partial_z \delta_n(w)| = \log b_2 + O(\rho_0^n)$$

for some $0 < \rho_0 < 1$. Let us choose the constant $\rho = \rho_0/2$. Then we obtain the following asymptotic.

$$\begin{aligned} \log |\, \partial_z \delta_n(w)| &= 2^n \log b_2 + O(\rho^n) \\ &= 2^n \log b_2 + \log(1 + O(\rho^n)) \\ &= \log\, b_2^{2^n}(1 + O(\rho^n)) \end{aligned} \tag{13.2.3}$$

Hence,

$$|\partial_z \delta_n| = b_2^{2^n}(1 + O(\rho^n)) \tag{13.2.4}$$

By the assumption, $\partial_z \delta$ is not zero at any point. Then we may assume that $\partial_z \delta$ is positive. ◻

Lemma 13.2.2. *Let $F \in \mathcal{N} \cap \mathcal{I}_B(\bar{\varepsilon})$ and $\delta_n(w)$ be the third coordinate map of F_n for each $n \in \mathbb{N}$. Then the following equation holds*

$$\begin{aligned} &\partial_y \delta_n(w) \cdot \partial_z \delta \circ \Psi^n_{v^n}(w) \\ =\ &\partial_z \delta_n(w) \cdot \Big[\, \partial_y \delta \circ \Psi^n_{v^n}(w) + \sum_{i=0}^{n-1} q_i \circ \big(\pi_y \circ \Psi^n_{i,\mathbf{v}}(w)\big) \cdot \partial_z \delta \circ \Psi^n_{v^n}(w) \Big] \end{aligned}$$

for each $n \in \mathbb{N}$.

$$\partial_y \delta \circ \Psi^n_{v^n}(w) + \sum_{i=0}^{n-1} q_i \circ \big(\pi_y \circ \Psi^n_{i,\mathbf{v}}(w)\big) \cdot \partial_z \delta \circ \Psi^n_{v^n}(w) \ \leq\ C\sigma^n(1 + O(\rho^n))$$

for some $C > 0$ and $0 < \rho < 1$. Moreover, $\partial_y \delta_n(w) \leq C\sigma^n b_2^{2^n}(1 + O(\rho^n))$ for each $n \in \mathbb{N}$.

Proof. By the equation (12.2.14), we see

$$\begin{aligned} \partial_y \delta_n(w) = \ &\partial_z \delta_{n-1} \circ (F_{n-1} \circ H^{-1}_{n-1}(\sigma_{n-1} w)) \cdot \Big[\, \partial_y \delta_{n-1} \circ H^{-1}_{n-1}(\sigma_{n-1} w)) \\ &+ \partial_z \delta_{n-1} \circ H^{-1}_{n-1}(\sigma_{n-1} w)) \cdot \frac{d}{dy}\, \delta_{n-1}\big(\sigma_{n-1} y, f^{-1}_{n-1}(\sigma_{n-1} y), 0\big) \Big] \end{aligned}$$

Recall the definition of $q_k(y)$ in the equation (12.2.3). Then

$$
\begin{aligned}
& \partial_y \delta_n(w) \\
= \ & \partial_z \delta_{n-1} \circ \psi_c^n(w) \cdot \partial_y \delta_{n-1} \circ \psi_v^n(w) + \partial_z \delta_n(w) \cdot q_{n-1} \circ (\pi_y \circ \psi_v^n(w)) \\
= \ & \partial_z \delta_{n-1} \circ \psi_c^n(w) \cdot \Big[\partial_z \delta_{n-2} \circ (\psi_c^{n-1} \circ \psi_v^n(w)) \cdot \partial_y \delta_{n-2} \circ (\psi_v^{n-1} \circ \psi_v^n(w)) \\
& \quad + \partial_z \delta_{n-1} \circ \psi_v^n(w) \cdot q_{n-2} \circ (\pi_y \circ (\psi_v^{n-1} \circ \psi_v^n(w)) \Big] \\
& \quad + \partial_z \delta_n(w) \cdot q_{n-1} \circ (\pi_y \circ \psi_v^n(w)) \\
= \ & \partial_z \delta_{n-1} \circ \psi_c^n(w) \cdot \partial_z \delta_{n-2} \circ (\psi_c^{n-1} \circ \psi_v^n(w)) \cdot \partial_y \delta_{n-2} \circ (\psi_v^{n-1} \circ \psi_v^n(w)) \\
& \quad + \partial_z \delta_n(w) \cdot \Big[q_{n-2} \circ (\pi_y \circ (\psi_v^{n-1} \circ \psi_v^n(w)) + q_{n-1} \circ (\pi_y \circ \psi_v^n(w)) \Big] \\
& \quad \vdots \\
= \ & \partial_z \delta_{n-1} \circ \psi_c^n(w) \cdot \partial_z \delta_{n-2} \circ (\psi_c^{n-1} \circ \psi_v^n(w)) \cdot \\
& \quad \cdots \cdot \partial_z \delta \circ (\psi_c^1 \circ \psi_v^2 \circ \cdots \circ \psi_v^n(w)) \cdot \partial_y \delta \circ \Psi_{v^n}^n(w) \\
& \quad + \partial_z \delta_n(w) \cdot \sum_{i=0}^{n-1} q_i \circ \big(\pi_y \circ \Psi_{i,\mathbf{v}}^n(w)\big)
\end{aligned} \tag{13.2.5}
$$

Thus let us multiply $\partial_z \delta \circ \Psi_{v^n}^n(w)$. Then

$$
\begin{aligned}
& \partial_y \delta_n(w) \cdot \partial_z \delta \circ \Psi_{v^n}^n(w) \\
= \ & \partial_z \delta_n(w) \cdot \partial_y \delta \circ \Psi_{v^n}^n(w) + \partial_z \delta_n(w) \cdot \sum_{i=0}^{n-1} q_i \circ \big(\pi_y \circ \Psi_{i,\mathbf{v}}^n(w)\big) \cdot \partial_z \delta \circ \Psi_{v^n}^n(w) \\
= \ & \partial_z \delta_n(w) \cdot \Big[\partial_y \delta \circ \Psi_{v^n}^n(w) + \sum_{i=0}^{n-1} q_i \circ \big(\pi_y \circ \Psi_{i,\mathbf{v}}^n(w)\big) \cdot \partial_z \delta \circ \Psi_{v^n}^n(w) \Big]
\end{aligned} \tag{13.2.6}
$$

By Lemma 13.2.1, the asymptotic of $\partial_z \delta_n(w)$ is as follows

$$
\partial_z \delta_n(w) = b_2^{2^n} (1 + O(\rho^n)) \tag{13.2.7}
$$

for each $n \in \mathbb{N}$.
Let us estimate the following expression.

$$
\partial_y \delta \circ \Psi_{v^n}^n(w) + \sum_{i=0}^{n-1} q_i \circ \big(\pi_y \circ \Psi_{i,\mathbf{v}}^n(w)\big) \cdot \partial_z \delta \circ \Psi_{v^n}^n(w) \tag{13.2.8}
$$

Recall the definition of the tip in (13.1.1) and the critical point as the limit in

(13.1.3)

$$\{\tau_{F_i}\} = \lim_{n\to\infty} \Psi^n_{i,\mathbf{v}}(B), \quad \{c_{F_i}\} = \lim_{n\to\infty} \Psi^n_{i,\mathbf{c}}(B)$$

for each $i \in \mathbb{N}$. Moreover, since $F_i(c_{F_i}) = \tau_{F_i}$ and the Hénon-like map F_i is of the following form

$$F_i(w) = (f_i(x) - \varepsilon_i(w),\, x,\, \delta_i(w))$$

and especially the second coordinate of $F_i(w)$ is the first coordinate of the point w, we see the equation

$$\pi_x(c_{F_i}) = \pi_y(\tau_{F_i})$$

for every $i \in \mathbb{N}$. By Proposition 13.1.1, we have the following equation.

$$\partial_x\delta(c_F) = \lim_{n\to\infty} \sum_{i=0}^{n-1} q_i \circ \big(\pi_x \circ \Psi^n_{i,\mathbf{c}}(w)\big)$$

Take the limit of (13.2.8).

$$\begin{aligned}
&\partial_y\delta(\tau_F) + \lim_{n\to\infty} \sum_{i=0}^{n-1}\ q_i \circ \big(\pi_y \circ \Psi^n_{i,\mathbf{v}}(w)\big)\cdot \partial_z\delta(\tau_F)\\
=\ &\partial_y\delta(\tau_F) + \lim_{n\to\infty} \sum_{i=0}^{n-1}\ q_i \circ \big(\pi_y(\tau_{F_i})\big)\cdot \partial_z\delta(\tau_F)\\
=\ &\partial_y\delta(\tau_F) + \lim_{n\to\infty} \sum_{i=0}^{n-1}\ q_i \circ \big(\pi_x(c_{F_i})\big)\cdot \partial_z\delta(\tau_F)\\
=\ &\partial_y\delta(\tau_F) + \partial_x\delta(c_F)\cdot \partial_z\delta(\tau_F)
\end{aligned} \tag{13.2.9}$$

Since the fact that $F \in \mathcal{N}$, $c_{F_i} \in \Psi^n_{i,\mathbf{c}}(B)$ and $\tau_{F_i} \in \Psi^n_{i,\mathbf{v}}(B)$ for all $n \in \mathbb{N}$, the above expression (13.2.9) is zero. Moreover, $\Psi^n_{v^n}(B)$ and $\Psi^n_{c^n}(B)$ converge to τ_F and c_F respectively as $n \to \infty$ with at least the exponential rat because the scaling map ψ^n_v (respectively ψ^n_c) is the composition of the linear contraction with $\sigma^n(1+O(\rho^n))$ and the horizontal map, $H_n^{-1}(w)$ (respectively vertical map, $F_n \circ H_n^{-1}(w)$).

The $\displaystyle\sum_{i=0}^{n-1}\ q_i \circ \big(\pi_y \circ \Psi^n_{i,\mathbf{v}}(w)\big)$ converges exponentially fast as $n \to \infty$ by Corollary A.0.5. Then the following asymptotic holds

$$\partial_y \delta \circ \Psi^n_{v^n}(w) + \sum_{i=0}^{n-1} q_i \circ \big(\pi_y \circ \Psi^n_{i,\mathbf{v}}(w)\big) \cdot \partial_z \delta \circ \Psi^n_{v^n}(w) \le C\sigma^n (1 + O(\rho^n)) \quad (13.2.10)$$

for some $C > 0$ where $0 < \rho < 1$. Hence, applying the equation (13.2.7) and (13.2.10) to the equation (13.2.6), we obtain that

$$\partial_y \delta_n(w) \le C\sigma^n b_2^{2^n} (1 + O(\rho^n))$$

for some $C > 0$ where $0 < \rho < 1$. □

Remark 13.2.1. By the definition of the class $\mathcal{N}$, if $F \in \mathcal{N}$, then

$$\partial_y \delta_n \circ \psi^{n+1}_c(w) = \; \partial_z \delta_n \circ \psi^{n+1}_c(w) \cdot \big(- \partial_x \delta_n \circ \psi^{n+1}_v(w) \big).$$

Then $\| \, \partial_y \delta_n \circ \psi^{n+1}_c \| \le C \bar{\varepsilon}^{2^n} b_2^{2^n}$ for some $C > 0$.

Corollary 13.2.3. *Let $F \in \mathcal{N} \cap \mathcal{I}_B(\bar{\varepsilon})$ for each $n \in \mathbb{N}$. Then the following asymptotic holds*

$$\partial_y \delta_k \circ \Psi^n_{k,\mathbf{v}}(w) + \sum_{i=k}^{n-1} q_i \circ \big(\pi_y \circ \Psi^n_{i,\mathbf{v}}(w)\big) \cdot \partial_z \delta_k \circ \Psi^n_{k,\mathbf{v}}(w) \;\le\; C_n \sigma^n \, b_2^{2^k}$$

for every $k < n$ and where $C_n = C(1 + O(\rho^n)) > 0$ for some $0 < \rho < 1$.

Proof. By the direct calculation of the recursive formula in (13.2.5) from level n to k, the expression (13.2.8) is generalized as follows.

$$\begin{aligned} & \partial_y \delta_n(w) \cdot \partial_z \delta_k \circ \Psi^n_{k,\mathbf{v}}(w) \\ = \;& \partial_z \delta_n(w) \cdot \Big[\partial_y \delta_k \circ \Psi^n_{k,\mathbf{v}}(w) + \sum_{i=k}^{n-1} q_i \circ \big(\pi_y \circ \Psi^n_{i,\mathbf{v}}(w)\big) \cdot \partial_z \delta_k \circ \Psi^n_{k,\mathbf{v}}(w) \Big] \end{aligned}$$

Lemma 13.2.2 and the asymptotic of $\partial_z \delta_n$ in Lemma 13.2.1 implies the following expression.

$$\begin{aligned} \partial_y \delta_n(w) \cdot b_2^{2^k} \;=\; & b_2^{2^n} \Big[\partial_y \delta_k \circ \Psi^n_{k,\mathbf{v}}(w) + \sum_{i=k}^{n-1} q_i \circ \big(\pi_y \circ \Psi^n_{i,\mathbf{v}}(w)\big) \cdot \partial_z \delta_k \circ \Psi^n_{k,\mathbf{v}}(w) \Big] \\ & \cdot (1 + O(\rho^n)) \end{aligned}$$

Hence,

$$
\begin{aligned}
& \partial_y \delta_k \circ \Psi^n_{k,\mathbf{v}}(w) + \sum_{i=k}^{n-1} q_i \circ \big(\pi_y \circ \Psi^n_{i,\mathbf{v}}(w)\big) \cdot \partial_z \delta_k \circ \Psi^n_{k,\mathbf{v}}(w) \\
\leq\ & \partial_y \delta_n(w) \cdot \frac{b_2^{2^k}}{b_2^{2^n}} \\
\leq\ & C\sigma^n b_2^{2^n} \frac{b_2^{2^k}}{b_2^{2^n}} \left(1 + O(\rho^n)\right) =\ C\sigma^n\, b_2^{2^k} \left(1 + O(\rho^n)\right)
\end{aligned}
$$

for some $C > 0$. □

13.3 Asymptotic of partial derivative of ε_n over y

Let us consider the Jacobian of the Hénon-like map F_n in the class $\mathcal{N}$ at $\psi_c(w)$. By the universality theorem we obtain the following asymptotic.

$$
\begin{aligned}
& \operatorname{Jac} F_n \circ (F_n \circ H_n^{-1}(\sigma_n w)) \\
=\ & \partial_y \varepsilon_n \circ (F_n \circ H_n^{-1}(\sigma_n w)) \cdot \partial_z \delta_n \circ (F_n \circ H_n^{-1}(\sigma_n w)) \\
& \quad - \partial_z \varepsilon_n \circ (F_n \circ H_n^{-1}(\sigma_n w)) \cdot \partial_y \delta_n \circ (F_n \circ H_n^{-1}(\sigma_n w)) \\
=\ & \partial_y \varepsilon_n \circ (F_n \circ H_n^{-1}(\sigma_n w)) \cdot \partial_z \delta_n \circ (F_n \circ H_n^{-1}(\sigma_n w)) \\
& \quad - \partial_z \varepsilon_n \circ (F_n \circ H_n^{-1}(\sigma_n w)) \\
& \qquad \cdot \big[- \partial_z \delta_n \circ (F_n \circ H_n^{-1}(\sigma_n w)) \cdot \partial_x \delta_n \circ (H_n^{-1}(\sigma_n w)) \big] \\
=\ & \Big[\partial_y \varepsilon_n \circ (F_n \circ H_n^{-1}(\sigma_n w)) + \partial_z \varepsilon_n \circ (F_n \circ H_n^{-1}(\sigma_n w)) \cdot \partial_x \delta_n \circ (H_n^{-1}(\sigma_n w)) \Big] \\
& \quad \cdot \partial_z \delta_n \circ (F_n \circ H_n^{-1}(\sigma_n w)) \\
=\ & b^{2^n} a(\sigma_n x)(1 + O(\rho^n))
\end{aligned}
\tag{13.3.1}
$$

where b is the average Jacobian of F.

Let us define the number b_1 satisfying the equation, $b = b_1 b_2$. Combine the equation (13.2.3) and (13.3.1). Then

$$
\begin{aligned}
& b^{2^n} a(\sigma_n x)(1 + O(\rho^n)) \\
=\ & \Big[\partial_y \varepsilon_n \circ (F_n \circ H_n^{-1}(\sigma_n w)) + \partial_z \varepsilon_n \circ (F_n \circ H_n^{-1}(\sigma_n w)) \cdot \partial_x \delta_n \circ (H_n^{-1}(\sigma_n w)) \Big] \\
& \quad \cdot b_2^{2^n} (1 + O(\rho^n))
\end{aligned}
$$

Then we get the new asymptotic.

$$\begin{aligned}
&\partial_y \varepsilon_n \circ (F_n \circ H_n^{-1}(\sigma_n w)) + \partial_z \varepsilon_n \circ (F_n \circ H_n^{-1}(\sigma_n w)) \cdot \partial_x \delta_n \circ (H_n^{-1}(\sigma_n w)) \\
= \; & b_1^{2^n} a(\sigma_n x)(1 + O(\rho^n))
\end{aligned} \tag{13.3.2}$$

By the similar calculation, we obtain another asymptotic.

$$\begin{aligned}
&\partial_y \varepsilon_n \circ (F_n^2 \circ H_n^{-1}(\sigma_n w)) + \partial_z \varepsilon_n \circ (F_n^2 \circ H_n^{-1}(\sigma_n w)) \cdot \partial_x \delta_n \circ (F_n \circ H_n^{-1}(\sigma_n w)) \\
= \; & b_1^{2^n} \cdot a \circ f_n(\sigma_n x) \, (1 + O(\rho^n))
\end{aligned} \tag{13.3.3}$$

Lemma 13.3.1. *Let the Hénon-like map F is in the space $\mathcal{N} \cap \mathcal{I}_B(\bar{\varepsilon})$ for sufficiently small $\bar{\varepsilon} > 0$. Then the following equation holds*

$$\begin{aligned}
&\partial_y \varepsilon_k \circ (\Psi^n_{k,\mathbf{v}} \circ F_n(w)) + \sum_{i=k}^{n-1} \; q_i \circ \big(\pi_y \circ \Psi^n_{i,\mathbf{v}} \circ F_n(w)\big) \cdot \partial_z \varepsilon_k \circ (\Psi^n_{k,\mathbf{v}} \circ F_n(w)) \\
= \; & b_1^{2^k} \cdot a \circ \big(\pi_x \circ \Psi^n_{k,\mathbf{v}} \circ F_n(w)\big) \, \big(1 + O(\rho^k)\big)
\end{aligned}$$

where $a(x)$ is the universal function of x for some $C > 0$, $0 < \rho < 1$ and for each big enough k and n such that $n \geq k + A$ and A is depends only on b_1 and $\bar{\varepsilon}$.

Proof. Recall the equation (13.3.3) on the restricted domain $\Psi^n_{k,\mathbf{c}}(B)$

$$\begin{aligned}
&\partial_y \varepsilon_k \circ (F_k \circ \psi_c^{k+1}(w')) + \partial_z \varepsilon_k \circ (F_k \circ \psi_c^{k+1}(w')) \cdot \partial_x \delta_k \circ (\psi_c^{k+1}(w')) \\
= \; & b_1^{2^k} \cdot a \circ \big(\pi_x \circ F_k \circ \psi_c^{k+1}(w')\big) \, (1 + O(\rho^k))
\end{aligned} \tag{13.3.4}$$

for every $w' \in \Psi^n_{k+1,\mathbf{c}}(B)$. The following equation can be shown by the direct calculation using definition of renormalization

$$\Psi^n_{k,\mathbf{v}} \circ F_n(w) = F_k \circ \Psi^n_{k,\mathbf{c}}(w) \tag{13.3.5}$$

for every $k < n$. See the proof of Corollary 12.2.3. Moreover, the formula of the Hénon-like map, F_k implies the following equation.

$$\pi_x \circ \big(\Psi^n_{k,\mathbf{c}}(w)\big) = \pi_y \circ \big(F_k \circ \Psi^n_{k,\mathbf{c}}(w)\big) = \pi_y \circ \big(\Psi^n_{k,\mathbf{v}} \circ F_n(w)\big)$$

Then according to Proposition 13.1.1

$$
\begin{aligned}
&\partial_y \varepsilon_k \circ \big(\Psi^n_{k,\mathbf{v}} \circ F_n(w)\big) \\
&\quad + \sum_{i=k}^{n-1} q_i \circ \big(\pi_y \circ \big(\Psi^n_{k,\mathbf{v}} \circ F_n(w)\big)\big) \cdot \partial_z \varepsilon_k \circ \big(\Psi^n_{k,\mathbf{v}} \circ F_n(w)\big) \\
=\ &\partial_y \varepsilon_k \circ \big(\Psi^n_{k,\mathbf{v}} \circ F_n(w)\big) + \sum_{i=k}^{n-1} q_i \circ \big(\pi_x \circ \Psi^n_{i,\mathbf{c}}(w)\big) \cdot \partial_z \varepsilon_k \circ \big(\Psi^n_{k,\mathbf{v}} \circ F_n(w)\big) \\
=\ &\partial_y \varepsilon_k \circ \big(\Psi^n_{k,\mathbf{v}} \circ F_n(w)\big) \\
&\quad + \big[\, \partial_x \delta_k \circ \Psi^n_{k,\mathbf{c}}(w) - \partial_x \delta_n(w) \,\big] \cdot \partial_z \varepsilon_k \circ \big(\Psi^n_{k,\mathbf{v}} \circ F_n(w)\big) \\
=\ &\partial_y \varepsilon_k \circ \big(\Psi^n_{k,\mathbf{v}} \circ F_n(w)\big) + \partial_x \delta_k \circ \Psi^n_{k,\mathbf{c}}(w) \cdot \partial_z \varepsilon_k \circ \big(\Psi^n_{k,\mathbf{v}} \circ F_n(w)\big) \\
&\quad - \partial_x \delta_n(w) \cdot \partial_z \varepsilon_k \circ \big(\Psi^n_{k,\mathbf{v}} \circ F_n(w)\big)
\end{aligned}
\tag{13.3.6}
$$

Observe that $\|\partial_x \delta_n \cdot \partial_z \varepsilon_k\| = O(\,\bar{\varepsilon}^{2^n} \bar{\varepsilon}^{2^k})$. Then for exponential convergence of the equation (13.3.6), we need $\bar{\varepsilon}^{2^n} \bar{\varepsilon}^{2^k} \lesssim b_1^{2^k}$.

$$
\begin{aligned}
\bar{\varepsilon}^{2^n} \bar{\varepsilon}^{2^k} \lesssim b_1^{2^k} &\iff (2^n + 2^k) \log \bar{\varepsilon} \lesssim 2^k \log b_1 \\
&\iff 2^n \geq 2^k \left(\frac{\log b_1}{\log \bar{\varepsilon}} - 1 \right) + C_0
\end{aligned}
$$

for some positive $C_0 > 0$. If $b_1 < \varepsilon$, then $\bar{\varepsilon}^{2^k} \lesssim b_1^{2^k}$ is always true. Let us suppose that $b_1 > \bar{\varepsilon}$.

$$
n \geq k + C \max \left\{ 0,\ \log_2 \left(\frac{\log b_1}{\log \bar{\varepsilon}} - 1 \right) \right\}
$$

for some $C > 0$. Then the number A in the lemma is $O\left(\log_2 \left(\dfrac{\log b_1}{\log \bar{\varepsilon}} - 1 \right) \right)$.

Hence, apply the equation (13.3.4) to (13.3.6), the asymptotic is true.

$$
\begin{aligned}
&\partial_y \varepsilon_k \circ (\Psi^n_{k,\mathbf{v}} \circ F_n(w)) + \sum_{i=k}^{n-1} q_i \circ \big(\pi_y \circ \Psi^n_{i,\mathbf{v}} \circ F_n(w)\big) \cdot \partial_z \varepsilon_k \circ (\Psi^n_{k,\mathbf{v}} \circ F_n(w)) \\
=\ &b_1^{2^k} \cdot a \circ \big(\pi_x \circ \Psi^n_{k,\mathbf{v}} \circ F_n(w)\big)\ \big(1 + O(\rho^k)\big)
\end{aligned}
$$

The proof is complete. ☐

Corollary 13.3.2. *Let the Hénon-like map F is in the space $\mathcal{N} \cap \mathcal{I}_B(\bar{\varepsilon})$. Then* *the following equation holds*

$$\begin{aligned} & \partial_y \varepsilon_k \circ \Psi^n_{k,\,\mathbf{v}}(w) + \sum_{i=k}^{n-1} q_i \circ \big(\pi_y \circ \Psi^n_{i,\,\mathbf{v}}(w)\big) \cdot \partial_z \varepsilon_k \circ \Psi^n_{k,\,\mathbf{v}}(w) \\ = \;& b_1^{2^k} \cdot a \circ \big(\pi_x \circ \Psi^n_{k,\,\mathbf{v}}(w)\big) \big(1 + O(\rho^k)\big) \end{aligned}$$

for every $k < n$ which is big enough and for some $C > 0$ where $a(x)$ is the universal function of x and $0 < \rho < 1$.

Proof. By Lemma 13.3.1, the asymptotic at the tip, τ_{F_n} holds as follows.

$$\begin{aligned} & \partial_y \varepsilon_k \circ \Psi^n_{k,\,\mathbf{v}}(\tau_{F_n}) + \sum_{i=k}^{n-1} q_i \circ \big(\pi_y \circ \Psi^n_{i,\,\mathbf{v}}(\tau_{F_n})\big) \cdot \partial_z \varepsilon_k \circ \Psi^n_{k,\,\mathbf{v}}(\tau_{F_n}) \\ = \;& \partial_y \varepsilon_k(\tau_{F_k}) + \lim_{n\to\infty} \sum_{i=k}^{n-1} q_i \circ \big(\pi_y \circ (\tau_{F_i})\big) \cdot \partial_z \varepsilon_k(\tau_{F_k}) \\ = \;& b_1^{2^k} \cdot a \circ \big(\pi_x(\tau_{F_k})\big) \big(1 + O(\rho^k)\big) \end{aligned}$$

Moreover, since $\Psi^n_{k,\,\mathbf{v}}(w) \to \{\tau_{F_k}\}$ as $n \to \infty$ exponentially fast, each $\partial_y \varepsilon_k \circ \Psi^n_{k,\,\mathbf{v}}(w)$ and $\partial_z \varepsilon_k \circ \Psi^n_{k,\,\mathbf{v}}(w)$ converge to $\partial_y \varepsilon_k(\tau_{F_k})$ and $\partial_z \varepsilon_k(\tau_{F_k})$ respectively as $n \to \infty$ exponentially fast. Additionally, the universal function $a \circ \big(\pi_x \circ \Psi^n_{k,\,\mathbf{v}}(w)\big)$ converges to $a \circ \big(\pi_x(\tau_{F_k})\big)$ as $n \to \infty$ exponentially fast.

The exponential convergence of the series, $\displaystyle\sum_{i=k}^{n-1} q_i \circ \big(\pi_y \circ \Psi^n_{i,\,\mathbf{v}}(w)\big)$ comes from Corollary A.0.5. □

Remark 13.3.1. The space $\mathcal{N} \cap \mathcal{I}_B(\bar{\varepsilon})$ allows that the lower bound of $\|\partial_z \varepsilon\|$ to be zero. If $F \in \mathcal{N} \cap \mathcal{I}_B(\bar{\varepsilon})$ is a diffeomorphism with the condition $\|\partial_z \varepsilon\| \equiv 0$ and additionally if $\partial_y \delta(w) = \partial_x \delta(w) \equiv 0$, then $\delta_n(w) = \delta_n(z)$ for every $n \in \mathbb{N}$. Furthermore, the renormalizability implies that $\partial_z \delta_n(z) = b_2^{2^n}\,(1 + O(\rho^n))$ for each $n \in \mathbb{N}$ where $0 < \rho < 1$ and $0 < b_2 \ll 1$. The set of these map is contained in the intersection of model maps and the class $\mathcal{N}$. We can call this space *trivial extension* of the infinitely renormalizable two dimensional Hénon-like maps.

Chapter 14

Unbounded geometry of the Cantor set

The unbounded geometry of a certain class would be proved by the calculation of the three dimensional asymptotic.

14.1 Horizontal overlap of two adjacent boxes

The proof of the (un)bounded geometry of the Cantor set requires to compare the diameter of the box and the minimal distance of two adjacent boxes in the boxing. In order to compare these quantities, we would use the maps, $\Psi^n_k(w)$ and $F_k(w)$ with the two points $w_1 = (x_1, y_1, z_1)$ and $w_2 = (x_2, y_2, z_2)$ in the domain of $F_n(w)$, namely, $\mathrm{Dom}(R^nF)$. Let us each successive image of w_j under $\Psi^n_k(w)$ and $F_k(w)$ be $\dot{w}_j$, $\ddot{w}_j$ and $\dddot{w}_j$ for $j = 1, 2$.

$$w_j \xmapsto{\Psi^n_k} \dot{w}_j \xmapsto{F_k} \ddot{w}_j \xmapsto{\Psi^k_0} \dddot{w}_j$$

For example, $\dot{w}_j = \Psi^n_k(w_j)$ and $\dot{w}_j = (\dot{x}_j, \dot{y}_j, \dot{z}_j)$ for $j = 1, 2$. Let S_1 and S_2 be the (path) connected set on $\mathbb{R}^3$. If $\pi_x(\overline{S_1}) \cap \pi_x(\overline{S_2})$ contains at least two points, then this intersection is called the *$x-$axis overlap* or *horizontal overlap* of S_1 and S_2. Moreover, we say S_1 *overlaps* S_2 on the $x-$axis or horizontally. Recall σ is the linear scaling of F_*, the fixed point of the renormalization operator and $\sigma_k = \sigma(1 + O(\rho^k))$ for each $k \in \mathbb{N}$.

Recall the map Ψ^n_k from $B(R^nF)$ to $B^{n-k}_{\mathbf{v}}(R^kF)$, $\alpha_{n,k} = \sigma^{2(n-k)}(1 + O(\rho^k))$ and $\sigma_{n,k} = (-\sigma)^{n-k}(1 + O(\rho^k))$.

$$\Psi_k^n(w) = \begin{pmatrix} 1 & t_{n,k} & u_{n,k} \\ & 1 & \\ & d_{n,k} & 1 \end{pmatrix} \begin{pmatrix} \alpha_{n,k} & & \\ & \sigma_{n,k} & \\ & & \sigma_{n,k} \end{pmatrix} \begin{pmatrix} x + S_k^n(w) \\ y \\ z + R_k^n(y) \end{pmatrix}$$

where $\mathbf{v} = v^{n-k} \in W^{n-k}$. Thus for any $w \in B(R^nF)$ we have the following equation.

$$\pi_x \circ \Psi_k^n(w) = \alpha_{n,k}(x + S_k^n(w)) + \sigma_{n,k}\big(t_{n,k}\, y + u_{n,k}(z + R_k^n(y))\big)$$

Let us find the sufficient condition of the horizontal overlapping. Horizontal overlapping means that there exist two points $w_1 \in B_v^1(R^nF)$ and $w_2 \in B_c^1(R^nF)$ satisfying the following.

$$\pi_x \circ \Psi_k^n(w_1) - \pi_x \circ \Psi_k^n(w_2) = 0$$

Equivalently,

$$\begin{aligned} &\alpha_{n,k}\Big[\big(x_1 + S_k^n(w_1)\big) - \big(x_2 + S_k^n(w_2)\big)\Big] \\ &\quad + \sigma_{n,k}\Big[\, t_{n,k}(y_1 - y_2) + u_{n,k}\big\{z_1 - z_2 + R_k^n(y_1) - R_k^n(y_2)\big\}\Big] = 0. \end{aligned} \tag{14.1.1}$$

Recall that $x + S_k^n(w) = v_*(x) + O(\bar{\varepsilon}^{2^k} + \rho^n)$ for some $0 < \rho < 1$ with C^1 convergence. Since the universal map $v_*(x)$ is a diffeomorphism and $|x_1 - x_2| = O(1)$, we have the following estimation by the mean value theorem.

$$|\, x_1 + S_k^n(w_1) - \big(x_2 + S_k^n(w_2)\big)| = O(1)$$

Then the $x-$axis overlapping of two boxes, namely, the equation (14.1.1) implies that

$$\sigma^{n-k} \asymp t_{n,k}(y_1 - y_2) + u_{n,k}\big\{z_1 - z_2 + R_k^n(y_1) - R_k^n(y_2)\big\}$$

for every sufficiently big $k \in \mathbb{N}$.

Proposition 14.1.1. *Let $F \in \mathcal{N} \cap \mathcal{I}_B(\bar{\varepsilon})$. Then*

$$b_1^{2^k} \asymp t_{n,k}(y_1 - y_2) + u_{n,k}\left\{z_1 - z_2 + R_k^n(y_1) - R_k^n(y_2)\right\}$$

for every big enough k and n such that $n > k + A$ where A is the number defined on Lemma 13.3.1.

Proof. Let us choose two points in $B(R^nF)$ as follows.

$$\begin{aligned} w_1 &= (x_1, y_1, z_1) \in B^1_v(R^nF) \cap R^nF(B) \\ w_2 &= (x_2, y_2, z_2) \in B^1_c(R^nF) \cap R^nF(B) \end{aligned}$$

Recall that $|x_1 - x_2| = O(1)$, $|y_1 - y_2| = O(1)$. Let us choose the points w_1 and w_2 in $F_n(B)$. In particular, we may assume that $w_j \in \mathcal{O}_{R^nF}$ for $j = 1, 2$. Then $|z_1 - z_2| = O(\bar{\varepsilon}^{2^n})$. Two points w_1 and w_2 has their pre-image under R^nF and let w'_1 and w'_2 be the pre-image of w_1 and w_2 respectively. Then

$$|z_1 - z_2| = |\,\delta_n(w'_1) - \delta_n(w'_2)| \leq C\|D\delta_n\| \cdot \|w_1 - w_2\| = O(\,\bar{\varepsilon}^{2^n})$$

for some $C > 0$. Then

$$\begin{aligned} &t_{n,\,k}(y_1 - y_2) + u_{n,\,k}\left\{z_1 - z_2 + R^n_k(y_1) - R^n_k(y_2)\right\} \\ \asymp\ &t_{n,\,k}(y_1 - y_2) + u_{n,\,k}\left\{R^n_k(y_1) - R^n_k(y_2)\right\} \end{aligned}$$

It suffice to show that

$$t_{n,\,k}(y_1 - y_2) + u_{n,\,k}\left\{R^n_k(y_1) - R^n_k(y_2)\right\} \asymp b_1^{2^k}$$

By Proposition A.0.2, we see that

$$t_{n,\,k} - u_{n,\,k}\,d_{n,\,k} = \sum_{i=k}^{n-1} \sigma^{i-k}\left[\,t_{i+1,\,i} - u_{i+1,\,i}\,d_{i+1,\,k}\right](1 + O(\rho^k))$$

Recall the fact if $A \asymp B$ and $A' \asymp B'$, then $A + A' \asymp B + B'$. Recall also that if a series is convergent exponentially fast, then the sum is comparable with the first term of the given series. Then we see the following asymptotic

$$\begin{aligned} &\Big[\,t_{n,\,k}(y_1 - y_2) + u_{n,\,k}\left\{R^n_k(y_1) - R^n_k(y_2)\right\}\Big](1 + O(\rho^k)) \\ =\ &\Big[\big(t_{n,\,k} - u_{n,\,k}\,d_{n,\,k}\big)(y_1 - y_2) + u_{n,\,k}\left\{d_{n,\,k}(y_1 - y_2) + R^n_k(y_1) - R^n_k(y_2)\right\}\Big] \\ &\cdot (1 + O(\rho^k)) \\ =\ &\sum_{i=k}^{n-1} \sigma^{i-k}\left[\,t_{i+1,\,i} - u_{i+1,\,i}\,d_{i+1,\,k}\right](y_1 - y_2) \\ &+ \sum_{i=k}^{n-1} \sigma^{i-k}\,u_{i+1,\,i}\left\{d_{n,\,k}(y_1 - y_2) + R^n_k(y_1) - R^n_k(y_2)\right\} \end{aligned}$$

$$
\begin{aligned}
&\asymp \big[\, t_{k+1,\,k} - u_{k+1,\,k}\, d_{k+1,\,k}\big](y_1 - y_2) \\
&\qquad + u_{k+1,\,k}\big\{d_{n,\,k}(y_1 - y_2) + R^n_k(y_1) - R^n_k(y_2)\big\} \\
(*) \;&\asymp \partial_y \varepsilon_k(\tau_{F_k})(y_1 - y_2) + \partial_z \varepsilon_k(\tau_{F_k}) \cdot \sum_{i=k}^{n-1} q_i \circ (\sigma_i\, \xi_{i+1}) \cdot (y_1 - y_2) \\
&= \Big[\partial_y \varepsilon_k(\tau_{F_k}) + \partial_z \varepsilon_k(\tau_{F_k}) \cdot \sum_{i=k}^{n-1} q_i \circ (\sigma_i\, \xi_{i+1})\Big](y_1 - y_2) \\
(**) \;&\asymp b_1^{2^k}(y_1 - y_2)
\end{aligned}
$$

where $\sigma_i\, \xi_{i+1}$ is some points in the line segment between $\pi_y \circ \Psi^n_i(w_1)$ and $\pi_z \circ \Psi^n_i(w_2)$ in $\pi_y \circ \Psi^n_i(B)$ for each $k \leq i \leq n-1$. Lemma 13.3.1 and Corollary 13.3.2 involve the $(*)$ and $(**)$. The proof is complete. □

14.2 Unbounded geometry of the critical Cantor set

Lemma 14.2.1. *Let $F \in \mathcal{N} \cap \mathcal{I}_B(\bar{\varepsilon})$. Let the box $B^n_{\mathbf{w}v}$ be $\Psi^n_{0,\,\mathbf{v}}(B^1_v(R^n F))$. Then*

$$
\operatorname{diam}(B^n_{\mathbf{w}v}) \geq |\, C_1\, \sigma^k \sigma^{2(n-k)} - C_2\, \sigma^k \sigma^{n-k} b_1^{2^k}\, |
$$

where $\mathbf{w} = v^k c\, v^{n-k-1} \in W^n$ for some positive C_1 and C_2.

Proof. Let us choose the two points

$$
w_j = (x_j, y_j, z_j) \in B^1_v(R^n F) \cap \mathcal{O}_{R^n F}
$$

where $j = 1, 2$. We may assume that $|x_1 - x_2| = O(1)$, $|y_1 - y_2| = O(1)$ and $|z_1 - z_2| = O(\bar{\varepsilon}^{2^n})$. Thus

$$
\pi_x \circ \Psi^n_k(w) = \alpha_{n,\,k}(x + S^n_k(w)) + \sigma_{n,\,k}\big(t_{n,\,k}\, y + u_{n,\,k}(z + R^n_k(y))\big).
$$

Then

$$
\begin{aligned}
\pi_x \circ \Psi^n_k(w_1) - \pi_x \circ \Psi^n_k(w_2) = \;& \alpha_{n,\,k}\Big[\big(x_1 + S^n_k(w_1)\big) - \big(x_2 + S^n_k(w_2)\big)\Big] \\
&+ \sigma_{n,\,k}\Big[\, t_{n,\,k}(y_1 - y_2) + u_{n,\,k}\big\{z_1 - z_2 + R^n_k(y_1) - R^n_k(y_2)\big\}\Big].
\end{aligned}
\tag{14.2.1}
$$

By the equation (14.2.1),

$$\begin{aligned}\dot{x}_1 - \dot{x}_2 = \ & \alpha_{n,\,k}\Big[\big(x_1 + S^n_k(w_1)\big) - \big(x_2 + S^n_k(w_2)\big)\Big] \\ & \qquad + \sigma_{n,\,k}\Big[\, t_{n,\,k}(y_1 - y_2) + u_{n,\,k}\big\{z_1 - z_2 + R^n_k(y_1) - R^n_k(y_2)\big\}\Big]\end{aligned}$$

Recall the estimation of the non linear part of $\pi_x \circ \Psi^n_k(w)$. Then $x + S^n_k(w) = v_*(x) + O(\bar{\varepsilon}^{2^k} + \rho^{n-k})$. Since $v_*(x)$ is a diffeomorphism,

$$|\, v_*(x_1) - v_*(x_2)| = |\, v'_*(\bar{x})(x_1 - x_2)| \geq C_0$$

for some $C_0 > 0$. The definition of the Hénon-like map F_k and the coordinate change map Ψ^k_0, we see the following equations.

$$\begin{aligned}\ddot{y}_1 - \ddot{y}_2 = \ & \dot{x}_1 - \dot{x}_2 \\ = \ & \alpha_{n,\,k}\Big[\big(x_1 + S^n_k(w_1)\big) - \big(x_2 + S^n_k(w_2)\big)\Big] \\ & \qquad + \sigma_{n,\,k}\Big[\, t_{n,\,k}(y_1 - y_2) + u_{n,\,k}\big\{z_1 - z_2 + R^n_k(y_1) - R^n_k(y_2)\big\}\Big] \\ = \ & \alpha_{n,\,k}\Big[\, v'_*(\bar{x}) + O(\bar{\varepsilon}^{2^k} + \rho^{n-k})\Big](x_1 - x_2) \\ & \qquad + \sigma_{n,\,k}\Big[\, t_{n,\,k}(y_1 - y_2) + u_{n,\,k}\big\{z_1 - z_2 + R^n_k(y_1) - R^n_k(y_2)\big\}\Big] \\ \dddot{y}_1 - \dddot{y}_2 = \ & \sigma_{k,\,0}(\ddot{y}_1 - \ddot{y}_2) = \ \sigma_{k,\,0}(\dot{x}_1 - \dot{x}_2)\end{aligned} \tag{14.2.2}$$

Clearly $\operatorname{diam}(B^n_{\mathbf{w}}) \geq |\dddot{y}_1 - \dddot{y}_2|$ where $\mathbf{w} = v^k c\, v^{n-k-1} \in W^n$. Hence, by Proposition 14.1.1, we have the estimation

$$\operatorname{diam}(B^n_{\mathbf{w}v}) \geq C\, \sigma^k \sigma^{2(n-k)}$$

where $\mathbf{w} = v^k c\, v^{n-k-1} \in W^n$ for some $C > 0$. ◻

Lemma 14.2.2. *Let $F \in \mathcal{N} \cap \mathcal{I}_B(\bar{\varepsilon})$. Let us choose two different points as follows.*

$$w_1 = (x, y, z) \in B^1_v(R^n F) \cap \mathcal{O}_{R^n F}, \quad w_2 = (x, y, z) \in B^1_c(R^n F) \cap \mathcal{O}_{R^n F}$$

Suppose that $B^{n-k}_{\mathbf{v}v}(R^k F)$ overlaps $B^{n-k}_{\mathbf{v}c}(R^k F)$ with respect to $\Psi^n_k(w_1)$ and $\Psi^n_k(w_2)$ on the $x-$axis for the word $\mathbf{v} = v^{n-k} \in W^{n-k}$. Then

$$\operatorname{dist}_{\min}(B^n_{\mathbf{w}v}, B^n_{\mathbf{w}c}) \leq C\big[\, \sigma^{2k}\sigma^{n-k} b_1^{2^k} + \sigma^{2k}\sigma^{2(n-k)} b_2^{2^k}\, \big]$$

where $\mathbf{w} = v^k c\, v^{n-k-1} \in W^n$ for some $C > 0$.

Proof. Recall the expression of the map Ψ^n_k from $B(R^nF)$ to $B^{n-k}_{\mathbf{v}}(R^kF)$.

$$\Psi^n_k(w) = \begin{pmatrix} 1 & t_{n,\,k} & u_{n,\,k} \\ & 1 & \\ & d_{n,\,k} & 1 \end{pmatrix} \begin{pmatrix} \alpha_{n,\,k} & & \\ & \sigma_{n,\,k} & \\ & & \sigma_{n,\,k} \end{pmatrix} \begin{pmatrix} x + S^n_k(w) \\ y \\ z + R^n_k(y) \end{pmatrix}$$

where $\mathbf{v} = v^{n-k} \in W^{n-k}$. Then the expression of Ψ^n_k on the above and the assumption of the overlapping on the $x-$axis, we obtain the following estimation.

$$\begin{aligned} \dot{x_1} - \dot{x_2} =&\ 0 \\ \dot{y_1} - \dot{y_2} =&\ \sigma_{n,\,k}(y_1 - y_2) \\ \dot{z_1} - \dot{z_2} =&\ \sigma_{n,\,k}\big[\, d_{n,\,k}(y_1 - y_2) + z_1 - z_2 + R^n_k(y_1) - R^n_k(y_2)\,\big] \end{aligned}$$

By the mean value theorem and the map R^kF, we obtain the following equations

$$\begin{aligned} \ddot{x}_1 - \ddot{x}_2 =&\ f(\dot{x_1}) - \varepsilon_k(\dot{w}_1) - [f(\dot{x_2}) - \varepsilon_k(\dot{w}_2)] \\ =&\ -\varepsilon_k(\dot{w}_1) + \varepsilon_k(\dot{w}_2) \\ =&\ -\partial_y\varepsilon_k(\eta)\cdot(\dot{y_1} - \dot{y_2}) - \partial_z\varepsilon_k(\eta)\cdot(\dot{z_1} - \dot{z_2}) \\ =&\ -\partial_y\varepsilon_k(\eta)\cdot\sigma_{n,\,k}(y_1 - y_2) \\ &\quad - \partial_z\varepsilon_k(\eta)\cdot\sigma_{n,\,k}\big[\, d_{n,\,k}(y_1 - y_2) + z_1 - z_2 + R^n_k(y_1) - R^n_k(y_2)\,\big] \\ \ddot{y}_1 - \ddot{y}_2 =&\ 0 \\ \ddot{z}_1 - \ddot{z}_2 =&\ \delta_k(\dot{w}_1) - \delta_k(\dot{w}_2) \\ =&\ \partial_y\delta_k(\zeta)\cdot(\dot{y_1} - \dot{y_2}) + \partial_z\delta_k(\zeta)\cdot(\dot{z_1} - \dot{z_2}) \\ =&\ \partial_y\delta_k(\zeta)\cdot\sigma_{n,\,k}(y_1 - y_2) \\ &\quad + \partial_z\delta_k(\zeta)\cdot\sigma_{n,\,k}\big[\, d_{n,\,k}(y_1 - y_2) + z_1 - z_2 + R^n_k(y_1) - R^n_k(y_2)\,\big] \end{aligned} \tag{14.2.3}$$

where η and ζ are some points in the line segment between $\dot{w}_1$ and $\dot{w}_2$ in $\Psi^n_k(B)$. Furthermore, by Proposition A.0.4, the distance $\ddot{x}_1 - \ddot{x}_2$ and $\ddot{z}_1 - \ddot{z}_2$ as follows.

$$
\begin{aligned}
\ddot{x}_1 - \ddot{x}_2 = & \ -\partial_y \varepsilon_k(\eta) \cdot \sigma_{n,k}(y_1 - y_2) - \partial_z \varepsilon_k(\eta) \cdot \sigma_{n,k} \cdot \sum_{i=k}^{n-1} q_i(\sigma_i\, \xi_{i+1}) \cdot (y_1 - y_2) \\
& - \partial_z \varepsilon_k(\eta) \cdot \sigma_{n,k} \cdot (z_1 - z_2) \\
= & \ -\Big[\, \partial_y \varepsilon_k(\eta) + \partial_z \varepsilon_k(\eta) \cdot \sum_{i=k}^{n-1} q_i(\sigma_i\, \xi_{i+1}) \,\Big] \cdot \sigma_{n,k}\, (y_1 - y_2) \\
& - \partial_z \varepsilon_k(\eta) \cdot \sigma_{n,k}\, (z_1 - z_2)
\end{aligned}
$$

$$
\begin{aligned}
\ddot{z}_1 - \ddot{z}_2 = & \ \partial_y \delta_k(\zeta) \cdot \sigma_{n,k}\, (y_1 - y_2) + \partial_z \delta_k(\zeta) \cdot \sigma_{n,k} \cdot \sum_{i=k}^{n-1} q_i(\sigma_i\, \xi_{i+1}) \cdot (y_1 - y_2) \\
& + \partial_z \delta_k(\zeta) \cdot \sigma_{n,k}\, (z_1 - z_2) \\
= & \ \Big[\, \partial_y \delta_k(\zeta) + \partial_z \delta_k(\zeta) \cdot \sum_{i=k}^{n-1} q_i(\sigma_i\, \xi_{i+1}) \,\Big] \cdot \sigma_{n,k}\, (y_1 - y_2) \\
& + \partial_z \delta_k(\zeta) \cdot \sigma_{n,k}\, (z_1 - z_2)
\end{aligned}
$$

Since $\eta, \zeta \in \Psi^n_k(B)$ and $\sigma_i\, \xi_{i+1} \in \pi_y \circ \Psi^n_{i+1}(B)$ for each $k \le i \le n-1$, the asymptotic in Corollary 13.2.3 bounds $|\, \ddot{z}_1 - \ddot{z}_2|$ and Corollary 13.3.2 bounds $|\, \ddot{x}_1 - \ddot{x}_2|$. Then

$$
\begin{aligned}
|\, \ddot{x}_1 - \ddot{x}_2| \lesssim & \ |\, \sigma^{n-k}\,| \cdot \big[\, |-b_1^{2^k} \cdot a \circ f_k(\sigma_k x)\,| \cdot |\, y_1 - y_2| + |\, \partial_z \varepsilon_k(\eta)| \cdot |\, z_1 - z_2|\,\big] \\
& (1 + O(\rho^k)) \\
|\, \ddot{z}_1 - \ddot{z}_2| \lesssim & \ |\, \sigma^n\, b_2^{2^k} \sigma^{n-k}| \cdot |\, y_1 - y_2| \ + b_2^{2^k} |\, \sigma^{n-k}| \cdot |\, z_1 - z_2|\ (1 + O(\rho^k))
\end{aligned}
\tag{14.2.4}
$$

Recall

$$
\pi_x \circ \Psi^n_k(w) = \alpha_{n,k}(x + S^n_k(w)) + \sigma_{n,k}\big(t_{n,k}\, y + u_{n,k}(z + R^n_k(y))\big).
$$

Then

$$
\begin{aligned}
\dddot{x}_1 - \dddot{x}_2 = & \ \pi_x \circ \Psi^k_0(\ddot{w}_1) - \pi_x \circ \Psi^k_0(\ddot{w}_2) \\
= & \ \alpha_{k,0}\big[\, (\ddot{x}_1 + S^k_0(\ddot{w}_1)) - (\ddot{x}_2 + S^k_0(\ddot{w}_2))\,\big] \\
& \quad + \sigma_{k,0}\, \big[\, t_{k,0}\, (\ddot{y}_1 - \ddot{y}_2) + u_{k,0}\big(\ddot{z}_1 - \ddot{z}_2 + R^k_0(\ddot{y}_1) - R^k_0(\ddot{y}_2)\big)\,\big] \\
= & \ \alpha_{k,0}\big[\, v'_*(\bar{x}) + O(\bar{\varepsilon} + \rho^k)\,\big](\ddot{x}_1 - \ddot{x}_2) + \sigma_{k,0} \cdot u_{k,0}\, (\ddot{z}_1 - \ddot{z}_2) \\
\dddot{y}_1 - \dddot{y}_2 = & \ \sigma_{k,0}\, (\ddot{y}_1 - \ddot{y}_2) = 0
\end{aligned}
\tag{14.2.5}
$$

$$
\begin{aligned}
\dddot{z}_1 - \dddot{z}_2 &= \pi_z \circ \Psi^k_0(\ddot{w}_1) - \pi_z \circ \Psi^k_0(\ddot{w}_2) \\
&= \sigma_{k,0}\,(\ddot{z}_1 - \ddot{z}_2) + \sigma_{k,0}\big[\, d_{k,0}(\ddot{y}_1 - \ddot{y}_2) + R^n_k(\ddot{y}_1) - R^n_k(\ddot{y}_2)\,\big] \\
&= \sigma_{k,0}\,(\ddot{z}_1 - \ddot{z}_2)
\end{aligned}
\tag{14.2.6}
$$

Moreover, let us apply the estimations in (14.2.4) to $\dddot{x}_1 - \dddot{x}_2$ and $\dddot{z}_1 - \dddot{z}_2$. Let us assume that both k and n are even numbers.

$$
\begin{aligned}
&\operatorname{dist}_{\min}(B^n_{\mathbf{w}v}, B^n_{\mathbf{w}c}) \\
\le\ & |\,\dddot{x}_1 - \dddot{x}_2| + |\,\dddot{z}_1 - \dddot{z}_2| \\
\le\ & \big[\, \sigma^{2k} \cdot |\,\ddot{x}_1 - \ddot{x}_2| \cdot v_*(\bar{x}) + \sigma^k \cdot (1 + u_{k,0})\,|\,\ddot{z}_1 - \ddot{z}_2|\,\big](1 + O(\rho^n)) \\
\le\ & C\Big\{ \big[\, \sigma^{2k}\sigma^{n-k}\, b_1^{2^k} \cdot a \circ (\pi_x \circ \Psi^n_{k,\mathbf{v}}(w)) + \sigma^k\sigma^n\sigma^{n-k}\, b_2^{2^k}\,\big] \cdot |\,y_1 - y_2| \\
& \quad + \big[\, \sigma^{2k}|\,\partial_z \varepsilon_k(\eta)| + \sigma^k\sigma^{n-k} \cdot b_2^{2^k}\,\big] \cdot |\,z_1 - z_2| \Big\}
\end{aligned}
\tag{14.2.7}
$$

for some $C > 0$.

Observe that $F_n(B^1_v(F_n)) \subset B^1_c(F_n)$ and $F_n(B^1_c(F_n)) \subset B^1_v(F_n)$. Let us measure the distance of each third coordinates of two given points

$$
w_1 = (x, y, z) \in B^1_v(F_n) \cap \mathcal{O}_{F_n}, \quad w_2 = (x, y, z) \in B^1_c(F_n) \cap \mathcal{O}_{F_n}
$$

as follows. Recall $|\,x_1 - x_2|$ and $|\,y_1 - y_2|$ is $O(1)$. By the mean value theorem, the $z-$coordinate distance of two points is estimated as follows.

$$
|\,\pi_z \circ F_n(w_1) - \pi_z \circ F_n(w_2)| = |\,\delta_n(w_1) - \delta_n(w_2)| \le \|D\delta_n\| \cdot \|\,w_1 - w_2\| = O(\,\bar{\varepsilon}^{2^n})
$$

Since the critical Cantor set $\mathcal{O}_{F_n}$ is an invariant compact set under F_n, we may assume that $|\,z_1 - z_2| \le C\varepsilon^{2^n}$. For sufficiently small $\bar{\varepsilon} > 0$ and big enough $n \gg k$, the number $\bar{\varepsilon}^{2^n}$ is very small, that is, $\bar{\varepsilon}^{2^n} \ll \sigma^{n-k}\, b_{\min}^{2^k}$ where $b_{\min} = \min\{b_1, b_2\}$. Then the estimation (14.2.7) is refined as follows.

$$
\begin{aligned}
\operatorname{dist}_{\min}(B^n_{\mathbf{w}v}, B^n_{\mathbf{w}c}) &\le |\,\dddot{x}_1 - \dddot{x}_2| + |\,\dddot{z}_1 - \dddot{z}_2| \\
&\le C\big[\, \sigma^{2k}\sigma^{n-k} b_1^{2^k} + \sigma^{2k}\sigma^{2(n-k)} b_2^{2^k}\,\big]
\end{aligned}
\tag{14.2.8}
$$

□

For the (un)bounded geometry of the Cantor set, both the level k and n travels through any big natural numbers toward the infinity with each fixed numbers

b_1 and b_2. Then by the comparison of the diameter of the box and minimal distance between adjacent boxes, $\mathcal{O}_F$ has the unbounded geometry.

Theorem 14.2.3. *Let $F_{b_1} \in \mathcal{N} \cap \mathcal{I}_B(\bar{\varepsilon})$ be in the set of parametrized family for $b_1 \in [0, 1]$. Suppose that $b_1 b_2 = b$ where b is the average Jacobian and b_2 is a fixed number. Then for some $\bar{b}_1 > 0$, the set of parameter values, a interval $[0, \bar{b}_1]$ on which F_{b_1} has no bounded geometry of $\mathcal{O}_{F_{b_1}}$ contains a dense G_δ set.*

Proof. Let us choose the two points as follows.

$$w_1 = (x_1, y_1, z_1) \in B^1_v(R^n F) \cap \mathcal{O}_{R^n F}, \quad w_2 = (x_2, y_2, z_2) \in B^1_c(R^n F) \cap \mathcal{O}_{R^n F}$$

By the choice of above two points, distances between each coordinates of w_1 and w_2 are as follows.

$$|x_1 - x_2| \asymp 1, \quad |y_1 - y_2| \asymp 1, \quad |z_1 - z_2| = O(\bar{\varepsilon}^{2^n})$$

Let $\dot{w}_j = (\dot{x}_j, \dot{y}_j, \dot{z}_j)$ be $\Psi^n_k(w_j)$ for $j = 1, 2$. By (14.2.1), we see that

$$\begin{aligned} \dot{x}_1 - \dot{x}_2 = \ & \alpha_{n,k}\Big[\big(x_1 + S^n_k(w_1)\big) - \big(x_2 + S^n_k(w_2)\big)\Big] \\ & + \sigma_{n,k}\left[\, t_{n,k}(y_1 - y_2) + u_{n,k}\big\{z_1 - z_2 + R^n_k(y_1) - R^n_k(y_2)\big\}\right]. \end{aligned} \tag{14.2.9}$$

Recall that $\alpha_{n,k} = \sigma^{2(n-k)}(1 + O(\rho^k))$, $\sigma_{n,k} = (-\sigma)^{n-k}(1 + O(\rho^k))$ and $x + S^n_k(w) = v_*(x) + O(\bar{\varepsilon}^{2^k} + \rho^n)$. Since v_* is a diffeomorphism and $|x_1 - x_2| \asymp 1$, then $|v_*(x_1) - v_*(x_2)| \asymp 1$ by the mean value theorem.

Moreover, Proposition 14.1.1 implies the following.

$$b_1^{2^k} \asymp t_{n,k}(y_1 - y_2) + u_{n,k}\left\{z_1 - z_2 + R^n_k(y_1) - R^n_k(y_2)\right\}$$

Then we express the equation (14.2.9) as follows.

$$\dot{x}_1 - \dot{x}_2 = \sigma^{2(n-k)}\big(v_*(x_1) - v_*(x_2)\big) \cdot \left[\, 1 + r_{n,k}\, b_1^{2^k}(-\sigma)^{-(n-k)}\right](1 + O(\rho^k))$$

Then $r_{n,k}$ depends uniformly on b_1. Let $r \leq r_{n,k} \leq \frac{1}{r}$. Let us take any number b_1^- in the parameter space $(0, \bar{b}_1)$ and any natural number $k \geq N$ for some big enough N.

Then we can find the biggest number n such that $n - k$ is odd and $\sigma^{n-k} > \dfrac{1}{r}\,(b_1^-)^{2^k}$, that is,

$$1 + r_{n,k} \cdot (b_1^-)^{2^k}(-\sigma)^{-(n-k)} \geq 1 + \frac{1}{r}\,(b_1^-)^{2^k}(-\sigma)^{-(n-k)} > 0$$

Let us increase the parameter from b_1^- to b_1^+ such that $(b_1^+)^{2^k} = \dfrac{2}{r}\, \sigma^{(n-k)}$. Then

$$1 + r_{n,\,k} \cdot (b_1^+)^{2^k} (-\sigma)^{-(n-k)} \leq 1 + r \cdot \frac{2}{r}\,(-1) = -1 < 0$$

Then there exists $b_1 \in (b_1^-, b_1^+)$ such that $\dot{x}_1 - \dot{x}_2 = 0$, that is, $\Psi^n_k(B^1_v(R^nF))$ and $\Psi^n_k(B^1_c(R^nF))$ overlaps over the $x-$axis with respect to $\dot{w}_1$ and $\dot{w}_2$. Moreover, $b_1^{2^k} \asymp \sigma^{n-k}$. For all big enough k, $b_1 \asymp b_1^-$. Thus $\log(b_1/b_1^-) = O(2^{-k})$. Then b_1 converges to b_1^- as $k \to \infty$. Then we obtain the dense subset of the parameter, $(0, \bar{b}_1)$ on which $\Psi^n_k(B^1_v(R^nF))$ and $\Psi^n_k(B^1_c(R^nF))$ overlaps over the $x-$axis. Moreover, there exists open subset, J_m of parameter $(0, \bar{b}_1)$ for each fixed level $k \geq m$. Then $\cap_m J_m$ is a G_δ subset of $(0, \bar{b}_1)$.

Let us compare the distance of two adjacent boxes and the diameter of the box for every big $k < n$. Let us take n such that $\sigma^{n-k} \asymp b_1^{2^k}$. We may assume that $B^{n-k}_{\mathbf{v}v}(R^kF)$ overlaps $B^{n-k}_{\mathbf{v}c}(R^kF)$ on the $x-$axis where $\mathbf{v} = v^{n-k-1} \in W^{n-k-1}$. By Lemma 14.2.1 and Lemma 14.2.2,

$$\begin{aligned} \operatorname{diam}(B^n_{\mathbf{w}v}) &\geq C_0\, \sigma^k \sigma^{2(n-k)} \\ \operatorname{dist}_{\min}(B^n_{\mathbf{w}v}, B^n_{\mathbf{w}c}) &\leq C_1 \big[\, \sigma^{2k}\sigma^{n-k} b_1^{2^k} + \sigma^{2k}\sigma^{2(n-k)} b_2^{2^k}\, \big] \end{aligned}$$

where $\mathbf{w} = v^k c\, v^{n-k-1} \in W^n$ for some positive C_0 and C_1.

Moreover, by Proposition 14.1.1, the condition of the overlapping of two adjacent boxes, $B^{n-k}_{\mathbf{v}v}(R^kF)$ and $B^{n-k}_{\mathbf{v}c}(R^kF)$ on the $x-$axis implies that

$$\sigma^{n-k} \asymp b_1^{2^k}$$

Hence,

$$\operatorname{dist}_{\min}(B^n_{\mathbf{w}v}, B^n_{\mathbf{w}c}) \leq C_2\, \sigma^k \operatorname{diam}(B^n_{\mathbf{w}v})$$

for every sufficiently large $k \in \mathbb{N}$. Then the critical Cantor set has the unbounded geometry. $\square$

Chapter 15

Non rigidity on the critical Cantor set

15.1 Distance between two points

Let us estimate the lower bounds of the distance. The diameter of the box $B^n_{\mathbf{w}}$ and $B^n_{\mathbf{w}v}$ has same bounds up to the constant $-\sigma_{n+1,\,n} = \sigma(1 + O(\rho^n))$. Then we obtain the following lemma similar to Lemma 14.2.1

Lemma 15.1.1. *Let $F \in \mathcal{N} \cap \mathcal{I}_B(\bar{\varepsilon})$. Then*

$$\operatorname{diam}(B^n_{\mathbf{w}}) \geq |\, C_1\, \sigma^k \sigma^{2(n-k)} - C_2\, \sigma^k \sigma^{n-k} b_1^{2^k}\, |$$

where $\mathbf{w} = v^k c\, v^{n-k-1} \in W^n$ for some positive C_1 and C_2.

Proof. See the proof of Lemma 14.2.1. □

Let us estimate the upper bound of the distance. The estimation does not contain the assumption of horizontal overlapping of some two points. Then the distance of all general two points has the larger upper bound than distance with horizontal overlapping.

Lemma 15.1.2. *Let $F \in \mathcal{N} \cap \mathcal{I}_B(\bar{\varepsilon})$. Then*

$$\operatorname{diam}(B^n_{\mathbf{w}}) \leq C \left[\, \sigma^k \sigma^{2(n-k)} + \sigma^k \sigma^{n-k} b_1^{2^k}\, \right]$$

where $\mathbf{w} = v^k c\, v^{n-k-1} \in W^n$ for some $C > 0$.

Proof. Recall the map Ψ^n_k from $B(R^n F)$ to $B^{n-k}_{\mathbf{v}}(R^k F)$.

$$\Psi^n_k(w) = \begin{pmatrix} 1 & t_{n,k} & u_{n,k} \\ & 1 & \\ & d_{n,k} & 1 \end{pmatrix} \begin{pmatrix} \alpha_{n,k} & & \\ & \sigma_{n,k} & \\ & & \sigma_{n,k} \end{pmatrix} \begin{pmatrix} x + S^n_k(w) \\ y \\ z + R^n_k(y) \end{pmatrix}$$

where $\mathbf{v} = v^{n-k} \in W^{n-k}$. Let us choose the two points

$$w_1 = (x_1, y_1, z_1) \in B^1_v(R^nF) \cap \mathcal{O}_{R^nF}, \quad w_2 = (x_2, y_2, z_2) \in B^1_c(R^nF) \cap \mathcal{O}_{R^nF}.$$

Recall $\dot{w}_j = \Psi^n_k(w_j)$, $\ddot{w}_j = F_k(\dot{w}_j)$ and $\dddot{w}_j = \Psi^k_0(\ddot{w}_j)$ for $j = 1, 2$. Let us express distances between each coordinates of $\dot{w}_1$ and $\dot{w}_2$, $\ddot{w}_1$ and $\ddot{w}_2$ and between $\dddot{w}_1$ and $\dddot{w}_2$. Observe that $|x_1 - x_2|$ and $|y_1 - y_2|$ is $O(1)$. Moreover, we may assume that $|z_1 - z_2| = O(\bar{\varepsilon}^{2^n})$ because $\mathcal{O}_{R^nF}$ is a completely invariant set under R^nF.

By the equation (14.2.2), we have the following expressions.

$$\begin{aligned}
\dot{x}_1 - \dot{x}_2 = \ & \alpha_{n,k}\Big[\big(x_1 + S^n_k(w_1)\big) - \big(x_2 + S^n_k(w_2)\big)\Big] \\
& + \sigma_{n,k}\Big[\ t_{n,k}(y_1 - y_2) + u_{n,k}\big\{z_1 - z_2 + R^n_k(y_1) - R^n_k(y_2)\big\}\Big] \\
= \ & \alpha_{n,k}\Big[\ v'_*(\bar{x}) + O(\bar{\varepsilon}^{2^k} + \rho^{n-k})\Big](x_1 - x_2) \\
& + \sigma_{n,k}\Big[\ t_{n,k}(y_1 - y_2) + u_{n,k}\big\{z_1 - z_2 + R^n_k(y_1) - R^n_k(y_2)\big\}\Big] \\
\leq \ & C\big[\,\sigma^{2(n-k)} + \sigma^{n-k} b_1^{2^k}\,\big]
\end{aligned} \tag{15.1.1}$$

for some $C > 0$.

$$\begin{aligned}
\dot{y}_1 - \dot{y}_2 = \ & \sigma_{n,k}(y_1 - y_2) \\
\dot{z}_1 - \dot{z}_2 = \ & \sigma_{n,k}\big[\,d_{n,k}(y_1 - y_2) + z_1 - z_2 + R^n_k(y_1) - R^n_k(y_2)\,\big]
\end{aligned}$$

Moreover, by the equation (14.2.3), we estimate the distance between each coordinates of $F_k(\dot{w}_1)$ and $F_k(\dot{w}_2)$ as follows.

$$\begin{aligned}
\ddot{x}_1 - \ddot{x}_2 = \ & f(\dot{x}_1) - \varepsilon_k(\dot{w}_1) - [f(\dot{x}_2) - \varepsilon_k(\dot{w}_2)] \\
= \ & f'(\bar{x})(\dot{x}_1 - \dot{x}_2) - \varepsilon_k(\dot{w}_1) + \varepsilon_k(\dot{w}_2) \\
= \ & [\,f'(\bar{x}) - \partial_x\varepsilon_k(\eta)\,](\dot{x}_1 - \dot{x}_2) - \partial_y\varepsilon_k(\eta)(\dot{y}_1 - \dot{y}_2) - \partial_z\varepsilon_k(\eta)(\dot{z}_1 - \dot{z}_2) \\
= \ & [\,f'(\bar{x}) - \partial_x\varepsilon_k(\eta)\,](\dot{x}_1 - \dot{x}_2) - \partial_y\varepsilon_k(\eta)\cdot\sigma_{n,k}(y_1 - y_2) \\
& - \partial_z\varepsilon_k(\eta)\cdot\sigma_{n,k}\big[\,d_{n,k}(y_1 - y_2) + z_1 - z_2 + R^n_k(y_1) - R^n_k(y_2)\,\big]
\end{aligned}$$

$$\ddot{y}_1 - \ddot{y}_2 = \ \dot{x}_1 - \dot{x}_2$$

$$
\begin{aligned}
\ddot{z}_1 - \ddot{z}_2 = \;& \delta_k(\dot{w}_1) - \delta_k(\dot{w}_2) \\
= \;& \partial_x \delta_k(\zeta) \cdot (\dot{x}_1 - \dot{x}_2) + \partial_y \delta_k(\zeta) \cdot (\dot{y}_1 - \dot{y}_2) + \partial_z \delta_k(\zeta) \cdot (\dot{z}_1 - \dot{z}_2) \\
= \;& \partial_x \delta_k(\zeta) \cdot (\dot{x}_1 - \dot{x}_2) + \partial_y \delta_k(\zeta) \cdot \sigma_{n,\,k}(y_1 - y_2) \\
& \quad + \partial_z \delta_k(\zeta) \cdot \sigma_{n,\,k} \big[\, d_{n,\,k}(y_1 - y_2) + z_1 - z_2 + R^n_k(y_1) - R^n_k(y_2) \,\big]
\end{aligned}
$$

where η and ζ are some points in the line segment between $\dot{w}_1$ and $\dot{w}_2$ in $\Psi^n_k(B)$. Recall the coordinate change map Ψ^k_0. Then the difference of each coordinates of $\Psi^k_0(\ddot{w}_1)$ and $\Psi^k_0(\ddot{w}_2)$ as follows.

$$
\begin{aligned}
\dddot{x}_1 - \dddot{x}_2 = \;& \pi_x \circ \Psi^k_0(\ddot{w}_1) - \pi_x \circ \Psi^k_0(\ddot{w}_2) \\
= \;& \alpha_{k,0} \big[\, (\ddot{x}_1 + S^k_0(\ddot{w}_1)) - (\ddot{x}_2 + S^k_0(\ddot{w}_2)) \,\big] \\
& \quad + \sigma_{k,\,0} \, \big[\, t_{k,\,0}\, (\ddot{y}_1 - \ddot{y}_2) + u_{k,\,0} \big(\ddot{z}_1 - \ddot{z}_2 + R^k_0(\ddot{y}_1) - R^k_0(\ddot{y}_2) \big) \,\big] \\
= \;& \alpha_{k,0} \big[\, v'_*(\bar{x}) + O(\bar{\varepsilon} + \rho^k) \,\big] (\ddot{x}_1 - \ddot{x}_2) + \sigma_{k,\,0} \cdot u_{k,\,0}\, (\ddot{z}_1 - \ddot{z}_2) \\
& \quad + \sigma_{k,\,0} \, \big[\, t_{k,\,0}\, (\ddot{y}_1 - \ddot{y}_2) + u_{k,\,0} \big(R^k_0(\ddot{y}_1) - R^k_0(\ddot{y}_2) \big) \,\big]
\end{aligned}
\tag{15.1.2}
$$

$$
\begin{aligned}
\dddot{y}_1 - \dddot{y}_2 = \;& \sigma_{k,\,0}\, (\ddot{y}_1 - \ddot{y}_2) = \sigma_{k,\,0}\, (\dot{x}_1 - \dot{x}_2) \\
\dddot{z}_1 - \dddot{z}_2 = \;& \pi_z \circ \Psi^k_0(\ddot{w}_1) - \pi_z \circ \Psi^k_0(\ddot{w}_2) \\
= \;& \sigma_{k,\,0}\, (\ddot{z}_1 - \ddot{z}_2) + \sigma_{k,\,0} \big[\, d_{k,0}(\ddot{y}_1 - \ddot{y}_2) + R^n_k(\ddot{y}_1) - R^n_k(\ddot{y}_2) \,\big]
\end{aligned}
\tag{15.1.3}
$$

Let us calculate a upper bound of the distance, $|\,\dddot{w}_1 - \dddot{w}_2|$.

$$
\begin{aligned}
& |\,\dddot{w}_1 - \dddot{w}_2| \\
\leq \;& |\,\dddot{x}_1 - \dddot{x}_2| + |\,\dddot{y}_1 - \dddot{y}_2| + |\,\dddot{z}_1 - \dddot{z}_2| \\
\leq \;& \Big|\, \alpha_{k,0} \big[\, v'_*(\bar{x}) + O(\bar{\varepsilon} + \rho^k) \,\big] (\ddot{x}_1 - \ddot{x}_2) + \sigma_{k,\,0} \cdot u_{k,\,0}\, (\ddot{z}_1 - \ddot{z}_2) \\
& \quad + \sigma_{k,\,0} \, \big[\, t_{k,\,0}\, (\ddot{y}_1 - \ddot{y}_2) + u_{k,\,0} \big(R^k_0(\dot{x}_1) - R^k_0(\ddot{y}_2) \big) \,\big] \,\Big| + \Big|\, \sigma_{k,\,0}\, (\ddot{y}_1 - \ddot{y}_2) \,\Big| \\
& \quad + \Big|\, \sigma_{k,\,0}\, (\ddot{z}_1 - \ddot{z}_2) + \sigma_{k,\,0} \big[\, d_{k,0}(\ddot{y}_1 - \ddot{y}_2) + R^n_k(\ddot{y}_1) - R^n_k(\ddot{y}_2) \,\big] \,\Big| \\
\leq \;& \big|\, \alpha_{k,0} \big[\, v'_*(\bar{x}) + O(\bar{\varepsilon} + \rho^k) \,\big] (\ddot{x}_1 - \ddot{x}_2) \,\big| + \big|\, \sigma_{k,\,0} \, [\, 1 + t_{k,\,0} + d_{k,\,0} \,]\, (\ddot{y}_1 - \ddot{y}_2) \,\big| \\
& \quad + \big|\, \sigma_{k,\,0} \, [\, 1 + u_{k,\,0} \,]\, (R^n_k(\ddot{y}_1) - R^n_k(\ddot{y}_2)) \,\big| + \big|\, \sigma_{k,\,0} \, [\, 1 + u_{k,\,0} \,]\, (\ddot{z}_1 - \ddot{z}_2) \,\big|
\end{aligned}
$$

$$
\begin{aligned}
&\leq\; C_1\,\sigma^{2k}\big|\,[\,f'(\bar{x})-\partial_x\varepsilon_k(\eta)\,](\dot{x}_1-\dot{x}_2)-\partial_y\varepsilon_k(\eta)\cdot\sigma_{n,\,k}(y_1-y_2)\\
&\qquad\quad -\;\partial_z\varepsilon_k(\eta)\cdot\sigma_{n,\,k}\big[\,d_{n,\,k}(y_1-y_2)+z_1-z_2+R^n_k(y_1)-R^n_k(y_2)\,\big]\,\big|\\
&\qquad +\;C_2\,\big|\,\sigma^k(\dot{x}_1-\dot{x}_2)\,\big|+C_3\,\big|\,\sigma^k\bar{\varepsilon}^{2^k}(\dot{x}_1-\dot{x}_2)\,\big|\\
&\qquad +\;C_4\,\sigma^k\big|\,\partial_x\delta_k(\zeta)\cdot(\dot{x}_1-\dot{x}_2)+\partial_y\delta_k(\zeta)\cdot\sigma_{n,\,k}(y_1-y_2)\\
&\qquad\quad +\;\partial_z\delta_k(\zeta)\cdot\sigma_{n,\,k}\big[\,d_{n,\,k}(y_1-y_2)+z_1-z_2+R^n_k(y_1)-R^n_k(y_2)\,\big]\,\big|\\
&\leq\; C_5\,\sigma^k|\,\dot{x}_1-\dot{x}_2|\\
&\qquad +\;C_6\,\sigma^{2k}\sigma^{n-k}\bigg|\,\partial_y\varepsilon_k(\eta)+\partial_z\varepsilon_k(\eta)\left[\,d_{n,\,k}+\frac{R^n_k(y_1)-R^n_k(y_2)}{y_1-y_2}\right]\bigg|\\
&\qquad +\;C_7\,\sigma^k\sigma^{n-k}\bigg|\,\partial_y\delta_k(\zeta)+\partial_z\delta_k(\zeta)\left[\,d_{n,\,k}+\frac{R^n_k(y_1)-R^n_k(y_2)}{y_1-y_2}\right]\bigg|\\
&\qquad +\;C_8\left[\,\sigma^{2k}\sigma^{n-k}\|\partial_z\varepsilon_k\|+\sigma^k\sigma^{n-k}b_2^{2^k}\,\right]|\,z_1-z_2|\\
(*)\quad &\leq\; C_9\,\sigma^k\big[\,\sigma^{2(n-k)}+\sigma^{n-k}b_1^{2^k}\,\big]+C_{10}\,\sigma^{2k}\sigma^{n-k}b_1^{2^k}+C_{11}\,\sigma^k\sigma^{n-k}\sigma^n b_2^{2^k}\\
&\qquad +C_{12}\,\sigma^k\sigma^{n-k}\bar{\varepsilon}^{2^n}\\
&\leq\; C_{13}\,\sigma^k\big[\,\sigma^{2(n-k)}+\sigma^{n-k}b_1^{2^k}\,\big]
\end{aligned}
$$

for some positive C_j, $1\leq j\leq 13$ independent of k and n. The second last line, $(*)$ holds by the estimation of (15.1.1), Corollary 13.2.3 and Corollary 13.3.2 with Proposition A.0.3 and Proposition A.0.4.

□

15.2 Non rigidity on the Cantor set with b_1

Theorem 15.2.1. *Let F and $\widetilde{F}$ are in $\mathcal{N}\cap\mathcal{I}_B(\bar{\varepsilon})$. Moreover, let b_1 be the ratio of the average Jacobian and the asymptotic number $b_2\asymp\partial_z\delta$ of F. The number $\widetilde{b}_1$ is defined by the similar way. Suppose that $b_1>\widetilde{b}_1$. Let $\phi\colon\mathcal{O}_{\widetilde{F}}\to\mathcal{O}_F$ be a homeomorphism which conjugate $F_{\mathcal{O}_F}$ and $\widetilde{F}_{\mathcal{O}_{\widetilde{F}}}$ and $\phi(\tau_{\widetilde{F}})=\tau_F$. Then the Hölder exponent of ϕ is not greater than $\dfrac{1}{2}\left(1+\dfrac{\log b_1}{\log\widetilde{b}_1}\right)$.*

Proof. For sufficiently large $k\in\mathbb{N}$, let us choose n depending on k which satisfies the following inequality

$$\sigma^{n-k+1}\leq\widetilde{b}_1^{2^k}<\sigma^{n-k}$$

Observe that $b_1^{2^k}\gg\widetilde{b}_1^{2^k}$. By Lemma 15.1.1 and Lemma 15.1.2, we have the following inequalities

$$\begin{aligned}
\operatorname{dist}(\ddot{w}_1, \ddot{w}_2) &\leq C_0 \left[\, \sigma^k \sigma^{2(n-k)} + \sigma^k \sigma^{n-k} b_1^{2^k} \,\right] \leq C_1\, \sigma^k \widetilde{b}_1^{2^k} \widetilde{b}_1^{2^k} \\
\operatorname{dist}(\dddot{\widetilde{w}}_1, \dddot{\widetilde{w}}_2) &\geq |\, C_2\, \sigma^k \sigma^{2(n-k)} - C_3\, \sigma^k \sigma^{n-k} \widetilde{b}_1^{2^k} \,| \geq C_4\, \sigma^k \widetilde{b}_1^{2^k} b_1^{2^k}
\end{aligned}$$

for some positive C_j where $j = 0, 1, 2, 3$ and 4.
The Hölder continuous function h with the Hölder exponent α has to satisfy

$$\operatorname{dist}(\dddot{\widetilde{w}}_1, \dddot{\widetilde{w}}_2) \leq C \left(\operatorname{dist}(\ddot{w}_1, \ddot{w}_2) \right)^{\alpha}$$

for some $C > 0$. Then we see that

$$\sigma^k \widetilde{b}_1^{2^k} b_1^{2^k} \leq C \left(\sigma^k \widetilde{b}_1^{2^k} \widetilde{b}_1^{2^k} \right)^{\alpha}$$

Take the logarithm both sides and divide them by 2^k. Passing the limit and after that divide both sides by the negative number, $2 \log \widetilde{b}_1$. Then the desired upper bound of the Hölder exponent is obtained.

$$\begin{aligned}
k \log \sigma + 2^k \log \widetilde{b}_1 + 2^k \log b_1 &\leq \log C + \alpha \big(k \log \sigma + 2^k \log \widetilde{b}_1 + 2^k \log \widetilde{b}_1 \big) \\
\frac{k}{2^k} \log \sigma + \log \widetilde{b}_1 + \log b_1 &\leq \frac{1}{2^k} \log C + \alpha \left(\frac{k}{2^k} \log \sigma + \log \widetilde{b}_1 + \log \widetilde{b}_1 \right) \\
\log \widetilde{b}_1 + \log b_1 &\leq \alpha \cdot 2 \log \widetilde{b}_1 \\
\alpha &\leq \frac{1}{2} \left(1 + \frac{\log b_1}{\log \widetilde{b}_1} \right)
\end{aligned}$$

□

The average Jacobian of the map in $\mathcal{N} \cap \mathcal{I}_B(\bar{\varepsilon})$ less affects the non rigidity than the number b_1. In the two dimensional Hénon-like map theory, when two average Jacobian of F, say b and $\widetilde{F}$, say $\widetilde{b}$ are same, the best possible regularity of homeomorphic conjugation between two critical Cantor set is unknown. However, the upper bound of Hölder exponent is worse than the two dimensional in general.
Let us consider a map in $\mathcal{I}_B(\bar{\varepsilon})$ as follows.

$$F(w) = (f(x) - \varepsilon(x, y),\, x,\, \delta(z))$$

We call the set of the map which is of the above form *trivial extension* of two dimensional Hénon-like maps. Let us denote this set to be $\mathcal{T}$. It seems to be worth notifying that $\mathcal{T} \cap \mathcal{I}_B(\bar{\varepsilon}) \subset \mathcal{N} \cap \mathcal{I}_B(\bar{\varepsilon})$ and $\mathcal{T} \cap \mathcal{I}_B(\bar{\varepsilon})$ a space which is invariant under renormalization. $\mathcal{T}$ is also contained in the set of model maps. Then if $F \in \mathcal{T} \cap \mathcal{I}_B(\bar{\varepsilon})$, then the n^{th} renormalized map of F, $F_n \equiv R^n F$ is of the following form

$$F_n(w) = (f_n(x) - a(x)b_1^{2^n} y\,(1 + O(\rho^n)),\, x,\, b_2^{2^n} z\,(1 + O(\rho^n)))$$

where b_1 is the average Jacobian of two dimensional map, $\pi_{xy} \circ F$ and $b_2 = b/b_1$ for some $0 < \rho < 1$. Let $\widetilde{F}$ be another map in $\mathcal{T} \cap \mathcal{I}_B(\bar{\varepsilon})$ with the corresponding numbers $\widetilde{b}_1$, $\widetilde{b}$ and $\widetilde{b}_2$. By Theorem 15.2.1 , if $b_1 > \widetilde{b}_1$, the upper bound of Hölder exponent is

$$\frac{1}{2}\left(1 + \frac{\log b_1}{\log \widetilde{b}_1}\right)$$

Let δ and $\widetilde{\delta}$ be the third coordinate map of F and $\widetilde{F}$ respectively. Since b_1 and b_2 are completely independent of each other for every map $F \in \mathcal{T} \cap \mathcal{I}_B(\bar{\varepsilon})$ and b_2 is the contracting rate on the third coordinate, the different b_2 from $\widetilde{b}_2$ may require the less regularity of the homeomorphic conjugacy between critical Cantor sets of F and $\widetilde{F}$. For example, let us assume that

$$b_1 b_2 = b = \widetilde{b} = \widetilde{b}_1 \widetilde{b}_2$$

with the condition $b_1 > \widetilde{b}_1$. It implies that $b_2 < \widetilde{b}_2$. Then Theorem 15.2.1 holds and the conjugacy between third coordinate map is also Hölder map because δ and $\widetilde{\delta}$ is asymptotically linear map with different contracting rates. Then the upper bound of Hölder exponent in Theorem 15.2.1 might not be even sharp even though the average Jacobian of F and $\widetilde{F}$ are same. Then the average Jacobian of the three dimensional Hénon-like map in $\mathcal{N} \cap \mathcal{I}_B(\bar{\varepsilon})$ is not corresponding invariant with the average Jacobian of two dimensional Hénon-like map in the sense of the critical Cantor set geometry.

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Appendix A

Recursive formula of Ψ_k^n

Proposition A.0.2. *Let $F \in \mathcal{I}_B(\bar{\varepsilon})$ and denote k^{th} and n^{th} renormalized map of F to be F_k and F_n respectively. The derivative of the non-linear conjugation at the tip, τ_{F_k} between $F_k^{2^{n-k}}$ and F_n from domain of the n^{th} level, $B(F_n)$ to the k^{th} level, $B(F_k)$ is called D_k^n. The expression of D_k^n is as follows*

$$D_k^n = \begin{pmatrix} \alpha_{n,\,k} & \sigma_{n,\,k}\,t_{n,\,k} & \sigma_{n,\,k}\,u_{n,\,k} \\ & \sigma_{n,\,k} & \\ & \sigma_{n,\,k}\,d_{n,\,k} & \sigma_{n,\,k} \end{pmatrix}$$

where $\sigma_{n,\,k}$ and $\alpha_{n,\,k}$ are linear scaling factors such that $\sigma_{n,\,k} = (-\sigma)^{n-k}(1 + O(\rho^k))$ and $\alpha_{n,\,k} = \sigma^{2(n-k)}(1 + O(\rho^k))$. Then

$$d_{n,\,k} = \sum_{i=k}^{n-1} d_{i+1,\,i}\,, \quad u_{n,\,k} = \sum_{i=k}^{n-1} \sigma^{i-k}\,u_{i+1,\,i}\,(1 + O(\rho^k))$$

$$t_{n,\,k} = \sum_{i=k}^{n-1} \sigma^{i-k}\left[\,t_{i+1,\,i} + u_{i+1,\,i}\,d_{n,\,i+1}\right](1 + O(\rho^k))$$

$$t_{n,\,k} - u_{n,\,k}\,d_{n,\,k} = \sum_{i=k}^{n-1} \sigma^{i-k}\left[\,t_{i+1,\,i} - u_{i+1,\,i}\,d_{i+1,\,k}\right](1 + O(\rho^k))$$

where $\sigma^{i-k}(1 + O(\rho^k)) = \displaystyle\prod_{j=k}^{i-1} \frac{\alpha_{j+1,\,j}}{\sigma_{j+1,\,j}}$. Moreover, $d_{n,\,k}$, $u_{n,\,k}$ and $t_{n,\,k}$ are convergent as $n \to \infty$ super exponentially fast.

Proof. $D_k^n = D_k^m \cdot D_m^n$ for any m between k and n because the image of the tip under $\Psi_k^n(\tau_{F_n})$ is the tip of k^{th} level. By the direct calculation,

$$
D_k^m \cdot D_m^n
= \begin{pmatrix} \alpha_{n,k} & L & \boxed{\alpha_{m,k}\sigma_{n,m}\, u_{n,m} + \sigma_{n,k}\, u_{m,k}} \\ & \sigma_{n,k} & \\ & \boxed{\sigma_{n,k}\, d_{m,k} + \sigma_{n,k}\, d_{n,m}} & \sigma_{n,k} \end{pmatrix}
$$

where $L = \alpha_{m,k}\,\sigma_{n,m}\,t_{n,m} + \sigma_{n,k}\,t_{m,k} + \sigma_{n,k}\,u_{m,k}\,d_{n,m}$. Then

$$
\begin{aligned}
\sigma_{n,k}\,t_{n,k} &= \alpha_{m,k}\,\sigma_{n,m}\,t_{n,m} + \sigma_{n,k}\,t_{m,k} + \sigma_{n,k}\,u_{m,k}\,d_{n,m} \\
\sigma_{n,k}\,u_{n,k} &= \alpha_{m,k}\,\sigma_{n,m}\,u_{n,m} + \sigma_{n,k}\,u_{m,k} \\
\sigma_{n,k}\,d_{n,k} &= \sigma_{n,k}\,d_{m,k} + \sigma_{n,k}\,d_{n,m}
\end{aligned}
$$

for any m between k and n. Recall that $\sigma_{n,k} = \sigma_{n,m} \cdot \sigma_{m,k}$ and $\alpha_{n,k} = \alpha_{n,m} \cdot \alpha_{m,k}$. Let m be $k+1$. Then

$$
\begin{aligned}
d_{n,k} &= d_{n,k+1} + d_{k+1,k} \\
&= d_{n,k+2} + d_{k+2,k+1} + d_{k+1,k} \\
&\quad\vdots \\
&= d_{n,n-1} + \cdots + d_{k+2,k+1} + d_{k+1,k} \\
&= \sum_{i=k}^{n-1} d_{i+1,i}
\end{aligned}
\tag{A.0.1}
$$

Moreover, the absolute value each term is super exponentially small. More precisely, each term is bounded by ε^{2^i} for each i, that is, $|\,d_{i+1,i}| \asymp |\,q_i(\pi_y(\tau_{i+1})| \leq \|D\delta_i\| = O(\bar{\varepsilon}^{2^i})$. Then $d_{n,k}$ converges to a number, say $d_{*,k}$ super exponentially fast.

Let us see the recursive formula of $u_{n,\,k}$.

$$
\begin{aligned}
u_{n,\,k} &= \frac{\alpha_{k+1,\,k}}{\sigma_{k+1,\,k}} u_{n,\,k+1} + u_{k+1,\,k} \\
&= \frac{\alpha_{k+1,\,k}}{\sigma_{k+1,\,k}} \left[\frac{\alpha_{k+2,\,k+1}}{\sigma_{k+2,\,k+1}} u_{n,\,k+2} + u_{k+2,\,k+1} \right] + u_{k+1,\,k} \\
&\quad\vdots \\
&= \sum_{i=k+1}^{n-1} \prod_{j=k}^{i-1} \frac{\alpha_{j+1,\,j}}{\sigma_{j+1,\,j}}\ u_{i+1,\,i} + u_{k+1,\,k} \\
&= \sum_{i=k}^{n-1} \sigma^{i-k} u_{i+1,\,i}\,(1 + O(\rho^k))
\end{aligned}
\tag{A.0.2}
$$

Moreover, $u_{i+1,\,i} \asymp \partial_z \varepsilon_i(\tau_{F_{i+1}})$. Then $u_{n,\,k}$ converges to a number, say $u_{*,\,k}$ super exponentially fast by the similar reason for $d_{n,\,k}$.

Let us see the recursive formula of $t_{n,\,k}$.

$$
\begin{aligned}
t_{n,\,k} &= \frac{\alpha_{k+1,\,k}}{\sigma_{k+1,\,k}}\ t_{n,\,k+1} + t_{k+1,\,k} + u_{k+1,\,k}\, d_{n,\,k+1} \\
&= \frac{\alpha_{k+1,\,k}}{\sigma_{k+1,\,k}} \left[\frac{\alpha_{k+2,\,k+1}}{\sigma_{k+2,\,k+1}}\ t_{n,\,k+2} + t_{k+2,\,k+1} + u_{k+2,\,k+1}\, d_{n,\,k+2} \right] \\
&\qquad + t_{k+1,\,k} + u_{k+1,\,k}\, d_{n,\,k+1} \\
&\qquad \vdots \\
&= \sum_{i=k+1}^{n-1} \prod_{j=k}^{i-1} \frac{\alpha_{j+1,\,j}}{\sigma_{j+1,\,j}}\ t_{i+1,\,i} + t_{k+1,\,k} + \sum_{i=k+1}^{n-1} \prod_{j=k}^{i-1} \frac{\alpha_{j+1,\,j}}{\sigma_{j+1,\,j}}\ u_{i+1,\,i}\, d_{n,\,i+1} \\
&\qquad + u_{k+1,\,k}\, d_{n,\,k+1} \\
&= \sum_{i=k}^{n-1} \sigma^{i-k} \big[\, t_{i+1,\,i} + u_{i+1,\,i}\, d_{n,\,i+1} \big](1 + O(\rho^k))
\end{aligned}
\tag{A.0.3}
$$

Moreover, by the equations (A.0.1), (A.0.2) and (A.0.3), we obtain the recursive formula of $t_{n,\,k} - u_{n,\,k}\, d_{n,\,k}$ as follows.

$$
\begin{aligned}
& t_{n,\,k} - u_{n,\,k}\, d_{n,\,k} \\
= \ & \sum_{i=k+1}^{n-1} \prod_{j=k}^{i-1} \frac{\alpha_{j+1,\,j}}{\sigma_{j+1,\,j}} \big[\, t_{i+1,\,i} + u_{i+1,\,i}\, d_{n,\,i+1} \,\big] + t_{k+1,\,k} + u_{k+1,\,k}\, d_{n,\,k+1} \\
& - \left[\, \sum_{i=k+1}^{n-1} \prod_{j=k}^{i-1} \frac{\alpha_{j+1,\,j}}{\sigma_{j+1,\,j}}\ u_{i+1,\,i} + u_{k+1,\,k} \right] d_{n,\,k} \\
= \ & \sum_{i=k+1}^{n-1} \prod_{j=k}^{i-1} \frac{\alpha_{j+1,\,j}}{\sigma_{j+1,\,j}} \big[\, t_{i+1,\,i} + u_{i+1,\,i}\, d_{n,\,i+1} - u_{i+1,\,i} d_{n,\,k} \,\big] \\
& + t_{k+1,\,k} + u_{k+1,\,k}\, d_{n,\,k+1} - u_{k+1,\,k} d_{n,\,k} \\
= \ & \sum_{i=k+1}^{n-1} \prod_{j=k}^{i-1} \frac{\alpha_{j+1,\,j}}{\sigma_{j+1,\,j}} \big[\, t_{i+1,\,i} - u_{i+1,\,i} d_{i+1,\,k} \,\big] + t_{k+1,\,k} - u_{k+1,\,k} d_{k+1,\,k} \\
= \ & \sum_{i=k}^{n-1} \sigma^{i-k} \big[\, t_{i+1,\,i} - u_{i+1,\,i}\, d_{i+1,\,k} \,\big] (1 + O(\rho^k))
\end{aligned}
$$

Recall the expression of the derivative of the coordinate change map at the tip on each level.

$$
\begin{aligned}
\sigma_k \cdot DH_k(\tau_{F_k}) &= (D^{k+1}_k)^{-1} \\
&= \begin{pmatrix} (\alpha_k)^{-1} & & \\ & (\sigma_k)^{-1} & \\ & & (\sigma_k)^{-1} \end{pmatrix} \cdot \begin{pmatrix} 1 & -t_k + u_k\, d_k & -u_k \\ & 1 & \\ & -d_k & 1 \end{pmatrix}
\end{aligned}
$$

Since $H_k(w) = (f_k(x) - \varepsilon_k(w),\, y,\, z - \delta_k(y, f_k^{-1}(y), 0))$, we see that $\partial_y \varepsilon_k(\tau_{F_k}) \asymp -t_k + u_k\, d_k$ for every $k \in \mathbb{N}$. Moreover, the fact that $t_{i+1,\,i} - u_{i+1,\,i}\, d_{i+1,\,i} \asymp \partial_y \varepsilon_i(\tau_{F_{i+1}})$ and $|\, u_{i+1,\,i}\, d_{n,\,i}|$ is super exponentially small for each $i < n$ implies that $t_{n,\,k}$ converges to a number, say $t_{*,\,k}$ super exponentially fast.

$\square$

Recall the expression of the map Ψ^n_k from $B(R^n F)$ to $B^{n-k}_{\mathbf{v}}(R^k F)$.

$$
\Psi^n_k(w) = \begin{pmatrix} 1 & t_{n,\,k} & u_{n,\,k} \\ & 1 & \\ & d_{n,\,k} & 1 \end{pmatrix} \begin{pmatrix} \alpha_{n,\,k} & & \\ & \sigma_{n,\,k} & \\ & & \sigma_{n,\,k} \end{pmatrix} \begin{pmatrix} x + S^n_k(w) \\ y \\ z + R^n_k(y) \end{pmatrix}.
$$

where $\mathbf{v} = v^{n-k} \in W^{n-k}$.

Proposition A.0.3. *Let $F \in \mathcal{I}_B(\bar{\varepsilon})$ and Ψ^n_k be the map from $B(R^nF)$ to $B^(R^kF)$ as the conjugation between $(R^kF)^{2^{n-k}}$ and R^nF. Moreover, $R^n_k(y)$ be the non linear part of $\pi_z \circ \Psi^n_k$ depending on the second variable y. Then both $R^n_k(y)$ and $(R^n_k)'(y)$ converges to zero exponentially fast as $n \to \infty$.*

Proof. Let $w = (x, y, z)$ be the point in $B(R^nF)$ and let $\Psi^n_{n-1}(w)$ be $w' = (x', y', z')$. Recall $\Psi^n_k = \Psi^n_{n-1} \circ \Psi^{n-1}_k$. Thus

$$\begin{aligned} z' = &\ \pi_z \circ \Psi^n_{n-1}(w) = \sigma_{n,\,n-1} \left[\, d_{n,\,n-1} y + z + R^n_{n-1}(y) \, \right] \\ y' = &\ \pi_y \circ \Psi^n_{n-1}(w) = \sigma_{n,\,n-1} \, y \end{aligned}$$

Then by the similar calculation and the composition of Ψ^{n-1}_k and Ψ^n_{n-1}, we obtain the recursive formula of $\pi_z \circ \Psi^n_k$ as follows.

$$\begin{aligned} &\ \pi_z \circ \Psi^n_k(w) = \sigma_{n,\,k} \left[\, d_{n,\,k} \, y + z + R^n_k(y) \, \right] \\ = &\ \pi_z \circ \Psi^{n-1}_k(w') = \sigma_{n-1,\,k} \left[\, d_{n-1,\,k} \, y' + z' + R^{n-1}_k(y') \, \right] \\ = &\ \sigma_{n-1,\,k} \left[\, d_{n-1,\,k} \, \sigma_{n,\,n-1} \, y + \sigma_{n,\,n-1} \left[\, d_{n,\,n-1} y + z + R^n_{n-1}(y) \, \right] \right. \\ &\ \left. + R^{n-1}_k(\sigma_{n,\,n-1} \, y) \, \right] \\ = &\ \sigma_{n,\,k} \, (d_{n-1,\,k} + d_{n,\,n-1}) + \sigma_{n,\,k} \, z + \sigma_{n,\,k} \, R^n_{n-1}(y) + \sigma_{n-1,\,k} \, R^{n-1}_k(\sigma_{n,\,n-1} \, y) \end{aligned} \tag{A.0.4}$$

By Proposition A.0.2, $d_{n,\,k} = d_{n-1,\,k} + d_{n,\,n-1}$. Let us compare the left side of (A.0.4) with the right side of it. Recall the equation $\sigma_{n,\,k} = \sigma_{n,\,n-1} \cdot \sigma_{n-1,\,k}$. Then

$$R^n_k(y) = R^n_{n-1}(y) + \frac{1}{\sigma_{n,\,n-1}} \; R^{n-1}_k(\sigma_{n,\,n-1} \, y)$$

Each $R^i_j(y)$ is the sum of second and higher order terms of $\pi_z \circ \Psi^i_j$ for $i > j$. Thus

$$R^n_k(y) = a_{n,\,k} \, y^2 + A_{n,\,k}(y) y^3$$

Moreover, $\| \, R^n_{n-1} \| = O(\bar{\varepsilon}^{2^{n-1}})$ because $R^n_{n-1}(y)$ is the second and higher order terms of the map $\delta_{n-1}(\sigma_{n,\,n-1} \, y, \, f^{-1}_{n-1}(\sigma_{n,\,n-1} \, y), \, 0)$. Then

$$R^n_k(y) = \frac{1}{\sigma_{n,\,n-1}} \; R^{n-1}_k(\sigma_{n,\,n-1} \, y) + c_{n,\,k} \, y^2 + O(\bar{\varepsilon}^{2^{n-1}} y^3)$$

where $c_{n,k} = O(\bar{\varepsilon}^{2^{n-1}})$. The recursive formula for $a_{n,k}$ and $A_{n,k}$ as follows.

$$R^n_k(y) = \frac{1}{\sigma_{n,\,n-1}} \Big(a_{n-1,\,k}(\sigma_{n,\,n-1}\,y)^2 + A_{n-1,\,k}(\sigma_{n,\,n-1}\,y)\cdot(\sigma_{n,\,n-1}\,y)^3 \Big) + O(\bar{\varepsilon}^{2^{n-1}} y^3)$$

Then $a_{n,k} = \sigma_{n,\,n-1}\,a_{n-1,\,k} + c_{n,\,k}$ and $\|A_{n,\,k}\| \leq \|\,\sigma_{n,\,n-1}\|^2 \|A_{n-1,\,k}\| + O(\bar{\varepsilon}^{2^{n-1}})$. Then for each fixed $k < n$, $a_{n,\,k} \to 0$ and $A_{n,\,k} \to 0$ exponentially fast as $n \to \infty$. $R^n_k(y)$ converges to zero as $n \to \infty$ exponentially fast.

Let us estimate $\|A'_{n,\,k}\|$ in order to measure how fast $(R^n_k)'(y)$ is convergent. By the similar method of the recursive formula of $R^n_k(y)$, we have the expression and recursive formula of $(R^n_k)'(y)$ as follows.

$$\begin{aligned} (R^n_k)'(y) &= \; 2\,a_{n,\,k}\,y + 3\,A_{n,\,k}(y)\,y^2 + A'_{n,\,k}(y)\,y^3 \\ \text{Moreover,} \quad (R^n_k)'(y) &= \; (R^n_{n-1})'(y) + R^{n-1}_k(\sigma_{n,\,n-1}\,y) \\ &= \; R^{n-1}_k(\sigma_{n,\,n-1}\,y) + 2\,c_{n,\,k}\,y + O(\bar{\varepsilon}^{2^{n-1}} y^2) \end{aligned}$$

Then

$$\begin{aligned} (R^n_k)'(y) = \; & 2\,a_{n-1,\,k}\,\sigma_{n,\,n-1}\,y + 3\,A_{n-1,\,k}(\sigma_{n,\,n-1}\,y)(\sigma_{n,\,n-1}\,y)^2 \\ & + A'_{n,\,k}(\sigma_{n,\,n-1}\,y)\,(\sigma_{n,\,n-1}\,y)^3 + 2\,c_{n,\,k}\,y + O(\bar{\varepsilon}^{2^{n-1}} y^2) \end{aligned}$$

Let us compare quadratic and higher order terms of $(R^n_k)'(y)$.

$$\begin{aligned} 3\,A_{n,\,k}(y)\,y^2 + A'_{n,\,k}(y)\,y^3 = \; & 3\,A_{n-1,\,k}(\sigma_{n,\,n-1}\,y)(\sigma_{n,\,n-1}\,y)^2 \\ & + A'_{n,\,k}(\sigma_{n,\,n-1}\,y)\,(\sigma_{n,\,n-1}\,y)^3 + O(\bar{\varepsilon}^{2^{n-1}} y^2) \end{aligned}$$

Thus

$$\begin{aligned} A'_{n,\,k}(y)\,y = {} & A'_{n,\,k}(\sigma_{n,\,n-1}\,y)\,\sigma^3_{n,\,n-1} y - 3\,A_{n,\,k}(y) + 3\,A_{n-1,\,k}(\sigma_{n,\,n-1}\,y)\,\sigma^2_{n,\,n-1} \\ & + O(\bar{\varepsilon}^{2^{n-1}}) \end{aligned}$$

Then

$$\begin{aligned} \|A'_{n,\,k}\| \leq \; & \|A'_{n-1,\,k}\| \|\sigma_{n,\,n-1}\|^3 + 3\|A_{n,\,k}\| + 3\|\,A_{n-1,\,k}\| \|\,\sigma_{n,\,n-1}\|^2 + O(\bar{\varepsilon}^{2^{n-1}}) \\ \leq \; & \|\,A'_{n-1,\,k}\| \|\,\sigma_{n,\,n-1}\|^3 + C\|\,\sigma_{n,\,n-1}\|^2 \end{aligned}$$

for some $C > 0$. Then $A'_{n,\,k} \to 0$ as $n \to \infty$ exponentially fast. Hence, so does $(R^n_k)'(y)$ exponentially fast. $\square$

Proposition A.0.4. *Let $F \in \mathcal{I}_B(\bar{\varepsilon})$. Then*

$$\begin{aligned} \dot{z_1} - \dot{z_2} &= \pi_z \circ \Psi^n_k(w_1) - \pi_z \circ \Psi^n_k(w_2) \\ &= \sigma_{n,\,k}(z_1 - z_2) + \sigma_{n,\,k} \sum_{i=k}^{n-1} q_i(\sigma_i\, \xi_{i+1}) \cdot (y_1 - y_2) \end{aligned}$$

where $\sigma_i\, \xi_{i+1}$ is some points in the line segment between $\pi_y \circ \Psi^n_i(w_1)$ and $\pi_z \circ \Psi^n_i(w_2)$ in $\pi_y \circ \Psi^n_i(B)$ for each $k \le i \le n-1$. Moreover,

$$\sum_{i=k}^{n-1} q_i(\sigma_i\, \xi_{i+1}) \cdot (y_1 - y_2) = d_{n,\,k}(y_1 - y_2) + R^n_k(y_1) - R^n_k(y_2)$$

Proof. Firstly, let us express $\pi_z \circ \Psi^n_k(w)$. Denote $\delta_i(y, f_i^{-1}(y), 0)$ to be $p_i(y)$ in order to simplify the expression. Recall the definition of $q_i(y)$, namely, $\frac{d}{dy}\, p_i(y) = q_i(y)$. Let $\Psi^n_i(w)$ be w_i for $k \le i \le n-1$ and let $w_i = (x_i, y_i, z_i)$. For notational compatibility, let $\Psi^i_i(B) = B$, that is, $\Psi^i_i = \mathrm{id}$ and let $\sigma_{i,\,i} = 1$ for every $i \in \mathbb{N}$. Let $w = w_n$. Recall $\pi_z \circ \psi^{i+1}_i(w_{i+1}) = \sigma_i\, z_{i+1} + p_i(\sigma_i\, y_{i+1})$. Since $\Psi^n_k = \psi^{k+1}_k \circ \Psi^n_{k+1}$, we estimate z_k using recursive formula

$$\begin{aligned} z_k =\ & \pi_z \circ \Psi^n_k(w) = \pi_z \circ \psi^{k+1}_k(w_{k+1}) \\ =\ & \sigma_k \cdot z_{k+1} + p_k(\sigma_k y_{k+1}) \\ =\ & \sigma_k\big(\sigma_{k+1} \cdot z_{k+2} + p_{k+1}(\sigma_{k+1} \cdot y_{k+2})\big) + p_k(\sigma_k \cdot y_{k+1}) \\ =\ & \sigma_k\sigma_{k+1} \cdot z_{k+2} + \sigma_k \cdot p_{k+1}(\sigma_{k+1} \cdot y_{k+2}) + p_k(\sigma_k \cdot y_{k+1}) \\ & \quad \vdots \\ =\ & \sigma_k\sigma_{k+1} \cdots \sigma_{n-1} \cdot z + \big[\, \sigma_k\sigma_{k+1} \cdots \sigma_{n-2} \cdot p_{n-1}(\sigma_{n-1} \cdot y) \\ & + \sigma_k\sigma_{k+1} \cdots \sigma_{n-3} \cdot p_{n-2}(\sigma_{n-2} \cdot y_{n-1}) + \cdots + p_k(\sigma_k \cdot y_{k+1}) \,\big] \qquad \text{(A.0.5)} \\ =\ & \sigma_{n,\,k} \cdot z + \sigma_{n-1,\,k} \cdot p_{n-1}(\sigma_{n-1} \cdot y) + \sigma_{n-2,\,k} \cdot p_{n-2}(\sigma_{n-2} \cdot y_{n-1}) + \\ & \cdots + p_k(\sigma_k \cdot y_{k+1}) \\ =\ & \sigma_{n,\,k} \cdot z + \sum_{i=k}^{n-1} \sigma_{i,\,k} \cdot p_i\,(\sigma_i \cdot y_{i+1}) \end{aligned}$$

where $\sigma_{k+1,\,k} = \sigma_k$. Moreover, by definition of w_i, $y_i = \pi_y \circ \Psi^n_i(w)$. Moreover, the second coordinate function of each $\psi^{i+1}_i(w)$ is just scaling map with σ_i by the definition, $H_i \circ \Lambda_i(w) = (\phi_i^{-1}(\sigma_i w),\, \sigma_i y,\, \bullet)$ for each $k \le i \le n-1$. Recall

$y = y_n$. Thus
$$\sigma_{n,\,i} \cdot y = \sigma_i \cdot y_{i+1} = y_i = \pi_y \circ \Psi_i^n(w)$$
Then the above equation, (A.0.5) is expressed as follows.

$$\pi_z \circ \Psi_k^n(w) = \sigma_{n,\,k} \cdot z + \sum_{i=k}^{n-1} \sigma_{i,\,k} \cdot p_i\,(\pi_y \circ \Psi_i^n(w)) \tag{A.0.6}$$

Secondly, let us estimate $\dot{z}_1 - \dot{z}_2 = \pi_z \circ \Psi_k^n(w_1) - \pi_z \circ \Psi_k^n(w_2)$ where $w_j \in B(R^n F)$ for $j = 1, 2$. By the equation (A.0.6) and Mean Value Theorem, we obtain that

$$\begin{aligned}
& \dot{z}_1 - \dot{z}_2 \\
= \;& \pi_z \circ \Psi_k^n(w_1) - \pi_z \circ \Psi_k^n(w_2) \\
= \;& \sigma_{n,\,k} \cdot (z_1 - z_2) + \sum_{i=k}^{n-1} \sigma_{i,\,k} \cdot \big[\, p_i\,(\pi_y \circ \Psi_i^n(w_1)) - p_i\,(\pi_y \circ \Psi_i^n(w_1))\,\big] \\
= \;& \sigma_{n,\,k} \cdot (z_1 - z_2) + \sum_{i=k}^{n-1} \sigma_{i,\,k} \cdot \sigma_i \cdot q_i\,(\sigma_i \cdot \xi_{i+1}) \cdot \big\{\sigma_{n,\,i+1} \cdot y_1 - \sigma_{n,\,i+1} \cdot y_2\big\} \\
= \;& \sigma_{n,\,k} \cdot (z_1 - z_2) + \sum_{i=k}^{n-1} \sigma_{i+1,\,k} \cdot q_i\,(\sigma_i \cdot \xi_{i+1}) \cdot \sigma_{n,\,i+1} \cdot (y_1 - y_2) \\
= \;& \sigma_{n,\,k} \cdot (z_1 - z_2) + \sigma_{n,\,k} \cdot \sum_{i=k}^{n-1} q_i\,(\sigma_i \cdot \xi_{i+1}) \cdot (y_1 - y_2)
\end{aligned} \tag{A.0.7}$$

where $\xi_{i+1} \in \pi_y \circ \Psi_i^n(B)$ for each $k \leq i+1 \leq n-1$. Moreover, by the expression of Ψ_k^n,
$$\pi_z \circ \Psi_k^n(w) = \sigma_{n,\ k}\big[\, d_{n,\,k}\, y + z + R_k^n(y)\,\big]$$
Then

$$\begin{aligned}
\dot{z}_1 - \dot{z}_2 = \;& \pi_z \circ \Psi_k^n(w_1) - \pi_z \circ \Psi_k^n(w_2) \\
= \;& \sigma_{n,\,k}\,\big[\, d_{n,\,k}\,(y_1 - y_2) + (z_1 - z_2) + R_k^n(y_1) - R_k^n(y_2)\,\big] \\
= \;& \sigma_{n,\,k} \cdot (z_1 - z_2) + \sigma_{n,\,k} \cdot \big[\, d_{n,\,k}\,(y_1 - y_2) + R_k^n(y_1) - R_k^n(y_2)\,\big]
\end{aligned} \tag{A.0.8}$$

Hence,

$$\sum_{i=k}^{n-1} q_i\,(\sigma_i \cdot \xi_{i+1}) \cdot (y_1 - y_2) = d_{n,\,k}\,(y_1 - y_2) + R_k^n(y_1) - R_k^n(y_2)$$

□

Corollary A.0.5. *Let $F \in \mathcal{I}_B(\bar{\varepsilon})$. Then*

$$\sum_{i=k}^{n-1} q_i(\pi_y \circ \Psi^n_{i,\mathbf{v}}(w)) = d_{*,k} + (R^n_k)'(\pi_y \circ \Psi^n_{i,\mathbf{v}}(w))\,(1 + O(\sigma^{2n}))$$

Proof. Let us compare the equation (A.0.7) and (A.0.8).

$$\begin{aligned} &\sum_{i=k}^{n-1} \sigma_{i,k} \cdot \left[\, p_i\,(\sigma_{n,i} \cdot y_1) - p_i\,(\sigma_{n,i} \cdot y_2)\,\right] \\ &= \; \sigma_{n,k} \cdot \left[\, d_{n,k}\,(y_1 - y_2) + R^n_k(y_1) - R^n_k(y_2)\,\right] \end{aligned}$$

$$\sum_{i=k}^{n-1} \sigma_{i,\;k} \cdot \frac{p_i\,(\sigma_{n,i} \cdot y_1) - p_i\,(\sigma_{n,i} \cdot y_2)}{y_1 - y_2} \;=\; \sigma_{n,k} \cdot \left[\, d_{n,k} + \frac{R^n_k(y_1) - R^n_k(y_2)}{y_1 - y_2}\,\right]$$

By the mean value theorem, we see the equation as follows.

$$\sigma_{n,k} \cdot \sum_{i=k}^{n-1} q_i\,(\sigma_i \cdot \xi_{i+1}) = \sigma_{n,k} \cdot \left[\, d_{n,k} + (R^n_k)'(\sigma_i \cdot \zeta_{i+1})\,\right]$$

where $\sigma_i\,\xi_{i+1}$ and $\sigma_i\,\zeta_{i+1}$ are some points in the line segment between $\sigma_{n,i}\,y_1$ and $\sigma_{n,i}\,y_2$ in $\pi_y \circ \Psi^n_{i,\mathbf{v}}(B)$ for each $k \leq i \leq n-1$. The points $\sigma_i\,\xi_{i+1}$ and $\sigma_i\,\zeta_{i+1}$. We choose the point $\sigma_i\,\xi_{i+1}$ arbitrarily in the domain $\pi_y \circ \Psi^n_{i,\mathbf{v}}(B)$ for each $k \leq i \leq n-1$ and $|\,(R^n_k)'| \leq C\sigma^{2n}$ for some $C > 0$. Moreover, $d_{n,k} \to d_{*,k}$ as $n \to \infty$ super exponentially fast by Proposition A.0.2. Hence,

$$\begin{aligned} \sum_{i=k}^{n-1} q_i(\pi_y \circ \Psi^n_{i,\mathbf{v}}(w)) = \;& \left[\, d_{n,k} + (R^n_k)'(\pi_y \circ \Psi^n_{i,\mathbf{v}}(w))\,\right](1 + O(\sigma^{2n})) \\ = \;& d_{*,k} + (R^n_k)'(\pi_y \circ \Psi^n_{i,\mathbf{v}}(w))\,(1 + O(\sigma^{2n})) \end{aligned}$$

□

Appendix B

Recursive formula of $\operatorname{Jac} R^n F$

Let $F_{2d} \in \mathcal{I}_B(\bar{\varepsilon})$ for sufficiently small $\bar{\varepsilon} > 0$. Let $R^n F_{2d} \equiv {}_{2d}F_n = (f_n(x) - \varepsilon_n(x, y), x)$ be the n^{th} renormalized map of F_{2d}. Then by the Universality theorem, $\varepsilon_n(x, y)$ has the universal expression, $\varepsilon_n(x, y) = b_1^{2^n} a(x)\, y((1 + O(\rho^n))$ for some $0 < \rho < 1$ where b_1 is the average Jacobian of F_{2d} and $a(x)$ is the universal function of x. Let us define the horizontal diffeomorphism $H_{2d,\,n}$ and its inverse map $H_{2d,\,n}^{-1}$ as follows.

$$
\begin{aligned}
H_{2d,\,n}(w) &= (f_n(x) - \varepsilon_n(x, y),\ y) \\
H_{2d,\,n}^{-1}(w) &= (\phi_{2d,\,n}^{-1}(w),\ y)
\end{aligned}
$$

Proposition B.0.6. *Let F_{2d} be the infinitely renormalizable two dimensional Hénon-like map with sufficiently small $\bar{\varepsilon} > 0$ where $\|\varepsilon\|_{C^3} \leq C\bar{\varepsilon}$. Let the n^{th} renormalized map be $F_n(x, y) = (f_n(x) - \varepsilon_n(x, y),\ x)$ and $\sigma_n = \sigma(1 + O(\rho^n))$ be the scaling factor for n^{th} renormalized map. Then $\varepsilon_n(x, y) = a(x)\, b_1^{2^n} y(1 + O(\rho^n))$ with the universal function $a(x)$. Moreover,*

$$
f_n' \circ f_n(\sigma_n x) \cdot a(\sigma_n x) \cdot (f_n^{-1})'(\sigma_n x) \cdot a \circ \phi_{2d,\,n}^{-1}(\sigma_n x)
$$

converges to $a(x)$ as $n \to \infty$ exponentially fast.

Proof. The first part of Proposition is the Universality theorem of two dimensional Hénon-like maps. Denote the point $w = (x, y)$. Let the inverse of the horizontal diffeomorphism be $H^{-1}(w) = (\phi_{2d}^{-1}(w), y)$. Then by the definition of $H^{-1}(w)$, we see the following equation

$$
\phi_{2d}^{-1}(w) = f^{-1}(x + \varepsilon \circ H^{-1}(w)).
$$

Thus

$$\begin{aligned}
&\partial_y \phi_{2d}^{-1}(w) \\
= \ & (f^{-1})'(x+\varepsilon \circ H^{-1}(w)) \cdot \partial_y(\varepsilon \circ H^{-1}(w)) \\
= \ & (f^{-1})'(x+\varepsilon \circ H^{-1}(w)) \cdot \big[\, \partial_x \varepsilon \circ H^{-1}(w) \cdot \partial_y \phi_{2d}^{-1}(w) + \partial_y \varepsilon \circ H^{-1}(w) \big]
\end{aligned}$$

Then,

$$\begin{aligned}
\partial_y \phi_{2d}^{-1}(w) = \ & \frac{(f^{-1})'(x+\varepsilon \circ H^{-1}(w))}{1-(f^{-1})'(x+\varepsilon \circ H^{-1}(w)) \cdot \partial_x \varepsilon \circ H^{-1}(w)} \cdot \partial_y \varepsilon \circ H^{-1}(w) \\
= \ & (f^{-1})'(x) \cdot \partial_y \varepsilon \circ (\phi_{2d}^{-1}(w), y)(1+O(\bar{\varepsilon}))
\end{aligned}$$

By the Universality theorem of the two dimensional Hénon-like maps, we can let

$$\varepsilon_n(x,y) = a(x)\, b_1^{2^n} y\ (1+O(\rho^n))$$

where b_1 is the average Jacobian of F_{2d} and for some positive $\rho < 1$.
Then using the definition of the pre-renormalization, let us define the map in the following.

$$\mathrm{Pre}\,[f_{n+1}(x) - \varepsilon_{n+1}(x,y)] = f_n(f_n(x) - \varepsilon_n \circ F_n \circ H_n^{-1}(w)) - \varepsilon_n \circ F_n^2 \circ H_n^{-1}(w)$$

Then up to the exponential convergence, we see that

$$\begin{aligned}
&\partial_y [\,\mathrm{Pre}\, \varepsilon_{n+1}] \\
= \ & f_n'(f_n(x) - \varepsilon_n(x, \phi_{n,\,2d}^{-1}(w)) \cdot \partial_y \varepsilon(x, \phi_{n,\,2d}^{-1}(w)) \\
& + \partial_y \varepsilon_n(f_n(x) - \varepsilon_n(x, \phi_{n,\,2d}^{-1}(w)),\ x) \\
= \ & f_n'(f_n(x) - \varepsilon_n(x, \phi_{n,\,2d}^{-1}(w)) \cdot \partial_y \varepsilon(x, \phi_{n,\,2d}^{-1}(w)) \\
& - \partial_x \varepsilon_n \circ (F_n^2 \circ H_n^{-1}(w)) \cdot \partial_y \varepsilon_n(x, \phi_{n,\,2d}^{-1}(w)) \\
= \ & \big[f_n'(f_n(x) - \varepsilon_n(x, \phi_{n,\,2d}^{-1}(w)) + \partial_x \varepsilon_n \circ (F_n^2 \circ H_n^{-1}(w)) \big] \cdot \partial_y \varepsilon(x, \phi_{n,\,2d}^{-1}(w)) \\
= \ & \big[f_n'(f_n(x) - \varepsilon_n(x, \phi_{n,\,2d}^{-1}(w)) + \partial_x \varepsilon_n \circ (F_n^2 \circ H_n^{-1}(w)) \big] \\
& \cdot \partial_y \varepsilon \circ (x, \phi_{n,\,2d}^{-1}(w)) \cdot \partial_y \phi_{n,\,2d}^{-1}(w) \\
= \ & f_n'(f_n(x)) \cdot a(x)\, b_1^{2^n} \cdot (f_n^{-1})'(x) \cdot a \circ \phi_{n,\,2d}^{-1}(w)\, b_1^{2^n} (1 + O(\bar{\varepsilon}^{2^n}))
\end{aligned}$$

Hence,

$$\begin{aligned}\partial_y \varepsilon_{n+1}(x,y) = \ & f_n' \circ f_n(\sigma_n x) \cdot a(\sigma_n x) \cdot (f_n^{-1})'(\sigma_n x) \cdot a \circ \phi_{2d,\, n}^{-1}(\sigma_n x)\ b_1^{2^{n+1}} \\ & \cdot (1 + O(\rho^n)) \\ = \ & a(x)\ b_1^{2^{n+1}} (1 + O(\rho^n))\end{aligned}$$

Therefore, by the Universality theorem of two dimensional Hénon-like map with exponential convergence, we obtain that

$$f_n' \circ f_n(\sigma_n x) \cdot a(\sigma_n x) \cdot (f_n^{-1})'(\sigma_n x) \cdot a \circ \phi_{2d,\, n}^{-1}(\sigma_n x) \longrightarrow a(x)$$

as $n \to \infty$ exponentially fast. □

The three dimensional map $\phi^{-1}(w)$ is also defined as the first coordinate map of $H^{-1}(w)$. Then we can estimate $\partial_y \phi^{-1}(w)$ and $\partial_z \phi^{-1}(w)$ in terms of $\partial_y \varepsilon$ and $\partial_z \varepsilon$.

Let us estimate $\partial_z \phi^{-1}(w)$.

$$\begin{aligned}\partial_z \phi^{-1}(w) = \ & (f^{-1})'(x + \varepsilon \circ H^{-1}(w)) \cdot \partial_z(\varepsilon \circ H^{-1}(w)) \\ = \ & (f^{-1})'(x + \varepsilon \circ H^{-1}(w)) \\ & \quad \cdot \left[\, \partial_x \varepsilon \circ H^{-1}(w) \cdot \partial_z \phi^{-1}(w) + \partial_z \varepsilon \circ H^{-1}(w) \right]\end{aligned}$$

Then

$$\begin{aligned}\partial_z \phi^{-1}(w) = \ & \frac{(f^{-1})'(x + \varepsilon \circ H^{-1}(w))}{1 - (f^{-1})'(x + \varepsilon \circ H^{-1}(w)) \cdot \partial_x \varepsilon \circ H^{-1}(w)} \ \cdot \partial_z \varepsilon \circ H^{-1}(w) \\ = \ & (f^{-1})'(x) \cdot \partial_z \varepsilon \circ H^{-1}(w)(1 + O(\bar{\varepsilon}))\end{aligned} \tag{B.0.1}$$

Let us estimate $\partial_y \phi^{-1}(w)$.

$$\begin{aligned}\partial_y \phi^{-1}(w) = \ & (f^{-1})'(x + \varepsilon \circ H^{-1}(w)) \cdot \partial_y(\varepsilon \circ H^{-1}(w)) \\ = \ & (f^{-1})'(x + \varepsilon \circ H^{-1}(w)) \cdot \Big[\, \partial_x \varepsilon \circ H^{-1}(w) \cdot \partial_y \phi^{-1}(w) \\ & + \partial_y \varepsilon \circ H^{-1}(w) + \partial_z \varepsilon \circ H^{-1}(w) \cdot \frac{d}{dy}\ \delta(y, f^{-1}(y), 0) \Big]\end{aligned}$$

Then

$$
\begin{aligned}
\partial_y \phi^{-1}(w) = \ & \frac{(f^{-1})'(x+\varepsilon\circ H^{-1}(w))}{1-(f^{-1})'(x+\varepsilon\circ H^{-1}(w))\cdot\partial_x\varepsilon\circ H^{-1}(w)} \\
& \cdot\left[\partial_y\varepsilon\circ H^{-1}(w)+\partial_z\varepsilon\circ H^{-1}(w)\frac{d}{dy}\ \delta(y,f^{-1}(y),0)\right] \\
= \ & (f^{-1})'(x)\cdot\left[\partial_y\varepsilon\circ H^{-1}(w)+\partial_z\varepsilon\circ H^{-1}(w)\cdot\frac{d}{dy}\ \delta(y,f^{-1}(y),0)\right] \\
& \cdot(1+O(\bar{\varepsilon}))
\end{aligned} \tag{B.0.2}
$$

On the above equations let us define the map $(f_\varepsilon^{-1})'(x)$ as follows

$$
(f_\varepsilon^{-1})'(x) = \frac{(f^{-1})'(x+\varepsilon\circ H^{-1}(w))}{1-(f^{-1})'(x+\varepsilon\circ H^{-1}(w))\cdot\partial_x\varepsilon\circ H^{-1}(w)} \tag{B.0.3}
$$

Jacobian of R^nF can be expressed as the formula using Jacobian of $R^{n-1}F$. In order to express of the Jacobian as the recursive formula, each partial derivatives of ε_n and δ_n should be expressed by the function of partial derivatives of ε_{n-1} and δ_{n-1} firstly.
Let us estimate $\partial_x \mathrm{Pre}\,\delta_1(w)$.

$$
\begin{aligned}
& \partial_x(\delta\circ F\circ H^{-1}(w)-\delta(x,f^{-1}(x),0)) \\
= \ & \partial_x\delta(x,\phi^{-1}(x),\delta\circ H^{-1}(w))-\frac{d}{dx}\ \delta(x,f^{-1}(x),0) \\
= \ & \partial_x\delta\circ(F\circ H^{-1}(w))+\partial_y\delta\circ(F\circ H^{-1}(w))\cdot\partial_x\phi^{-1}(w) \\
& +\partial_z\delta\circ(F\circ H^{-1}(w))\cdot\partial_x(\delta\circ H^{-1}(w))-\frac{d}{dx}\ \delta(x,f^{-1}(x),0) \\
= \ & \partial_x\delta\circ(F\circ H^{-1}(w))+\partial_y\delta\circ(F\circ H^{-1}(w))\cdot\partial_x\phi^{-1}(w) \\
& +\partial_z\delta\circ(F\circ H^{-1}(w))\cdot\partial_x\delta\circ H^{-1}(w)\cdot\partial_x\phi^{-1}(w)-\frac{d}{dx}\ \delta(x,f^{-1}(x),0) \\
= \ & \boxed{\left[\partial_y\delta\circ(F\circ H^{-1}(w))+\partial_z\delta\circ(F\circ H^{-1}(w))\cdot\partial_x\delta\circ H^{-1}(w)\right]}\cdot\partial_x\phi^{-1}(w) \\
& +\partial_x\delta\circ(F\circ H^{-1}(w))-\frac{d}{dx}\ \delta(x,f^{-1}(x),0)
\end{aligned} \tag{B.0.4}
$$

Let us estimate $\partial_y \mathrm{Pre}\,\delta_1(w)$.

$$
\begin{aligned}
&\partial_y(\delta\circ F\circ H^{-1}(w)-\delta(x,f^{-1}(x),0))=\partial_y\delta(x,\phi^{-1}(x),\delta\circ H^{-1}(w))\\
=\;&\partial_y\delta\circ(F\circ H^{-1}(w))\cdot\partial_y\phi^{-1}(w)+\partial_z\delta\circ(F\circ H^{-1}(w))\cdot\partial_y(\delta\circ H^{-1}(w))\\
=\;&\partial_y\delta\circ(F\circ H^{-1}(w))\cdot\partial_y\phi^{-1}(w)+\partial_z\delta\circ(F\circ H^{-1}(w))\\
&\quad\cdot\Big[\partial_x\delta\circ H^{-1}(w)\cdot\partial_y\phi^{-1}(w)\\
&\qquad+\partial_y\delta\circ H^{-1}(w)+\partial_z\delta\circ H^{-1}(w)\cdot\frac{d}{dy}\,\delta(y,f^{-1}(y),0)\Big]\\
=\;&\boxed{\Big[\partial_y\delta\circ(F\circ H^{-1}(w))+\partial_z\delta\circ(F\circ H^{-1}(w))\cdot\partial_x\delta\circ H^{-1}(w)\Big]}\cdot\partial_y\phi^{-1}(w)\\
&\quad+\partial_z\delta\circ(F\circ H^{-1}(w))\\
&\qquad\cdot\Big[\partial_y\delta\circ H^{-1}(w)+\partial_z\delta\circ H^{-1}(w)\cdot\frac{d}{dy}\,\delta(y,f^{-1}(y),0)\Big]
\end{aligned}
\tag{B.0.5}
$$

Similarly, we can estimate $\partial_z \mathrm{Pre}\,\delta_1(w)$.

$$
\begin{aligned}
&\partial_z(\delta\circ F\circ H^{-1}(w)-\delta(x,f^{-1}(x),0))=\partial_z\delta(x,\phi^{-1}(x),\delta\circ H^{-1}(w))\\
=\;&\partial_y\delta\circ(F\circ H^{-1}(w))\cdot\partial_z\phi^{-1}(w)+\partial_z\delta\circ(F\circ H^{-1}(w))\cdot\partial_z(\delta\circ H^{-1}(w))\\
=\;&\partial_y\delta\circ(F\circ H^{-1}(w))\cdot\partial_z\phi^{-1}(w)\\
&\quad+\partial_z\delta\circ(F\circ H^{-1}(w))\cdot\big[\partial_x\delta\circ H^{-1}(w)\cdot\partial_z\phi^{-1}(w)+\partial_z\delta\circ H^{-1}(w)\big]\\
=\;&\boxed{\big[\partial_y\delta\circ(F\circ H^{-1}(w))+\partial_z\delta\circ(F\circ H^{-1}(w))\cdot\partial_x\delta\circ H^{-1}(w)\big]}\cdot\partial_z\phi^{-1}(w)\\
&\quad+\partial_z\delta\circ(F\circ H^{-1}(w))\cdot\partial_z\delta\circ H^{-1}(w)
\end{aligned}
\tag{B.0.6}
$$

In order to estimate $\partial_y \mathrm{Pre}\,\varepsilon_1(w)$, we need to estimate $\partial_y(\varepsilon\circ F\circ H^{-1}(w))$ and $\partial_y(\varepsilon\circ F^2\circ H^{-1}(w))$ first.
Let us estimate $\partial_y(\varepsilon\circ F\circ H^{-1}(w))$.

$$
\begin{aligned}
&\partial_y(\varepsilon\circ F\circ H^{-1}(w))=\partial_y\varepsilon(x,\phi^{-1}(x),\delta\circ H^{-1}(w))\\
=\;&\partial_y\varepsilon\circ(F\circ H^{-1}(w))\cdot\partial_y\phi^{-1}(w)+\partial_z\varepsilon\circ(F\circ H^{-1}(w))\cdot\partial_y(\delta\circ H^{-1}(w))\\
=\;&\partial_y\varepsilon\circ(F\circ H^{-1}(w))\cdot\partial_y\phi^{-1}(w)+\partial_z\varepsilon\circ(F\circ H^{-1}(w))\\
&\quad\cdot\Big[\partial_x\delta\circ H^{-1}(w)\cdot\partial_y\phi^{-1}(w)\\
&\qquad+\partial_y\delta\circ H^{-1}(w)+\partial_z\delta\circ H^{-1}(w)\cdot\frac{d}{dy}\,\delta(y,f^{-1}(y),0)\Big]
\end{aligned}
$$

$$
\begin{aligned}
&= \left[\partial_y\varepsilon \circ (F\circ H^{-1}(w)) + \partial_z\varepsilon\circ(F\circ H^{-1}(w))\cdot\partial_x\delta\circ H^{-1}(w)\right]\cdot\partial_y\phi^{-1}(w) \\
&\quad + \partial_z\varepsilon\circ(F\circ H^{-1}(w)) \\
&\qquad \cdot \left[\partial_y\delta\circ H^{-1}(w) + \partial_z\delta\circ H^{-1}(w)\cdot\frac{d}{dy}\,\delta(y, f^{-1}(y), 0)\right]
\end{aligned}
$$

Moreover, we can express $\partial_y(\varepsilon\circ F^2\circ H^{-1}(w))$ in terms of $\partial_y(\varepsilon\circ F\circ H^{-1}(w))$ and $\partial_y(\delta\circ F\circ H^{-1}(w))$.

$$
\begin{aligned}
\partial_y(\varepsilon\circ F^2\circ H^{-1}(w)) &= \partial_y\varepsilon(f(x) - \varepsilon\circ F\circ H^{-1}(w),\, x,\, \delta\circ F\circ H^{-1}(w)) \\
&= -\partial_x\varepsilon\circ(F^2\circ H^{-1}(w))\cdot\partial_y(\varepsilon\circ F\circ H^{-1}(w)) \\
&\quad + \partial_z\varepsilon\circ(F^2\circ H^{-1}(w))\cdot\partial_y(\delta\circ F\circ H^{-1}(w))
\end{aligned}
\tag{B.0.7}
$$

Denote the function $f'(f(x) - \varepsilon\circ F\circ H^{-1}(w) - \partial_x\varepsilon\circ(F^2\circ H^{-1}(w)))$ to be $f'(f_\varepsilon(x))$. Then $\partial_y\mathrm{Pre}\,\varepsilon_1(w)$ can be estimated in terms of partial derivatives of $\varepsilon(w)$ and $\delta(w)$ as follows.

$$
\begin{aligned}
&\partial_y\mathrm{Pre}\,\varepsilon_1(w) \\
&= -\partial_y\left[f(f(x) - \varepsilon\circ F\circ H^{-1}(w)) - \varepsilon\circ F^2\circ H^{-1}(w)\right] \\
&= f'(f(x) - \varepsilon\circ F\circ H^{-1}(w))\cdot\partial_y(\varepsilon\circ F\circ H^{-1}(w)) + \partial_y(\varepsilon\circ F^2\circ H^{-1}(w)) \\
&= \left[f'(f(x) - \varepsilon\circ F\circ H^{-1}(w)) - \partial_x\varepsilon\circ(F^2\circ H^{-1}(w))\right]\cdot\partial_y(\varepsilon\circ F\circ H^{-1}(w)) \\
&\quad + \partial_z\varepsilon\circ(F^2\circ H^{-1}(w))\cdot\partial_y(\delta\circ F\circ H^{-1}(w)) \\
&= \Big[f'(f_\varepsilon(x))\cdot\{\partial_y\varepsilon\circ(F\circ H^{-1}(w)) + \partial_z\varepsilon\circ(F\circ H^{-1}(w))\cdot\partial_x\delta\circ H^{-1}(w)\} \\
&\quad + \partial_z\varepsilon\circ(F^2\circ H^{-1}(w)) \\
&\qquad \cdot\{\boxed{\partial_y\delta\circ(F\circ H^{-1}(w)) + \partial_z\delta\circ(F\circ H^{-1}(w))\cdot\partial_x\delta\circ H^{-1}(w)}\}\Big] \\
&\qquad\quad \cdot\partial_y\phi^{-1}(w) \\
&\quad + \Big[f'(f_\varepsilon(x))\cdot\partial_z\varepsilon\circ(F\circ H^{-1}(w)) \\
&\qquad + \partial_z\varepsilon\circ(F^2\circ H^{-1}(w))\cdot\partial_z\delta\circ(F\circ H^{-1}(w))\Big] \\
&\qquad \cdot\left[\partial_y\delta\circ H^{-1}(w) + \partial_z\delta\circ H^{-1}(w)\cdot\frac{d}{dy}\,\delta(y, f^{-1}(y), 0)\right]
\end{aligned}
\tag{B.0.8}
$$

Let us estimate $\partial_z(\varepsilon\circ F\circ H^{-1}(w))$.

$$
\begin{aligned}
&\partial_z(\varepsilon \circ F \circ H^{-1}(w)) = \partial_z \varepsilon(x, \phi^{-1}(x), \delta \circ H^{-1}(w)) \\
= \;& \partial_y \varepsilon \circ (F \circ H^{-1}(w)) \cdot \partial_z \phi^{-1}(w) + \partial_z \varepsilon \circ (F \circ H^{-1}(w)) \cdot \partial_z(\delta \circ H^{-1}(w)) \\
= \;& \partial_y \varepsilon \circ (F \circ H^{-1}(w)) \cdot \partial_z \phi^{-1}(w) \\
& + \partial_z \varepsilon \circ (F \circ H^{-1}(w)) \cdot \big[\, \partial_x \delta \circ H^{-1}(w) \cdot \partial_z \phi^{-1}(w) + \partial_z \delta \circ H^{-1}(w) \big] \\
= \;& \big[\, \partial_y \varepsilon \circ (F \circ H^{-1}(w)) + \partial_z \varepsilon \circ (F \circ H^{-1}(w)) \cdot \partial_x \delta \circ H^{-1}(w) \big] \cdot \partial_z \phi^{-1}(w) \\
& + \partial_z \varepsilon \circ (F \circ H^{-1}(w)) \cdot \partial_z \delta \circ H^{-1}(w)
\end{aligned}
\tag{B.0.9}
$$

Moreover, we can express $\partial_y(\varepsilon \circ F^2 \circ H^{-1}(w))$ in terms of $\partial_y(\varepsilon \circ F \circ H^{-1}(w))$ and $\partial_y(\delta \circ F \circ H^{-1}(w))$.

$$
\begin{aligned}
&\partial_z(\varepsilon \circ F^2 \circ H^{-1}(w)) = \partial_z \varepsilon(f(x) - \varepsilon \circ F \circ H^{-1}(w),\, x,\, \delta \circ F \circ H^{-1}(w)) \\
= \;& -\partial_x \varepsilon \circ (F^2 \circ H^{-1}(w)) \cdot \partial_z(\varepsilon \circ F \circ H^{-1}(w)) \\
& + \partial_z \varepsilon \circ (F^2 \circ H^{-1}(w)) \cdot \partial_z(\delta \circ F \circ H^{-1}(w))
\end{aligned}
\tag{B.0.10}
$$

Then $\partial_z \mathrm{Pre}\, \varepsilon_1(w)$ can be estimated in terms of partial derivatives of $\varepsilon(w)$ and $\delta(w)$.

$$
\begin{aligned}
&\partial_z \mathrm{Pre}\, \varepsilon_1(w) = -\partial_z \big[\, f(f(x) - \varepsilon \circ F \circ H^{-1}(w)) - \varepsilon \circ F^2 \circ H^{-1}(w) \big] \\
= \;& f'(f(x) - \varepsilon \circ F \circ H^{-1}(w)) \cdot \partial_y(\varepsilon \circ F \circ H^{-1}(w)) + \partial_y(\varepsilon \circ F^2 \circ H^{-1}(w)) \\
= \;& \big[\, f'(f(x) - \varepsilon \circ F \circ H^{-1}(w)) - \partial_x \varepsilon \circ (F^2 \circ H^{-1}(w)) \big] \cdot \partial_z(\varepsilon \circ F \circ H^{-1}(w)) \\
& + \partial_z \varepsilon \circ (F^2 \circ H^{-1}(w)) \cdot \partial_z(\delta \circ F \circ H^{-1}(w)) \\
= \;& \Big[\, f'(f_\varepsilon(x)) \cdot \{\partial_y \varepsilon \circ (F \circ H^{-1}(w)) + \partial_z \varepsilon \circ (F \circ H^{-1}(w)) \cdot \partial_x \delta \circ H^{-1}(w)\} \\
& + \partial_z \varepsilon \circ (F^2 \circ H^{-1}(w)) \\
& \quad \cdot \{\, \boxed{\partial_y \delta \circ (F \circ H^{-1}(w)) + \partial_z \delta \circ (F \circ H^{-1}(w)) \cdot \partial_x \delta \circ H^{-1}(w)} \,\} \Big] \\
& \quad \cdot \partial_z \phi^{-1}(w) \\
& + \Big[\, f'(f_\varepsilon(x)) \cdot \partial_z \varepsilon \circ (F \circ H^{-1}(w)) \\
& + \partial_z \varepsilon \circ (F^2 \circ H^{-1}(w)) \cdot \partial_z \delta \circ (F \circ H^{-1}(w)) \Big] \cdot \partial_z \delta \circ H^{-1}(w)
\end{aligned}
\tag{B.0.11}
$$

Lemma B.0.7. *Let F be an infinitely renormalizable three dimensional Hénon-*

like map. Then

$$
\begin{aligned}
&\operatorname{Jac} R^n F(w) \\
= \;& (f_{n-1}^{-1})'(\sigma_{n-1}x) \cdot f_{n-1}'(f_{n-1}(\sigma_{n-1}x)) \\
&\cdot \operatorname{Jac} R^{n-1}F \circ (H_{n-1}^{-1}(\sigma_{n-1}w)) \cdot \operatorname{Jac} R^{n-1}F \circ (F_{n-1} \circ H_{n-1}^{-1}(\sigma_{n-1}w))
\end{aligned}
$$

Proof. Let us calculate $\operatorname{Jac} RF(w)$ in terms of partial derivatives of ε and δ. Recall the equations (B.0.5), (B.0.6), (B.0.8) and (B.0.11). Let us express $\operatorname{Jac} RF$ in terms of these.

$$
\begin{aligned}
&\operatorname{Jac} RF(w) = \partial_y \varepsilon_1(w) \cdot \partial_z \delta_1(w) - \partial_z \varepsilon_1(w) \cdot \partial_y \delta_1(w) \\
= \;& \Big[\big\{ f'(f_\varepsilon(\sigma_0 x)) \cdot \{ \partial_y \varepsilon \circ (F \circ H^{-1}(\sigma_0 w)) + \partial_z \varepsilon \circ (F \circ H^{-1}(\sigma_0 w)) \\
&\quad \cdot \partial_x \delta \circ H^{-1}(\sigma_0 w) \} \\
&\quad + \partial_z \varepsilon \circ (F^2 \circ H^{-1}(\sigma_0 w)) \\
&\quad\quad \cdot \{ \partial_y \delta \circ (F \circ H^{-1}(\sigma_0 w)) + \partial_z \delta \circ (F \circ H^{-1}(\sigma_0 w)) \cdot \partial_x \delta \circ H^{-1}(\sigma_0 w) \} \big\} \\
&\quad\quad\quad \cdot \partial_y \phi^{-1}(\sigma_0 w) \\
&\quad + \big\{ f'(f_\varepsilon(\sigma_0 x)) \cdot \partial_z \varepsilon \circ (F \circ H^{-1}(\sigma_0 w)) + \partial_z \varepsilon \circ (F^2 \circ H^{-1}(\sigma_0 w)) \\
&\quad\quad\quad \cdot \partial_z \delta \circ (F \circ H^{-1}(\sigma_0 w)) \big\} \\
&\quad\quad\quad \cdot \Big\{ \partial_y \delta \circ H^{-1}(\sigma_0 w) + \partial_z \delta \circ H^{-1}(\sigma_0 w) \cdot \frac{d}{dy}\, \delta(\sigma_0 y, f^{-1}(\sigma_0 y), 0) \Big\} \Big] \\
&\quad \cdot \Big[\big\{ \partial_y \delta \circ (F \circ H^{-1}(\sigma_0 w)) + \partial_z \delta \circ (F \circ H^{-1}(\sigma_0 w)) \cdot \partial_x \delta \circ H^{-1}(\sigma_0 w) \big\} \\
&\quad\quad\quad \cdot \partial_z \phi^{-1}(\sigma_0 w) + \partial_z \delta \circ (F \circ H^{-1}(\sigma_0 w)) \cdot \partial_z \delta \circ H^{-1}(\sigma_0 w) \Big]
\end{aligned}
\tag{B.0.12}
$$

$$
\begin{aligned}
&-\Big[\Big\{\, f'(f_\varepsilon(\sigma_0 x)) \cdot \{\, \partial_y \varepsilon \circ (F \circ H^{-1}(\sigma_0 w)) \\
&\qquad + \partial_z \varepsilon \circ (F \circ H^{-1}(\sigma_0 w)) \cdot \partial_x \delta \circ H^{-1}(\sigma_0 w)\} \\
&\quad + \partial_z \varepsilon \circ (F^2 \circ H^{-1}(\sigma_0 w)) \\
&\quad \cdot \{\, \partial_y \delta \circ (F \circ H^{-1}(\sigma_0 w)) + \partial_z \delta \circ (F \circ H^{-1}(\sigma_0 w)) \cdot \partial_x \delta \circ H^{-1}(\sigma_0 w)\,\}\Big\} \\
&\qquad \cdot \partial_z \phi^{-1}(\sigma_0 w) \\
&\;+ \Big\{\, f'(f_\varepsilon(\sigma_0 x)) \cdot \partial_z \varepsilon \circ (F \circ H^{-1}(\sigma_0 w)) \\
&\qquad + \partial_z \varepsilon \circ (F^2 \circ H^{-1}(\sigma_0 w)) \cdot \partial_z \delta \circ (F \circ H^{-1}(\sigma_0 w))\Big\} \\
&\qquad \cdot \partial_z \delta \circ H^{-1}(\sigma_0 w)\Big] \\
&\;\cdot \Big[\Big\{\partial_y \delta \circ (F \circ H^{-1}(\sigma_0 w)) + \partial_z \delta \circ (F \circ H^{-1}(\sigma_0 w)) \cdot \partial_x \delta \circ H^{-1}(\sigma_0 w)\Big\} \\
&\qquad \cdot \partial_y \phi^{-1}(\sigma_0 w) \\
&\quad + \partial_z \delta \circ (F \circ H^{-1}(\sigma_0 w)) \cdot \Big\{\, \partial_y \delta \circ H^{-1}(\sigma_0 w) + \partial_z \delta \circ H^{-1}(\sigma_0 w) \\
&\qquad \cdot \frac{d}{dy}\, \delta(\sigma_0 y, f^{-1}(\sigma_0 y), 0)\Big\}\Big]
\end{aligned}
$$

On the above equation, let us denote some factors to be A, B, C and D as follows.

$$
\begin{aligned}
A =&\; f'(f_\varepsilon(\sigma_0 x)) \cdot \{\, \partial_y \varepsilon \circ (F \circ H^{-1}(\sigma_0 w)) \\
&\quad + \partial_z \varepsilon \circ (F \circ H^{-1}(\sigma_0 w)) \cdot \partial_x \delta \circ H^{-1}(\sigma_0 w)\} \\
&+ \partial_z \varepsilon \circ (F^2 \circ H^{-1}(\sigma_0 w)) \\
&\quad \cdot \{\, \partial_y \delta \circ (F \circ H^{-1}(\sigma_0 w)) + \partial_z \delta \circ (F \circ H^{-1}(\sigma_0 w)) \cdot \partial_x \delta \circ H^{-1}(\sigma_0 w)\,\} \\
B =&\; f'(f_\varepsilon(\sigma x)) \cdot \partial_z \varepsilon \circ (F \circ H^{-1}(\sigma_0 w)) \\
&\quad + \partial_z \varepsilon \circ (F^2 \circ H^{-1}(\sigma_0 w)) \cdot \partial_z \delta \circ (F \circ H^{-1}(\sigma_0 w)) \\
C =&\; \partial_y \delta \circ (F \circ H^{-1}(\sigma_0 w)) + \partial_z \delta \circ (F \circ H^{-1}(\sigma_0 w)) \cdot \partial_x \delta \circ H^{-1}(\sigma_0 w) \\
D =&\; \partial_y \delta \circ H^{-1}(\sigma_0 w) + \partial_z \delta \circ H^{-1}(\sigma_0 w) \cdot \frac{d}{dy}\, \delta(\sigma_0 y, f^{-1}(\sigma_0 y), 0)
\end{aligned}
\tag{B.0.13}
$$

Let us calculate $A \cdot \partial_z \delta \circ (F \circ H^{-1}(\sigma w)) - BC$ for later use.

$$
\begin{aligned}
& A \cdot \partial_z \delta \circ (F \circ H^{-1}(\sigma_0 w)) - BC \\
= & \Big[f'(f_\varepsilon(\sigma_0 x)) \cdot \{ \partial_y \varepsilon \circ (F \circ H^{-1}(\sigma_0 w)) \\
& \quad + \partial_z \varepsilon \circ (F \circ H^{-1}(\sigma_0 w)) \cdot \partial_x \delta \circ H^{-1}(\sigma_0 w) \} \\
& + \partial_z \varepsilon \circ (F^2 \circ H^{-1}(\sigma_0 w)) \\
& \cdot \{ \partial_y \delta \circ (F \circ H^{-1}(\sigma_0 w)) + \partial_z \delta \circ (F \circ H^{-1}(\sigma_0 w)) \cdot \partial_x \delta \circ H^{-1}(\sigma_0 w) \} \Big] \\
& \quad \cdot \partial_z \delta \circ (F \circ H^{-1}(\sigma_0 w)) \\
& - \Big[f'(f_\varepsilon(\sigma_0 x)) \cdot \partial_z \varepsilon \circ (F \circ H^{-1}(\sigma_0 w)) \\
& \qquad + \partial_z \varepsilon \circ (F^2 \circ H^{-1}(\sigma_0 w)) \cdot \partial_z \delta \circ (F \circ H^{-1}(\sigma_0 w)) \Big] \\
& \cdot \Big[\partial_y \delta \circ (F \circ H^{-1}(\sigma_0 w)) + \partial_z \delta \circ (F \circ H^{-1}(\sigma_0 w)) \cdot \partial_x \delta \circ H^{-1}(\sigma_0 w) \Big] \\
= & \; f'(f_\varepsilon(\sigma_0 x)) \cdot \big[\partial_y \varepsilon \circ (F \circ H^{-1}(\sigma_0 w)) \cdot \partial_z \delta \circ (F \circ H^{-1}(\sigma_0 w)) \\
& \qquad\qquad - \partial_z \varepsilon \circ (F \circ H^{-1}(\sigma_0 w)) \cdot \partial_y \delta \circ (F \circ H^{-1}(\sigma_0 w)) \big]
\end{aligned}
\tag{B.0.14}
$$

Then the above equation of $\operatorname{Jac} RF$ is expressed as follows.

$$
\begin{aligned}
& \partial_y \varepsilon_1(w) \cdot \partial_z \delta_1(w) - \partial_z \varepsilon_1(w) \cdot \partial_y \delta_1(w) \\
= & \Big[A \cdot \partial_y \phi^{-1}(\sigma_0 w) + BD \Big] \\
& \quad \cdot \Big[C \cdot \partial_z \phi^{-1}(\sigma_0 w) + \partial_z \delta \circ (F \circ H^{-1}(\sigma_0 w)) \cdot \partial_z \delta \circ H^{-1}(\sigma_0 w) \Big] \\
& - \Big[A \cdot \partial_z \phi^{-1}(\sigma_0 w) + B \cdot \partial_z \delta \circ H^{-1}(\sigma_0 w) \Big] \\
& \quad \cdot \Big[C \cdot \partial_y \phi^{-1}(\sigma_0 w) + \partial_z \delta \circ (F \circ H^{-1}(\sigma_0 w)) \cdot D \Big] \\
= & \; A \cdot \partial_y \phi^{-1}(\sigma_0 w) \cdot \partial_z \delta \circ (F \circ H^{-1}(\sigma_0 w)) \cdot \partial_z \delta \circ H^{-1}(\sigma_0 w) \\
& \quad + BCD \cdot \partial_z \phi^{-1}(\sigma_0 w) \\
& - \big[AD \cdot \partial_z \phi^{-1}(\sigma_0 w) \cdot \partial_z \delta \circ (F \circ H^{-1}(\sigma_0 w)) \\
& \quad + BC \cdot \partial_z \delta \circ H^{-1}(\sigma_0 w) \cdot \partial_y \phi^{-1}(\sigma_0 w) \big]
\end{aligned}
\tag{B.0.15}
$$

Recall the equations (B.0.2) and (B.0.1) for $\partial_y \phi^{-1}(\sigma_0 w)$ and $\partial_z \phi^{-1}(\sigma_0 w)$ respectively. Let us expand D in the equation (B.0.13). Then above equation is continued as follows.

$$
\begin{aligned}
= \ & A \cdot (f_\varepsilon^{-1})'(\sigma_0 x) \cdot \big\{ \, \partial_y \varepsilon \circ H^{-1}(\sigma_0 w) + \partial_z \varepsilon \circ H^{-1}(\sigma_0 w) \\
& \quad \cdot \frac{d}{dy} \, \delta(\sigma_0 y, f^{-1}(\sigma_0 y), 0) \, \big\} \\
& \quad \cdot \partial_z \delta \circ (F \circ H^{-1}(\sigma_0 w)) \cdot \partial_z \delta \circ H^{-1}(\sigma_0 w) \\
& + BC \cdot \big\{ \, \partial_y \delta \circ H^{-1}(\sigma_0 w) + \partial_z \delta \circ H^{-1}(\sigma_0 w) \cdot \frac{d}{dy} \, \delta(\sigma_0 y, f^{-1}(\sigma_0 y), 0) \big\} \\
& \quad \cdot (f_\varepsilon^{-1})'(\sigma_0 x) \cdot \partial_z \varepsilon \circ H^{-1}(\sigma_0 w) \\
& - \Big[\, A \cdot \big\{ \partial_y \delta \circ H^{-1}(\sigma_0 w) + \partial_z \delta \circ H^{-1}(\sigma_0 w) \cdot \frac{d}{dy} \, \delta(\sigma_0 y, f^{-1}(\sigma_0 y), 0) \big\} \\
& \quad \cdot (f_\varepsilon^{-1})'(\sigma_0 x) \cdot \partial_z \varepsilon \circ H^{-1}(\sigma_0 w) \cdot \partial_z \delta \circ (F \circ H^{-1}(\sigma_0 w)) \\
& + BC \cdot \partial_z \delta \circ H^{-1}(\sigma_0 w) \cdot (f_\varepsilon^{-1})'(\sigma_0 x) \\
& \quad \cdot \big\{ \, \partial_y \varepsilon \circ H^{-1}(\sigma_0 w) + \partial_z \varepsilon \circ H^{-1}(\sigma_0 w) \cdot \frac{d}{dy} \, \delta(\sigma_0 y, f^{-1}(\sigma_0 y), 0) \, \big\} \Big] \\
= \ & A \cdot (f_\varepsilon^{-1})'(\sigma_0 x) \cdot \partial_z \delta \circ (F \circ H^{-1}(\sigma_0 w)) \\
& \cdot \big[\, \partial_y \varepsilon \circ H^{-1}(\sigma_0 w) \cdot \partial_z \delta \circ H^{-1}(\sigma_0 w) - \partial_z \varepsilon \circ H^{-1}(\sigma_0 w) \cdot \partial_y \delta \circ H^{-1}(\sigma_0 w) \, \big] \\
& - BC \cdot (f_\varepsilon^{-1})'(\sigma_0 x) \cdot \big[\, \partial_y \varepsilon \circ H^{-1}(\sigma_0 w) \cdot \partial_z \delta \circ H^{-1}(\sigma_0 w) - \partial_z \varepsilon \circ H^{-1}(\sigma_0 w) \\
& \quad \cdot \partial_y \delta \circ H^{-1}(\sigma_0 w) \, \big] \\
= \ & \big[\, A \cdot \partial_z \delta \circ (F \circ H^{-1}(\sigma_0 w)) - BC \, \big] \cdot (f_\varepsilon^{-1})'(\sigma_0 x) \\
& \cdot \big[\, \partial_y \varepsilon \circ H^{-1}(\sigma_0 w) \cdot \partial_z \delta \circ H^{-1}(\sigma_0 w) - \partial_z \varepsilon \circ H^{-1}(\sigma_0 w) \cdot \partial_y \delta \circ H^{-1}(\sigma_0 w) \, \big]
\end{aligned}
$$

By the equation (B.0.14), the above equation is continued as follows.

$$
\begin{aligned}
= \ & (f_\varepsilon^{-1})'(\sigma_0 x) \\
& \cdot \big[\, \partial_y \varepsilon \circ H^{-1}(\sigma_0 w) \cdot \partial_z \delta \circ H^{-1}(\sigma_0 w) - \partial_z \varepsilon \circ H^{-1}(\sigma_0 w) \cdot \partial_y \delta \circ H^{-1}(\sigma_0 w) \, \big] \\
& \cdot f'(f_\varepsilon(\sigma_0 x)) \cdot \big[\, \partial_y \varepsilon \circ (F \circ H^{-1}(\sigma_0 w)) \cdot \partial_z \delta \circ (F \circ H^{-1}(\sigma_0 w)) \\
& \qquad - \partial_z \varepsilon \circ (F \circ H^{-1}(\sigma_0 w)) \cdot \partial_y \delta \circ (F \circ H^{-1}(\sigma_0 w)) \, \big] \\
= \ & f'(f_\varepsilon(\sigma_0 x)) \cdot (f_\varepsilon^{-1})'(\sigma_0 x) \cdot \operatorname{Jac} F \circ (H^{-1}(\sigma_0 w)) \cdot \operatorname{Jac} F \circ (F \circ H^{-1}(\sigma_0 w))
\end{aligned}
\tag{B.0.16}
$$

Similarly, $\operatorname{Jac} R^n F(w)$ is expressed in terms of the partial derivatives of ε_{n-1}

and δ_{n-1} as follows.

$$\begin{aligned}
&\operatorname{Jac} R^n F(w) \\
= \; &(f_{n-1,\,\varepsilon}^{-1})'(\sigma_{n-1}x) \cdot f_{n-1}'(f_{n-1,\,\varepsilon}(\sigma_{n-1}x)) \\
&\cdot \operatorname{Jac} F_{n-1} \circ (H_{n-1}^{-1}(\sigma_{n-1}w)) \cdot \operatorname{Jac} F_{n-1} \circ (F_{n-1} \circ H_{n-1}^{-1}(\sigma_{n-1}w))
\end{aligned}$$

$\square$

Appendix C

Further research topics

The theory of three dimensional renormalizable Hénon-like maps has open problems. Recall that $\mathcal{I}_B(\bar{\varepsilon})$ is the set of infinitely renormalizable three dimensional Hénon-like maps. Let us consider subsets of $\mathcal{I}_B(\bar{\varepsilon})$ appearing on the previous sections. Let us define each set using the curly alphabet and consider their basic properties.

- $\mathcal{T}$ — set of the Hénon-like maps such that

$$\partial_z \varepsilon \equiv 0, \quad \partial_x \delta \equiv 0, \quad \text{and} \quad \partial_y \delta \equiv 0$$

  Let the maps satisfying above conditions be *trivial extension of two dimensional Hénon-like map* or simply trivially extended map, which is of the following form.

$$(x, y, z) \mapsto (f(x) - \varepsilon(x, y),\ x,\ \delta(z))$$

- $\mathcal{M}$ — set of model maps. See §9.1.

$$\partial_z \varepsilon \equiv 0$$

- $\mathcal{SM}$ — set of small perturbation of model maps with the condition $b_1 \gg b_2$. In particular, $(b_1)^r \gg b_2$ is assumed for the given finite number $r \geq 3$. See §9.3 and §10.2.

- $\mathcal{N}$ — set of maps the following identical equation of partial derivatives of the third coordinate map

$$\partial_y \delta \circ F(w) + \partial_z \delta \circ F(w) \cdot \partial_x \delta(w) \equiv 0$$

where $w = (x, y, z) \in B^1_v \cup B^1_c$. See §12.1.

Then clearly we observe the following inclusion property of each sets.

$$\mathcal{T} \subsetneq \mathcal{M} \cap \mathcal{N}$$

Moreover, the set differences, $\mathcal{M} \setminus \mathcal{N}$ and $\mathcal{N} \setminus \mathcal{M}$ are non empty sets. Furthermore, each of the following sets

$$\mathcal{T} \cap \mathcal{I}_B(\bar{\varepsilon}), \quad \mathcal{M} \cap \mathcal{I}_B(\bar{\varepsilon}), \quad \text{and} \quad \mathcal{N} \cap \mathcal{I}_B(\bar{\varepsilon})$$

are invariant under renormalization. The set $\mathcal{SM} \cap \mathcal{I}_B(\bar{\varepsilon})$ with the condition $(b_1)^r \gg b_2$ is invariant under renormalization in the sense that there exist invariant surfaces under renormalized map for each level.
We can call each of those sets a subspace of $\mathcal{I}_B(\bar{\varepsilon})$. The renormalized map of $F_\bullet \in \mathcal{T} \cap \mathcal{I}_B(\bar{\varepsilon})$ is the following due to the universality theorem of two dimensional Hénon-like maps.

$$R^n F_\bullet(x, y, z) = \big(f_n(x) - b_1^{2^n} a(x)\, y(1 + O(\rho^n)),\ x,\ b_2^{2^n}(z - z_n)(1 + O(\rho^n))\big)$$

where $|z_n| = O(\bar{\varepsilon}^{2^n})$ for some $\rho \in (0,1)$. Then $R^n F_\bullet$ has invariant plane parallel to $xy-$plane for each $n \in \mathbb{N}_+$. However, the Hénon-like maps in the space $\mathcal{SM} \cap \mathcal{I}_B(\bar{\varepsilon})$ has C^r invariant surfaces. Then by the diffeomorphism between surface and $xy-$plane two dimensional renormalizable C^r Hénon-like maps are defined. Moreover, it is shown that this renormalization is the same as the usual definition of Hénon renormalization by the conjugation of the horizontal diffeomorphism and dilation.

Problem I
There are open problems related to invariant surfaces and two dimensional C^r Hénon-like maps.

(1) Are there C^∞ or C^ω invariant surfaces different from plane under $R^n F \in \mathcal{I}_B(\bar{\varepsilon})$ for each $n \in \mathbb{N}_+$?

(2) Suppose that the map, $F_*(x, y) = (f_*(x),\ x)$ is the fixed point under renormalization operator of infinitely renormalizable two dimensional C^r Hénon-like maps for big enough $r < \infty$, say $\mathcal{I}^r$. Is F_* the hyperbolic fixed point under renormalization?

(3) Invariant surfaces under $R^n F \in \mathcal{I}_B(\bar{\varepsilon})$ in three dimension can define a subset of infinitely renormalizable two dimensional C^r Hénon-like maps, say $\mathcal{I}^r_Q$. Clearly $\mathcal{I}^r_Q \subset \mathcal{I}^r$. Is the set $\mathcal{I}^r_Q$ dense or open in $\mathcal{I}^r$?

Problem II

There some open problems about the Hénon-like maps in the space $\mathcal{N} \cap \mathcal{I}_B(\bar{\varepsilon})$.

(1) Recall that $b_2^{2^n} = \partial_z \delta_n(w)(1 + O(\rho^n))$ and the average Jacobian and b_1 satisfies that $b = b_1 b_2$. Are $\log b_1$ and $\log b_2$ Lyapunov exponents on the Cantor attractor? It might be yes.

(2) Does the *continuous* invariant line field exist on the Cantor attractor of $F \in \mathcal{N} \cap \mathcal{I}_B(\bar{\varepsilon})$?

(3) The set of parametrized Hénon-like maps by b_1 in $\mathcal{N} \cap \mathcal{I}_B(\bar{\varepsilon})$ has the parameters $(0, \bar{b_1}]$ for some $\bar{b_1} > 0$. There exists parametrized subfamily of Hénon-like maps of which Cantor attractor has unbounded geometry. Then the corresponding parameters of the above subfamily contains G_δ subset of $(0, \bar{b_1}]$. Can this parameters contain the points of the full Lebesgue measure?

Problem III

The extension of the Hénon renormalization to the larger space is a further research topic.

(1) Does there exist invariant subspace of $\mathcal{I}_B(\bar{\varepsilon})$ which contains $\mathcal{M} \cup \mathcal{N}$? Can a subspace of $\mathcal{I}_B(\bar{\varepsilon})$ invariant under renormalization describe the map of the whole family $\mathcal{I}_B(\bar{\varepsilon})$ generically?

(2) For the maps in $\mathcal{M} \cup \mathcal{N}$, $\partial_z \delta_n = b_2^{2^n}(1 + O(\rho^n))$ for some positive small number b_2. Is it true for all maps in $\mathcal{I}_B(\bar{\varepsilon})$? If not, is there a map whose Cantor attractor has dynamical properties which cannot be induced from the two dimensional Hénon-like maps?

(3) Extend three dimensional theory to the arbitrary finite dimensional map. The Hénon-like map in the general dimension is of the following form.

$$(x, y, \mathbf{z}) \mapsto (f(x) - \varepsilon(x, y, \mathbf{z}),\ x,\ \delta(x, y, \mathbf{z}))$$

where $\mathbf{z} = (z_1, z_2, \ldots, z_m)$ for any fixed $m \in \mathbb{N}$.