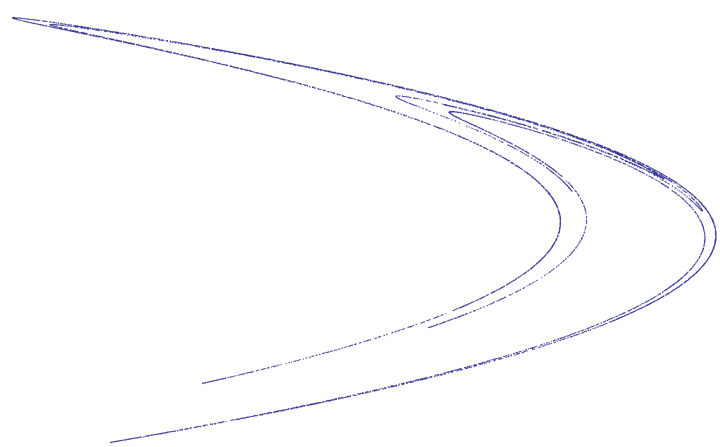
Hénon-like Maps

and

Renormalisation



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RIJKSUNIVERSITEIT GRONINGEN

Hénon-like Maps and Renormalisation

Proefschrift

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Agreement of Joint Program

The following is a dissertation submitted in partial fulfillment of the requirements for the degree Doctor of Philosophy in Mathematics awarded jointly by Rijksuniversiteit Groningen, The Netherlands and Stony Brook University, USA. It has been agreed that neither institution shall award a full doctorate. It has been agreed by both institutions that the following are to be the advisors and reading committee.

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Both institutions agree that the defense of the above degree will take place on Monday 8th December 2008 at 11:00am at the Academiegebouw, RuG, Groningen, The Netherlands and that, if successful, the degree Doctor of Philosophy in Mathematics will be awarded jointly by Rector Magnificus, dr. F. Zwarts, RuG, Groningen, and Prof. L. Martin, Graduate School Dean, Stony Brook University.

Hénon-like Maps

and

Renormalisation

A Dissertation Presented

by

Peter Edward Hazard

to

The Graduate School

in Partial Fulfillment of the

Requirements

for the Degree of

Doctor of Philosophy

in

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Peter Edward Hazard

We, the dissertation committee for the above candidate for the
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Abstract of the Dissertation

Hénon-like Maps and Renormalisation

by

Peter Edward Hazard

Doctor of Philosophy

in

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The aim of this dissertation is to develop a renormalisation theory for the Hénon family

$$F_{a,b}(x,y) = (a - x^2 - by, x) \tag{†}$$

for combinatorics other than period-doubling in a way similar to that for the standard unimodal family $f_a(x) = a - x^2$. This work breaks into two parts. After recalling background needed in the unimodal renormalisation theory, where a space $\mathcal{U}$ of unimodal maps and an operator $\mathcal{R}_{\mathcal{U}}$ acting on a subspace of $\mathcal{U}$ are considered, we construct a space $\mathcal{H}$ –the strongly dissipative Hénon-like maps– and an operator $\mathcal{R}$ which acts on a subspace of $\mathcal{H}$. The space $\mathcal{U}$ is canonically embedded in the boundary of $\mathcal{H}$. We show that $\mathcal{R}$ is a dynamically-defined continuous operator which continuously extends $\mathcal{R}_{\mathcal{U}}$ acting on $\mathcal{U}$. Moreover the classical renormalisation picture still holds: there exists a unique renormalisation fixed point which is hyperbolic, has a codimension one stable manifold, consisting of all infinitely renormalisable maps, and a dimension one local unstable manifold.

Infinitely renormalisable Hénon-like maps are then examined. We show, as in the unimodal case, that such maps have invariant Cantor sets supporting a unique invariant probability. We construct a metric invariant, the average Jacobian. Using this we study the dynamics of infinitely renormalisable maps around a prescribed point, the 'tip'. We show, as in the unimodal case, universality exists at this point. We also show the dynamics at the tip is non-rigid: any two maps with differing average Jacobians cannot be C^1-conjugated by a tip-preserving diffeomorphism.

Finally it is shown that the geometry of these Cantor sets is, metrically and generically, unbounded in one-parameter families of infinitely renormalisable maps satisfying a transversality condition.

Dedicated to my Mother, Father

Sister and Brother

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Chapter 1

Introduction

1.1 Background on Hénon-like Maps

This work aims to describe some of the dynamical properties of Hénon-like maps. These are maps of the square to itself which 'bend' at a unique place. The prototype for these maps is the Hénon family of maps, given by

$$F_{a,b}(x,y) = (a - x^2 - by, x). \tag{1.1.1}$$

In [24], Hénon gave numerical evidence which suggested, for particular values of parameters[1] a and b, there exists a strange attractor for this map (see the front cover for a picture). Since that time much work has been done in studying the properties of such maps and the bifurcations the family exhibits in the (a,b)-plane.

Showing that the attractor actually existed for certain parameter values turned out to be a significant achievement. This was first done in the work of Benedicks and Carleson [2]. They showed, for a large set of parameters that the unstable manifold is attracting and that it has a definite basin of attraction. Their breakthrough was to compare the dynamics of $F_{a,b}$ with that of the one-dimensional unimodal map $f_a(x) = a - x^2$ (their parametrisation was different but we state the equivalent formulation, see below). The tools they developed in their proof of Jakobson's Theorem allowed them to get very precise information about a specific point whose orbit turns out to be dense in the attractor. We will return with a precise formulation of their results later.

Let us finally remark that this application of the one-dimensional unimodal theory is one of the driving forces in current investigations of these systems. As far as we are aware this was first suggested by Feigenbaum (see the book [7]

[1] Hénon actually studied the family

$$H_{a,b}(x,y) = (1 - ax^2 + y, by) \tag{1.1.2}$$

but the two families are affinely conjugate. He found this interesting behaviour for the parameter values $a = 1.4, b = 0.3$.

by Collet and Eckmann). This is a leitmotif that drives the present work, and one which will be developed in this introduction. Before we describe Hénon-like maps in more detail let us consider the development of dynamics from a more global viewpoint.

1.2 Uniform Hyperbolicity and Topological Dynamics

First let us set up some notation. Given manifolds M and N and any $r = 0, 1, \ldots, \infty, \omega$, let $C^r(M, N)$ denote the space of all C^r-smooth maps from M to N, let $C^r_0(M, N)$ denote the subspace of maps with compact support and let $\mathrm{Emb}^r(M, N)$ denote the subspace of all C^r-embeddings from M to N. We let $\mathrm{End}^r(M)$ denote the space of C^r-endomorphisms of M and we let $\mathrm{Diff}^r(M)$ denote the space of C^r-diffeomorphisms on M. We will denote the usual C^r-norm on $C^r(M, N)$ by $|-|_{C^r(M,N)}$. If the spaces M, N are understood we will simply write $|-|_{C^r}$. In the special case when $r = 0$, the sup-norm will be denoted $|-|_M$. We will reserve the notation $\|-\|$ or $\|-\|_E$ to denote the operator norm of a linear operator on the Banach space E.

Given $f \in \mathrm{Diff}^r(M)$ we will denote the set of its periodic points by $\mathrm{Per}(f)$ and the the orbit of $x \in M$ under f by $\mathrm{orb}_f(x)$. The set of non-wandering points is denoted by $\Omega(f)$. Given a periodic point $x \in M$ we will denote its table and unstable manifolds by $W^s(x)$ and $W^u(x)$ respectively.

In the late 1950's Smale initiated the study of uniformly hyperbolic dynamical systems. The aim was to show such systems were generic and structurally stable. If this were shown a reasonable topological or differential topological classification of dynamical systems would be achieved. Systems such as *Morse-Smale*, *Kupka-Smale* and *Axiom A* were considered in detail.

Definition 1.2.1 (Kupka-Smale, Morse-Smale)**.** Let M be a manifold and $f \in \mathrm{Diff}^r(M)$ a diffeomorphism. If f satisfies the following properties,

(i) each $p \in \mathrm{Per}(f)$ is hyperbolic;

(ii) $W^u(p) \pitchfork W^s(q)$ for each $p, q \in \mathrm{Per}(f)$;

then we say f is a *Kupka-Smale* diffeomorphism on M. If f satisfies the additional properties,

(iii) $\mathrm{Per}(f)$ has finite cardinality;

(iv) $\bigcup_{p \in \mathrm{Per}(f)} W^s(p) = M$;

(v) $\bigcup_{p \in \mathrm{Per}(f)} W^u(p) = M$;

then we say f is a *Morse-Smale* diffeomorphism on M.

Definition 1.2.2 (Axiom A)**.** Let M be a manifold and $f \in \mathrm{Diff}^r(M)$ a diffeomorphism. If f satisfies the following properties,

(i) the nonwandering set $\Omega(f)$ is hyperbolic;

(ii) $\mathrm{Per}(f)$ is dense in $\Omega(f)$;

then we say f is an *Axiom A* diffeomorphism on M.

The hope was, for a long time, that, Axiom A maps would be dense. This was shown not to be the case, most conclusively by Newhouse. The following two results were shown by him in [39] and [40]. We refer the reader to chapter 6 of the book [42] by Palis and Takens for more details.

Theorem 1.2.3 (Newhouse)**.** *For any two dimensional manifold M there exists an open set $U \subset \mathrm{Diff}^2(M)$, and a dense subset $B \subset U$ such that every map $f \in B$ possesses a homoclinic tangency.*

Theorem 1.2.4 (Newhouse)**.** *For any two dimensional manifold M, and any $r \geq 2$, there exists an open set $U \subset \mathrm{Diff}^r(M)$ and a residual subset $B \subset U$ such that every map $f \in B$ has infinitely many hyperbolic periodic attractors.*

Let us also recall the following result of Katok, which acts as a nice counterpoint to the first of these two theorems.

Theorem 1.2.5 (Katok)**.** *For any compact two dimensional manifold M, let $f \in \mathrm{Diff}^{1+\alpha}(M)$ preserve the Borel probability measure μ and also satisfy the following properties,*

(i) the support of μ is not concentrated on a single periodic orbit;

(ii) μ is f-ergodic;

(iii) f has non-zero characteristic exponents with respect to μ;

then f having a transversal homoclinic point implies $h_{top}(f) > 0$, where $h_{top}(f)$ denotes the topological entropy of f.

This shows that the dense set B constructed by Newhouse lives close to the region of 'chaotic' maps. We will consider this in more detail later when outlining the renormalisation picture.

1.3 Non-Uniform Hyperbolicity and Measurable Dynamics

In the late 1960's Oseledets and Pesin, among others, initiated the study of non-uniformly hyperbolic systems, i.e. ones for which the tangent bundle does not split into factors which contract or expand at a uniform rate. The key observation was that it was the asymptotic behaviour of the action of f on elements of the tangent bundle that was significant. By considering the long term behaviour only it was discovered that there still existed a splitting, but a measure zero set of "irregular" points needed to be removed first. More precisely, Oseledets proved the following Theorem, for a proof we refer the reader to the book [31] of Mañé.

Theorem 1.3.1 (Oseledets)**.** *Let M be smooth, compact, boundary-free Riemannian manifold of dimension n. Let $f \in \mathrm{Diff}(M)$ and for each $p \in M$ let E_p^λ denote the subspace of T_pM whose elements have characteristic exponent λ. Then there exists an f-invariant Borel subset $R \subset M$ and for each $\varepsilon > 0$ a Borel function $r_\varepsilon \colon R \to (1, \infty)$ such that for all $p \in R, v \in E_p^\lambda$ and each integer n, the following properties hold,*

(i) $\bigoplus_\lambda E_p^\lambda = T_pM$;

(ii) $\frac{1}{r_\varepsilon(p)(1+\varepsilon)^{|n|}} \leq \frac{\|\mathrm{D}_p f^{\circ n}(v)\|}{\lambda^n \|v\|} \leq r_\varepsilon(p)(1+\varepsilon)^{|n|}$;

(iii) $\angle(E_p^\Lambda, E_p\Lambda') \geq r_\varepsilon(p)^{-1}$ *if* $\Lambda \cap \Lambda' = \emptyset$;

(iv) $\frac{1}{1+\varepsilon} \leq \frac{r_\varepsilon(p)}{r_\varepsilon(p)} \leq 1 + \varepsilon$.

Moreover R has total probability, in that $\mu(R) = 1$ for any f invariant Borel probability measure μ on M. Also, the characteristic exponents, characteristic subspaces and their dimensions are Borel functions of the base space R.

Using this result as his starting point Pesin was then able to construct much of what was known for uniformly hyperbolic systems but in a measurable context. In particular he was able to prove the following Stable Manifold Theorem: there exists a partition of the space into stable manifolds which, moreover, is absolutely continuous[2] and induce conditional measures on local unstable manifolds of almost every point. For more details we recommend [16] and [43].

1.4 The Palis Conjecture

For many properties of uniformly hyperbolic systems it is reasonable to expect they occur in other systems, at least on a large scale. For example, the property of having finitely many indecomposable sets, the so-called basic sets in the hyperbolic setting, and the property that an open dense set of orbits in each indecomposable set is attracted to a subset, called the attractor, of the indecomposable set, both hold for hyperbolic systems. These are topological notions, but the results developed by Oseledets and Pesin suggested they could be carried over to a topological/measurable framework for a larger class of systems. In [41], Palis proposed that this was indeed the case - by changing the topological notions to measurable ones in the right places he conjectures that we will be able to describe all dynamical behaviour generically. We will state this conjecture more precisely below. The most topologically significant part of this conjecture is that finitude of attractors holds generically, especially since the results of Newhouse seem to suggest this should not be possible. However, the

[2]This means the holonomy maps which transport, locally, points from one unstable manifold to another are measurable and do not send zero measure sets to positive measure sets or vice versa.

notion of attractor and basic set in the measurable setting requires careful attention. For example we have the two following definitions (see the articles [35] and [36] by Milnor).

Definition 1.4.1 (Measure Attractor)**.** Let M be a Riemannian manifold and let $f \in \mathrm{Diff}^r(M)$. A closed subset $A \subset M$ is a *measure attractor* if the following properties hold,

(i) the realm of attraction $\rho(A)$, defined to be the set of all points $x \in M$ such that $\omega(x) \subset A$, has strictly positive measure (with respect to the Riemannian volume form on M);

(ii) there is no strictly smaller closed set $A' \subset A$ such that $\rho(A')$ differs from $\rho(A)$ by a set of zero measure only.

Measure attractors are sometimes called *Milnor attractors*.

Definition 1.4.2 (Statistical Attractor)**.** A closed subset $A \subset M$ is a *statistical attractor* if the following properties hold,

(i) the orbit of almost every $x \in M$ converges statistically to A, this means $\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathrm{dist}(f^{\circ i}(x), A) = 0$;

(ii) there is no strictly smaller closed set $A' \subset A$ with the same property.

Another notion that was shown to be useful in the uniformly hyperbolic case was that of a physical measure These are also referred to as SRB, BRS, or SBR-measures, named after Sinai, Ruelle and Bowen.

Definition 1.4.3 (Physical Measure)**.** Assume we are given a measurable Borel space M and a Borel transformation $T\colon M \to M$. Endow M with a background measure μ (for example, Lebesgue). A measure ν on M is a *physical measure* if it is T-invariant and for a set B_ν of positive μ-measure, $z \in B_\nu$ implies

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi \circ T^n(z) = \int_M \phi d\nu, \tag{1.4.1}$$

for any $\phi \in C^0(M, \mathbb{R})$. The set B_ν is called the *basin* of the physical measure ν.

We make the following remarks. Typically we require that the basin of attraction, B_ν, of the measure ν has full measure in an open set which contains it. Compare this definition with Birkhoff's Ergodic Theorem: in that situation ergodicity and measure preservation was required which allowed us to use L^1-observables ϕ but here we have removed ergodicity and measure preservation with the restriction that the observable be continuous.

Before we state the Palis Conjecture let us consider the following. Let M be a manifold, $\mathrm{End}^r(M)$ the space of C^r-endomorphisms. Let $\mathcal{P}^r(M)$ denote the subspace of $\mathrm{End}^r(M)$ consisting of maps with the following properties:

(i) there are finitely many attractors $A_0, A_1, \ldots, A_k$;

(ii) each attractor A_i supports a physical measure ν_i;

(iii) $\sum \mu(B_{\nu_i}) = \mu(M)$, where μ denotes the Riemannian volume of M;

The Palis Conjecture then states that for any manifold M and any degree of regularity $r \geq 1$ the space $\mathcal{P}^r(M)$ is generic in $\mathrm{End}^r(M)$. Actually it states more. Firstly given a generic, finite dimensional family f_t in $\mathrm{End}^r(M)$ for typical parameter values, there is a neighbourhood of this parameter such that for almost all parameters in that neighbourhood the corresponding endomorphism also has finitely many attractors which support a physical measure *and* for each attractor of the initial map there are finitely many attractors for the perturbation whose union of basins is 'nearly equal' to the basin of the initial map. Secondly each attractor is stochastically stable.

1.5 Renormalisation of Unimodal maps

Towards the end of the 1970's a new phenomenon in the dynamics of one dimensional unimodal maps was discovered by Feigenbaum [17], [18], and independently by Collet and Tresser [9], [10]. They observed that in many one-parameter families of unimodal maps, specifically maps with a quadratic critical point, the sequence of period doubling bifurcations accumulate to a specific parameter value and asymptotically the ratio between successive bifurcations is universal (i.e. independent of the one-parameter family). Feigenbaum's explanation of this was then (after paraphrasing) as follows:

> There exists an operator $\mathcal{R}_{\mathcal{U}}$, called the period-doubling renormalisation operator, acting on a subspace of the space of unimodal maps $\mathcal{U}$, which has a unique fixed point, which is hyperbolic with codimension-one stable manifold and dimension one local unstable manifold.

The relation to the observed phenomena is as follows. The space of unimodal maps is foliated by codimension-one manifolds whose kneading sequence is the same. The stable manifold is one of the leaves of this foliation. If the renormalisation operator is defined on one point of a leaf it is defined on the whole leaf. Moreover renormalisation will permute these leaves. Generically a one parameter family, or curve in the space of unimodal maps, intersecting the stable manifold will intersect it transversely, and hence all leaves sufficiently close will also be intersected transversely. Each period doubling bifurcation has a uniquely prescribed kneading sequence, and so they correspond to the intersection of our curve with certain singular leaves. In a neigbourhood of the fixed point each leaf, except the unstable manifold, will be pushed away from the fixed point at a geometric rate corresponding to the unstable eigenvalue. Hence these singular leaves accumulate on the unstable manifold at a geometric

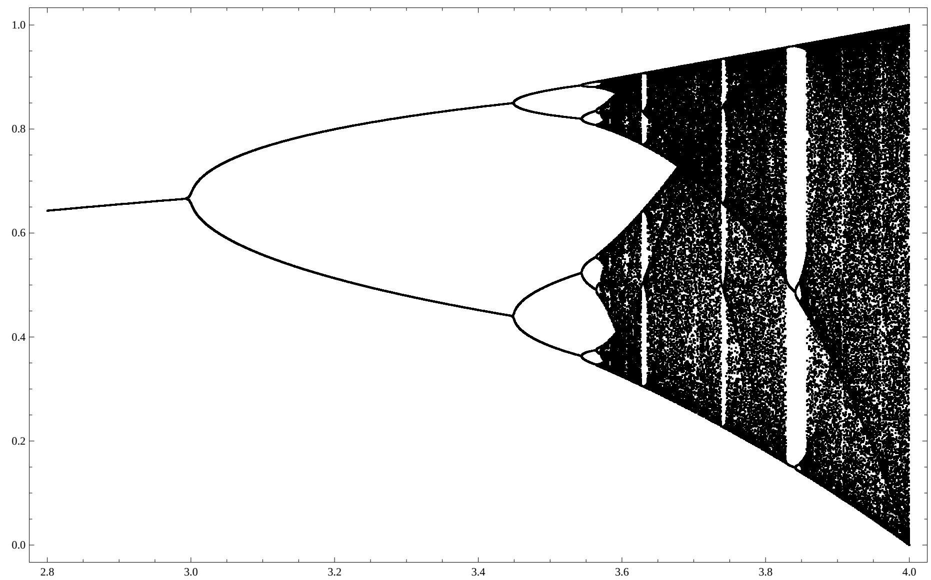

Figure 1.1: The bifurcation diagram for the family $f_\mu(x) = \mu x(1-x)$ on the interval $[0,1]$ for parameter values $2.8 \le \mu \le 4$.

rate. This means the ratios between successive bifurcations will converge to the unstable eigenvalue of the renormalisation operator.

The second aspect of renormalisation, fittingly, deals with the second aspect of the bifurcation diagram, namely what happens after the accumulation of period doubling? The picture suggests regions where the attractor consists of infinitely many points (so-called stochastic regions) and regions where there are only finitely many (regular regions). However it appears these regions are intricately interlaced. Again let us return to the kneading theoretic point of view. Firstly the period doubling bifurcations occur typically because of a monotone increase in the critical value. It was shown by Milnor and Thurston, [37], that in the particular case of the standard family, this monotone increase in critical value creates a monotone increase in the topological entropy (for details see [7] and [13]). It turns out that the onset of positive topological entropy occurs precisely at the unstable manifold of the renormalisation operator- and hence we may say renormalisation is the boundary of chaos. This is shown in two steps: first, it needs to be shown that the stochastic regions accumulate on the unstable manifold of the renormalisation operator; second, we need to show each map in this region possesses an absolutely continuous invariant measure with positive measurable entropy. Finally we invoke the variational principle.

The first conceptual proof of the first part of the Feigenbaum conjecture was given by Sullivan (see the article by Sullivan [46] or Chapter 6 of the book [13] by de Melo and van Strien). In his approach he considered a renormalisation operator acting on a the space of certain quadratic-like maps which was first constructed by Douady and Hubbard in [14]. The renormalisation of a quadratic-like map which is unimodal when restricted to a real interval coincides with the usual unimodal renormalisation of the quadratic-like map restricted to this real interval. The main tools he developed were the real and complex a priori bounds, which allows us to control the geometry of central intervals and domains respectively, and the pullback argument, which allows you to construct a quasiconformal conjugacy between two maps with the same (bounded) combinatorics. We note that the pullback argument requires real a priori bounds. Using these tools he was then able to show that two infinitely renormalisable quadratic-like maps f, g with the same (bounded) combinatorics must satisfy

$$\lim_{n\to\infty} \mathrm{dist}_{J-T}(\mathcal{R}^n_{\mathcal{U}} f, \mathcal{R}^n_{\mathcal{U}} g) = 0 \tag{1.5.1}$$

where dist_{J-T} denotes the so-called Julia-Teichmüller metric.

The equivalence of the universal (real and complex a priori bounds) and rigid (pullback argument) properties were significant for many results in unimodal dynamics, see for example [26, 29, 27, 28]. Together with works such as [33], which used real methods, this culminated in a proof of the Palis Conjecture on the space of unimodal maps with quadratic critical point and negative Schwarzian derivative, see [1] and the survey article [30] for more details.

1.6 From Dimension One to Two: Hénon maps

Period-doubling cascades were also considered by Bowen and Franks at around the same time as Feigenbaum, but in a more constructive way and on the disk instead of the interval. In [5], Bowen and Franks constructed a C^1-smooth Kupka-Smale mapping of the disk to itself such that all its periodic points were saddles. In [20], Franks and Young increased the degree of regularity to C^2-smoothness. Their motivation was a question of Smale in [44], which asked if there was a Kupka-Smale diffeomorphism of the sphere without sinks or sources. An obvious surgery, gluing two disks together, gave a map with these properties. The biggest problem with this approach was that of regularity: could this construction be extended from a C^2-smooth map to a C^∞-smooth one?

Such a map was given by Gambaudo, Tresser and van Strien in [21], but using a different strategy - instead of constructing a map combinatorially via surgery and then smoothing they considered families of maps that were already smooth and tried to locate a parameter with the desired properties. The family of maps they consider was first discussed in the paper by Collet, Eckmann and Koch [8]. Namely, they considered infinitely renormalisable unimodal maps, with doubling combinatorics, embedded in a higher dimensional space so the dynamics is preserved and examined a neighbourhood of such maps intersected with the space of embeddings. It turns out that many properties of a unimodal map are shared by those maps close by.

A complementary approach to the study of embeddings of the disk was initiated by Benedicks and Carleson in [2] at about the same time as the work by

Gambaudo, Tresser and van Strien. This was done using the tools constructed by the same authors in their proof of Jakobson's Theorem on the existence of absolutely continuous invariant measures in the standard family, see [3]. As was mentioned before, their main result was the proof of the existence of an attractor for a large set of parameters. More specifically they showed the following.

Theorem 1.6.1. *Let $F_{a,b}(x,y)=(1+y-ax^2,bx)$. Let $W_{a,b}$ denote the unstable manifold of the fixed point lying in $\mathbb{R}_+\times\mathbb{R}_+$. Then for all $c<\log 2$ there exists a $b_0>0$ such that for all $b\in(0,b_0)$ there exists a set E_b of positive (one-dimensional) Lebesgue measure such that for all $a\in E_b$ the following holds:*

(i) There exists an open set $U_{a,b}\subset\mathbb{R}_+\times\mathbb{R}_+$ such that for all $z\in U_{a,b}$,

$$\lim_{n\to\infty}\operatorname{dist}(F_{a,b}^{\circ n}(z),\overline{W}_{a,b})=0; \tag{1.6.1}$$

(ii) There exists a point $z_{a,b}^0\in W_{a,b}$ such that $\operatorname{orb}(z_{a,b}^0)$ is dense in $W_{a,b}$ and,

$$\left\|\mathrm{D}_{z_{a,b}^0}F_{a,b}^{\circ n}(0,1)\right\|\geq e^{cn}. \tag{1.6.2}$$

The first statement tells us there is a realm of attraction for the unstable manifold, and the second tells us the unstable manifold is minimal and, in some sense, expansive. The existence of a physical measure is not shown, but it is suggested by the final theorem in [21], albeit in a slightly different setting. Together these suggested the Palis Conjecture should be true for a large family of Hénon maps.

1.7 Hénon Renormalisation

In [12], de Carvalho, Lyubich and Martens constructed a period-doubling renormalisation operator for Hénon-like mappings of the form

$$F(x,y)=(f(x)-\varepsilon(x,y),x). \tag{1.7.1}$$

Here f is a unimodal map and ε was a real-valued map from the square to the positive real numbers of small size (we shall be more explicit about the maps under consideration in Sections 2 and 3). They showed that for $|\varepsilon|$ sufficiently small the unimodal renormalisation picture carries over to this case. Namely, there exists a unique renormalisation fixed point (which actually coincides with unimodal period-doubling renormalisation fixed point) which is hyperbolic with codimension one stable manifold, consisting of infinitely renormalisable period-doubling maps, and dimension one local unstable manifold. They later called this regime *strongly dissipative*.

In the period doubling case, de Carvalho, Lyubich and Martens then studied the dynamics of infinitely renormalisable Hénon-like maps F. They showed that such a map has an invariant Cantor set, $\mathcal{O}$, upon which the map acts like an adding machine. This allowed them to define the *average Jacobian* given by

$$b = \exp \int_{\mathcal{O}} \log |\mathrm{Jac}_z F| d\mu(z) \tag{1.7.2}$$

where μ denotes the unique F-invariant measure on $\mathcal{O}$ induced by the adding machine. This quantity played an important role in their study of the local behaviour of such maps around the Cantor set. They took a distinguished point, τ, of the Cantor set called the *tip*. They examined the dynamics and geometry of the Cantor set asymptotically taking smaller and smaller neighbourhoods around τ. Their two main results can then be stated as follows.

Theorem 1.7.1 (Universality at the tip)**.** *There exists a universal constant $0 < \rho < 1$ and a universal real-analytic real-valued function $a(x)$ such that the following holds: Let F be a strongly dissipative, period-doubling, infinitely renormalisable Hénon-like map. Then*

$$\mathcal{R}^n F(x, y) = (f_n(x) - b^{2^n} a(x) y(1 + \mathrm{O}(\rho^n)), x) \tag{1.7.3}$$

where b denotes the average Jacobian of F and f_n are unimodal maps converging exponentially to the unimodal period-doubling renormalisation fixed point.

Theorem 1.7.2 (Non-rigidity around the tip)**.** *Let F and $\tilde{F}$ be two strongly dissipative, period-doubling, infinitely renormalisable Hénon-like maps. Let their average Jacobians be b and $\tilde{b}$ and their Cantor sets be $\mathcal{O}$ and $\tilde{\mathcal{O}}$ respectively. Then for any conjugacy $\pi\colon \mathcal{O} \to \tilde{\mathcal{O}}$ between F and $\tilde{F}$ the Hölder exponent α satisfies*

$$\alpha \leq \frac{1}{2}\left(1 + \frac{\log b}{\log \tilde{b}}\right) \tag{1.7.4}$$

In particular if the average Jacobians b and $\tilde{b}$ differ then there cannot exist a C^1-smooth conjugacy between F and $\tilde{F}$.

For a long time it was assumed that the properties satisfied by the one dimensional unimodal renormalisation theory would also be satisfied by any renormalisation theory in any dimension. In particular, the equivalence of the universal (real and complex a priori bounds) and rigid (pullback argument) properties in this setting made it natural to think that such a relation would be realised for any reasonable renormalisation theory. That is, if universality controls the geometry of an attractor and we have a conjugacy mapping one attractor to another[3] it seems reasonable to think that we could extend such a conjugacy in a “smooth” way, since the geometry of infinitesimally close pairs of orbits cannot differ too much. The above shows that this intuitive reasoning is incorrect.

In section 3 we generalise this renormalisation operator to other combinatorial types. We show that in this case too the renormalisation picture holds

[3]this requires only combinatorial information

if $|\bar{\varepsilon}|$ is sufficiently small. Namely, for any stationary combinatorics there exists a unique renormalisation fixed point, again coinciding with the unimodal renormalisation fixed point, which is hyperbolic with codimension one stable manifold, consisting of infinitely renormalisable period-doubling maps, and dimension one local unstable manifold.

We then study the dynamics of infinitely renormalisable maps of stationary combinatorial type and show that such maps have an F-invariant Cantor set $\mathcal{O}$ on which F acts as an adding machine. We would like to note that the strategy to show that the limit set is a Cantor set in the period-doubling case does not carry over to maps with general stationary combinatorics. The reason is that in both cases the construction of the Cantor set is via 'Scope Maps', defined in sections 2 and 3, which we approximate using the so-called 'Presentation function' of the renormalisation fixed point. In the period-doubling case this is known to be contracting as the renormalisation fixed point is convex (see the result of Davie [11]). In the case of general combinatorics this is unlikely to be true. The work of Eckmann and Wittwer [15] suggests the convexity of fixed points for sufficiently large combinatorial types does not hold. The problem of contraction of branches of the presentation function was also asked in [25].

Once this is done we are in a position to define the average Jacobian and the tip of an infinitely renormalisable Hénon-like map in a way completely analogous to the period-doubling case. This then allows us, in Section 4, to generalise the universality and non-rigidity results stated above to the case of arbitrary combinatorics. We also generalise another result from [12], namely the Cantor set of an infinitely renormalisable Hénon-like map cannot support a continuous invariant line field. Our proof, though, is significantly different. This is because in the period-doubling case the argument a 'flipping' phenonmenon was observed where orientations were changed purely because of combinatorics. This argument clearly breaks down in the more general case.

Another facet of the renormalisation theory for unimodal maps is the notion of a priori bounds and bounded geometry. In chapter 5 we study the geometry of Cantor sets for infinitely renormalisable Hénon-like maps in more detail. Recall that, in the unimodal case, a priori bounds states there are uniform or eventually uniform bounds for the geometry of the images of the central interval at each renormalisation step. Namely at each renormalisation level there is a bounded decrease in size of these interval and their gaps. More precisely if J is an image of the i-th central interval, and J' is an image of the $i+1$-st central interval contained in J, then $|J'|/|J|, |L'|/|J|$ and $|R'|/|J|$ are (eventually) uniformly bounded, where L', R' are the left and right connected components of $J \setminus J'$.

Several authors have worked on consequences of a similar notion of a priori bounds in the two dimensional case. For example, in the papers of Catsigeras, Moreira and Gambaudo [6], and Moreira [38], they consider common generalisations of the model introduced by Bowen and Franks, in [5], and Franks and Young, in [20], and of the model introduced by Gambaudo, Tresser and van Strien in [21] and [22]. In [6] it is shown that given a dissipative infinitely renormalisable diffeomorphism of the disk with bounded combinatorics and bounded geometry, there is a dichotomy: either it has positive topological entropy or it is eventually period doubling. In [38] a comparison is made between the smoothness and combinatorics of the two models using the asymptotic linking number: given a period doubling, C^∞-smooth, dissipative, infinitely renormalisable diffeomorphism of the disk with bounded geometry the convergents of the asymptotic linking number cannot converge monotonically. This should be viewed as a kind of combinatorial rigidity result which, in particular, implies that Bowen-Franks-Young maps cannot be C^∞.

We would like to note,*as of yet*, there are no known examples of infinitely renormalisable Hénon-like maps with bounded geometry. In the more general case of infinitely renormalisable diffeomorphisms of the disk considered in [6] and [38], we know of no example with bounded geometry either. In fact, at least for the Hénon-like case, we will show the following result:

Theorem 1.7.3. *Let F_b be a one parameter family of infinitely renormalisable Hénon-like maps, parametrised by the average Jacobian $b = b(F_b) \in [0, b_0)$. Then there is a subinterval $[0, b_1] \subset [0, b_0)$ for which there exists a dense G_δ subset $S \subset [0, b_1)$ with full relative Lebesgue measure such that the Cantor set $\mathcal{O}(b) = \mathcal{O}(F_b)$ has unbounded geometry for all $b \in S$.*

This is the main result of chapter 5. We conclude with a discussion of future directions of research and some open problems which the current work suggests.

1.8 Notations and Conventions

First let us introduce some standard definitions. We will denote the integers by $\mathbb{Z}$, the real numbers by $\mathbb{R}$ and the complex numbers by $\mathbb{C}$. We will denote by $\mathbb{Z}_+$ the set of strictly positive integers and by $\mathbb{R}_+$ the set of strictly positive real numbers. Given real-valued functions $f(x)$ and $g(x)$ we say that $f(x)$ is $\mathrm{O}(g(x))$ if there exists $\delta > 0$ and $C > 0$ such that $|f(x)| \leq C|g(x)|$ whenever $|x| < \delta$. We say that $f(x)$ is $\mathrm{o}(g(x))$ if $\lim_{x\to 0} |f(x)/g(x)| = 0$.

Given a topological space M and a subspace $S \subset M$ we will denote its interior by $\mathrm{int}(S)$ and its closure by $\mathrm{cl}(S)$. If M is also a metric space with metric d we define the distance between subsets S and S' of M by

$$\mathrm{dist}(S, S') = \inf_{s\in S, s'\in S'} d(s, s'). \tag{1.8.1}$$

For S, S' both compact we define the *Hausdorff distance* between S and S' by

$$d_{Haus}(S, S') = \max\left\{\sup_{s\in S}\inf_{s'\in S'} d(s, s'), \sup_{s'\in S'}\inf_{s\in S} d(s, s')\right\}. \tag{1.8.2}$$

If M also has a linear structure we denote the convex hull of S by $\mathrm{Hull}(S)$.

For an integer $p \geq 2$ we set $W_p = \{0, 1, \ldots, p-1\}$. When p is fixed we will simply denote this by W. We denote by W^n the space of all words of length n and by W^* the totality of all finite words over W. We will use juxtapositional notation to denote elements of W^*, so if $\mathbf{w} \in W^*$ then $\mathbf{w} = w_0 \ldots w_n$ for some

$w_0, \dots, w_n \in W$. For all $w \in W$ and $n > 0$ we will let w^n denote $w \dots w$, where the juxtaposition is taken n times. Given $\mathbf{w} \in W$ we will denote the m-th word from the left by $\mathbf{w}(m)$ whenever it exists.

We endow W^* with the structure of a topological semi-group as follows. First endow W^* with the topology whose bases are the cylinder sets

$$[w_1 \dots w_n]_m = \{\mathbf{w} \in W^* \colon \mathbf{w}(m) = w_1, \dots, \mathbf{w}(m+n) = w_n\} \tag{1.8.3}$$

Now consider the map $m\colon W^* \times W^* \to \mathbb{Z}_+^*$, where $\mathbb{Z}_+^*$ denotes the set of words of arbitrary length over the positive integers $\mathbb{Z}_+$, given by $m(\mathbf{x}, \mathbf{y})(i) = \mathbf{x}(i) + \mathbf{y}(i)$. Then we define the map $s\colon \mathbb{Z}_+^* \to W^*$ inductively by

$$s(\mathbf{w})(i) = \begin{cases} \mathbf{w}(i) & \mathbf{w}(i-1) \in W_p \text{ and } \mathbf{w}(i) \in W_p \\ \mathbf{w}(i) + 1 & \mathbf{w}(i-1) \notin W_p \text{ and } \mathbf{w}(i) + 1 \in W_p \\ 0 & \text{otherwise} \end{cases} \tag{1.8.4}$$

The addition on W^* is given by $+\colon W^* \times W^* \to W^*, \mathbf{x} + \mathbf{y} = s \circ m(\mathbf{x}, \mathbf{y})$. Let $\mathbf{1} = (1, 0, 0, \dots)$ and let $T\colon W^* \to W^*$ be given by $T(\mathbf{w}) = \mathbf{1} + \mathbf{w}$. This map is called *addition with infinite carry*[4]. The pair (W^*, T) is called the *adding machine* over W^*. The set of all infinite words will be denoted by $\overline{W}$. Observe that T can be extended to $\overline{W}$.

Typically, we will treat the adding machine as an index set for cylinder sets of a Cantor set. The following definition[5] will also be useful.

Definition 1.8.1. Let $\mathcal{O} \subset S$ be a Cantor set, where S is a metrizable space. A *presentation* for $\mathcal{O}$ is a collection $\{B^{\mathbf{w}}\}_{\mathbf{w} \in W^*}$ of closed topological disks $B^{\mathbf{w}}$ such that, if $B^d = \bigcup_{\mathbf{w} \in W^n} B^{\mathbf{w}}$,

(i) $\operatorname{int} B^{\mathbf{w}} \cap \operatorname{int} B^{\tilde{\mathbf{w}}} = \emptyset$ for all $\mathbf{w} \neq \tilde{\mathbf{w}} \in W^*$ of the same length;

(ii) $B^d \supset B^{d+1}$ for each $n \geq 0$;

(iii) $\bigcap_{d \geq 0} B^d = \mathcal{O}$.

For $\mathbf{w} \in W^d$ we call $B^{\mathbf{w}}$ a *piece of depth d*.

Now let us describe indexing issues in some detail. Given a presentation of a Cantor set $\mathcal{O}$ we could give the pieces the indexing above or we could have given them the ordering $B^{d,i}$, where d denotes the depth and i corresponds to a linear ordering $i = 0, \dots, p^d - 1$ of all the pieces of depth d. Typically this ordering has the property that if $B^{d+1,i} \subset B^{d,j}$ then $B^{d+1,i+1} \subset B^{d+1,j+1}$. Let $\mathbf{q}\colon W^* \to \mathbb{Z}_+ \times \mathbb{Z}_+$ denote the correspondence between these two indexings.

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[4]Explicitly this is defined by

$$T(\mathbf{w}) = \begin{cases} (1 + x_0, x_1, \dots) & x_0 < p - 1 \\ (0, 0, \dots, 0, 1 + x_k, \dots) & x_0, \dots, x_{k-1} = p - 1, x_k \neq p - 1 \end{cases}$$

[5]An equivalent definition is given in [13, Chapter VI, Section 3], the only difference being the indexing. However, their definition is more general as it allows combinatorial types other than stationary type.

Given a function F we will denote its domain by $\mathrm{Dom}(F)$. Typically this will be a subset of $\mathbb{R}^n$ or $\mathbb{C}^n$. If $F\colon \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at a point $z \in \mathbb{R}^n$ we will denote the derivative of F at z by $\mathrm{D}_z F$. The *Jacobian* of F is given by

$$\mathrm{Jac}_z F = \det \mathrm{D}_z F \tag{1.8.5}$$

Given a bounded region $S \subset \mathbb{R}^n$ we will define the *distortion* of F on S by

$$\mathrm{Dis}(F; S) = \sup_{z,\tilde{z} \in S} \log \left| \frac{\mathrm{Jac}_z F}{\mathrm{Jac}_{\tilde{z}} F} \right| \tag{1.8.6}$$

and the *variation* of F on S by

$$\mathrm{Var}(F; S) = \sup_{G \in C_0^1(S) : |G(z)| \leq 1} \int_S F \mathrm{div} G dz. \tag{1.8.7}$$

According to [23], when $S \subset \mathbb{R}^2$ this coincides with

$$\mathrm{Var}(F; S) = \max \left\{ \int_{S_x} \mathrm{Var}(F; S_y) dx, \int_{S_y} \mathrm{Var}(F; S_x) dy \right\}, \tag{1.8.8}$$

i.e. the integral of the one-dimensional variations, restricted to vertical or horizontal slices, is taken in the orthogonal direction.

Given a domain $S \subset \mathbb{R}^n$ and a map $F\colon S \to \mathbb{R}^n$ we will denote its i-th iterate by $F^{\circ i}$ and, if it is a diffeomorphism onto its image, its i-th preimage by $F^{\circ -i}\colon F^{\circ i}(S) \to \mathbb{R}^n$. If F is not a map we are iterating (for example if it is a change of coordinates) then we will denote its inverse by $\bar{F}$ instead. It will become clear when considering Hénon-like maps why we need to make this distinction. It is to make our indexing conventions consistent.

Now we will restrict our attention to the one- and two-dimensional cases, both real and complex. Let $\pi_x, \pi_y \colon \mathbb{R}^2 \to \mathbb{R}$ denote the projections onto the x- and y- coordinates. We will identify these with their extensions to $\mathbb{C}^2$. (In fact we will identify all real functions with their complex extensions whenever they exist.)

Given $a, b \in \mathbb{R}$ we will denote the closed interval between them by $[a, b] = [b, a]$. We will denote $[0, 1]$ by J. For any interval $T \subset \mathbb{R}$ we will denote its boundary by ∂T, its left endpoint by $\partial^- T$ and its right endpoint by $\partial^+ T$. Given two intervals $T_0, T_1 \subset J$ we will denote an affine bijection from T_0 to T_1 by $\iota_{T_0 \to T_1}$. Typically it will be clear from the situation whether we are using the unique orientation preserving or orientation reversing bijection.

Let us denote the square $[0, 1] \times [0, 1] = J^2$ by B. We call $S \subset B$ a *rectangle* if it is the Cartesian product of two intervals. Given two points $z, \tilde{z} \in B$, the *closed rectangle spanned by z and $\tilde{z}$* is given by

$$[\![z, \tilde{z}]\!] = [\pi_x(z), \pi_x(\tilde{z})] \times [\pi_y(z), \pi_y(\tilde{z})], \tag{1.8.9}$$

and the *straight line segment between z and $\tilde{z}$* is denoted by $[z, \tilde{z}]$. Given two rectangles $B_0, B_1 \subset B$ we will denote an affine bijection from B_0 to B_1 preserving horizontal and vertical lines by $I_{B_0 \to B_1}$. Again the orientations of its components will be clear from the situation.

Let S denote the interval J or the square B. Let S' be a closed sub-interval or sub-square of S respectively. Let $\mathcal{D}_p^\omega(S') \subset \mathrm{End}^\omega(S)$ denote the subspace of endomorphisms F such that $F^{\circ p}(S') \subset S'$. Then the *zoom operator* $\mathcal{Z}_{S'} \colon \mathcal{D}_p(S') \to \mathrm{End}^\omega(S)$ is given by

$$\mathcal{Z}_{S'}F = I_{S' \to S} \circ F^{\circ p} \circ I_{S \to S'} \colon S \to S \tag{1.8.10}$$

where $I_{S \to S'} \colon S \to S'$ denotes the orientation-preserving affine bijection between S and S' which preserves horizontal and vertical lines. We note that in certain situations it will be more natural to change orientations but in these cases we shall be explicit.

Let $\Omega_x \subseteq \Omega_y \subset \mathbb{C}$ be simply connected domains compactly containing J and let $\Omega = \Omega_x \times \Omega_y$ denote the resulting polydisk containing B.

Appendices

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Samenvatting

Het doel van deze scriptie is het ontwikkelen van een renormalisatietheorie voor de Hénon familie,

$$F_{a,b}(x,y) = (a = x^2 - by, x), \tag{†}$$

voor combinatoriek die verschilt van periode-verdubbelen, op een manier die vergelijkbaar is met die voor de standaard unimodale familie $f_a(x) = a - x^2$. Dit werk bestaat uit twee delen. Na het bespreken van de achtergrond die nodig is in de unimodale renormalisatietheorie, waar een ruimte $\mathcal{U}$ van unimodale afbeeldingen en een operator $\mathcal{R}_{\mathcal{U}}$ die werkt op een deelruimte van $\mathcal{U}$ worden beschouwd, construeren we een ruimte $\mathcal{H}$ –de sterk dissipatieve Hénon-achtige afbeeldingen– en een operator $\mathcal{R}$ die werkt op een deelruimte van $\mathcal{H}$. De ruimte $\mathcal{U}$ wordt op een kanonieke manier afgebeeld in de rand van $\mathcal{H}$. We laten zien dat $\mathcal{R}$ een dynamisch-gedefinieerde continue operator is die een continue uitbreiding is van $\mathcal{R}_{\mathcal{U}}$, de operator die werkt op $\mathcal{U}$. Het klassieke renormalisatieplaatje klopt nog steeds: er bestaat een uniek renormalisatie vast punt dat hyperbolisch is, een stabiele variëteit van codimensie één heeft die bestaat uit alle afbeeldingenen die oneindig vaak gerenormaliseerd kunnen worden, en een lokaal onstabiele variëteit van dimensie één heeft.

Dan worden Hénon afbeeldingen bestudeerd die oneindig vaak gerenormaliseerd kunnen worden. We laten zien dat, net als in het unimodale geval, zulke afbeeldingen invariante Cantor verzalimingen hebben waarop een unique invariante kansmaat leeft. We construeren een metriek invariant, de gemiddelde Jacobiaan. Met gebruik hiervan bestuderen we het dynamische gedrag van afbeeldingen die oneindig vaak renormaliseerd kunnen worden rond een voorgeschreven punt, de 'tip'. We laten zien dat er net als in het unimodale geval universaliteit bestaat bij dit tip. We laten ook zien dat het dynamische gedrag bij de punt niet-regide is: twee afbeeldingen met verschillende gemiddelde Jacobianen kunnen nooit C^1 worden geconjugeerd met een diffeomorfisme dat de tip vast houdt.

Tot slot laten we zien dat de meetkunde van deze Cantor verzamelingen, zowel generiek als qua metriek, onbegrensd is in één-parameter families van afbeeldingen die oneindig vaak renormaliseerd kunnen worden en aan een transversale eis voldoen.

Summary

The aim of this thesis is to develop a renormalisation theory for the Hénon family

$$F_{a,b}(x,y) = (a - x^2 - by, x) \tag{$\dagger$}$$

for combinatorics other than period-doubling in a way similar to that for the standard unimodal family $f_a(x) = a - x^2$. This work breaks into two parts. After recalling background needed in the unimodal renormalisation theory, where a space $\mathcal{U}$ of unimodal maps and an operator $\mathcal{R}_{\mathcal{U}}$ acting on a subspace of $\mathcal{U}$ are considered, we construct a space $\mathcal{H}$ –the strongly dissipative Hénon-like maps– and an operator $\mathcal{R}$ which acts on a subspace of $\mathcal{H}$. The space $\mathcal{U}$ is canonically embedded in the boundary of $\mathcal{H}$. We show that $\mathcal{R}$ is a dynamically-defined continuous operator which continuously extends $\mathcal{R}_{\mathcal{U}}$ acting on $\mathcal{U}$. Moreover the classical renormalisation picture still holds: there exists a unique renormalisation fixed point which is hyperbolic, has a codimension one stable manifold, consisting of all infinitely renormalisable maps, and a dimension one local unstable manifold.

Infinitely renormalisable Hénon-like maps are then examined. We show, as in the unimodal case, that such maps have invariant Cantor sets supporting a unique invariant probability. We construct a metric invariant, the average Jacobian. Using this we study the dynamics of infinitely renormalisable maps around a prescribed point, the 'tip'. We show, as in the unimodal case, universality exists at this point. We also show the dynamics at the tip is non-rigid: any two maps with differing average Jacobians cannot be C^1-conjugated by a tip-preserving diffeomorphism.

Finally it is shown that the geometry of these Cantor sets is, metrically and generically, unbounded in one-parameter families of infinitely renormalisable maps satisfying a transversality condition.