Final Exam SOLUTIONS

MAT 131

Fall 2011

1. Compute the following limits.

(a) $\displaystyle\lim_{x\to 1^-} \frac{x^2+1}{x^2-1}$

The numerator is always positive, whereas the denominator is negative for numbers slightly smaller than 1. Also, as $x \to 1^-$, the numerator approaches 2 while the denominator approaches 0. The limit is therefore $\boxed{-\infty}$.

(b) $\displaystyle\lim_{x\to 0} \frac{\sin(x)}{2^x-1}$

Both the numerator and the denominator approach 0 as $x \to 0$. We apply L'Hôpital's Rule:

$$\lim_{x\to 0} \frac{\sin(x)}{2^x-1} = \lim_{x\to 0} \frac{\cos(x)}{(\ln 2)2^x} = \boxed{\frac{1}{\ln 2}}.$$

(c) $\displaystyle\lim_{h\to 0} \frac{\sqrt{3+h}-\sqrt{3-h}}{h}$

Again both the numerator and the denominator tend to 0 as $h \to 0$. This time, we perform an algebraic manipulation:

$$\begin{aligned}\lim_{h\to 0} \frac{\sqrt{3+h}-\sqrt{3-h}}{h} &= \lim_{h\to 0} \frac{(3+h)-(3-h)}{h(\sqrt{3+h}+\sqrt{3-h})} \\ &= \lim_{h\to 0} \frac{2h}{h(\sqrt{3+h}+\sqrt{3-h})} \\ &= \lim_{h\to 0} \frac{2}{(\sqrt{3+h}+\sqrt{3-h})} = \boxed{\frac{1}{\sqrt{3}}}.\end{aligned}$$

(d) $\displaystyle\lim_{x\to\infty} \frac{x^3+2x+1}{3x^2+17x+219}$

The degree of the numerator in this rational function is higher than the degree of the denominator, and so the limit is $\boxed{\infty}$.

2. Compute the derivatives of the following functions.

(a) $f(x) = \log_5(x^2+3)$

$$f'(x) = \frac{1}{\ln 5}\cdot\frac{2x}{x^2+3}$$

(b) $f(x) = 2^x x^{3/2}$

$$f'(x) = \frac{3}{2}2^x x^{1/2} + (\ln 2)2^x x^{3/2} = 3\cdot 2^{x-1}\sqrt{x} + (\ln 2)2^x x^{3/2}$$

(c) $f(x) = \dfrac{\cos(x)+1}{\sin(x)-1}$

$$f'(x) = \frac{(\sin(x)-1)(-\sin(x)) - (\cos(x)+1)(\cos(x))}{(\sin(x)-1)^2} = \frac{\sin(x)-\cos(x)-1}{(\sin(x)-1)^2}$$

(d) $f(x) = \arctan\sqrt{x^2-1}$

$$f'(x) = \frac{1}{1+(\sqrt{x^2-1})^2}\cdot\frac{2x}{2\sqrt{x^2-1}} = \frac{1}{x^2}\cdot\frac{x}{\sqrt{x^2-1}} = \frac{1}{x\sqrt{x^2-1}}$$

(e) $g(u) = \dfrac{1}{u^2} + \sqrt{u} + u^3$

$$g'(u) = -\frac{2}{u^3} + \frac{1}{2\sqrt{u}} + 3u^2$$

(f) $f(t) = (\cos t)^{\tan t}$

We use logarithmic differentiation. First set $y = f(t)$, so that $\ln y = (\tan t)(\ln\cos t)$. Then

$$\frac{y'}{y} = \tan t\cdot\frac{-\sin t}{\cos t} + \sec^2 t\cdot\ln\cos t = -\tan^2 t + \sec^2 t\cdot\ln\cos t$$

Finally, multiply both sides by y to get

$$y' = (\cos t)^{\tan t}\big(-\tan^2 t + \sec^2 t\cdot\ln\cos t\big).$$

3. Let F be the function

$$F(x) = \frac{x^2}{x^3+4}.$$

(a) Find $\lim_{x\to\infty} F(x)$ and $\lim_{x\to-\infty} F(x)$.

Because the degree of the denominator is greater than the degree of the numerator, both limits are $\boxed{0}$.

(b) The domain of F consists of all real numbers except one number a. What is a? Find $\lim_{x\to a^-} F(x)$ and $\lim_{x\to a^+} F(x)$.

The number a is the where the denominator equals zero, i.e., $a = -4^{1/3}$. For values of x greater than a, the denominator is positive, whereas for values of x less than a, the denominator is negative. In all cases, the numerator is ≥ 0. Thus

$$\lim_{x\to a^-} F(x) = -\infty \qquad \text{and} \qquad \lim_{x\to a^+} F(x) = \infty.$$

(c) Compute the first derivative of F.

The first derivative of F is $F'(x) = \dfrac{(x^3+4)(2x) - x^2(3x^2)}{(x^3+4)^2} = \boxed{\dfrac{-x^4+8x}{(x^3+4)^2}}$.

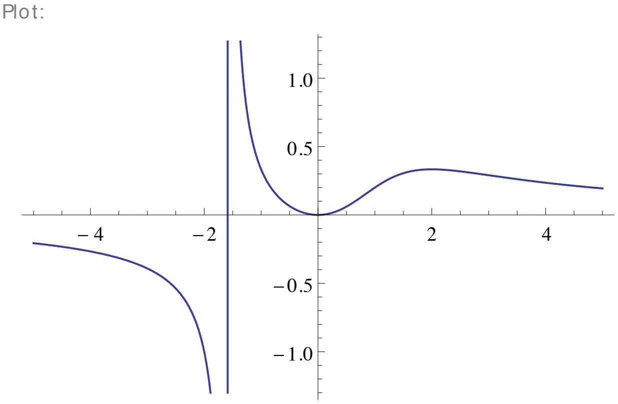
For future reference, the second derivative of F is $F''(x) = \dfrac{2(x^6 - 28x^3 + 16)}{(4+x^3)^3}$.

(d) Find all the local extreme values of F and where they occur. For each of these values, specify whether it is a local maximum or a local minimum.

$F'(x)$ equals zero when its numerator is zero, i.e., when $x(-x^3+8) = 0$. The real solutions to this equation are $x = 0$ and $x = 2$.
We compute $F(0) = 0$ and $F(2) = \dfrac{4}{8+4} = \dfrac{1}{3}$.
Because $F''(0) = \dfrac{2(16)}{4^3} > 0$ and $F''(2) = \dfrac{2(2^6 - 28(2^3)+16)}{(4+2^3)^3} < 0$, we conclude by the Second Derivative Test that $\boxed{\text{0 is a local minimum}}$ at $x = 0$ and $\boxed{\text{1/3 is a local maximum}}$ at $x = 2$.

(e) Sketch the graph of F. (Plot below made using Wolfram|Alpha.)

Plot:



There is a vertical asymptote at $x = -4^{1/3}$, and a horizontal asymptote at $y = 0$.

4. A TV camera is positioned 600 meters away from the rocket launch pad. The rocket is launched, rising vertically up, and the camera is kept aimed at the rocket, tracking its motion. At the moment the rocket's altitude is 600 meters, its speed is 100 m/s. How fast is the camera's angle of elevation changing at this moment? (The answer should be written in rad/s.)

Let y be the altitude of the rocket and θ the camera's angle of elevation. Then we have the relation

$$\tan\theta = \frac{y}{600}.$$

Differentiating with respect to time, we obtain

$$(\sec^2\theta)\frac{d\theta}{dt} = \frac{1}{600}\frac{dy}{dt}.$$

We are interested in the moment in time when $y = 600$ and $dy/dt = 100$. At this point, $\tan\theta = 1$, and so

$$\sec^2\theta = \left(\sqrt{2}\right)^2 = 2.$$

Thus we have the equation

$$2 \cdot \frac{d\theta}{dt} = \frac{1}{600} \cdot 100,$$

or

$$\boxed{\frac{d\theta}{dt} = \frac{1}{12}\ \text{rad/s}}.$$

5. Find the linearization of the function $f(x) = e^{-x}\sin(\pi x)$ at the point $a = 2$.

First we compute

$$f'(x) = e^{-x}(\pi\cos(\pi x)) - e^{-x}\sin(\pi x).$$

At $a = 2$, this has the value

$$f'(2) = \pi e^{-2}\cos(2\pi) - e^{-2}\sin(2\pi) = \frac{\pi}{e^2}.$$

At $a = 2$, we also have

$$f(2) = e^{-2}\sin(2\pi) = 0.$$

The linearization of f at a is $L(x) = f(a) + f'(a)(x-a)$, which becomes

$$L(x) = f(2) + f'(2)(x-2) = 0 + \frac{\pi}{e^2}(x-2) = \boxed{\frac{\pi(x-2)}{e^2}}.$$

6. Maximize the area of a rectangle that can be inscribed in the ellipse

$$\frac{x^2}{4} + y^2 = 1.$$

A rectangle inscribed in the above ellipse has corners (x, y), $(-x, y)$, $(-x, -y)$, and $(x, -y)$. Thus we want to maximize the function $A = (2x)(2y) = 4xy$. Solving the given equation for y and replacing it in the area function, we get (assuming $x \geq 0$)

$$A(x) = 4x\sqrt{1 - \frac{x^2}{4}}.$$

We will find the critical points of the square of this function. Set

$$f(x) = \big(A(x)\big)^2 = 16x^2\left(1 - \frac{x^2}{4}\right) = 16x^2 - 4x^4.$$

Then

$$f'(x) = 32x - 16x^3 = 16x(2 - x^2).$$

This vanishes when $x = 0$ (yielding an area of 0) or when $2 - x^2 = 0$, i.e., when $x = \pm\sqrt{2}$. Clearly $f(\sqrt{2}) = f(-\sqrt{2})$ is the local (in fact, global) maximum of f. Therefore the maximum area is

$$A(\sqrt{2}) = 4\sqrt{2} \cdot \sqrt{1 - \frac{2}{4}} = 4\sqrt{2} \cdot \frac{1}{\sqrt{2}} = \boxed{4}.$$

7. Compute each of the following (definite or indefinite) integrals.

(a) $\displaystyle\int_0^2 \big(x^3 + 2x + 1\big)\, dx$

$$= \left[\frac{x^4}{4} + x^2 + x\right]_0^2 = \frac{2^4}{4} + 2^2 + 2 - 0 = 4 + 4 + 2 = \boxed{10}$$

(b) $\displaystyle\int \big(3e^x + 2\sin x\big)\, dx$

$$= \boxed{3e^x - 2\cos x + C}$$

(c) $\displaystyle\int \sqrt{3x+1}\, dx$

First we substitute $u = 3x + 1$, so that $du = 3\,dx$. The integral becomes

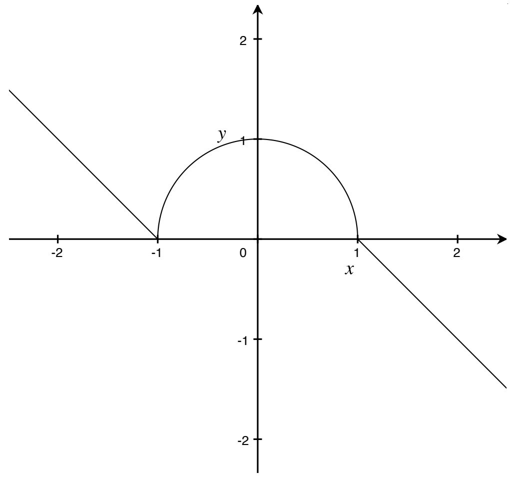
$$\frac{1}{3}\int \sqrt{u}\, du = \frac{1}{3} \cdot \frac{2}{3} \cdot u^{3/2} + C = \boxed{\frac{2}{9}(3x+1)^{3/2} + C}$$

(d) $\displaystyle\int_0^{\sqrt{\pi}} x\sin(x^2)\, dx$

We set $u = x^2$, so that $du = 2x\,dx$. When $x = 0$, $u = 0$, and when $x = \sqrt{\pi}$, $u = \pi$, so the integral becomes

$$\frac{1}{2}\int_0^{\pi} \sin u\, du = \frac{1}{2}\big[-\cos u\big]_0^{\pi} = \frac{1}{2}(-\cos\pi + \cos 0) = \frac{1}{2}(1+1) = \boxed{1}.$$

8. Below is the graph of a continuous function f. The graph is composed of the upper half of the unit circle and two half-lines with slope -1.



(a) Compute $\int_0^2 f(x)\,dx$. (*Hint:* You do not need to find an explicit antiderivative for f.)

The integral is equal to the area of one-quarter of the unit circle (the portion of the graph of f above the x-axis) minus the area of a right triangle with legs of length 1 (the portion of the graph below the x-axis). Thus

$$\int_0^2 f(x)\,dx = \boxed{\frac{\pi}{4} - \frac{1}{2}}.$$

(b) Define $g(x) = \int_{-3}^{x} f(t)\,dt$. What are the critical points of g? For each critical point c, determine whether $g(c)$ is a local minimum, a local maximum, or neither.
(*Note:* Critical points are also called "critical numbers".)

By the Fundamental Theorem of Calculus, g is differentiable everywhere and its critical points occur when $f = 0$. The critical points of g are therefore $\boxed{1}$ and $\boxed{-1}$. The value $g(1)$ is a $\boxed{\text{local maximum}}$ because just before 1 f is positive (g is increasing) and just after 1 f is negative (g is decreasing). The value $g(-1)$ is $\boxed{\text{neither a maximum nor a minimum}}$ because f does not change sign from positive to negative, or vice versa, at -1.