SEMI-STABLE LOCUS OF A GROUP COMPACTIFICATION

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Abstract. In this paper, we consider the diagonal action of a connected semisimple group of adjoint type on its wonderful compactification. We show that the semi-stable locus is a union of the G-stable pieces and we calculate the geometric quotient.

0.1. **Introduction.** Let G be a connected, semisimple algebraic group of adjoint type over an algebraically closed field and X be its wonderful compactification. We will give an explicit description of the semi-stable locus of X (for the diagonal G-action) using Lusztig's G-stable pieces and calculate the geometric quotient $X//G$. We also deal with the case where the G-action is twisted by a diagram automorphism.

The results will be used by the first author [He] to study character sheaves on the wonderful compactification.

During the time the article was writing, we learned that De Concini, Kanna and Maffei [CKM] described the semi-stable locus and geometric quotient for complete symmetric varieties (which includes as a special case the non-twisted conjugation action of G on its wonderful compactification).

0.2. **Geometric invariant theory.** The foundations of geometric invariant theory are developed in [MFK94]. We quickly review that part which we use. Let k be a field. The setup for geometric invariant theory over k consists of $(G, X, \tau, \mathcal{L}, \psi)$ where

(i) G is a reductive algebraic group over k,
(ii) X is a separated, finite type k-scheme,
(iii) $\tau : G \times X \to X$ is an algebraic action of G on X,
(iv) $\mathcal{L}$ is an invertible sheaf on X, and
(v) $\psi : \tau^*\mathcal{L} \to \mathrm{pr}_X^*\mathcal{L}$ is a *G-linearization* of $\mathcal{L}$ (where $\mathrm{pr}_X : G \times X \to X$ is the projection), i.e., an isomorphism of invertible sheaves on $G \times X$ which defines a lifting of the action τ to an action of G on $\mathrm{Spec}_X \mathrm{Sym}^\bullet(\mathcal{L})$.

The fundamental theorem of geometric invariant theory, [MFK94, Theorem 1.10, p. 38], associates to this datum a pair $(X^{\mathrm{ss}}(\mathcal{L}), \phi)$. Here $X^{\mathrm{ss}}(\mathcal{L})$ is the union X_s over all positive integers n and all G-invariant sections s of $\Gamma(X, \mathcal{L}^{\otimes n})$, provided X_s is affine (recall, X_s is defined to be the maximal open subscheme of X on which s is a generator of $\mathcal{L}^{\otimes n}$).

And ϕ is a G-invariant k-morphism

$$\phi : X^{\mathrm{ss}}(\mathcal{L}) \to X//_{\mathcal{L}}G$$

which is a *uniform categorical quotient* of the action of G on $X^{\mathrm{ss}}(\mathcal{L})$. Moreover the following hold.

(i) The morphism ϕ is affine and universally submersive.

(ii) For some integer $n > 0$, there exists an ample invertible sheaf $\mathcal{M}$ on $X//_{\mathcal{L}}G$ such that $\phi^*\mathcal{M}$ is isomorphic to $\mathcal{L}^{\otimes n}$ as G-linearized invertible sheaves (in particular, $X//_{\mathcal{L}}G$ is quasi-projective).

(iii) There exists a unique open subscheme U of $X//_{\mathcal{L}}G$ such that $\phi^{-1}(U)$ is the *stable locus.* And the induced morphism $\phi : \phi^{-1}(U) \to U$ is a *uniform geometric quotient* of $\phi^{-1}(U)$.

Since we do not make use of them, we will not make precise the definitions of uniform categorical quotient, stable locus and uniform geometric quotient. But we will use a few other known facts about geometric invariant theory.

Fact 1. When X is projective and $\mathcal{L}$ is ample, every open X_s is affine. Thus $X//_{\mathcal{L}}G$ is canonically isomorphic to

$$X//_{\mathcal{L}}G = \mathrm{Proj} \oplus_{n\geq 0} \Gamma(X, \mathcal{L}^{\otimes n})^G$$

and $X^{\mathrm{ss}}(\mathcal{L})$ is the maximal open subscheme of X on which the natural rational map from X to $X//_{\mathcal{L}}G$ is defined, [Ses77].

Fact 2. Again when X is proper, every G-orbit O in $X^{\mathrm{ss}}(\mathcal{L})$ contains a unique closed G-orbit in its closure (in $X^{\mathrm{ss}}(\mathcal{L})$). And two G-orbits O_1 and O_2 in $X^{\mathrm{ss}}(\mathcal{L})$ are in the same fiber of ϕ if and only if the associated closed G-orbits are equals. In particular, ϕ establishes a natural bijection between the points of $X//_{\mathcal{L}}G$ and the closed G-orbits in $X^{\mathrm{ss}}(\mathcal{L})$, [Ses77].

Fact 3.(Matsushima's criterion) A G-orbit O is affine if and only if the stabilizer group of one (and hence every) closed point is itself reductive, [Ric77]. In particular, since the fibers of ϕ are affine, every closed G-orbit in $X^{\mathrm{ss}}(\mathcal{L})$ is affine, and hence has reductive stabilizer group.

Fact 4. If X is normal (or if $X^{\mathrm{ss}}(\mathcal{L})$ is normal), then ϕ factors through the normalization of the target. Thus by the universal property , the target $X//_{\mathcal{L}}G$ is normal.

0.3. **Notations.** Now we fix the notations used in the rest of this article. Let G be a connected semisimple algebraic group of adjoint type over an algebraically closed field k. Let B be a Borel subgroup of G, B^- be an opposite Borel subgroup and $T = B \cap B^-$. Let $(\alpha_i)_{i\in I}$ be the set of simple roots determined by (B, T). We denote by W the Weyl group $N(T)/T$. For $w \in W$, we choose a representative $\dot{w}$ in $N(T)$. For $i \in I$, we denote by ω_i and s_i the fundamental weight and the simple reflection corresponding to α_i.

For $J \subset I$, let $P_J \supset B$ be the standard parabolic subgroup defined by J and let $P_J^- \supset B^-$ be the parabolic subgroup opposite to P_J. Set $L_J = P_J \cap P_J^-$. Then L_J is a Levi subgroup of P_J and P_J^-. The semisimple quotient of L_J of adjoint type will be denoted by G_J. We denote by π_{P_J} (resp. $\pi_{P_J^-}$) the projection of P_J (resp. P_J^-) onto G_J. Let W_J be the subgroup of W generated by $\{s_j \mid j \in J\}$ and W^J be the set of minimal length coset representatives of W/W_J.

0.4. **Wonderful compactification of G.** We consider G as a $G \times G$-variety by left and right translation. Then there exists a canonical $G \times G$-equivariant embedding X of G which is called the *wonderful compactification* ([DCP83], [Str87]). The variety X is an irreducible, smooth projective $(G \times G)$-variety with finitely many $G \times G$-orbits Z_J indexed by the subsets J of I. The boundary $X - G$ is a union of smooth divisors $\overline{Z_{I-\{i\}}}$ (for $i \in I$), with normal crossing. The $G \times G$-variety Z_J is isomorphic to the product $(G \times G) \times_{P_J^- \times P_J} G_J$, where $P_J^- \times P_J$ acts on $G \times G$ by $(q, p) \cdot (g_1, g_2) = (g_1 q^{-1}, g_2 p^{-1})$ and on G_J by $(q, p) \cdot z = \pi_{P_J^-}(q) z \pi_{P_J}(p)^{-1}$. We denote by h_J the image of $(1, 1, 1)$ in Z_J under this isomorphism.

0.5. **Twisted actions.** We follow the approach in [HT06, Section 3]. Let σ be an automorphism on G such that $\sigma(B) = B$ and $\sigma(T) = T$. We also assume that σ is a diagram automorphism, i.e., the order of σ coincides with the order of the associated permutation on I.

Let G_σ (resp. X_σ) be the $(G \times G)$-variety which as a variety is isomorphic to G (resp. X), but the $G \times G$-action is twisted by $(g, g') \mapsto (g, \sigma(g'))$. Then G_σ is an open $G \times G$-subvariety of X_σ and we call X_σ the wonderful compactification of G_σ.

Under the natural bijection between X and X_σ, we may identify the $G \times G$-orbits on X with the $G \times G$-orbits on X_σ. We denote by $Z_{J,\sigma}$ the $G \times G$-orbit on X_σ that corresponds to $Z_{\sigma(J)} \subset X$. Accordingly, we denote by $h_{J,\sigma}$ the base point in $Z_{J,\sigma}$ which corresponds to the base point $h_{\sigma(J)}$ of $Z_{\sigma(J)}$.

0.6. **σ-semisimple elements in G_σ.** We follow the notation of [Spr06]. An element $g \in G_\sigma$ is called *σ-semisimple* if it is conjugated to an element in T. We have the following result.

Theorem 0.1. *Let $g \in G_\sigma$. Then the following conditions are equivalent:*

(1) The element g is σ-semisimple.
(2) The G-orbit of g is closed in G_σ.
(3) The isotropy subgroup of g in G is reductive.

The equivalence of (1) and (2) can be found in [Lus03, 1.4 (e)] (in terms of disconnected groups instead of twisted conjugation action). In the case of simply connected group, the equivalence is also proved in [Spr06, Proposition 3]. The equivalence of (2) and (3) follows from Fact 3, Matsushima's criterion.

0.7. **G-stable-piece decomposition.** Let G_Δ be the diagonal image of G in $G \times G$. The classification of the G_Δ-orbits on X was obtained by Lusztig [Lus04] in terms of G-stable pieces. A similar result also occurs in [EL06]. We list some known results which will be used later.

For $J \subset I$ and $w \in W^{\sigma(J)}$, set

$$Z_{J,\sigma;w} = G_\Delta(B\dot{w}, B) \cdot h_{J,\sigma}.$$

We call $Z_{J,\sigma;w}$ a *G-stable piece* of X_σ. By [Lus04, 12.3] and [He06, Proposition 2.6], X_σ is a disjoint union of the G-stable pieces.

Fact 5. $X_\sigma = \bigsqcup_{J \subset I} \bigsqcup_{w \in W^{\sigma(J)}} Z_{J,\sigma;w}$.

Set $I(J, \sigma; w) = \max\{K \subset J \mid w\sigma(K) = K\}$. Then the subvariety $L_{I(J,\sigma;w)}\dot{w}$ of G_σ is stable under the action of $L_{I(J,\sigma;w)} \times L_{I(J,\sigma;w)}$ and in particular, is stable under the conjugation action of $L_{I(J,\sigma;w)}$. Moreover, by [Lus04, 12.3(a)] and [He07, Lemma 1.4],

$$Z_{J,\sigma;w} = G_\Delta(L_{I(J,\sigma;w)}\dot{w}, 1) \cdot h_{J,\sigma}$$

and there exists a natural bijection between the G_Δ-orbits on $Z_{J,\sigma;w}$ and the $L_{I(J,\sigma;w)}$-orbits on $L_{I(J,\sigma;w)}\dot{w}/Z^0(L_J) \subset G_\sigma/Z^0(L_J)$ (for the conjugation action of $L_{I(J,\sigma;w)}$).

For any point z in $Z_{J,\sigma;w}$, the isotropy subgroup

$$G_z = \{g \in G \mid (g, g) \cdot z = z\}$$

was described explicitly in [EL06, Theorem 3.13]. We only need the following special case in our paper.

Fact 6. Let $z = (gl\dot{w}, g) \cdot h_{J,\sigma}$ for $g \in G$ and $l \in L_{I(J,\sigma;w)}$. Then G_z is reductive if and only if $w = 1$ and l is a σ-semisimple element in $L_{I(J,\sigma;1)}$.

By [He07, Theorem 4.5], the closure of each G-stable piece is a union of G-stable pieces and the closure relation can be described explicitly. More precisely, for $J \subset I$, $w \in W^{\sigma(J)}$ and $w' \in W$, we write $w' \leqslant_{J,\sigma} w$ if there exists $u \in W_J$ such that $w' \geqslant uw\sigma(u)^{-1}$. Then

$$\overline{Z_{J,\sigma;w}} = \sqcup_{J' \subset J} \sqcup_{w' \in W^{J'}, w' \leqslant_{J,\sigma} w} Z_{J',\sigma;w'}.$$

Notice that if $1 \leqslant_{J,\sigma} w$, then we must have $w = 1$. Therefore,

Fact 7. $\sqcup_{J \subset I} Z_{J,\sigma;1}$ is open in X_σ.

0.8. **Nilpotent Cone of X.** For any dominant weight λ, let $\mathrm{H}(\lambda)$ be the dual Weyl module for G_{sc} with lowest weight $-\lambda$. Let ${}^{\sigma}\mathrm{H}(\lambda)$ be the G_{sc}-module which as a vector space is $\mathrm{H}(\lambda)$, but the G_{sc}-action is twisted by the automorphism σ on G_{sc}. Then there exists (up to a nonzero constant) a unique G_{sc} isomorphism ${}^{\sigma}\mathrm{H}(\lambda) \to \mathrm{H}(\sigma(\lambda))$. In particular, if $\lambda = \sigma(\lambda)$, then we have an isomorphism $f_\lambda : {}^{\sigma}\mathrm{H}(\lambda) \to \mathrm{H}(\lambda)$.

By [DCS99, 3.9], there exists a $G \times G$-equivariant morphism

$$\rho_\lambda : X \to \mathbb{P}\big(\mathrm{End}(\mathrm{H}(\lambda))\big)$$

which extends the morphism $G_\sigma \to \mathbb{P}\big(\mathrm{End}(\mathrm{H}(\lambda))\big)$ defined by $g \mapsto g[\mathrm{Id}_\lambda]$, where $[\mathrm{Id}_\lambda]$ denotes the class representing the identity map on $\mathrm{H}(\lambda)$ and g acts by the left action. We denote by $\mathcal{L}_X(\lambda)$ the $G_{\mathrm{sc}} \times G_{\mathrm{sc}}$-linearized invertible sheaf on X which is the pullback under $\rho_\lambda$of $\mathcal{O}(1)$ with its canonical linearization. This is the "usual" linearized invertible sheaf on X associated to the weight λ, e.g., as defined in [BP00, p. 100]. For sufficiently divisible and positive n, the $G_{\mathrm{sc}} \times G_{\mathrm{sc}}$-linearization of $\mathcal{L}_X(\lambda)^{\otimes n} = \mathcal{L}_X(n \cdot \lambda)$ factors through a $G \times G$-linearization. This induces a G_Δ-linearization of $\mathcal{L}_X(\lambda)^{\otimes n}$. If moreover, λ is regular, then $\mathcal{L}_X(\lambda)$ is ample (see [Str87, section 2]).

The morphism ρ_λ induces a $G \times G$-equivariant morphism $X_\sigma \to \mathbb{P}\big(\mathrm{Hom}({}^{\sigma}\mathrm{H}(\lambda), \mathrm{H}(\lambda))\big)$. When $\lambda = \sigma(\lambda)$, we may apply the isomorphism $f_\lambda : {}^{\sigma}\mathrm{H}(\lambda) \to \mathrm{H}(\lambda)$ to obtain the $G \times G$-equivariant morphism

$$\rho_{\lambda,\sigma} : X_\sigma \to \mathbb{P}\big(\mathrm{End}(\mathrm{H}(\lambda))\big).$$

As above, $\mathcal{L}_{X_\sigma}(\lambda, \sigma)$ denotes the $G_{\mathrm{sc}} \times G_{\mathrm{sc}}$-linearized invertible sheaf on X_σ which is the pullback under $\rho_{\lambda,\sigma}$ of $\mathcal{O}(1)$ with its canonical linearization. Of course X_σ equals X as varieties, and $\mathcal{L}_X(\lambda)$ equals $\mathcal{L}_{X_\sigma}(\lambda, \sigma)$ as invertible sheaves on this variety. But the $G \times G$-actions are not the same, and thus the $G \times G$-linearized invertible sheaves are not the same.

For $\lambda = \sigma(\lambda)$, let $\mathcal{N}(\lambda)_\sigma$ be the subvariety of X_σ consisting of elements that may be represented by a nilpotent endomorphism of $\mathrm{H}(\lambda)$. We call $\mathcal{N}(\lambda)_\sigma$ the *nilpotent cone* of X_λ associated to the dominant weight λ. We have an explicit description of $\mathcal{N}(\lambda)$ which was obtained in [HT06, Proposition 4.4]

$$\mathcal{N}(\lambda)_\sigma = \bigsqcup_{J \subset I} \bigsqcup_{\substack{w \in W^{\sigma(J)} \\ I(\lambda) \cap \mathrm{supp}(w) \neq \varnothing}} Z_{J,\sigma;w},$$

where $I(\lambda) = \{i \in I \mid a_i \neq 0\}$ of I for $\lambda = \sum_{i \in I} a_i \omega_i$ and $\mathrm{supp}(w) \subset I$ is the set of simple roots whose associated simple reflections occur in some (or equivalently, any) reduced decomposition of w.

Two subvarieties of X related to the nilpotent cones of X are of special interest. One is

$$\cap_{\lambda \text{ is dominant}} \mathcal{N}(\lambda)_\sigma = \sqcup_{J \subset I} \sqcup_{w \in W^{\sigma(J)}, \text{supp}(w)=I} Z_{J,\sigma;w}.$$

This subvariety is actually the boudary of the closure in X_σ of unipotent subvariety of G_σ in the case where G is simple (See [He06, Theorem 4.3] and [HT06, Theorem 7.3]).

The other one is $X_\sigma - \cup_{\lambda \text{ is dominant}} \mathcal{N}(\lambda)_\sigma = \sqcup_{J \subset I} Z_{J,\sigma;1}$, which is the complement of $\mathcal{N}(\lambda)_\sigma$ for any σ-stable dominant regular weight. By the next theorem, this subvariety is actually the semi-stable locus of X_σ for the G_Δ-action.

Theorem 0.2. *For λ as above, i.e., σ-stable, dominant and regular, the semistable locus $(X_\sigma)^{ss}(\mathcal{L}_X(\lambda)^{\otimes n})$ equals $\sqcup_{J \subset I} Z_{J,\sigma;1}$. In particular, the semistable locus is independent of the choice of weight λ.*

Proof. We simply write the semistable locus $(X_\sigma)^{\text{ss}}(\mathcal{L}_X(\lambda)^{\otimes n})$ as X_σ^{ss}. On $\text{End}(H(\lambda))$ the characteristic polynomial map

$$\chi : \text{End}(H(\lambda)) \to k[t], \qquad (f : H(\lambda) \to H(\lambda)) \mapsto \chi_f(t)$$

is a morphism which is invariant under the conjugation action. The coefficients of the characteristic polynomial define homogeneous polynomials on $\text{End}(H(\lambda))$ which are invariant under the conjugation action. Also the degree is positive except for the leading coefficient (which is 1). Thus each non-leading coefficient defines a G_Δ-invariant sections of positive power $\mathcal{O}(n)$ on $\mathbb{P}(\text{End}(H(\lambda)))$. The pullbacks of these sections are G_Δ-invariant sections of positive powers $\mathcal{L}^{\otimes n}$. By Fact 1, the nonvanishing locus of each of these sections is in the semistable locus. Equivalently, the non-semistable locus is contained in the common zero locus of all of these sections. But the common zero locus of these pullback sections on X_σ equals the inverse image of the common zero locus of the original sections on $\mathbb{P}(\text{End}(H(\lambda)))$. And this common zero locus is precisely the nilpotent cone in $\mathbb{P}(\text{End}(H(\lambda)))$. Thus the non-semistable locus is contained in $\mathcal{N}(\lambda)_\sigma$. So X_σ^{ss} contains $X_\sigma - \mathcal{N}(\lambda)_\sigma$, i.e., X_σ^{ss} contains $\sqcup_{J \subset I} Z_{J,\sigma;1}$.

Also, by Fact 7, $X_\sigma^{ss} - \sqcup_{J \subset I} Z_{J,\sigma;1}$ is closed in X_σ^{ss}. If X_σ^{ss} strictly contains $\sqcup_{J \subset I} Z_{J,\sigma;1}$, then there exists a closed G_Δ-orbit in X_σ^{ss} that is not contained in $\sqcup_{J \subset I} Z_{J,\sigma;1}$. Let z be an element in that orbit. By Fact 3 above, the isotropy subgroup of z, $\{g \in G \mid (g,g) \cdot z = z\}$, is reductive. By Fact 5 above, z is in $Z_{J,\sigma;w}$ for some $J \subset I$ and $w \in W^{\sigma(J)}$ with $w \neq 1$. But this contradicts Fact 6 above. Therefore X_σ^{ss} equals $\sqcup_{J \subset I} Z_{J,\sigma;1}$. ◻

0.9. Set $\bar{T}^0 = \sqcup_{J \subset I} \cdot (T,1) h_{J,\sigma}$ and $\bar{T}' = (N_\sigma)_\Delta \cdot \bar{T}^0$, where N_σ is the inverse image of W^σ under the map $N_G(T) \to N_G(T)/T = W$. In the case where σ acts trivially on W, $\bar{T}'$ is the closure of T in X and it is just the toric variety associated with the fan of Weyl chambers (see [BK05, Lemma 6.1.6 (ii)]).

Lemma 0.3. *A G_Δ-orbit $\mathcal{O}$ in X^{ss} is closed in X^{ss} if and only if it intersects $\bar{T}^0$.*

Proof. By Fact 6, the closed G_Δ-orbits in X^{ss} are of the form $\{(gl, g) \cdot h_J \mid g \in G\}$ for some σ-semisimple element $l \in L_J$. By Theorem 0.1, l is σ-conjugate to an element in T. Thus a G_Δ-orbit $\mathcal{O} \in X^{ss}$ is closed if and only if $\mathcal{O} \cap \bar{T}^0 \neq \emptyset$ (or equivalently, $\mathcal{O} \cap \bar{T}' \neq \emptyset$). □

Lemma 0.4. *For every element z in $\bar{T}'$, the intersection $G_\Delta \cdot z \cap \bar{T}'$ of the G_Δ-orbit with $\bar{T}'$ equals the $(N_\sigma)_\Delta$-orbit $(N_\sigma)_\Delta \cdot z$.*

Proof. Obviously $(N_\sigma)_\Delta \cdot z$ is contained in $G_\Delta \cdot z \cap \bar{T}'$. The content of the lemma is the opposite inclusion.

Since $\bar{T}'$ equals $(N_\sigma)_\Delta \cdot \bar{T}^0$, we may assume without loss of generality that z has the form $z = (t, 1) \cdot h_{J,\sigma}$. Suppose that $(g, g) \cdot z$ equals $(t', 1) \cdot h_{J,\sigma}$ for some $t' \in T$, i.e., $(g, g) \cdot z$ is a point of $G_\Delta \cdot z \cap \bar{T}'$.

Denote $\cap_i \sigma^i(J)$ by J_σ. Let $F_{J,\sigma} = (P_{J_\sigma}, P_{J_\sigma}) \cdot h_{J,\sigma}$, then by [He07, Proposition 1.10], the action of G on X induces an isomorphism of $Z_{J,1;\sigma}$ with $G \times_{P_{J_\sigma}} F_{J,\sigma}$. Thus g is in P_{J_σ}. Also both t and t' are contained in the same P_{J_σ}-orbit in $L_{J_\sigma}\sigma / Z^0(L_J)$, i.e. in the same G_J-conjugacy class. Hence there exists an element n in N'_σ such that t' equals ntn^{-1} (see [Lus03, 1.14(d)]). Therefore $G_\Delta \cdot z \cap \bar{T}^0$ is a subset of $(N_\sigma)_\Delta \cdot z$. Now also

$$\begin{aligned} G_\Delta \cdot z \cap \bar{T}' &= G_\Delta \cdot z \cap (N_\sigma)_\Delta \cdot \bar{T}^0 = (N_\sigma)_\Delta \cdot (G_\Delta \cdot z \cap \bar{T}^0) \\ &\subset (N_\sigma)_\Delta \cdot ((N_\sigma)_\Delta \cdot z) = (N_\sigma)_\Delta \cdot z \end{aligned}$$

proving the lemma. □

Corollary 0.5. *The embedding $\bar{T}' \to X_\sigma$ induces a morphism*

$$i : \bar{T}'/N_\sigma \to X_\sigma//G$$

which is bijective on points. If char(k) equals 0, i is an isomorphism. Also, if σ is the identity map (and char(k) is arbitrary), then $\bar{T}'/N_\sigma$ equals $\bar{T}/W$ and the induced morphism

$$i : \bar{T}/W \to X//G$$

is an isomorphism.

Proof. The morphism $\bar{T}' \to X_\sigma \to X_\sigma//G$ is N_σ-invariant, and hence factors through a morphism

$$i : \bar{T}'/N_\sigma \to X_\sigma//G.$$

By Lemma 0.3, every closed G-orbit in X_σ^{ss} intersects $\bar{T}'$. Thus i is surjective. For every element z in $\bar{T}$, the G-orbit of z is closed in X_σ^{ss}. Thus two elements z, z' in $\bar{T}$ have the same image under i if and only if they lie in the same G-orbit. On the other hand, by Lemma 0.4, two elements z, z' in $\bar{T}$ lie in the same G-orbit if and only if they lie in the same N_σ-orbit. Hence i is a bijection on points.

Assume for the moment that char(k) is 0. By Fact 4, X_σ^{ss} is a normal k-variety. And i is a dominant morphism of varieties which is a bijection on points. Since char(k) is zero, this implies that i is birational. By Zariski's Main Theorem, a bijective, birational morphism of varieties is an isomorphism if the target is normal. Thus i is an isomorphism when char(k) is 0. In positive characteristic, the possibility remains that i may be a purely inseparable morphism.

Next assume that s is the identity map, but char(k) may be arbitrary. Then $\bar{T}'/N_\sigma$ equals $\bar{T}/W$ for the natural W-action on $\bar{T}$ which extends the W-action on T. By [Ste65, section 6], the restriction of i to the open subvariety T/W of $\bar{T}/W$ gives an isomorphism $T/W \cong G//G$. Hence, as above, i is a bijective, birational morphism of varieties whose target is a normal variety. So again by Zariski's Main Theorem, i is an isomorphism. □

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