THE MAXIMAL FREE RATIONAL QUOTIENT

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Abstract. This short, expository note proves existence of the maximal quotient of a variety by free rational curves.

1. Definition of a maximal free rational quotient

Definition 1.1. Let V be a Deligne-Mumford stack over a field k, and denote the smooth locus by $V^{\mathrm{sm}} \subset V$. A 1-morphism $f : \mathbb{P}^1_k \to V^{\mathrm{sm}}$ is a *free rational curve to* V if f^*T_V is generated by global sections and has positive degree.

Let S be an irreducible algebraic space, let $\pi_{\overline{X}} : \overline{X} \to S$ be a proper, locally finitely presented 1-morphism of Deligne-Mumford stacks with integral geometric generic fiber, and let $X \subset \overline{X}$ be a normal dense open substack. Denote by $\pi_X : X \to S$ the restriction of $\pi_{\overline{X}}$.

Definition 1.2. A *free rational quotient* of π_X is a triple (X^*, Q^*, ϕ) where $X^* \subset X$ is a dense open substack, where Q^* is a normal algebraic space, finitely presented over S with integral geometric generic fiber, and where $\phi : X^* \to Q^*$ is a dominant 1-morphism of S-stacks satisfying,

(i) the geometric generic fiber F of ϕ is integral, and
(ii) a general pair of distinct points of F is contained in the image of a free rational curve.

A free rational quotient is *trivial* if $\phi : X^* \to Q^*$ is birational, and *nontrivial* otherwise.
A free rational quotient $(X^*, Q^*, \phi : X^* \to Q^*)$ is *maximal* if for every free rational quotient $(X_1^*, Q_1^*, \phi_1 : X_1^* \to Q_1^*)$ there exists a dense open subset $U \subset Q_1^*$ and a smooth morphism $\psi : U \to Q^*$ such that $\phi|_{\phi_1^{-1}(U)} = \psi \circ \phi_1$.

Theorem 1.3. *There exists a maximal free rational quotient.*

It is not true that a maximal free rational quotient is unique, but it is unique up to unique birational equivalence of Q^*.

2. Proof of Theorem 1.3

The proof is very similar to the proofs of existence of the rational quotient in [1] and [2]. Existence of the maximal free rational quotient can be deduced from theorems there. However there are 2 special features of this case: The relation of containment in a free rational curve is already a rational equivalence relation, so existence of the quotient is less technical than the general case. Also, unlike the

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general case, there is no need to perform a purely inseparable base-change of S to define the quotient. Of course if $\mathcal{O}_S$ contains $\mathbb{Q}$ and if π is smooth and proper, [3, 1.1] implies the free rational quotient is the rational quotient.

If there is no free rational curve in the geometric generic fiber of π_X, then the trivial rational quotient $(X, X, \mathrm{Id}_X : X \to X)$ is a maximal free rational quotient. Therefore assume there is a free rational curve to X whose image is contained in the smooth locus of π_X (equivalently, there is a free rational curve in the geometric generic fiber of π_X). Use will be made of the flat, proper morphism $C \to S$ obtained from 2 copies of $\mathbb{P}^1_S$ by identifying the 0 section in the first copy to the 0 section in the second copy.

Denote by $\mathrm{Hom}_S(\mathbb{P}^1_S, X)$ the Deligne-Mumford stack constructed in [4]. Define $H \subset \mathrm{Hom}_S(\mathbb{P}^1_S, X)$ to be the open substack parametrizing free rational curves in fibers of the smooth locus of π_X. By hypothesis, H is nonempty.

Let $H_i \subset H$ be a connected component. Denote by,

$$u_i : H_i \times_S \mathbb{P}^1_S \to \overline{X},$$

and by,

$$u_i^{(2)} : H_i \times_S \mathbb{P}^1_S \times_S \mathbb{P}^1_S \to \overline{X} \times_S \overline{X},$$

the obvious 1-morphisms.

Lemma 2.1. *The* 1*-morphism* u_i *is smooth.*

Proof. The proof is the same as in the case that π_X is a projective morphism, [2, Cor. II.3.5.4]. $\square$

Denote by $W_i \subset \overline{X} \times_S \overline{X}$ the closed image of $u_i^{(2)}$, i.e., W_i is the minimal closed substack such that $u_i^{(2)}$ factors through W_i. For any geometric point x of $\overline{X}$ with residue field $\kappa(x)$, denote by $(W_i)_x \subset X \otimes_{\mathcal{O}_S} \kappa(x)$ the scheme $\mathrm{pr}_2(\mathrm{pr}_1^{-1}(x) \cap W_i)$.

Because $H_i \times_S \mathbb{P}^1_S \times_S \mathbb{P}^1_S$ is irreducible and reduced, also W_i is irreducible and reduced. Because u_i is smooth, $\mathrm{pr}_1 : W_i \to \overline{X}$ is surjective on geometric points. The geometric generic fiber has pure dimension

$$d_i = \dim(W_i \otimes_{\mathcal{O}_S} K(S)) - \dim(X \otimes_{\mathcal{O}_S} K(S)).$$

Of course d_i is bounded by $\dim(X \otimes_{\mathcal{O}_S} K(S))$. Let H_i be a connected component such that d_i is maximal.

Let H_j be any connected component of H. Denote by $V_{i,j}$ the 2-fiber product,

$$V_{i,j} = (H_i \times_S \mathbb{P}^1_S) \times_{u_i, X, u_j} (H_j \times_S \mathbb{P}^1_S).$$

In other words $V_{i,j}$ parametrizes data $(([f_i], t_i), ([f_j], t_j), \theta)$ where f_i, resp. f_j, is an object of H_i, resp. H_j, where t_i, t_j are points of $\mathbb{P}^1$, and where $\theta : f_i(t_i) \to f_j(t_j)$ is an equivalence of objects. Denote by $F_{i,j}$ the 1-morphism of S-stacks,

$$F_{i,j} : V_{i,j} \times_S \mathbb{P}^1_S \times_S \mathbb{P}^1_S \to \overline{X} \times_S \overline{X},$$

that sends a datum $((([f_i], t_i), ([f_j], t_j), \theta), t'_i, t'_j)$ to $(f_i(t'_i), f_j(t'_j))$. Alternatively $F_{i,j}$ is the 1-morphism whose domain is,

$$(H_i \times_S \mathbb{P}^1_S \times_S \mathbb{P}^1_2) \times_{u_i \circ \mathrm{pr}_{1,3}, X, u_j \circ \mathrm{pr}_{1,2}} (H_j \times_S \mathbb{P}^1_S \times_S \mathbb{P}^1_S),$$

such that $\mathrm{pr}_1 \circ F_{i,j}$ is $u_i \circ \mathrm{pr}_{1,2}$ and such that $\mathrm{pr}_2 \circ F_{i,j}$ is $u_j \circ \mathrm{pr}_{1,3}$.

Proposition 2.2. *The image of $F_{i,j}$ is contained in W_i.*

Proof. Note that $V_{i,j}$ is smooth over S. Moreover, because u_i and u_j are smooth, $V_{i,j}$ is nonempty. Let $((f_i, t_i), (f_j, t_j), \theta)$ be a point of $V_{i,j}$. There is a reducible, connected genus 0 curve C obtained by identifying t_i in one copy of $\mathbb{P}^1$ to t_j in a second copy of $\mathbb{P}^1$. The morphisms f_i, f_j and the equivalence θ induce a 1-morphism $f : C \to X$ whose restriction to the first irreducible component is f_i and whose restriction to the second irreducible component is f_j. In a suitable sense, $f : C \to X$ is still a free rational curve, and it deforms to free rational curves $f' : \mathbb{P}^1 \to X$. After examining these deformations, the proposition easily follows.

Let $0 : S \to \mathbb{A}^1_S \times_S \mathbb{P}^1_S$ be the morphism whose projection to each factor is the zero section. Denote by $Z \subset \mathbb{A}^1_S \times_S \mathbb{P}^1_S$ the image of 0. Denote by P the blowing up of $\mathbb{A}^1_S \times_S \mathbb{P}^1_S$ along Z. The projection morphism $\mathrm{pr}_{\mathbb{A}^1} : P \to \mathbb{A}^1_S$ is flat and projective. Moreover, the restriction of P over $\mathbb{G}_{m,S} \subset \mathbb{A}^1_S$ is canonically isomorphic to $\mathbb{G}_{m,S} \times_S \mathbb{P}^1_S$. And the restriction of P over the zero section of $\mathbb{A}^1_S$ is canonically isomorphic to the curve C over S obtained by identifying 0 in one copy of $\mathbb{P}^1$ to 0 in a second copy of $\mathbb{P}^1$.

Denote $S' = \mathbb{A}^1_S$. Denote by $\mathrm{Hom}_{S'}(P, X_{S'})$ the Deligne-Mumford stack constructed in [4]. Denote by $H' \subset \mathrm{Hom}_{S'}(P, X_{S'})$ the open substack parametrizing morphisms to the smooth locus of $(\pi_X)'$ such that the pullback of $T_{(\pi_X)'}$ restricts to a globally generated sheaf of positive degree on every irreducible component of every geometric fiber. The morphism $H' \to S'$ is smooth for reasons similar to [2, Cor. II.3.5.4].

Let $V_{i,j,k}$ be a connected component of $V_{i,j}$. Define $f_{V,i} : V_{i,j,k} \times_S \mathbb{P}^1_S \to X$ to be the composition

$$u_i \circ (\mathrm{pr}_{H_i} \circ \mathrm{pr}_{H_i \times_S \mathbb{P}^1_S}, \mathrm{Id}_{\mathbb{P}^1_S}) : V_{i,j,k} \times_S \mathbb{P}^1_S \to H_i \times_S \mathbb{P}^1_S \to X.$$

Define $f_{V,j} : V_{i,j,k} \times_S \mathbb{P}^1_S \to X$ to be the composition $u_j \circ (\mathrm{pr}_{H_j} \circ \mathrm{pr}_{H_j \times_S \mathbb{P}^1_S}, \mathrm{Id}_{\mathbb{P}^1_S})$. Define $s_i : V_{i,j,k} \to V_{i,j,k} \times_S \mathbb{P}^1_S$ to be the unique $V_{i,j,k}$-morphism such that $\mathrm{pr}_{\mathbb{P}^1_S} \circ s_i = \mathrm{pr}_{\mathbb{P}^1_S} \circ \mathrm{pr}_{H_i \times_S \mathbb{P}^1_S}$. Define $s_j : V_{i,j,k} \to V_{i,j,k} \times_S \mathbb{P}^1_S$ similarly.

Replacing $V_{i,j,k}$ by a dense open subset, there exist 2 isomorphisms of $V_{i,j,k}$-schemes,

$$\alpha_i, \alpha_j : V_{i,j,k} \times_S \mathbb{P}^1_S \to V_{i,j,k} \times_S \mathbb{P}^1_S,$$

such that $s_i = \alpha_i \circ 0$ and $s_j = \alpha_j \circ 0$ where 0 is the zero section of $V_{i,j,k} \times_S \mathbb{P}^1_S \to V_{i,j,k}$. There is a unique 1-morphism of S-stacks,

$$f_{i,j,k} : V_{i,j,k} \times_S C \to X,$$

such that the restriction of $f_{i,j,k}$ to the first irreducible component of $V_{i,j,k} \times_S C$ is $f_{V,i} \circ \alpha_i$, and the restriction to the second irreducible component is $f_{V,j} \circ \alpha_j$. The image of $f_{i,j,k}$ is contained in the smooth locus of π_X, and the restriction of $f^*_{i,j,k} T_{\pi_X}$ to each irreducible component is generated by global sections relative to $V_{i,j,k}$. Denote by,

$$f^{(2)}_{i,j,k} : V_{i,j,k} \times_S C \times_S C \to \overline{X} \times_S \overline{X},$$

the obvious 1-morphism.

By definition of H' there is a 1-morphism,

$$q : V_{i,j,k} \to S \times_{0,S'} H',$$

such that the pullback by q of the universal morphism is 2-equivalent to $f_{i,j,k}$. Because $V_{i,j,k}$ is connected, the image of q is contained in a connected component H'_l of H'. By definition of H there is a 1-morphism of S-stacks,

$$r : \mathbb{G}_{m,S} \times_{S'} H'_l \to H,$$

such that the restriction to $\mathbb{G}_{m,S} \times_{S'} H'_l$ of the universal morphism over H'_l is 2-equivalent to the pullback by r of the universal morphism over H. Denote by H_l the connected component of H dominated by r. The morphism r dominates a connected component $H_l \subset H$. There exists a 1-isomorphism,

$$i : \mathbb{G}_{m,S} \times_{S'} H'_l \to \mathbb{G}_{m,S} \times_S H_l,$$

unique up to unique 2-equivalence, such that $\mathrm{pr}_{\mathbb{G}_m} \circ i = \mathrm{pr}_{\mathbb{G}_m}$ and such that $\mathrm{pr}_{H_l} \circ i$ is 2-equivalent to r.

Denote by,

$$v_l : H'_l \times_{S'} P \to \overline{X},$$

and by,

$$v_l^{(2)} : H'_l \times_{S'} P \times_{S'} P \to \overline{X} \times_S \overline{X},$$

the obvious morphisms. Denote by $W'_l \subset \overline{X} \times_S \overline{X}$ the minimal closed substack through which $v_l^{(2)}$ factors. Because $\mathrm{pr}_{S'} : H'_l \times_{S'} P \times_{S'} P \to S'$ is flat, the preimage of $\mathbb{G}_{m,S} \subset S'$ is dense. Thus W'_l is the image of the restriction of $v_l^{(2)}$ over $\mathbb{G}_{m,S}$. The restriction of $v_l^{(2)}$ is 2-equivalent to $u_l^{(2)} \circ (r, \mathrm{Id}_{\mathbb{P}^1}, \mathrm{Id}_{\mathbb{P}^1})$. Therefore the image of $v_l^{(2)}$ equals the image of $u_l^{(2)}$, i.e., $W'_l = W_l$.

On the other hand, the pullback of $v_l^{(2)}$ to $V_{i,j,k} \times_S C \times_S C$ is 2-equivalent to $f_{i,j,k}^{(2)}$. There are 2 irreducible components of C, and thus 4 irreducible components of $C \times_S C$. Restrict $f_{i,j,k}^{(2)}$ to the irreducible component of $C \times_S C$ that is the product of the first irreducible component of C and the first irreducible component of C. This is 2-equivalent to the pullback of $u_i^{(2)}$, hence W_l contains W_i. Because W_l is an integral stack of dimension at most d_i containing the d_i-dimensional stack W_i, W_l equals W_i.

Finally, restrict $f_{i,j,k}^{(2)}$ to the irreducible component of $C \times_S C$ that is the product of the first irreducible component of C and the second irreducible component of C. This is 2-equivalent to the pullback of $F_{(i,j)}$, hence $W_i = W_l$ contains the image of $F_{(i,j)}$. □

Lemma 2.3. *The geometric generic fiber of $pr_1 : W_i \to \overline{X}$ is integral.*

Proof. Denote by K the algebraic closure of the function field of $\overline{X}$. Since $u_i : H_i \times_S \mathbb{P}^1_S \to \overline{X}$ is smooth, the geometric generic fiber $(H_i \times_S \mathbb{P}^1_S) \otimes_{\mathcal{O}_{\overline{X}}} K$ is smooth over K. There is an induced 1-morphism,

$$(u_i^{(2)})_K : (H_i \times_S \mathbb{P}^1_S \otimes_{\mathcal{O}_{\overline{X}}} K) \times_{\mathrm{Spec}\ (K)} \mathbb{P}^1_K \to (\overline{X} \times_S \overline{X}) \otimes_{\mathrm{pr}_1, \overline{X}} \mathrm{Spec}\ (K) \cong \overline{X} \otimes_{\mathcal{O}_S} K$$

Formation of the closed image is compatible with flat base change. Therefore the closed image of $(u_i^{(2)})_K$ is $W_i \otimes_{\mathrm{pr}_1, \overline{X}} \mathrm{Spec}\ (K)$. In particular, $W_i \otimes_{\mathrm{pr}_1, \overline{X}} \mathrm{Spec}\ (K)$ is reduced since the closed image of a reduced stack is reduced.

The proof that the geometric generic fiber is irreducible is essentially the same as the proof of Proposition 2.2. Let $W' \subset W_i \otimes_{\mathrm{pr}_1,\overline{X}} \mathrm{Spec}\ (K)$ be an irreducible component. To prove that $W' = W_i \otimes_{\mathrm{pr}_1,\overline{X}} \mathrm{Spec}\ (K)$, it suffices to prove that it contains the image of $(u_i^{(2)})_K$. Let $x \in \overline{X} \otimes_{\mathcal{O}_S} K$ be the K-point corresponding to the diagonal; x is contained in $\mathrm{Image}(u_i^{(2)})_K$. Let y_1 be a K-point of $W' \cap \mathrm{Image}(u_i^{(2)})_K$ and let y_2 be a K-point of $\mathrm{Image}(u_i^{(2)})_K$. There are free K-morphisms, $f_1, f_2 : \mathbb{P}^1_K \to X \otimes_{\mathcal{O}_S} K$ such that $f_1(0) = f_2(0) = x$ and $f_1(\infty) = y_1, f_2(\infty) = y_2$. This defines a morphism from $C \otimes_{\mathcal{O}_S} K$ to $X \otimes_{\mathcal{O}_S} K$. As in the proof of Proposition 2.2, deformations of this morphism are free rational curves that come from a connected component H_l of H. By construction, there is an irreducible component of $W_l \otimes_{\mathrm{pr}_1,X} \mathrm{Spec}\ (K)$ that contains W and y_2. Since the dimension of W_l is at most d_i, this irreducible component equals W. Therefore $y_2 \in W$, i.e., $W = W_i \otimes_{\mathrm{pr}_1,X} \mathrm{Spec}\ (K)$. ☐

Consider the projection $\mathrm{pr}_1 : W_i \to \overline{X}$. By [5, Thm. 3.2], there exists a dense open subset $X^{\mathrm{flat}} \subset X$ over which W_i is flat. Denote $W_i^{\mathrm{flat}} = W_i \times_{\mathrm{pr}_1,X} X^{\mathrm{flat}}$. By Lemma 2.3, there is a dense open substack $X^0 \subset X^{\mathrm{flat}}$ such that every geometric fiber of $W_i \times_{\mathrm{pr}_1,X} X^0 \to X^0$ is integral. Denote $W_i^0 = W_i \times_{\mathrm{pr}_1,\overline{X}} X^0$.

Let $H_j \subset H$ be a connected component and denote by $G_{i,j}$ the unique 1-morphism,

$$G_{i,j} : (H_j \times_S \mathbb{P}^1_S \times_S \mathbb{P}^1_S) \times_{u_j \circ \mathrm{pr}_{1,3},\overline{X},\mathrm{pr}_1} W_i \to \overline{X} \times_S \overline{X},$$

such that,

$$\mathrm{pr}_1 \circ G_{i,j} = u_j \circ \mathrm{pr}_{1,2} \circ \mathrm{pr}_{H_j \times \mathbb{P}^1 \times \mathbb{P}^1},$$

and such that,

$$\mathrm{pr}_2 \circ G_{i,j} = \mathrm{pr}_2 \circ \mathrm{pr}_{W_i}.$$

Corollary 2.4. (i) *The image of $G_{i,j}$ is contained in W_i.*

(ii) *For every geometric point $s \in S$, for every free morphism $f : \mathbb{P}^1_s \to X_s$ and for every point $x \in X_s$ such that $f(\mathbb{P}^1) \cap (W_i)_x$ is nonempty, $f(\mathbb{P}^1)$ is contained in $(W_i)_x$.*

(iii) *The image of the "composition morphism",*

$$c : W_i \times_{pr_2,\overline{X},pr_1} W_i^0 \to \overline{X} \times_S \overline{X},$$

is contained in W_i.

(iv) *For every geometric point $s \in S$, for every pair of closed points $(x, y) \in \overline{X}_s \times X_s^0$, if $y \in (W_i)_x$ then $(W_i)_y \subset (W_i)_x$.*

(v) *For every geometric point $s \in S$, for every pair of closed points $(x, y) \in X_s^0 \times X_s^0$, $y_i \in (W_i)_x$ iff $x \in (W_i)_y$ iff $(W_i)_x = (W_i)_y$.*

Proof. **(i):** First of all, the projection morphism,

$$\mathrm{pr}_{W_i} : (H_j \times_S \mathbb{P}^1_S \times_S \mathbb{P}^1_S) \times_{\overline{X}} W_i \to W_i,$$

is smooth. Therefore every connected component of the domain is integral and dominates W_i. So to prove W_i contains the image of $G_{i,j}$ it suffices to first base-change by,

$$u_i^{(2)} : H_i \times_S \mathbb{P}^1_S \times_S \mathbb{P}^1_S \to W_i.$$

The base-change of $G_{i,j}$ by $u_i^{(2)}$ is 2-equivalent to $F_{i,j}$. By Proposition 2.2 the W_i contains the image of $F_{i,j}$. Therefore W_i contains the image of $G_{i,j}$.

(ii): By construction, $W_i \subset \overline{X} \times_S \overline{X}$ is symmetric with respect to permuting the factors. Let $t' \in \mathbb{P}^1$ be a point such that $x' = f(t')$ is in $(W_i)_x$. Let H_j be the connected component of H that contains $[f]$. Then the subset,

$$\left\{(([f], t, t'), (x', x)) \in (H_j \times_S \mathbb{P}^1_S \times_S \mathbb{P}^1_S) \times W_i | t \in \mathbb{P}^1_s\right\},$$

is contained in,

$$(H_j \times_S \mathbb{P}^1_S \times_S \mathbb{P}^1_S) \times_{\overline{X}} W_i.$$

Therefore by (i), W_i contains the image under $G_{i,j}$. Because W_i is symmetric, this implies that $(W_i)_x$ contains $f(\mathbb{P}^1)$,

(iii): The 1-morphism c satisfies $\mathrm{pr}_1 \circ c = \mathrm{pr}_1 \circ \mathrm{pr}_1$ and $\mathrm{pr}_2 \circ c = \mathrm{pr}_2 \circ \mathrm{pr}_2$, up to 2-equivalence. Because $\mathrm{pr}_1 : W_i^{\mathrm{flat}} \to \overline{X}$ is flat and the geometric fibers are integral also the projection,

$$\mathrm{pr}_1 : W_i \times_{\mathrm{pr}_2, \overline{X}, \mathrm{pr}_1} W_i^{\mathrm{flat}} \to W_i,$$

is flat and the geometric fibers are integral. In particular the domain is integral. Hence to prove W_i contains the image of c it suffices to first base-change by,

$$u_i^{(2)} : H_i \times_S \mathbb{P}^1_S \times_S \mathbb{P}^1_S \to W_i.$$

After base-change, this morphism factors through $G_{i,i}$. By (i), the W_i contains the image of $G_{i,i}$. Therefore W_i contains the image of c.

(iv) and (v): Item (iv) follows immediately from (iii), and Item (v) follows from (iv) and symmetry of W_i. □

Lemma 2.5. *Let k be a field, let $g : Y \to Z$ be a morphism of smooth Deligne-Mumford stacks over k, and let $f : \mathbb{P}^1_k \to Y$ be a free morphism such that $f(\mathbb{P}^1)$ is contained in a fiber of g. Denote by Y^{sm} the smooth locus of g. If $f(\mathbb{P}^1) \cap Y^{sm}$ is nonempty, then $f(\mathbb{P}^1) \subset Y^{sm}$.*

Proof. There is a morphism of locally free sheaves on $\mathbb{P}^1_k$,

$$dg : f^*T_Y \to g^*f^*T_Z.$$

Because $f(\mathbb{P}^1) \cap Y^{\mathrm{sm}}$ is nonempty, the cokernel of dg is torsion. Because $f(\mathbb{P}^1)$ is contained in a fiber of g, $g^*f^*T_Z \cong \mathcal{O}^r_{\mathbb{P}^1_k}$ for some nonnegative integer r. Because f^*T_Y is generated by global sections, also the image of dg is generated by global sections. But the only coherent subsheaf of $\mathcal{O}^r_{\mathbb{P}^1_k}$ whose cokernel is torsion and that is generated by global sections is *all* of $\mathcal{O}^r_{\mathbb{P}^1_k}$. Therefore dg is surjective, i.e., $f(\mathbb{P}^1) \subset Y^{\mathrm{sm}}$. □

By [5, Thm. 1.1], the Hilbert functor of $\overline{X} \to S$ is represented by an algebraic space that is separated and locally finitely presented, $\mathrm{Hilb}_{\overline{X}/S}$. And $W_i^{\mathrm{flat}} \subset X^{\mathrm{flat}} \times_S \overline{X}$ is a closed substack that is proper, flat and finitely presented over X^{flat}. Therefore there is a 1-morphism of S-stacks,

$$\phi^{\mathrm{flat}} : X^{\mathrm{flat}} \to \mathrm{Hilb}_{\overline{X}/S},$$

such that W_i^{flat} is the pullback by ϕ^{flat} of the universal closed substack. Denote by $X^* \subset X$ the maximal open substack over which ϕ^{flat} extends to a morphism. Denote by $Q^* \to \mathrm{Hilb}_{X/S}$ the Stein factorization of $X^* \to \mathrm{Hilb}_{X/S}$, i.e., the integral closure of the image in the function field of the coarse moduli space $|X^*|$. Denote by $\phi : X^* \to Q^*$ the induced morphism.

Proposition 2.6. *The morphism $\phi : X^* \to Q^*$ is a free rational quotient.*

Proof. By construction, Q^* is normal, $Q^* \to S$ is a finitely presented morphism whose geometric generic fiber is integral, and ϕ is a dominant 1-morphism whose geometric generic fiber is integral. It remains to prove Definition 1.2 (ii).

By [5], $\mathrm{Hilb}_{\overline{X}/S}$ satisfies the valuative criterion of properness (but it is not necessarily proper since it is not necessarily quasi-compact). And X is normal. Therefore every irreducible component of $X - X^*$ has codimension ≥ 2 in X; more precisely, the geometric generic fiber over S has codimension ≥ 2 in the geometric generic fiber of X over S. For reasons similar to [2, Prop.II.3.7], X^* contains $f(\mathbb{P}^1)$ for every H_j and general $[f] \in H_j$.

There is a dense open subspace $Q^0 \subset Q^*$ over which ϕ is flat and the geometric fibers are integral. Replace X^0 by $X^0 \cap \phi^{-1}(Q^0)$. By Corollary 2.4 (v), the subscheme $X^0 \times Q^* X^0$ equals $W_i \cap (X^0 \times_S X^0)$. In particular, for every $x \in X^0$ the fiber of ϕ containing x is $(W_i)_x$. Therefore for a general fiber of ϕ, for a general pair of points in the fiber, there is a free rational curve in H_i whose image is contained in the fiber and contains the two points. By Lemma 2.5, the image is contained in the smooth locus of ϕ, i.e. this is a free rational curve in the fiber. This proves Definition 1.2 (ii). □

Proposition 2.7. *The morphism $\phi : X^* \to Q^*$ is a maximal free rational quotient.*

Proof. Let $\phi_1 : X_1^* \to Q_1^*$ be a free rational quotient. If this is a trivial free rational quotient, the morphism ψ is trivial. Therefore assume it is a nontrivial free rational quotient.

There exists a dense open $U \subset Q_1^*$ such that,

$$\chi : U' \to U,$$

is faithfully flat and quasi-compact and the geometric fibers are integral, where $U' = X^0 \cap \phi_1^*(U)$ and where χ is the restriction of ϕ_1. By faithfully flat descent, to construct $\psi : U \to Q^*$, it is equivalent to construct a morphism $\psi' : U' \to Q^*$ satisfying a cocycle condition: indeed, the morphism ψ is equivalent to the graph of ψ, which is equivalent to a certain kind of quasi-coherent sheaf on $U \times Q^*$, so faithfully flat descent for quasi-coherent sheaves applies to $(\chi, 1) : U' \times Q^* \to U \times Q^*$.

Define ψ' to be the restriction of ϕ. For each geometric point x in U', define Y_x to be the fiber of χ containing x. The cocycle condition for ψ' is that the fiber product $U'' = U' \times_{\chi,U,\chi} U'$ is contained in $U' \times_{\psi',Q^*,\psi'} U'$. Now $U'' \to U'$ is a flat morphism whose geometric fibers are integral. Thus U'' is integral. So it suffices to prove it is set-theoretically contained in W_i, i.e., for a general geometric point x of U', $U''_x \subset (W_i)_x$.

By hypothesis, there is a dense subset of U''_x consisting of points y contained in a free morphism $f : \mathbb{P}^1 \to X$ such that $f(0) = x$ and $f(\infty) = y$. By Corollary 2.4 (ii), $f(\mathbb{P}^1) \subset (W_i)_x$, in particular $y \in (W_i)_x$. Therefore $U''_x \subset (W_i)_x$. □

References

[1] O. Debarre. *Higher-dimensional algebraic geometry.* Universitext. Springer-Verlag, New York, 2001.

[2] J. Kollár. *Rational curves on algebraic varieties*, volume 32 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 1996.

[3] J. Kollár, Y. Miyaoka, and S. Mori. Rational connectedness and boundedness of Fano manifolds. *J. Differential Geom.*, 36(3):765–779, 1992.

[4] M. Olsson. Hom stacks and restriction of scalars. preprint, 2005.

[5] M. Olsson and J. Starr. Quot functors for Deligne-Mumford stacks. *Comm. Algebra*, 31(8):4069–4096, 2003. Special issue in honor of Steven L. Kleiman.

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