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MAT 614 Problem Set 2

Problems.

Problem 1. Let $f : X \to Y$ be a proper, birational morphism from a smooth projective surface X to a normal projective surface Y over an algebraically closed field k.

(a) Prove that for every $q \geq 0$, the $\mathcal{O}_Y$-module $R^q f_* \mathcal{O}_X$ is coherent, prove that $R^q f_* \mathcal{O}_X$ is the zero sheaf for $q > 2$, and prove that the natural map

$$(R^q f_* \mathcal{O}_X) \otimes_{\mathcal{O}_Y} \mathcal{E} \to R^q f_*(f^* \mathcal{E})$$

is an isomorphism for every locally free $\mathcal{O}_Y$-module $\mathcal{E}$. In particular, the natural map $\mathcal{E} \to f_*(f^*\mathcal{E})$ is an isomorphism for every locally free $\mathcal{E}$.

(b) Let $\mathcal{A}$ be an invertible sheaf on Y that is sufficiently ample so that for every $q \geq 0$, the $\mathcal{O}_Y$-module $(R^q f_* \mathcal{O}_X) \otimes_{\mathcal{O}_Y} \mathcal{A}$ is globally generated and $h^p(Y, (R^q f_* \mathcal{O}_X) \otimes_{\mathcal{O}_Y} \mathcal{A})$ equals 0 for all $p > 0$. Using the Leray spectral sequence for f, prove that the edge map

$$H^2(X, f^*\mathcal{A}) \to H^0(Y, (R^2 f_* \mathcal{O}_X) \otimes_{\mathcal{O}_Y} \mathcal{A})$$

is an isomorphism. In particular, if $R^2 f_* \mathcal{O}_X$ is nonzero, then $h^2(X, f^*\mathcal{A})$ is nonzero. Finally, prove that for $\mathcal{A}$ sufficiently ample, for any ample Cartier divisor class H on X, $(c_1 f^*\mathcal{A} \cap H)_X$ is larger than $(K_X \cap H)_X$ so that $h^2(X, f^*\mathcal{A})$ is zero. Conclude that $R^2 f_* \mathcal{O}_X$ is zero.

(c) Since $R^2 f_* \mathcal{O}_X$ is zero, show that the edge map

$$H^1(X, f^*\mathcal{A}) \to H^0(Y, (R^1 f_* \mathcal{O}_X) \otimes_{\mathcal{O}_Y} \mathcal{A})$$

is an isomorphism. Conclude that $R^1 f_* \mathcal{O}_X$ is nonzero if and only if $h^1(X, f^*\mathcal{A})$ is nonzero. Finally, since $h^0(X, f^*\mathcal{A}) = h^0(Y, f_* f^*\mathcal{A})$ equals $h^0(Y, \mathcal{A})$, and since $h^1(Y, \mathcal{A})$ equals 0, conclude that $R^1 f_* \mathcal{O}_X$ is zero if and only if $\chi(X, f^*\mathcal{A})$ equals $\chi(Y, \mathcal{A})$.

(d) Assume now that Y is smooth. Apply Riemann-Roch on Y to $\mathcal{A}$ and on X to $f^*\mathcal{A}$. Use this to prove that $\chi(X, f^*\mathcal{A})$ equals $\chi(Y, \mathcal{A})$. Conclude that $R^1 f_* \mathcal{O}_X$ is zero. A contraction is called *rational* if $R^q f_* \mathcal{O}_X$ is the zero sheaf for all $q > 0$ (this is really a property of the singularities of Y). This shows that every contraction between smooth surfaces is rational (the same is true in higher dimensions by a similar argument).

(e) Give an example where Y is singular and $R^1 f_* \mathcal{O}_X$ is nonzero, i.e., Y has a non-rational singularity.

Problem 2. Read about the theorem on formal functions. For $f : X \to Y$ the blowing up of a smooth point on Y, use the theorem on formal functions to prove that $R^q f_* \mathcal{O}_X$ is the zero sheaf for all $q > 0$. Combine this with Castelnuovo's theorem to prove that for every proper, birational morphism $f : X \to Y$ of smooth, projective surfaces, $R^q f_* \mathcal{O}_X$ is the zero sheaf for all $q > 0$. Also use the theorem on formal functions to prove that for every contraction of a $(-m)$-curve (which is isomorphic to $\mathbb{P}^1$ by definition), $R^q f_* \mathcal{O}_X$ is the zero sheaf for all $q > 0$.

Problem 3. Supply the missing proof of Lemma 1.1 in the obstruction theory notes.

Problem 4. Supply the missing proof of Lemma 2.4 in the obstruction theory notes.

Problem 5. Prove the missing converse direction in the proof of Lemma 3.2 in the obstruction theory notes.

Problem 6. Check that the obstruction elements defined in Section 3 of the obstruction theory notes are compatible with morphisms of deformation situations.

Problem 7. Let Λ be a complete, local, Noetherian ring with residue field k. As in the obstruction theory notes, denote by $\mathcal{C}_\Lambda$ the category of local, Artin Λ-algebras A whose residue field equals k. Denote by $F : \mathcal{C}_\Lambda \to \text{Sets}$ the pointed functor that associates to every A the set of isomorphism classes of pairs $(\mathcal{E}_A, \phi)$ of a locally free sheaf $\mathcal{E}_A$ on $\mathbb{P}^1_A$ and an isomorphism $\phi : \mathcal{E}_k \to \mathcal{O}_{\mathbb{P}^1_k}(+1) \oplus \mathcal{O}_{\mathbb{P}^1_k}(-1)$. An *equivalence* $u_A : (\mathcal{E}_A, \phi) \to (\mathcal{E}'_A, \phi')$ is an isomorphism of locally free sheaves $u_A : \mathcal{E}_A \to \mathcal{E}'_A$ such that $\phi' \circ u_k$ equals ϕ.

Prove that F is *not* pro-representable. On the other hand, prove that there exists a formally smooth natural transformation $\theta : h_{\mathcal{C},\Lambda[\![t]\!]} \Rightarrow F$.

Problem 8. Let x_0, x_1, x_2, x_3 be the usual basis of $H^0(\mathbb{P}^3_k, \mathcal{O}_{\mathbb{P}^3}(1))$. Let $Z \subset \text{Spec } k[\![t]\!] \times \mathbb{P}^3_k$ be the closed subscheme whose homogeneous ideal is generated by

$$\langle x_1x_2, x_1x_3, tx_0x_2 + x_2x_3, tx_0x_3 + x_3^2 \rangle.$$

Similarly, let $W \subset \text{Spec } k[\![u]\!] \times \mathbb{P}^3_k$ be the closed subscheme whose homogeneous ideal is generated by

$$\langle x_1x_2, x_1x_3, x_2x_3, x_3(x_3 + tx_0) \rangle.$$

(a) Prove that Z is flat over $\text{Spec } k[\![t]\!]$ and that W is flat over $\text{Spec } k[\![u]\!]$.

(b) Prove that the closed fibers Z_k and W_k are equal.

(c) Prove that the geometric generic fiber of Z is a reduced curve that is a union of two disjoint lines. Compute that the Hilbert scheme of $\mathbb{P}^3$ at the corresponding point is smooth of dimension $2 \times 4 = 8$.

(e) Prove that the geometric generic fiber of W is a union of a (singular) plane conic and a disjoint, reduced point. Compute that the Hilbert scheme of $\mathbb{P}^3$ at the corresponding point is smooth of dimension $(3 + 5) + 3 = 11$.

(e) Conclude that the Hilbert scheme is reducible at the point corresponding to $Z_k = W_k$. Therefore the Hilbert scheme is not smooth at this point.

(f) If you feel ambitious, compute the dimension n of the Zariski tangent space of the Hilbert scheme at $Z_k = W_k$. Can you compute a nonzero "obstruction element" in the kernel of the corresponding homomorphism

$$f : k\,[\![v_1, \dots, v_n]\!] \to \widehat{\mathcal{O}}_{\text{Hilb},[Z_k]}?$$

Problem 9. For every separated, flat, finitely presented Λ-scheme X_Λ and for every k-point x of X_k, consider the pointed Hilbert functor $\text{Hilb}_{X_\Lambda/\Lambda,\{x\}}$ as defined in the obstruction theory notes.

(a) Prove that this functor is pro-representable by $\widehat{R} = \widehat{\mathcal{O}}_{X_\Lambda,x}$.

(b) Compute that the "standard" obstruction group is $\text{Ext}^1_{\widehat{\mathcal{O}}_{X_k,x}}(\mathfrak{m}_{X,x}, k)$, where $\mathfrak{m}_{X,x} \subset \widehat{\mathcal{O}}_{X_k,x}$ is the maximal ideal.

(c) Let $f : \Lambda\,[\![t_1, \dots, t_n]\!] \to \widehat{\mathcal{O}_{X_\Lambda,x}}$ be a surjection such that $\text{Ker}(f)$ is contained in $\mathfrak{m}^2$. By considering elements in Ext^1 as Yoneda extensions, prove that there is an injection

$$\text{Ext}^1_{\widehat{\mathcal{O}}_{X_k,x}}(\mathfrak{m}_{X,x}, k) \hookrightarrow \text{Ext}^1_{k[\![t_1,\dots,t_n]\!]}(\mathfrak{m}_{X,x}, k).$$

Thus, every obstruction element can be considered as an element of the second group.

(d) Denote by $\mathfrak{m}_n$ the maximal ideal in $k\,[\![t_1, \dots, t_n]\!]$ so that there is a short exact sequence of $k\,[\![t_1, \dots, t_n]\!]$-modules,

$$0 \to \text{Ker}(f_k) \to \mathfrak{m}_n \to \mathfrak{m}_{X,x} \to 0.$$

There is an associated long exact sequence of Ext, part of which is

$$\text{Hom}_{k[\![t_1,\dots,t_n]\!]}(\mathfrak{m}_n, k) \xrightarrow{0} \text{Hom}_{k[\![t_1,\dots,t_n]\!]}(\text{Ker}(f_k), k) \to \text{Ext}^1_{k[\![t_1,\dots,t_n]\!]}(\mathfrak{m}_{X,x}, k) \to \text{Ext}^1_{k[\![t_1,\dots,t_n]\!]}(\mathfrak{m}_n, k)$$

Use Exercise II.8.6, p. 188, of Hartshorne's book to prove that the image of every obstruction element in the third group is zero. Thus every obstruction element lifts uniquely to an element in $\text{Hom}_{k[\![t_1,\dots,t_n]\!]}(\text{Ker}(f_k), k)$. This is the canonical obstruction group associated to the representable functor $h_{\mathcal{C},\widehat{R}}$.

Problem 10. Let Λ be a complete, discrete valuation ring whose fraction field has characteristic 0 and whose residue field k is a finite field with $q = p^e$ elements for some prime p. Let C_Λ be a smooth, projective scheme over Spec Λ whose closed fiber C_k is a geometrically integral curve of genus $g > 1$. Let $F^e_{C_k} : \mathcal{O}_{C_k} \to \mathcal{O}_{C_k}$ be the *absolute Frobenius* morphism that is the identity on underlying topological spaces and with

$$(F^e_{C_k})^{\#}(U) : \mathcal{O}_{C_k}(U) \to \mathcal{O}_{C_k}(U),\ g \mapsto g^q.$$

(a) Check that $F^e_{C_k}$ is a k-morphism from C_k to C_k.

(b) For the corresponding Hom scheme $\mathrm{Hom}_\Lambda(C_\Lambda, C_\Lambda, F^e_{C_k})$, prove that the pro-representing Λ-algebra $\widehat{R}$ must have $\mathfrak{m}_\Lambda^N \widehat{R} = \{0\}$ for some integer $N \gg 0$.

(c) Compute that the Zariski tangent space of the functor is $\{0\}$, hence $\widehat{R}$ is isomorphic to Λ/J for some ideal J containing $\mathfrak{m}_\Lambda^N$ for $N \gg 0$.

(d) Compute the obstruction group. Does this alone give you any indication of the smallest integer N as above? (No, it does not.)

(e) Consider the complete, local Λ-algebra, $\Lambda_{3g-3} := \Lambda\llbracket t_1, \dots, t_{3g-3}\rrbracket$. There exists a flat, projective morphism $Y_{\Lambda_{3g-3}} \to \mathrm{Spec}\ \Lambda_{3g-3}$ whose base change by the natural quotient $\Lambda_{3g-3} \to \Lambda$ equals C_Λ and that is *versal*, i.e., the restriction of the family over $k\llbracket t_1, \dots, t_{3g-3}\rrbracket$ has isomorphic Kodiara-Spencer map $\mathfrak{m}/\mathfrak{m}^2 \to H^0(C_k, \omega_{C_k}^{\otimes 2})$. Since Y_k equals C_k, the Frobenius morphism $F^e_{C_k}$ gives rise to a Hom functor, $\mathrm{Hom}_{\Lambda_{3g-3}}(C_{\Lambda_{3g-3}}, Y_{\Lambda_{3g-3}}, F^e_k)$. The obstruction group is the same as in the previous case. Is the dimension of the obstruction group larger than the dimension $1 + (3g-3)$ of Λ_{3g-3}? Does there exist a lifting of $F^e_{C_k}$ to some morphism of curves in characteristic 0, $C \to Y$? (What does Riemann-Hurwitz say?)

(f) Do the same computations for $\mathbb{P}^1_\Lambda$ instead of a curve of genus g. Does your computation make sense? Can you explicitly write down a lift of Frobenius to characteristic 0?

(g) What about the case where C_k is an elliptic curve? Note that the versal family in this case is over $\Lambda\llbracket j\rrbracket$, which is 2-dimensional, so there is a lift of Frobenius to some morphism of genus 1 curves in characteristic 0, $C \to Y$. Given C, can you guess what is Y and the morphism $C \to Y$?