MAT 589 Lecture Summaries

Homework. These are the problems from the assigned Problem Set which can be completed using the material from that date's lecture.

Lecture 1. January 23, 2017

Homework. Problem Set 1 Write up solutions for 5 problems from the problem sets of a classical algebraic geometry course (explained on the webpage).

Explained the Nullstellensatz. Explained the ideal-variety correspondence.

Lecture 2. January 25, 2017

Explained some basic constructions in commutative algebra such as modules, tensor product and Hom, fraction rings, and localization at a prime. Finished the discussion of the ideal-variety correspondence.

Lecture 3. January 31, 2017

Homework. Problem Set 2 Part I: (a), (b), (c), (e); Part II: Problems 2 and 3.

Discussed the category of quasi-affine algebraic k-sets and the contravariant functor of global regular functions. Discussed the glueing lemma and how it leads to the notion of sheaves. Defined presheaves and sheaves. Stated the problem of sheafification.

Fun problem 1. Let k be an algebraically closed field whose characteristic is not 2. Let (u, v, w) be a triple of elements in k such that $2uvw(u-v)(v-w)(w-u)$ is nonzero. In $\mathbb{P}^2$, how many conics are tangent to the 5 lines $L_1 = \mathbb{V}(x)$, $L_2 = \mathbb{V}(y)$, $L_3 = \mathbb{V}(z)$, $L_4 = \mathbb{V}(x+y+z)$ and $L_5 = \mathbb{V}(ux+vy+wz)$? What is the equation of this line? Use the equation to find the number of conics tangent to $L_1, \dots, L_4$ and containing $[x_0, y_0, z_0]$ for any triple such that $x_0y_0z_0(x_0+y_0+z_0)$ is nonzero.

Lecture 4. February 2, 2017

Homework. Problem Set 2 Part I: (d); Part II: Problem 1.

Discussed limits and colimits, please confer MAT 536 Problem Set 8, Problem 0. Discussed stalks of sheaves. Mentioned adjoint pairs of functors, please confer MAT 536 Problem Set 2, Problem 1. Described the espace étalé of a presheaf. Used this to construct the sheafification of a presheaf of sets.

Hint for Fun Problem 1. Use duality between conics in $\mathbb{P}^2$ and conics in the dual $\mathbb{P}^2$ of lines.

Fun Problem 2. Let k be a finite field and let $f(x, y, z)$ be a quadratic, homogeneous polynomial with coefficients in k. Show that f has a nonzero solution in k^3. This is tricky! It is best to start with $k = \mathbb{F}_2$, $\mathbb{F}_3$, and maybe $\mathbb{F}_5$. Given a point p in $\mathbb{P}^2$, how many conics with coefficients in k contain p? How many smooth conics contain p? How many smooth conics are there in total in $\mathbb{P}^2$? How many points are contained on a smooth conic? What happens when you compare these numbers?

Lecture 5. February 7, 2017

Homework. Problem Set 2

Defined adjoint pairs of functors. Used adjoint functors to construct the associated sheaf of a presheaf. Proved criterion for a morphism of sheaves to be a monomorphism, resp. isomorphism, epimorphism, in terms of the morphisms of stalks being monomorphisms, resp. isomorphisms, epimorphisms.

Sheaves of Abelian groups form an Abelian category. Associated to a continuous map $f : X \to Y$, there are functors,

$$f_* : \text{Sheaves}_X \to \text{Sheaves}_Y,$$
$$f^{-1} : \text{Sheaves}_Y \to \text{Sheaves}_X.$$

These functors form an *adjoint pair*.

Defined ringed spaces and locally ringed spaces. Gave examples of locally ringed spaces coming from "spaces with functions". Quickly stated the GAGA principle, which gives an equivalence between two full subcategories of the category of locally ringed spaces (over Spec $\mathbb{C}$).

Fun Problem 3. Let k be an algebraically closed field of characteristic different from 2, 3. Let $F(s, t, u) \in k[s, t, u]$ be a homogeneous polynomial of degree 3 such that the common zero set of $\partial F/\partial s$, $\partial F/\partial t$, and $\partial F/\partial u$ is the empty set inside the projective plane $\mathbb{P}^2_k$. What are all of the possible orders for the subgroup $\text{Aut}(\text{Zero}(F)) \subset \text{Aut}(\mathbb{P}^2_k) = \mathbf{PGL}_3(k)$?