18.725 SOLUTIONS TO PROBLEM SET 9

Due date: Friday, December 3 in lecture. Late work will be accepted only with a medical note or for another Institute-approved reason. You are strongly encouraged to work with others, but the final write-up should be entirely your own and based on your own understanding.

Read through all the problems. Write solutions to the "Required Problems", 1, 2, 3, and 4 together with 1 more problem to a total of 5.

Required Problem 1, Intersection Multiplicity: This problem is essentially (Hartshorne, Exer. I.5.4). Let $F, G \in k[X_0, X_1, X_2]$ be non-constant, irreducible, homogeneous polynomials, and denote $C = \mathbb{V}(F), D = \mathbb{V}(G)$ in $\mathbb{P}^2_k$. Let $p \in C \cap D$ be an element such that $\dim(C \cap D, p) = 0$, i.e., p is an isolated point of $C \cap D$. The *intersection multiplicity of C and D at p*, $i(C, D; p)$, is defined to be,

$$i(C, D; p) = \dim_k(\mathcal{O}_{\mathbb{P}^2,p}/\langle F_p, G_p\rangle),$$

where $F_p, G_p \in \mathcal{O}_{\mathbb{P}^2,p}$ are germs of dehomogenizations of F and G at p.

Let $P \subset k[X_0, X_1, X_2]$ be the homogeneous ideal corresponding to p. Form the graded $k[X_0, X_1, X_2]$-module, $M = \text{Image}(\phi_p)$, where ϕ_p is the homomorphism of graded modules,

$$\phi_p : k[X_0, X_1, X_2]/\langle F, G\rangle \to (k[X_0, X_1, X_2]/\langle F, G\rangle)_P.$$

(a) Prove that the Hilbert polynomial of M equals $i(C, D; p)$, i.e., for all $l \gg 0$, $\dim_k M_l = i(C, D; p)$. **Hint:** You may assume existence of a *Jordan-Hölder filtration of M*: a filtration of M by graded submodules, $M = M^0 \supset M^1 \supset \cdots \supset M^r = \{0\}$, such that for every $i = 1, \dots, r$, $M^{i-1}/M^i \cong (k[X_0, X_1, X_2]/P)(d_i)$ for some integer d_i. For every $X \in k[X_0, X_1, X_2]_1 - P$, the dehomogenization of M with respect to X equals $\mathcal{O}_{\mathbb{P}^2,p}/\langle F_p, G_p\rangle$ and has an induced Jordan-Hölder filtration whose associated graded pieces are the dehomogenizations of the graded modules M^{i-1}/M^i. Relate the length of the dehomogenization of M, the Hilbert polynomial of M and the integer r.

Solution: The definition of M given above is incorrect. The module M should be the image of the *graded localization*: $k[X_0, X_1, X_2]/\langle F, G\rangle \to S^{-1}k[X_0, X_1, X_2]/\langle F, G\rangle$, where $S = \cup_{e \geq 0}(k[X_0, X_1, X_2]_e - P_e)$. In the problem, the existence of a Jordan-Hölder filtation was given as a hypothesis. For completeness, the existence will be proved – this makes the solution a bit longer. The solution of this problem in Hartshorne's *Algebraic geometry* does not use all of the properties of the filtration (and so is more elementary).

Lemma 0.1. *For every pair of polynomials $F, G \in k[X_0, X_1, X_2]$, for every $p \in \mathbb{P}^2_k$ with associated homogeneous ideal P, if $p \in \mathbb{V}(F, G)$ is an isolated point then*

(i) *M is a P-primary module, i.e., the only associated prime of M is P, and*
(ii) *there is a filtration of M by graded submodules, $M = M^0 \supset \cdots \supset M^r = (0)$, and a collection of integers $d_0, \dots, d_{r-1}$ such that for every $i = 0, \dots, r-1$, $M^i/M^{i+1} \cong (k[X_0, X_1, X_2]/P)(d_i)$ as graded modules.*

Proof. **(i)** Consider the ideal $J = \langle F, G\rangle$. This is contained in the prime ideal P. By the existence of a primary decomposition, there exists a collection of associated primes of J, $Q_1, \dots, Q_m$, and a collection of homogeneous ideals, $J_1, \dots, J_m$, such that

(i) for every $i = 1, \dots, m$, J_i is Q_i-primary, i.e., for some integer $a_i > 0$, $Q_i^{a_i} \subset J_i \subset Q_i$, and,
(ii) $J = J_1 \cap \cdots \cap J_m$.

Because $J \subset P$ and P is prime, there exists i such that $J_i \subset P$, which in turn implies $Q_i \subset P$, i.e., $\mathbb{V}(P) \subset \mathbb{V}(Q_i) \subset \mathbb{V}(F, G)$. By hypothesis p is an isolated point of $\mathbb{V}(F, G)$ so that $\mathbb{V}(Q_i) = \mathbb{V}(P) = \{p\}$. By the Nullstellensatz, $Q_i = P$, i.e., P is an associated prime of J. **Note:** This is just the usual argument that the minimal primes that contain J are the same as the minimal primes among the associated primes of J.

The module M is a quotient of $k[X_0, X_1, X_2]/J$, i.e., $M = k[X_0, X_1, X_2]/I$ for a homogeneous ideal I containing J. In fact $I = \{a \in k[X_0, X_1, X_2] | \exists s \in S, sa \in J\}$, i.e., $I/J \subset k[X_0, X_1, X_2]/J$ is the submodule of elements annihilated by an element in S. Because P is an associated prime of J, $P = \text{ann}(f)$ for some element $f \in k[X_0, X_1, X_2]/J$. The element f can be chosen homogeneous. The annihilator of the image $\overline{f} \in k[X_0, X_1, X_2]/I$ is $\{a \in k[X_0, X_1, X_2] | \exists s \in S, saf = 0\} = \{a \in k[X_0, X_1, X_2] | \exists s \in S, sa \in P\}$. Because P is a prime and $S \cap P = \emptyset$, if $sa \in P$, then $a \in P$. Therefore the annihilator of $\overline{f}$ is P. In particular, P is an associated prime of M.

Let Q be an associated prime of M. By construction, every element of S acts as a non-zero-divisor on $S^{-1}(k[X_0, X_1, X_2]/\langle F, G\rangle)$, and thus on M as well. There is a homogeneous element $m \in M$ such that every homogeneous element of Q annihilates m, so the element is not in S which implies it is a homogeneous element of P. Because M is a graded module, every associated prime is a homogeneous ideal. Therefore $Q \subset P$. As proved above, P is a minimal prime containing J, and $J \subset Q$ so that $Q = P$. Therefore P is the unique associated prime of M, i.e., I is a P-primary homogeneous ideal.

(ii) For every integer $i \geq 0$, define $M^i \subset M$ to be the kernel of the homomorphis of graded modules that is the composition,

$$M \to S^{-1}M \to S^{-1}M/P^i(S^{-1}M).$$

Of course $PM^i \subset M^{i+1}$, so M^i/M^{i+1} is a finitely-generated graded module over $k[X_0, X_1, X_2]/P \cong k[T]$. Also, since I is a P-primary ideal, $P^e M = (0)$ for some integer e so that $M^e = (0)$, i.e., the filration stabilizes to (0). Moreover, M^i/M^{i+1} is a submodule of $P^i S^{-1}M/P^{i+1}(S^{-1}M) = S^{-1}(P^i M/P^{i+1}M)$. By construction, ever nonzero homogeneous element of $k[X_0, X_1, X_2]/P$ acts as a non-zero-divisor on $S^{-1}(P^i M/P^{i+1}M)$, thus also on M^i/M^{i+1}. So M^i/M^{i+1} is a torsion-free finitely-generated $k[T]$-module, i.e., it is a finite free $k[T]$-module. Every finite free $k[T]$-module is free; likewise every graded finite free $k[T]$-module is of the form $k[T](d_1) \oplus \cdots \oplus k[T](d_m)$ for a sequence of integers $d_1, \dots, d_m$. Thus M^i/M^{i+1} has a filtration by graded submodules (in fact a direct sum decomposition) where the associated subquotients are of the form $k[T](d_i)$.

The induced filtration of each M^i/M^{i+1} determines a refinement of the original filtration to a filtration by graded submodules, $M = M^0 \supset \cdots \supset M^r = (0)$, such that for every $i = 0, \dots, r-1$ the associated subquotient of the new filtration, M^i/M^{i+1}, is isomorphic to $k[T](d_i)$ for some integer d_i. $\square$

By the additivity of Hilbert polynomials, the Hilbert polynomial of M is the sum of the Hilbert polynomials of the associated graded pieces M^{i-1}/M^i. For every integer $i = 0, \dots, r-1$, $M^{i-1}/M^i \cong k[T](d_i)$. So the Hilbert polynomial of M^{i-1}/M^i is 1. Therefore the Hilbert polynomial of M is r.

Consider the functor $*_{(P)}$ that associates to a graded $k[X_0, X_1, X_2]$-module N the $(S^{-1}k[X_0, X_1, X_2])_0$-module,

$$N_{(P)} = (S^{-1}N)_0.$$

Localization is exact, as is the functor assigning to a graded module its degree 0 graded part, thus $*_{(P)}$ is an exact functor. In particular, there is an induced filtration of $M_{(P)}$, $(M_{(P)})^i = (M^i)_{(P)}$. For every $i = 0, dots, r-1$, the associated subquotient of this filration is $(M^i/M^{i+1})_{(P)} \cong (k[t](d_i))_{(t)}$. Now $S^1 k[t](d_i) \cong k[t, t^{-1}](d_i)$, and the degree 0 graded summand is just $k\{t^{d_i}\}$, the 1-dimensional k-vector space spanned by the monomial t^{d_i}. Now $\dim_k M_{(P)}$ is the sum of the dimensions of the associated subquotients, which is r.

For every $X \in k[X_0, X_1, X_2]_1 - P_1$, the ring $(k[X_0, X_1, X_2][1/X])_0$ is the coordinate ring $k[D_+(X)]$ of the affine neighborhood of p, $D_+(X) \subset \mathbb{P}^2_k$. For every $s \in k[X_0, X_1, X_2]_d - P_d$, the dehomogenization of s with respect to X is an element of $k[D_+(X)] - \mathfrak{m}_p$, and vice versa every element of $k[D_+(X)] - \mathfrak{m}_p$ is the dehomogenization of a homogeneous element in $k[X_0, X_1, X_2] - P$. It follows that $(k[X_0, X_1, X_2])_{(P)} \cong k[D_+(X)]_{\mathfrak{m}_p} = \mathcal{O}_{\mathbb{P}^2_k, p}$. Moreover, $(k[X_0, X_1, X_2]/\langle F, G\rangle)_{(P)} \cong \mathcal{O}_{\mathbb{P}^2_k, p}/\langle F_p, G_p\rangle$. So the intersection multiplicity $i(C, D; p)$ equals the dimension of $M_{(P)}$. Therefore $i(C, D; p)$ equals the Hilbert polynomial of M.

(b) This problem is rather difficult. Attempt it, but you don't have to solve it. Denote by $e(C; p)$, resp. $e(D; p)$, the Hilbert-Samuel multiplicity of C at p, resp. of D at p. Prove that $i(C, D; p) \geq e(C; p)e(D; p)$. **Hint:** Work in affine coordinates for which $p = (0, 0)$. First consider the case that $C = \mathbb{V}(f), D = \mathbb{V}(g)$ where f and g are relatively prime homogeneous polynomials in x, y. Next deduce the case where f and g are not necessarily homogeneous, but the tangent cones of C and D at p have no common irreducible component. The general case can be deduced from this one by an "semicontinuity" argument.

Solution: A more complete, but less elementary, solution than the following is in Chapter 12 of Fulton's *Intersection Theory* (there are also solutions in most textbooks on algebraic curves).

Lemma 0.2. *Let $f_1, f_2, g \in k[\![x, y]\!]$ be elements in $\mathfrak{m} = \langle x, y\rangle$ such that for $i = 1, 2$, f_i and g have no common factor. Then $\langle f_1, g\rangle$, $\langle f_2, g\rangle$ and $\langle f_1 f_2, g\rangle$ are $\mathfrak{m}$-primary, and*

$$dim_k(k[\![x, y]\!]/\langle f_1 f_2, g\rangle) = dim_k(k[\![x, y]\!]/\langle f_1, g\rangle) + dim_k(k[\![x, y]\!]/\langle f_2, g\rangle).$$

Proof. For $f = f_1$, f_2 or $f_1 f_2$, because f and g have no common factor and because $k[\![x, y]\!]$ is a Unique Factorization Domain, f, g are a regular sequence. In particular, every prime over $\langle f, g\rangle$ has height 2. The only prime in $k[\![x, y]\!]$ of height 2 is $\mathfrak{m}$, so $\langle f, g\rangle$ is a $\mathfrak{m}$-primary ideal.

There is a short exact sequence of k-vector spaces,

$$0 \longrightarrow \langle f_1, g\rangle/\langle f_1 f_2, g\rangle \longrightarrow k\llbracket x,y\rrbracket/\langle f_1 f_2, g\rangle \longrightarrow k\llbracket x,y\rrbracket/\langle f_1, g\rangle \longrightarrow 0.$$

So to prove the equation of dimensions, it suffices to prove that $\langle f_1, g\rangle/\langle f_1 f_2, g\rangle$ is isomorphic to $k\llbracket x,y\rrbracket/\langle f_2, g\rangle$ as a module. There is a $k\llbracket x,y\rrbracket$-module homomorphism $\phi : k\llbracket x,y\rrbracket/\langle f_2, g\rangle \to \langle f_1, g\rangle/\langle f_1 f_2, g\rangle$ by $\phi(1) = f_1$. Of course ϕ is surjective. For every $h \in \ker(\phi)$, $f_1 h \in \langle f_1 f_2, g\rangle$, i.e., $f_1 h = a f_1 f_2 + bg$ for some $a, b \in k\llbracket x,y\rrbracket$. This can be rewritten as $bg = f_1(h - af_2)$; in particular f_1 divides bg. By hypothesis, f_1 and g have no common factor. The ring $k\llbracket x,y\rrbracket$ is a Unique Factorization Domain, thus f_1 divides b, i.e., $b = f_1 c$. Then $h = af_2 + cg$, which is in $\langle f_2, g\rangle$, i.e., ϕ is injective. □

Lemma 0.3. *Let $F, G \in k[x,y]$ be homogeneous polynomials of degrees d and e respectively. If F and G have no common factor, then $k[x,y]/\langle F, G\rangle$ is a k-vector space of dimension de.*

Proof. Both F and G factor as products of homogeneous linear polynomials, $F = L_1 \cdots L_d$, $G = M_1 \cdots M_e$ such that for every i, j, L_i and M_j are linearly independent. Of course $\mathbb{V}(F,G) = \cup_{(i,j)} \mathbb{V}(L_i, M_j) = \cup_{(i,j)}\{(0,0)\} = \{(0,0)\}$. By the Nullstellensatz, $\langle F, G\rangle$ is $\langle x, y\rangle$-primary, i.e., $\langle x,y\rangle^e \subset \langle F, G\rangle$ for some integer $e \geq 0$. Therefore,

$$k[x,y]\langle F,G\rangle \cong (k[x,y]/\langle x,y\rangle^e)/\langle F,G\rangle \cong (k\llbracket x,y\rrbracket/\langle x,y\rangle^e)/\langle F,G\rangle \cong k\llbracket x,y\rrbracket/\langle F,G\rangle.$$

The same goes when F and G are replaced by any L_i and M_j. Thus Lemma 0.2 applies and gives,

$$\dim_k(k\llbracket x,y\rrbracket/\langle F,G\rangle) = \sum_{i=1}^{d}\sum_{j=1}^{e} \dim_k(k\llbracket x,y\rrbracket/\langle L_i, M_j\rangle) = de.$$

□

Now let $f, g \in k\llbracket x,y\rrbracket$ with $f \in \mathfrak{m}^d - \mathfrak{m}^{d+1}$ and $g \in \mathfrak{m}^e - \mathfrak{m}^{e+1}$. Let $F = \overline{f} \in k[x,y]_d$ and $G = \overline{g} \in k[x,y]_e$.

Lemma 0.4. *If F and G have no common factor, then $k\llbracket x,y\rrbracket/\langle f,g\rangle$ is a k-vector space of dimension de.*

Proof. First of all, by Lemma 0.3, $k[x,y]/\langle F,G\rangle$ is a finite-dimensional k-vector space. Hence there exists an integer $r > 0$ such that $\langle x,y\rangle^r k[x,y] \subset \langle F,G\rangle k[x,y]$. It follows that $\mathfrak{m}^r \subset \langle f,g\rangle + \mathfrak{m}^{r+1}$. By Krull's Intersection Theorem, $\mathfrak{m}^r \subset \langle f,g\rangle$.

Let $\mathcal{B} \subset k[x,y] \subset k\llbracket x,y\rrbracket$ be a collection of homogeneous elements that map to a k-basis for $k[x,y]/\langle F,G\rangle$. The claim is that the images of the elements in $\mathcal{B}$ form a k-basis for $k\llbracket x,y\rrbracket$.

Linear independence: Suppose given a nontrivial k-linear relation among the images of the elements $\mathcal{B}$, i.e., a collection $(c_b | b \in \mathcal{B})$ of elements of k such that

$$\sum_{b\in\mathcal{B}} c_b b = uf + vg.$$

Of course u and v can be chosen so that either $u = 0$ or else $u \notin \langle g\rangle$; if u is in $\langle g\rangle$, replace u by 0 and replace v by $v + (u/g)f$. Moreover, the factors u and v can be chosen so that either $u = 0$ or else the lowest degree nonzero homogeneous part of u

is not divisible by G. If $u = 0$, this is trivial. If $u \neq 0$, then u is not in $\langle g \rangle$, which by Krull's Intersection Theorem equals $\cap_{N>0}(\mathfrak{m}^N + \langle g \rangle)$. So there exists some integer N such that $u \notin \mathfrak{m}^N + \langle g \rangle$. Let n be the largest integer such that $u \in \mathfrak{m}^n + \langle g \rangle$, i.e., $u = u_0 g + u_1$ where $u_1 \in \mathfrak{m}^n$. If the lowest degree part of u_1 is divisible by G, say $u_1 = u_2 G + u_3$ where $u_3 \in \mathfrak{m}^{n+1}$, then $u = (u_0 + u_2)g + u_2(G - g) + u_3$, and $u_2(G - g), u_3 \in \mathfrak{m}^{n+1}$ contradicting that $u \notin \mathfrak{m}^{n+1} + \langle g \rangle$. Therefore the lowest degree homogeneous part of u is not in $\langle g \rangle$.

Let n be the least integer such that either $\deg(b) = n$ for some b with $c_b = 0$, or $u \in k\llbracket x, y \rrbracket - \mathfrak{m}^{n-d+1}$, or $v \in k\llbracket x, y \rrbracket - \mathfrak{m}^{n-e+1}$. Then the linear relation above gives a k-linear relation modulo $\mathfrak{m}^{n+1}$ which is a nontrivial k-linear relation,

$$\sum_{b \in \mathcal{B}} c'_b b = UF + VG,$$

where $c'_b = c_b$ if $\deg(b) = n$ and $c'_b = 0$ otherwise, and where U, V are homogeneous polynomials of degrees $n - d$ and $n - e$ respectively. By hypothesis, $\mathcal{B}$ is linearly independent in $k[x, y]/\langle F, G \rangle$, so every $c'_b = 0$. Therefore at least one of U and V is nonzero and there is a relation $UF + VG = 0$, i.e., $UF = -VG$. Since F and G have no common factor, G divides U. By construction, U, the lowest degree graded part of u, is divisible by G iff $u = 0$. Therefore $U = 0$, which implies also $V = 0$. This contradicts the construction of n, proving the only linear relation among $\mathcal{B}$ in $k\llbracket x, y \rrbracket / \langle f, g \rangle$ is the trivial linear relation, i.e., $\mathcal{B}$ is linearly independent in $k\llbracket x, y \rrbracket / \langle f, g \rangle$.

Spanning: Let a be an element in $k\llbracket x, y \rrbracket / \langle f, g \rangle$. The claim is that $a \in \mathrm{Span}(\mathcal{B})$. If not then there exist a largest integer $n \geq 0$ such that $a \in \mathfrak{m}^n + \mathrm{Span}(B)$. Up to adding an element in $\mathrm{Span}(B)$, $a \in \mathfrak{m}^n$ and $a \notin \mathfrak{m}^{n+1} + \mathrm{Span}(B)$. Consider the associated homogeneous element,

$$A := \overline{a} \in \mathfrak{m}^n k\llbracket x, y \rrbracket / (\mathfrak{m}^n + 1 + \langle f, g \rangle) \cong (k[X, Y]/\langle F, G \rangle)_n.$$

Because $\mathcal{B}$ spans $k[X, Y]/\langle F, G \rangle$, there exists an expression for A as the sum of a k-linear combination of the elements in $\mathcal{B}$ and an element in $\langle F, G \rangle$. This gives an expression for a as the sum of a k-linear combination of the elements in $\mathcal{B}$, and an element in $\mathfrak{m}^{n+1}$ contrary to hypothesis. Therefore $\mathcal{B}$ spans $k\llbracket x, y \rrbracket / \langle f, g \rangle$. ☐

Corollary 0.5. *Let S be a surface, let $p \in S$ be a smooth point, and let $C, D \subset S$ be curves such that $p \in C \cap D$ is an isolated point of $C \cap D$.*

- (i) *If the tangent cone of C at p and the tangent cone of D at p have no common irreducible component, then $i(C, D; p)$ equals $e(C; p)e(D; p)$.*
- (ii) *In every case, $i(C, D; p) \geq e(C; p)e(D; p)$.*

Proof. **(i)** Let $\mathbb{I}(C)\mathcal{O}_{S,p} = \langle f \rangle \mathcal{O}_{S,p}$ and $\mathbb{I}(D)\mathcal{O}_{S,p} = \langle g \rangle \mathcal{O}_{S,p}$. Because p is an isolated point of $C \cap D$, $\mathfrak{m}^r \subset \langle f, g \rangle$ for some integer $r > 0$. Thus,

$$\mathcal{O}_{S,p}/\langle f, g \rangle \cong (\mathcal{O}_{S,p}/\mathfrak{m}^r)/\langle f, g \rangle \cong (\widehat{\mathcal{O}}_{S,p}/\mathfrak{m}^r)/\langle f, g \rangle \cong \widehat{\mathcal{O}}_{S,p}/\langle f, g \rangle.$$

Because p is a smooth point of S, $\widehat{\mathcal{O}}_{S,p}$ is isomorphic to $k\llbracket x, y \rrbracket$. Of course $e(C; p) = d$ where $f \in \mathfrak{m}^d - \mathfrak{m}^{d+1}$ and $e(D; p) = e$ where $g \in \mathfrak{m}^e - \mathfrak{m}^{e+1}$. By Lemma 0.4, $i(C, D; p) = \dim_k(\mathcal{O}_{S,p}/\langle f, g \rangle)$ equals $e(C; p)e(D; p)$.

(ii) The notation is as in Lemma 0.4. The hypothesis that $F, G \in k[x, y]$ are relatively prime is not necessarily satisfied. Let K denote the algebraic closure of the field $k(t)$. Then there exist $F' \in k[x, y]_d, G' \in k[x, y]_e$ such that $F + tF' \in K[x, y]_d$

and $G + tG' \in K[x,y]_e$ are relatively prime elements in $K[x,y]$. In $k[t]\llbracket x,y \rrbracket$, form the ideal $I = \langle f + tF', g + tG' \rangle$ and denote $M = \langle x,y \rangle$. Because $p \in C \cap D$ is an isolated point, there exists an integer $r \geq 0$ such that $\mathfrak{m}^r k\llbracket x,y \rrbracket \subset \langle f,g \rangle k\llbracket x,y \rrbracket$. Therefore $(I + M^r)/(I + M^{r+1})$ is a finitely-generated $k[t]$ module and modulo t this module is 0. By Nakayama's lemma, there exists a polynomial $a \in tk[t]$ such that $(1+a)$ annihilates this $k[t]$-module. Therefore, inverting $1+a$, $I + M^{r+1} \subset I + M^r$. By Krull's intersection theorem, $M^r \subset I$. Therefore $A := k[t]\llbracket x,y \rrbracket / \langle I = (k[t]\llbracket x,y \rrbracket / M^r)/I$ is a finitely-generated $k[t][1/(1+a)]$-module. By the structure theorem for finitely-generated modules over a PID, this is the direct sum of a finitely-generated torsion-module and a finite free module of some rank i. In particular, $\dim_k(A/tA) \geq i$. To compute i, tensor the module with K over $k[t][1/(1+a)]$. This gives $K\llbracket x,y \rrbracket / \langle f + tF', g + tG' \rangle$. By (i), this is a finite-dimensional K-vector space of dimension de. Therefore $i = de$. Since $A/tA = k\llbracket x,y \rrbracket / \langle f,g \rangle$, this gives,

$$i(C,D;p) = \dim_k(k\llbracket x,y \rrbracket / \langle f,g \rangle) \geq de = e(C;p)e(D;p).$$

□

Remark 0.6. A geometric interpretation of Lemma 0.2 is that if C and D are curves on S, $p \in C \cap D$ is an isolated point, and if the "completion" of C at p factors into "branches" with no common irreducible component, $C = C_1 \cup C_2$, then $i(C,D;p) = i(C_1,D;p) + i(C_2,D;p)$. This is "fictitious" since the factorization $f = f_1 f_2$ may not make sense in $\mathcal{O}_{S,p}$, only in $\widehat{\mathcal{O}}_{S,p}$. However, if $f = f_1 f_2$ is a factorization in $\mathcal{O}_{S,p}$, this does make sense. In any case, it is a useful fiction whose rigorous version is Lemma 0.2.

(c) Let X be a plane curve and $p \in X$ an element. Prove that for all but finitely many lines L in $\mathbb{P}^2$ containing p, $i(X,L;p) = e(X;p)$.

Solution: The tangent cone to X at p is a union of finitely many lines. Let L be any line containing p whose tangent cone is not one of these finitely many lines. By Corollary 0.5(i), $i(X,L;p) = e(X;p)e(L;p)$. Of course $e(L;p) = 1$, so $i(X,L;p) = e(X;p)$.

Required Problem 2, Bézout's Theorem in the Plane: This problem continues the previous problem. Let $d = \deg(F)$ and let $e = \deg(G)$. Assume $C \cap D = \{p_1, \dots, p_m\}$, i.e., $C \cap D$ has no irreducible component of dimension 1. Define $M = k[X_0,X_1,X_2]/\langle F,G \rangle$ as a graded module. For every $i = 1, \dots, m$, define $M_i = \text{Image}(\phi_{P_i})$ where P_i is the homogeneous ideal of p_i and where $\phi_{P_i} : k[X_0,X_1,X_2]/\langle F,G \rangle \to (k[X_0,X_1,X_2]/\langle F,G \rangle)_{P_i}$ is the localization homomorphism.

For the following homomorphism of graded modules, prove both the kernel and cokernel have finite length:

$$\phi : M \to \oplus_{i=1}^m M_i.$$

Hint: This requires more about the Jordan-Hölder filtration and associated primes. For a graded module M, there exists a filtration of M, $M = M^0 \supset \cdots \supset M^r = \{0\}$, such that for every $j = 1, \dots, r$, $M^{j-1}/M^j \cong (k[X_0,X_1,X_2]/Q_j)(d_j)$ where Q_j is an associated prime of M. If Q is a minimal associated prime, then $(M^{j-1}/M^j)_P$ is nonzero iff $P_j = P$. So the graded pieces in the filtration of M_i are the associated graded pieces in the filtration of M such that $Q_j = P_i$.

Solution: As in the previous problem, the localizations should have been the graded localizations. Again, at the expense of making the solution longer, the Jordan-Hölder filtration will be constructed.

In Problem 1, for every $i = 1, \dots, m$ a Jordan-Hölder filtration is constructed for the module M_i. Denote $K^0 = M$ and for every $j = 1, dots, m$, denote by $K^j \subset M$ the kernel of the projection to the first i factors, $K^j = \ker(M \to \oplus_{i=1}^j M_i)$. Then K^i/K^{i+1} is a graded submodule of M_i. By construction, there exists a filtration of M_i by graded submodules, $M_i^0 \supset \cdots \supset M_i^{r_i} = (0)$, such that every term M_i^l/M_i^{l+1} is isomorphic to $(k[X_0, X_1, X_2]/P_i)(d_l)$ for some integer d_l. For every $l = 1, \dots, r_i$, define $K^{i,l} \subset K^i$ to be the unique graded submodule containing K^{i+1} such that $K^{i,l}/K^{i+1} = (K^i/K^{i+1}) \cap M_i^l$. Then $K^{i,l}/K^{i,l+1}$ is a graded submodule of $(k[X_0, X_1, X_2]/P_i)(d_l) \cong k[t](d_l)$. Such a submodule is clearly either (0) or else $k[t](e_l)$ for some integer $e_l \geq d_l$. Concatenating these filtrations gives a filtration on M which stabilizes at $K^m = \ker(\phi)$ and such that the associated filtration on M/K^m has the properties mentioned above.

The claim is that $\ker(\phi)$ has finite length. If $\ker(\phi) = (0)$, this is trivial. Therefore assume $\ker(\phi) \neq (0)$. Every associated prime of $\ker(\phi)$ is an associated prime of M, thus it contains a minimal prime P_i. On the other hand, ϕ_{P_i} is an isomorphism by construction. Since localization is left exact $(\ker(\phi))_{P_i} = (0)$. Therefore the associated prime of $\ker(\phi)$ is not P_i, i.e., it properly contains P_i. Since M is a graded module, the assocated prime is a homogeneous prime that properly contains P_i. The only such prime is $\langle X_0, X_1, X_2 \rangle$. Since the only associated prime of $\ker(\phi)$ is $\langle X_0, X_1, X_2 \rangle$, $\ker(\phi)$ is $\langle X_0, X_1, X_2 \rangle$ primary, i.e., $\langle X_0, X_1, X_2 \rangle^e \ker(\phi) = (0)$ for some integer e. Therefore $\ker(\phi)$ is a finitely generated module over the local, Artinian k-algebra $k[X_0, X_1, X_2]/\langle X_0, X_1, X_2 \rangle^e$. It follows that $\ker(\phi)$ has finite length. The filtration of $\ker(\phi)$ by powers of the maximal ideal can be refined to a Jordan-Hölder filtration whose subquotients are all $k[X_0, X_1, X_2]/\langle X_0, X_1, X_2 \rangle$. Concatenating with the filtration above gives a Jordan-Hölder filtration on M.

More is true. As above, there is a filtration on $\oplus_i M_i$: $(\oplus M_i)^0 \supset \cdots \supset (\oplus M_i)^r = (0)$ and the homomorphism ϕ is *strict* for this filtration, i.e., $\phi^{-1}(\oplus_i M_i)^l = M^l$ for $l = 0, \dots, r$ (of course $M^l = \ker(\phi)$ has a further filtration, but this isn't relevant). Because ϕ is strict, the induced homomorphism $\phi : M^i/M^{i+1} \to (\oplus_i M_i)^l/(\oplus_i M_i)^{l+1}$ is injective for every l. As discussed above, the target is $(k[X_0, X_1, X_2]/P_j)(d_l)$ for some $j = 1, \dots, m$ and some integer d_l.

The claim is that the image of ϕ is nonzero. To see this, localize both sides by P_j. There is an induced filtration of the localization. Because localization is exact, $(M_{P_j})^l/(M_{P_j})^{l+1} \cong (M^l/M^{l+1})_{P_j}$ and

$$((\oplus_i M_i)_{P_j})^l/((\oplus_i M_i)_{P_j})^{l+1} \cong ((\oplus_i M_i)^l/(\oplus_i M_i)^{l+1})_{P_j} \cong (k[X_0, X_1, X_2]/P_j)_{P_j}.$$

By construction, ϕ_{P_j} is an isomorphism, and ϕ_{P_j} is strict, therefore $\phi_{P_j} : M_{P_j}^l/M_{P_j}^{l+1} \to (\oplus_i M_i)_{P_j}^l/(\oplus_i M_i)_{P_j}^l$ is an isomorphism. It follows that $(M^l/M^{l+1})_{P_j}$ is nonzero, and therefore M^l/M^{l+1} is nonzero, proving the claim. In particular, $M^l/M^{l+1} = (k[X_0, X_1, X_2]/P_j)(e_l)$ for some integer e_l, as a submodule of $(k[X_0, X_1, X_2]/P_j)(d_l)$. So the cokernel is isomorphic to $k[T](d_l)/k[T](e_l) \cong k[T]/\langle T^{e_l - d_l}$, which has finite length. Since this holds for every $l = 1, \dots, r$, the cokernel of ϕ has finite length.

This proves there exist Jordan-Hölder filrations for M and $\oplus_i M_i$ such that ϕ is a strict map of the filrations whose kernel and cokernel both have finite length.

Remark: It follows that the Hilbert polynomial of M equals the sum over i of the Hilbert polynomial of M_i. On the one hand, there is an exact sequence of graded modules,

$$0 \to k[X_0,X_1,X_2](-d-e) \xrightarrow{(G,-F)^\dagger} k[X_0,X_1,X_2](-d) \oplus k[X_0,X_1,X_2](-e) \\ \xrightarrow{(F,G)} k[X_0,X_1,X_2] \xrightarrow{k} [X_0,X_1,X_2]/\langle F,G\rangle \to 0,$$

from which it easily follows the Hilbert polynomial of M is de. On the other hand, by Problem 1, the Hilbert polynomial of each M_i is the intersection multiplicity $i(C,D;p_i)$. This gives *Bézout's theorem in the plane*,

$$\deg(C)\cdot\deg(D) = \sum_{p_i \in C\cap D} i(C,D;p_i).$$

Required Problem 3: This is essentially (Hartshorne, Exer. I.7.5). Let $C \subset \mathbb{P}^2_k$ be a plane curve of degree $d \geq 1$.

(a) If there exists $p \in C$ such that $e(C;p) = d$, prove C is a union of lines containing p.

Solution: Consider the projection $\pi_p : (C - \{p\}) \to \mathbb{P}^1$. Let $q \in C - \{p\}$ be a point, denote $q' = \pi_p(q)$, and let $L \subset \mathbb{P}^2_k$ be the line containing p corresponding to q'. Because L is irreducible, if L is not contained in C then $C \cap L$ is a proper closed subset of L which is a finite set. Then by Bézout's theorem, $\deg(C)\deg(L) \geq i(C,L;p) + i(C,L;q)$. Of course $\deg(L) = 1$ and $e(L;p) = e(L;q) = 1$. By Problem 1(b), $i(C,L;p) \geq e(C;p) = d$ and $i(C,L;q) \geq e(C;q) \geq 1$. So Bézout's theorem gives $d \geq d+1$, which is absurd. Therefore $L \subset C$. So C is a union of lines containing p.

(b) If C is irreducible, and $p \in C$ is a point such that $e(C;p) = d-1$, prove the projection from p is birational: $\pi_p : (C - \{p\}) \to \mathbb{P}^1_k$.

Solution: First of all, C is not a line since for every line L containing p, $e(L;p) = 1 = \deg(L)$ and $e(C;p) < \deg(C)$. So, continuing the argument from (a), for every $q \in C - \{p\}$, $L \cap C$ is a finite set and Bézout's theorem gives

$$d \geq e(C;p) + i(C,L;q) + \sum_{q' \in C\cap L - \{p,q\}} e(C;q') = d-1+i(C;q) + \sum_{q' \in C\cap L-\{p,q\}} e(C;q').$$

It follows that the fiber of π_p containing q equals $\{q\}$ and $i(C,L;q) = 1$. Let t be a uniformizer for $\mathcal{O}_{\mathbb{P}^1,q'}$, let s be a uniformizer for $\mathcal{O}_{C,q}$, and let $\pi_p^\#(t) = us^i$ for a unit u and an integer i. The algebra computing $i(C,L;q)$, namely $\mathcal{O}_{\mathbb{P}^2,q}/\langle F_q, G_q\rangle$ equals $\mathcal{O}_{C,q}/\langle G_q\rangle \cong \mathcal{O}_{C,q}\langle s^i\rangle$, which has length i. Therefore i equals q, i.e., $\pi_p^\# t$ is a uniformizer for C at q. In particular, $C - \{p\}$ is smooth and $\pi_p : C - \{p\} \to \mathbb{P}^1$ is injective, and more, the derivative $d\pi_p$ is everywhere an isomorphism. Since π_p is generically finite, there exists a dense open subset $U \subset \mathbb{P}^1$ such that $\pi_p : \pi_p^{-1}(U) \to U$ is finite. Because π_p is injective, the corresponding field extension $k(\mathbb{P}^1) \to k(C)$ is purely inseparable. Because $d\pi_p$ is not identically zero, this field extension is in fact an isomorphism, i.e., $\pi_p : C - \{p\} \to \mathbb{P}^1_k$ is birational.

Required Problem 4: Find an example of a weakly projective morphism $F : X \to Y$ that is not strongly projective. If you are ambitious, find an example where X and Y are quasi-compact and separated.

Solution: There are elementary examples if Y is not quasi-compact. For instance, let Y be the disjoint union $Y = \sqcup_{n=0}^{\infty} Y_n$ where $Y_n \cong \mathbb{A}_k^0$, let $X = \sqcup_{n=0}^{\infty} X_n$ where $X_n \cong \mathbb{P}_k^n$, and let $f : X \to Y$ be the locally constant morphism such that $f(X_n) = Y_n$. This is weakly projective because for every $p \in Y$ there exists an n with $p \in Y_n \subset Y$, the subset $Y_n \subset Y$ is an open affine subset, and $F : F^{-1}(Y_n) \to Y_n$ is strongly projective. But F is not strongly projective. Indeed, if there were a closed immersion $i : X \to Y \times \mathbb{P}_k^r$, then every irreducible component $X_n \cong \mathbb{P}_k^n$ of X would have a closed immersion into $Y_n \times \mathbb{P}_k^r \cong \mathbb{P}_k^r$. For $n > r$, $\dim(X_n) = n > r = \dim(\mathbb{P}_k^r)$, so there is no closed immersion $i : X_n \to \mathbb{P}_k^r$.

A more ambitious example is Hironaka's example, described in lecture. Here is an explicit version of the example from lecture. Denote homogeneous coordinates on $\mathbb{P}_k^3$ by X_0, X_1, X_2, Y. For every integer n, denote by X_n the variable X_a where $a \in \{0, 1, 2\}$ is the unique integer such that $n - a$ is divisible by 3. Let $U = \mathbb{P}_k^3 - \{[0, 0, 0, 1]\}$. For every integer n, denote $U_n = D_+(X_n) \subset U$, and denote by $F_n : V_n \to U_n$ the blowing up of the ideal $I_n = \langle (Y/X_n)^2, (Y/X_n)(X_{n+1}/X_n), (X_{n+1}/X_n)^2(X_{n+2}/X_n) \rangle$. Of course identify $V_n = V_m$ if $n - m$ is divisible by 3.

Because X_{n+1}/X_n is invertible on $U_n \cap U_{n+1}$, $F_n : F_n^{-1}(U_n \cap U_{n+1}) \to U_n \cap U_{n+1}$ is the blowing up of the ideal $I'_n = \langle (Y/X_n), (X_{n+2}/X_n) \rangle$. Similarly, $F_{n+1} : F_{n+1}^{-1}(U_n \cap U_{n+1}) \to U_n \cap U_{n+1}$ is the blowing up of the ideal $\langle (Y/X_{n+1})^2, (Y/X_{n+1})(X_{n+2}/X_{n+1}), (X_{n+2}/X_{n+1})^2 \rangle$, which is the same as the ideal $I''_n = \langle (Y/X_n)^2, (Y/X_n)(X_{n+2}/X_n), (X_{n+2}/X_n)^2 \rangle = (I'_n)^2$. Because the pullback of I'_n to $F_n^{-1}(U_n \cap U_{n+1})$ is a locally principal ideal, the pullback of $(I'_n)^2$ is a locally principal ideal whose generator is the square of the generator of the pullback of I'_n. By the universal property, there is an induced morphism $\phi_{n+1,n} : F_n^{-1}(U_n \cap U_{n+1}) \to F_{n+1}^{-1}(U_n \cap U_{n+1})$. Similarly, at every point of $F_{n+1}^{-1}(U_n \cap U_{n+1})$, because the pullback of I''_n is a principal ideal $\langle t \rangle$, one of the fractions of the consecutive generators of I''_n is a regular function. Since each such fraction is either $(Y/X_n)/(X_{n+2}/X_n)$ or $(X_{n+2}/X_n)/(Y/X_n)$, it follows that either the pullback of I'_n is generated by the pullback of (X_{n+2}/X_n) or it is generated by the pullback of (Y/X_n). Thus the pullback of I'_n is a locally principal ideal. By the universal property, there is an induced morphism $\phi_{n,n+1} : F_{n+1}^{-1}(U_n \cap U_{n+1}) \to F_n^{-1}(U_n \cap U_{n+1})$.

Of course $F_{n+1} \circ \phi_{n+1,n} = F_n$ and $F_n \circ \phi_{n,n+1} = F_{n+1}$. Therefore $F_{n+1} \circ (\phi_{n+1,n} \circ \phi_{n,n+1}) = F_{n+1}$. Because F_{n+1} is birational, $\phi_{n+1,n} \circ \phi_{n,n+1}$ equals the identity morphism over a dense open subset. Because V_n is separated, $\phi_{n+1,n} \circ \phi_{n,n+1}$ is the identity morphism. Similarly $\phi_{n,n+1} \circ \phi_{n+1,n}$ is the identity morphism, i.e., $\phi_{n,n+1}$ and $\phi_{n+1,n}$ are inverse isomorphisms. For essentially the same reason, the collection $((V_n)_n, (F_n^{-1}(U_n \cap U_{n+1}))_n, (\phi_{n,n+1}))$ satisfy the gluing lemma for varieties. Denote by V be the associated variety. And the collection (F_n) satisfies the gluing lemma for morphisms. Denote by $F : V \to U$ the associated morphism.

By construction, $F : V \to U$ is weakly projective: for every n, $F_n : V_n \to U_n$ is a blowing up which is strongly projective. Because U is quasi-projective, if F is strongly projective, then V is also quasi-projective. But by the argument in lecture comparing the degrees of various irreducible components of F, V is not quasi-projective. Thus F is weakly projective, but F is not strongly projective.

Problem 5: Assume $\text{char}(k)$ does not divide 6. Combine Problem 2 with Problem 2 from Problem Set 6 to deduce that every smooth plane curve C of degree $d \geq 3$ has at most $3d(d-2)$ flex lines.

Problem 6: If $\text{char}(k) = 3$, give an example of a smooth plane curve C of degree $d \geq 3$ having infinitely many flex lines. If you get stuck, look up (Hartshorne, Exer. IV.2.4).

Problem 7: Find two homogeneous polynomials $F_2 \in k[X_0, X_1, X_2, X_3]_2, F_3 \in k[X_0, X_1, X_2, X_3]_3$ such that $\mathbb{V}(F_2, F_3)$ is the rational normal curve $C = \{[s_0^3, s_0^2 s_1, s_0 s_1^2, s_1^3] \in \mathbb{P}^3_k | [s_0, s_1] \in \mathbb{P}^1_k\}$ Note that F_2, F_3 do *not* generate the homogeneous ideal $\mathbb{I}(C)$.

Problem 8: For every integer $n \geq 3$, find $n-1$ homogeneous polynomials $F_i \in k[X_0, \dots, X_n]_i$, $i = 2, \dots, n$, such that $\mathbb{V}(F_2, \dots, F_n)$ is the rational normal curve $C = \{[s_0^n, s_0^{n-1} s_1, \dots, s_0 s_1^{n-1}, s_1^n] \in \mathbb{P}^n_k | [s_0, s_1] \in \mathbb{P}^1_k\}$.

Solution: For every integer $m = 2, \dots, n$, define

$$F_m(X_0, \dots, X_n) = \sum_{k=0}^{m-1} (-1)^k \binom{m-1}{k} X_k X_{m-1}^k X_m^{m-1-k}.$$

This is homogeneous of degree m. Moreover,

$$\begin{aligned} F_m(s_0^n, s_0^{n-1} s_1, \dots, s_1^n) &= \textstyle\sum_{k=0}^{m-1} (-1)^k \binom{m-1}{k} (s_0^{n-k} s_1^k)(s_0^{n+1-m} s_1^{m-1})^k (s_0^{n-m} s_1^m)^{m-1-k} \\ &= (\textstyle\sum_{k=0}^{m-1} (-1)^k \binom{m-1}{k}) s_0^{m(n+1-m)} s_1^{m(m-1)} \\ &= (1-1)^{m-1} s_0^{m(n+1-m)} s_1^{m(m-1)} = 0. \end{aligned}$$

So $C \subset \mathbb{V}(F_2, \dots, F_n)$. Let $p = [a_0, a_1, \dots, a_n]$ be an element of $\mathbb{V}(F_2, \dots, F_n)$. First consider the case that $a_0 = 0$. The claim is that for every $i = 0, \dots, n-1$, $a_i = 0$. This is proved by induction on i. If $i = 0$, this is the hypothesis. By way of induction, assume $i > 0$ and assume $a_0 = \dots = a_{i-1} = 0$. Plugging in to F_{i+1},

$$0 = F_{i+1}(p) = 0 + \dots + (-1)^i \binom{i}{i} X_i^{i+1} = (-1)^i X_i^{i+1}.$$

Therefore $a_i = 0$, proving the claim by induction. So $p = [0, \dots, 0, 1]$, which is in C.

Next consider the case that $a_0 \neq 0$. Define $b = a_1/a_0$. The claim is that for every $i = 1, \dots, n$, $a_i = a_0 b^i$. This is proved by induction on i. If $i = 1$, this is the definition of b. By way of induction, assume $i > 0$ and assume $a_j = a_0 b^j$ for $j = 1, \dots, i-1$. Plugging in to F_i,

$$\begin{gathered} F_i(p) = \textstyle\sum_{k=0}^{i-1} (-1)^k \binom{i-1}{k} (a_0 b^k)(a_0 b^{i-1})^k a_i^{m-k} = \\ a_0 \textstyle\sum_{k=0}^{i-1} (-1)^k \binom{i-1}{k} (a_0 b^i)^k X_i^{m-k} = a_0 (a_i - a_0 b^i)^{i-1}. \end{gathered}$$

Since $i \geq 1$, since $a_0 \neq 0$ and since $F_i(p) = 0$, it follows that $(a_i - a_0 b^i)^{i-1} = 0$, i.e., $a_i = a_0 b^i$. The claim is proved by induction. So $p = [a_0, a_0 b, a_0 b^2, \dots, a_0 b^n]$, which is in C. Therefore $\mathbb{V}(F_2, \dots, F_n) \subset C$, proving that $C = \mathbb{V}(F_2, \dots, F_n)$.

Problem 9: Let $C \subset \mathbb{P}^n_k$ be an irreducible curve contained in no hyperplane. Let $p \in C$ be any point, and let $\pi_p : C - \{p\} \to \mathbb{P}^{n-1}_k$ be projection from p. Denote by D the closure of the image of C. Prove that D is contained in no hyperplane and $\deg(D) \leq \deg(C) - 1$.

Problem 10: This problem continues Problem 9. Prove that the only irreducible curve $C \subset \mathbb{P}^n_k$ of degree 1 is a line and use this to prove that $\deg(C) \geq n$ if C is an irreducible curve contained in no hyperplane.