18.725 PROBLEM SET 7

Due date: Friday, November 12 in lecture. Late work will be accepted only with a medical note or for another Institute-approved reason. You are strongly encouraged to work with others, but the final write-up should be entirely your own and based on your own understanding.

Read through all the problems. Write solutions to the "Required Problems", 1, 2, 3, and 4. There will be more problems posted soon, and you will be asked to do 2 more problems to a total of 6.

Required Problem 1: This problem consists of reading the following description. Everybody will receive full credit for this problem. The description will also be needed to do the following problem.

There is an inductive procedure for resolving singularities of plane curves called *embedded resolution*. Soon we will see another procedure by normalization, but embedded resolutions are often easier to compute. The input of the algorithm is a datum $(C,(p_1,\dots,p_r),(\phi_i : U_i \to \mathbb{A}^2_k)$ consisting of

(i) a nonempty algebraic variety C of pure dimension 1 (i.e., $\dim(C,p) = 1$ for every $p \in C$),
(ii) a finite collection of points $(p_1,\dots,p_r)$ of C, and
(iii) for every $i = 1,\dots,r$, an open subset $p_i \in U_i \subset C$ together with a closed immersion $\phi_i : U_i \to \mathbb{A}^2_k$ such that $\phi_i(p_i) = (0,0)$.

The output of the algorithm is another such datum $(C',(p'_1,\dots,p'_s),(\phi'_j : U'_j \to \mathbb{A}^2_k))$ together with a dominant, (weakly) projective morphism $G : C' \to C$ such that,

(i) $G(p'_1,\dots,p'_s) = (p_1,\dots,p_r)$, and
(ii) $G : C' - \{p'_1,\dots,p'_s\} \to C - \{p_1,\dots,p_r\}$ is an isomorphism.

The curve C' is "less singular" at the distinguished points $p'_1,\dots,p'_s$ in a sense that won't be made precise. After a finite number of repetitions of the algorithm, C' is smooth at every point p'_j (this will be easier to prove after the discussion on normalization).

Here is the algorithm. For every $i = 1,\dots,r$, if p_i is a smooth point of C, define $p'_{i,1} = p_i$, define $U'_{i,1} = U_i$, and define $\psi'_{i,1} = \psi_i$. Denote by $G_{i,1} : U'_{i,1} \to C$ the obvious open immersion.

Assume p_i is a singular point of C. Using the closed immersion ϕ_i, identify U_i with a plane curve so that $p_i = (0,0)$. Let $H : Y \to \mathbb{A}^2_k$ be the blowing up of $\mathbb{A}^2_k$ at the ideal $\langle x,y\rangle k[x,y]$. Recall from Problem 6 or Problem 8 on Problem Set 6 that $Y = \{((x,y),[u,v]) \in \mathbb{A}^2_k \times \mathbb{P}^1_k | xv - yu = 0\}$. Denote $E = H^{-1}(0)$, and denote by $\widetilde{U}_i \subset Y$ the *strict transform of* U_i, $\widetilde{U}_i = \overline{H^{-1}(U_i - \{(0,0)\})}$. Now Y has a covering by two open affines: $D_+(u)$ is isomorphic to $\mathbb{A}^2_k$ with coordinates $(x, v/u)$, and $D_+(v)$ is isomorphic to $\mathbb{A}^2_k$ with coordinates $(y, u/v)$. If $\mathbb{I}(U_i) = \langle f(x,y)\rangle$, then $\mathbb{I}(\widetilde{U}_i \cap D_+(u))$ is the principal ideal generated by the unique polynomial $g(x, v/u)$ not divisible by x such that $f(x, x\cdot v/u) = x^e g(x, v/u)$ for some integer $e \geq 0$, i.e., $g(x, v/u)$ is obtained by factoring as many copies of x as possible from $f(x, x\cdot v/u)$.

There is a similar description of the ideal of $\widetilde{U}_i \cap D_+(v)$. What is important is this is algorithmic – you can compute the two open affine pieces $\widetilde{U}_i \cap D_+(u)$ and $\widetilde{U}_i \cap D_+(v)$ by simply factoring a polynomial.

Denote by $p'_{i,1}, \dots, p'_{i,s_i}$ the elements of $\widetilde{U}_i \cap E_i$. These are precisely the points of $\widetilde{U}_i$ in the preimage of $(0,0)$. For every $j = 1, \dots, s_i$, let $U'_{i,j}$ be one of $\widetilde{U}_i \cap D_+(u)$ or $\widetilde{U}_i \cap D_+(v)$ such that $p'_{i,j} \in U'_{i,j}$. Denote by $\psi'_{i,j} : U'_{i,j} \to \mathbb{A}^2_k$ the closed immersion obtained by composing the closed immersion $\widetilde{U}_i \cap D_+(u) \subset D_+(u)$, resp. $\widetilde{U}_i \cap D_+(v) \subset D_+(v)$, with the identification of the open subset $D_+(u)$, resp. $D_+(v)$, with $\mathbb{A}^2_k$, followed by the translation of $\mathbb{A}^2_k$ sending $p'_{i,j}$ to $(0,0)$. Denote by $G_{i,j} : U'_{i,j} \to C$ the restriction to $U'_{i,j}$ of H.

Finally, denote $U'_0 = C - \{p_1, \dots, p_r\}$ and denote $G_0 : U'_0 \to C$ the obvious open immersion. It is straightforward to define the overlap open sets $U'_{(i_1,j_1),(i_2,j_2)} \subset U'_{i_1,j_1}$ and isomorphisms $\phi_{(i_1,j_1),(i_2,j_2)} : U'_{(i_1,j_1)} \to U'_{(j_1,i_1)}$ such that the datum satisfies the gluing lemma for objects giving an algebraic variety C', and such that the morphisms $G_{i,j}$ satisfy the gluing lemma for morphisms giving a dominant, (weakly) projective morphism $G : C' \to C$. The set $(p'_1, \dots, p'_s)$ is the concatenation of the sets $(p'_{i,j} | j = 1, \dots, s_i)$, and the open subset $U'_{i,j}$ and $\psi'_{i,j}$ are as defined above.

Example, the node: Let $C = \mathbb{V}(xy) \subset \mathbb{A}^2_k$, let $p_1 = (0,0)$, let $U_1 = C$ and let $\psi_1 : U_1 \to \mathbb{A}^2_k$ be the obvious closed immersion. The polynomial $x(x \cdot v/u) = x^2 \cdot (v/u)$. Therefore $\widetilde{U}_1 \cap D_+(u) = \mathbb{V}(v/u) \subset D_+(u)$. Similarly $\widetilde{U}_1 \cap D_+(v) = \mathbb{V}(u/v)$. The intersection $\widetilde{U}_1 \cap D_+(u) \cap E$ consists of one point $p'_{1,1}$, $(x, v/u) = (0,0)$. The intersection $\widetilde{U}_1 \cap D_+(v) \cap E$ consists of one point $p'_{1,2}$, $(y, u/v) = (0,0)$. The curve C' is the union of $U'_{1,1} = \widetilde{U}_1 \cap D_+(u)$ and $U'_{1,2} = \widetilde{U}_1 \cap D_+(v)$, which happens to be a disjoint union of two copies of $\mathbb{A}^1_k$. The morphism $G : C' \to C$ maps $U'_{1,1}$ isomorphically to the x-axis $\mathbb{V}(y) \subset C$ and maps $U'_{1,2}$ isomorphically to the y-axis $\mathbb{V}(x) \subset C$. The two points $p'_{1,1}$ and $p'_{1,2}$ are the inverse images of $(0,0)$ on each axis.

Required Problem 2: In each of the following cases, compute for yourself the complete embedded resolution of the curve, i.e., repeat the algorithm above until C' is smooth. *Do not write up all your computations.* In each case, sketch C, sketch the normal cone of C at p, say how many iterations of the algorithm are necessary to obtain the embedded resolution, and say how many points in the embedded resolution map to p.

(a), The cusp $C = \mathbb{V}(y^2 - x^3) \subset \mathbb{A}^2_k$, $p = (0,0)$.

(b), The tacnode $C = \mathbb{V}(y(y - x^2)) \subset \mathbb{A}^2_k$, $p = (0,0)$.

(c), The ramphoid cusp $C = \mathbb{V}(y^2 - x^5) \subset \mathbb{A}^2_k$, $p = (0,0)$.

(d), The triple point $C = \mathbb{V}(xy(y - x)) \subset \mathbb{A}^2_k$, $p = (0,0)$.

Required Problem 3, The Frobenius morphism I: Let k be an algebraically closed field of characteristic $p > 0$. For every integer $\nu > 0$, define the *Frobenius morphism*, $\mathrm{Frob}_\nu : \mathbb{A}^n_k \to \mathbb{A}^n_k$ to be the unique regular morphism $(a_1, \dots, a_n) \mapsto (a_1^{p^\nu}, \dots, a_n^{p^\nu})$.

(a) Prove that Frob_ν is a homeomorphism.

(b) Prove that Frob_ν is a dominant, finite morphism.

(c) For this morphism, compute the module of relative differentials $\Omega_{k[X]/k[Y]}$. For which points in $X = \mathbb{A}^n_k$ does the Jacobian criterion predict the fiber of Frob_ν is smooth?

Required Problem 4, The Frobenius morphism II: Let $X \subset \mathbb{A}^n_k$ be an irreducible affine algebraic set that is normal, i.e., $k[X]$ is integrally closed in $k(X)$. Define $Y = \mathrm{Frob}_\nu^{-1}(X) \subset \mathbb{A}^n_k$.

(a) Prove that $Y \subset \mathbb{A}^n_k$ is an irreducible affine algebraic set. **Hint:** Use Problem 2(a).

(b) Prove that Y is normal, i.e., $k[Y]$ is integrally closed in $k(Y)$. **Hint:** Let $y_1, \dots, y_n$ be coordinates on the domain, and let $x_1, \dots, x_n$ be coordinates on the target. Every element in $k(Y)$ can be written as $h = f(y_1, \dots, y_n)/\mathrm{Frob}_\nu^\#(g(x_1, \dots, x_n))$. If this is integral over $k[Y]$, prove h^{p^ν} is integral over $k[X]$, and so equals $h(x_1, \dots, x_n)$. Now directly construct $e(y_1, \dots, y_n) \in k[Y]$ such that $(e(y_1, \dots, y_n))^{p^\nu} = \mathrm{Frob}_\nu^\#(h(x_1, \dots, x_n))$.

(c) Give a characterization (somewhat up to you) of the image of the $k(X)$-field extension $k(Y)$ inside of the algebraic closure $\overline{k(X)}$.

Problem 5: Assume $\mathrm{char}(k) \neq 2$. Let $A = A^\dagger$ be a symmetric $(n+1) \times (n+1)$ matrix with coefficients in k. Define $Q_A \in k[x_0, \dots, x_n]_2$ to be the homogeneous quadratic polynomial,

$$Q_A(x_0, \dots, x_n) = \mathbf{x}^\dagger A \mathbf{x},$$
$$\mathbf{x} = \begin{pmatrix} x_0 \\ \dots \\ x_n \end{pmatrix}.$$

Assume $\mathrm{rank}(A) > 1$.

(a) Prove that $\mathbb{V}(Q_A) \in \mathbb{P}^n_k$ is smooth iff $\mathrm{rank}(A) = n + 1$.

(b) For every integer $2 \leq r \leq n - 1$, if $\mathrm{rank}(A) = r$, describe the set of singular points in $\mathbb{V}(Q_A)$.

Problem 6: For every integer $d \geq 3$, define the Fermat hypersurface $\mathbf{X}_d = \mathbb{V}(x_0^d + x_1^d + x_2^d + x_3^d) \subset \mathbb{P}^3_k$.

(a) If $\mathrm{char}(k)$ does not divide d, prove X_d is smooth.

(b) If $\mathrm{char}(k)$ does divide d, what is $\mathbb{I}(X_d)$?

Problem 7: Follow the same procedure in Problem 9 from Problem Set 10 to find $3d^2$ lines on X_d if $\mathrm{char}(k)$ does not divide d.

Difficult Problem 8: Follow the same procedure in Problem 10 from Problem Set 10 to prove the $3d^2$ lines from Problem 7 are the only lines on X_d. What hypotheses are necessary on $\mathrm{char}(k)$?

Very Difficult Problem 9: Assume $\mathrm{char}(k) = 0$. Find an irreducible quartic polynomial $F \in k[x_0, x_1, x_2, x_3]_4$ such that $X = \mathbb{V}(F) \subset \mathbb{P}^3_k$ is smooth, and X contains *more* than 48 lines (the number for the Fermat hypersurface). **Remark:** For $d = 4$, the maximum number of lines on a smooth quartic surface was computed by B. Segre: *The maximum number of lines lying on a quartic surface*, Quarterly J. of Math., 14, 1943, 86-96. For $d > 4$, as far as I am aware, it is unknown what is the maximum number of lines (in all but a few cases, there is no example that improves on the Fermat).

Problem 10: Assume $\mathrm{char}(k) \neq 2, 3$. As $[s,t] \in \mathbb{P}^1_k$ varies, consider the following family of plane cubics,

$$X_{[s,t]} = \mathbb{V}(tY^2Z - X(X-Z)(sX-tZ)) \subset \mathbb{P}^2_k,$$

where X, Y, Z are homogeneous coordinates on $\mathbb{P}^2_k$. Determine all values of $[s,t]$ such that $X_{[s,t]}$ is singular, and for each $[s,t]$, determine all singular points of $X_{[s,t]}$.