18.725 PROBLEM SET 5

Due date: Friday, October 29 in lecture. Late work will be accepted only with a medical note or for another Institute-approved reason. You are strongly encouraged to work with others, but the final write-up should be entirely your own and based on your own understanding.

Read through all the problems. Write solutions to the "Required Problems", 1, 2, 3, and 4, together with 2 others of your choice to a total of 6 problems. One or two more optional problems may be added to the problem set soon.

Required Problem 1: Let $F : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a dominant, separated morphism of irreducible algebraic varieties, i.e., $F(X) \subset Y$ is dense. The morphism F is *generically finite* if the induced map of fields of fractions, $F^\# : k(Y) \to k(X)$, is a finite, algebraic field extension. The next two problems prove the following proposition. This proposition reduces to the case that $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are affine varieties.

Proposition 0.1. *For every generically finite morphism $F : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$, there exists a dense open subset $U \subset Y$ such that $F : F^{-1}(U) \to U$ is a finite morphism.*

(a) Prove it suffices to consider the case when $(Y, \mathcal{O}_Y)$ is an affine variety.

(b) Prove the following lemma.

Lemma 0.2. *Let $F : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a separated morphism. If $Z \subset X$ is a locally closed subset such that $F|_Z : (Z, \mathcal{O}_Z) \to (Y, \mathcal{O}_Y)$ is proper, then $Z \subset X$ is closed.*

Sketch: Prove $\Delta_{X/Y} : X \to X \times_Y X$ is closed and $\pi_1 : X \times_Y Z \to X$ are closed. Deduce that $\Delta_{X/Y}|_Z : Z \to X \times_Y Z$ is closed, thus $Z = \pi_1(\Delta_{X/Y}|_Z(Z)) \subset X$ is closed.

(c) Back to the proposition, let $V \subset X$ be a dense open such that $F|_V : V \to Y$ is finite. By Required Problem 4(c) from Problem Set 4, $F|_V$ is proper. Use (b) to prove $V \subset X$ is open and closed, thus all of X. Deduce it suffices to prove the proposition after replacing X by a dense open affine $W \subset X$ (with $V = W \cap F^{-1}(U)$).

Required Problem 2: This is the follow-up to Required Problem 1. You may assume all parts of that problem. Thus $F : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a generically finite morphism of affine varieties. Let $a_1, \dots, a_n \in k[X]$ be generators for $k[X]$ as a k-algebra. Because $k(X)/k(Y)$ is an algebraic extension, each a_i satisfies a polynomial equation with coefficients in $k[Y]$. Clearing denominators, each a_i satisfies a polynomial equation $f_i(t)$ with coefficient in $k[Y]$,

$$f_i(t) = c_{i,d_i} t^{d_i} + \cdots + c_{i,1} t + c_{i,0},$$

where each $c_{i,d_i} \neq 0$. Define $c = c_{1,d_1} \cdots c_{n,d_n}$. Prove that for $U = D(c) \subset Y$ and $F^{-1}(U) = D(F^\#(c)) \subset X$, $F : F^{-1}(U) \to U$ is a finite morphism.

Required Problem 3: Let $F : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a quasi-compact, separated morphism of algebraic varieties. For every $p \in X$, denote $X_p = F^{-1}(F(p))$. For every integer $e \geq 0$, define $U_e(X, F) = \{p \in X | \dim(X_p, p) \leq e\}$. In this problem, you will prove the following proposition (if you get stuck, the reference in Eisenbud is Theorem 14.8). This problem has many parts. Do the parts you can; it will be graded generously.

Proposition 0.3. *For every integer $e \geq 0$, $U_e(X, F) \subset X$ is an open subset.*

(a) Prove it suffices to consider the case where $(Y, \mathcal{O}_Y)$ is affine and $(X, \mathcal{O}_X)$ is affine.

(a) Prove the following simple lemma from topology.

Lemma 0.4. *Let $U \subset X$ be a subset of a topological space such that for every closed subset $C \subset X$ containing $X - U$ and intersecting U, there is a nonempty relatively open subset $O \subset C$ contained in $U \cap C$. Then $U \subset X$ is open.*

Hint: For the closure C of $X - U$, prove $C \cap U = \emptyset$, hence $U = X - C$ is open.

(b) Let $Z \subset X$ be a closed set containing $X - U_e(X, F)$. Prove $U_e(Z, F|_Z) = U_e(X, F) \cap Z$. Combined with (a), reduce Proposition 0.3 to the proposition.

Proposition 0.5. *For every integer $e \geq 0$, if $U_e(X, F) \subset X$ is nonempty, it contains a nonempty open subset of X.*

Sketch: Denote $V = X - Z$. For every $p \in X$ and every irreducible component $T \subset X_p$ containing p, if $T \cap V$ is nonempty, prove $\dim(T) \leq e$. Conclude if $p \in U_e(Z, F|_Z)$, every irreducible component of X_p has dimension $\leq e$.

(c) By considering the restriction of F to each irreducible component of X, reduce Proposition 0.5 to the case that X and Y are irreducible and F is dominant.

(d), $e = 0$ If $U_0(X, F)$ is nonempty, use Corollary 17.4 (Eisenbud, Theorem 10.10) to prove $\dim(X) = \dim(Y)$, thus $k(X)/k(Y)$ is algebraic by Corollary 18.5 (Eisenbud, Theorem A). Assuming Required Problem 2, prove $U_0(X, F)$ contains a nonempty open subset of X. Deduce Proposition 0.3 for $e = 0$.

(e), $e > 0$ Let $p \in U_e(X, F)$. Let $f_1, \dots, f_e \in \mathfrak{m}_p X_p$ be a system of parameters. There exists an open subset $p \in V \subset X$ and elements $g_1, \dots, g_e \in \mathcal{O}_X(V)$ whose germs at p are $f_1, \dots, f_e$. Define $G = (g_1, \dots, g_e) : V \to \mathbb{A}^e_k$, and consider $F \times G : X \to Y \times \mathbb{A}^e_k$. Prove $p \in U_0(X, F \times G) \subset U_e(X, F)$. Assuming (d) which proves Proposition 0.3 for $e = 0$, prove $U_e(X, G)$ contains an open subset of X, thus proving Proposition 0.3.

Required Problem 4: Before solving this problem, read through Problem 5 (although you don't have to solve it). Let X be an algebraic variety. Let $n \geq 0$ be an integer.

Definition 0.6. An Abelian cone ζ is a *vector bundle of rank n on X* if for every point $p \in X$ there exists an open subset $p \in U \subset X$ such that $\pi^{-1}(U)$ is isomorphic to $U \times \mathbb{A}^n$ as an Abelian cone over U.

(a) Let $\zeta = (\pi : E \to X, +, \cdot, 0)$ be a vector bundle of rank n on X and let $F : Y \to X$ be a regular morphism. Denote $E_Y := Y \times_X E$, and $\pi_Y : E_Y \to Y$ is the projection. Prove there are "natural" choices of $+$, $\cdot$, and 0 such that $(\pi_Y : E_Y \to Y, +, \cdot, 0)$ is a vector bundle of rank n on Y. This is the *pullback vector bundle*, denoted $F^*\zeta$. Indicate why F^* is a functor from the category of vector bundles on X to the category of vector bundles on Y (but you don't have to prove this). Of course the same is true for Abelian cones as well – prove this if you prefer.

(b) Given a second regular morphism $G : Z \to Y$, prove there is a natural isomorphism of functors from the category of vector bundles on X to the category of vector bundles on Z, $\theta_{G,F}(\zeta) : G^*F^*\zeta \to (F \circ G)^*\zeta$. Given a third regular morphism $H : W \to Z$, prove that,

$$\theta_{H,G\circ F}(\zeta) \circ H^*(\theta_{G,F}(\zeta)) = \theta_{H\circ G,F}(\zeta) \circ \theta_{H,G}(F^*\zeta).$$

(c) For every integer $n \geq 0$, let $X = \mathbb{P}^n_k$, let ζ be the trivial vector bundle of rank $n+1$, i.e., $\mathbb{P}^n_k \times \mathbb{A}^{n+1}_k$, and denote,

$$E = \{([a_0, \ldots, a_n], (b_0, \ldots, b_n)) \in \mathbb{P}^n_k \times \mathbb{A}^{n+1}_k | \text{for every } 0 \leq i < j \leq n,\ a_j b_i - a_i b_j = 0\}.$$

Prove that E is a sub-Abelian cone of ζ and is, in fact, a vector bundle of rank 1 on X. This is the *tautological rank 1 subbundle* on $\mathbb{P}^n_k$.

Problem 5, Abelian cones: Let X be an algebraic variety.

Definition 0.7. An *Abelian cone over* X is a datum $\zeta = (\pi, +, \cdot, 0)$ of a regular morphism of algebraic varieties $\pi : E \to X$, a regular morphism $+ : E \times_X E \to E$, denoted $(e_1, e_2) \mapsto e_1 + e_2$, a regular morphism $\cdot : \mathbb{A}^1_k \times E \to E$, denoted $(\lambda, e) \mapsto \lambda \cdot e$, and a regular morphism $0 : X \to E$, denoted $x \mapsto 0_x$, satisfying the following axioms.

(i) For every $(e_1, e_2) \in E \times_X E$, $\pi(e_1 + e_2) = \pi(e_1) = \pi(e_2)$; for every $e \in E$ and $\lambda \in \mathbb{A}^1_k$, $\pi(\lambda \cdot e) = \pi(e)$; and for every $x \in X$, $\pi(0_x) = x$.
(ii) For every $x \in X$, denoting $E_x = \pi^{-1}(x)$, there exists an integer $n \geq 0$ (depending on x) such that the datum $(E_x, +, \cdot, 0_x)$ is isomorphic to the standard datum $(\mathbb{A}^n, +, \cdot, 0)$.

The variety E is called the *total space*, the morphism π is the *projection*, and the morphism 0 is the *zero section*. If ζ, η are Abelian cones over X, a *homomorphism of cones from* ζ *to* η is a regular morphism $F : E_\zeta \to E_\eta$ such that $\pi_\eta \circ F = \pi_z eta$, such that $F(e_1 + e_2) = F(e_1) + F(e_2)$, and such that $F(\lambda \cdot e) = \lambda \cdot F(e)$, for every e_1, e_2, e and λ.

(a) For every integer $n \geq 0$, for $E = X \times \mathbb{A}^n_k$ and $\pi = \pi_1 : X \times \mathbb{A}^n_k \to X$, prove there is a "natural" choice of $+$, $\cdot$ and 0 so that $(\pi, +, \cdot, 0)$ is an Abelian cone. This is called the *trivial vector bundle of rank n*.

Let ζ be an Abelian cone over X. The *sheaf of sections of* ζ is the sheaf of sets $\mathcal{E}_{\text{sec}}$ on X whose sections over each open $U \subset X$ are the regular morphisms $s : U \to \pi^{-1}(U)$ such that $\pi \circ s = \text{Id}_U$. The *sheaf of functionals of* ζ is the sheaf of sets $\mathcal{E}_{\text{func}}$ on X whose sections over each open $U \subset X$ are the Abelian cone morphisms $F : \pi^{-1}(U) \to U \times \mathbb{A}^1_k$.

(b) Prove that for every open subset $U \subset X$, the morphisms $+$ and $\cdot$ naturally determine a structure of $\mathcal{O}_X(U)$-module on $\mathcal{E}_{\text{sec}}(U)$ and $\mathcal{E}_{\text{func}}(U)$, and for every inclusion $V \subset U$, the restriction maps $\mathcal{E}_{\text{sec}}(U) \to \mathcal{E}_{\text{sec}}(V)$ and $\mathcal{E}_{\text{func}}(U) \to \mathcal{E}_{\text{func}}(V)$ are homomorphisms of $\mathcal{O}_X(U)$-modules. Such a sheaf is called a *sheaf of $\mathcal{O}_X$-modules.*

(c) Let $F : E_\zeta \to E_\eta$ be a homomorphism of Abelian cones over X. Prove there are induced homomorphisms of sheaves of $\mathcal{O}_X$-modules, $F_* : \mathcal{E}_{\zeta,\text{sec}} \to \mathcal{E}_{\eta,\text{sec}}$ and $F^* : \mathcal{E}_{\eta,\text{func}} \to \mathcal{E}_{\zeta,\text{func}}$.

(d) Let $X = \mathbb{A}^1_k$, let ξ be the trivial vector bundle of rank 1, $X \times \mathbb{A}^1_k$, whose total space is just $\mathbb{A}^2_k$. Denote by $E \subset \mathbb{A}^2_k$ the closed subvariety $\mathbb{V}(xy)$. Prove that E is a sub-Abelian cone of ζ. Denote this by η.

(e) Denote by ζ the trivial vector bundle on X of rank 0, i.e., $X \times \mathbb{A}^0_k$, and denote by $F : \zeta \to \eta$ the unique homomorphism of Abelian cones over X. Prove that F_* is an isomorphism, but F^* is not an isomorphism. Because the sheaves of functionals "detect" homomorphisms of Abelian cones that sheaves of sections do not detect, they are used more often in algebraic geometry (sheaves of sections are frequently used for vector bundles, especially in other branches of geometry, but rarely used for Abelian cones that are not vector bundles).

Problem 6: Let X be a variety and let $\zeta = (\pi : E \to X, +, \cdot, 0)$ be an Abelian cone on X. For every open set $U \subset X$ there is a pairing $\langle -, - \rangle_U : \mathcal{E}_{\text{func}}(U) \times \mathcal{E}_{\text{sec}}(U) \to \mathcal{O}_X(U)$ which maps a pair (F, s) of a functional $F : \pi^{-1}(U) \to U \times \mathbb{A}^1_k$ and a section $s : U \to \pi^{-1}(U)$ to the regular function $\text{pr}_2 \circ F \circ s$.

(a) Prove $\langle -, - \rangle_U$ is bilinear for the action of $\mathcal{O}_X(U)$ on each module.

(b) Prove that for every inclusion $V \subset U$, $\langle F|_V, s|_V \rangle_V = (\langle F, s \rangle_U)|_V$. Deduce that for every element $x \in X$, there is a pairing of stalks $\langle -, - \rangle_x : (\mathcal{E}_{\text{func}})_x \times (\mathcal{E}_{\text{sec}})_x \to \mathcal{O}_{X,x}$.

(c) If ζ is a vector bundle, prove that for every $x \in X$, the pairing $\langle -, - \rangle_x$ is a *perfect pairing*, i.e., the following induced homomorphisms of $\mathcal{O}_{X,x}$-modules are isomorphisms,

$$(\mathcal{E}_{\text{sec}})_x \to \text{Hom}_{\mathcal{O}_{X,x}}((\mathcal{E}_{\text{func}})_x, \mathcal{O}_{X,x}),$$
$$(\mathcal{E}_{\text{func}})_x \to \text{Hom}_{\mathcal{O}_{X,x}}((\mathcal{E}_{\text{sec}})_x, \mathcal{O}_{X,x}).$$

Problem 7, The universal property of projective space: Let $n \geq 0$ be an integer, and let $\eta = (\pi : E \to \mathbb{P}^n, +, \cdot, 0)$ be the tautological rank 1 subbundle on $\mathbb{P}^n_k$, and let $\phi : E \to \mathbb{P}^n_k \times \mathbb{A}^{n+1}_k$ be the inclusion. This is a homomorphism of vector bundles on $\mathbb{P}^n_k$ such that for every $x \in \mathbb{P}^n_k$, the induced map $\phi_x : E_x \to \mathbb{A}^{n+1}_k$ is injective.

(a) Let X be a variety, let $\zeta = (\pi : L \to X, +, \cdot, 0)$ be a vector bundle of rank 1 on X, and let $\psi : L \to X \times \mathbb{A}^{n+1}_k$ be a homomorphism of vector bundles on X such that for every $x \in X$, the induced map $\phi_x : E_x \to \mathbb{A}^{n+1}_k$ is injective. Let $U = L - 0(X)$, the complement of the zero section, and denote by $G : U \to \mathbb{P}^n_k$ the composition,

$$U \xrightarrow{\phi} X \times (\mathbb{A}^{n+1}_k - \{0\}) \xrightarrow{\pi_2} (\mathbb{A}^{n+1}_k - \{0\}) \xrightarrow{\pi} \mathbb{P}^n_k.$$

Prove there exists a unique morphism $F : X \to \mathbb{P}^n_k$ such that $F \circ \pi = G$. **Hint:** Use the gluing lemma to reduce to the case that $L \cong X \times \mathbb{A}^1_k$ and compose G with any section disjoint from the zero section.

(b) Prove there exists a unique isomorphism $\theta : L \to F^*E$ such that $\psi = F^*\phi \circ \theta$. This is the *universal property of projective space*: morphisms from a variety X to $\mathbb{P}^n_k$

are in natural bijective correspondence to the set of pairs (L, ψ) up to precomposing ψ with an automorphism of L.

Problem 8: Let X be a variety, and let ζ, resp. ζ and η, be vector bundles on X. For each of the following functors, resp. bifunctors, on the category of k-vector spaces, define a natural analogue for ζ, resp. ζ and η, i.e., an analogous functor, resp. bifunctor, on the category of vector bundles on X. Observe that even if X, the total space of ζ and the total space of η are quasi-projective, it is not obvious that the total space of each of the new vector bundles is quasi-projective (although this turns out to be true). This is one justification of working in the larger category of algebraic varieties.

(a) Duals, $V \mapsto V^\vee := \mathrm{Hom}_k(V, k)$.

(b) Hom, $(V, W) \mapsto \mathrm{Hom}_k(V, W)$.

(c) Tensor product, $(V, W) \mapsto V \otimes_k W$.

Problem 9, Another definition of dimension I: Let $X \subset \mathbb{P}^n_k$ be a nonempty projective algebraic subset.

(a) Prove that if $\dim(X) > 0$, then for every $f \in k[x_0, \dots, x_n]_1 - \{0\}$, the corresponding hyperplane $H = \mathbb{V}(f) \subset \mathbb{P}^n_k$ intersects X. **Hint:** If $X \cap H = \emptyset$ then $X \subset D(f) \cong \mathbb{A}^n_k$, which, combined with universal closedness, implies X is a finite set.

(b) If $\dim(X) = d$, combine (a) with Krull's Hauptidealsatz to conclude that for every d hyperplanes $H_1, \dots, H_d \subset \mathbb{P}^n_k$, the intersection $H_1 \cap \cdots \cap H_d \cap X$ is nonempty.

Problem 10, Another definition of dimension II: This continues Problem 9, which you may now assume. If $\dim(X) = d$, prove there exist hyperplanes $H_1, \dots, H_{d+1} \subset \mathbb{P}^n_k$ such that $H_1 \cap \cdots \cap H_{d+1} \cap X$ is empty. Deduce the following,

$$\dim(X) = \min\{d \geq 0 | \exists \text{ hyperplanes } H_1, \dots, H_{d+1} \subset \mathbb{P}^n_k \text{ s.t. } H_1 \cap \cdots \cap H_{d+1} \cap X = \emptyset\}.$$

Hint: Work by induction on d. Let $S \subset X$ be a finite set of elements of X intersecting every irreducible component of X, and let $H \subset \mathbb{P}^n_k$ be a hyperplane not intersecting S. Prove that $H \cap X$ is either empty (if $d = 0$), or else has smaller dimension than X.

Problem 11, An irreducible, separated variety that is not quasi-compact: Thanks to Genya for inspiring this problem. For every integer $n \geq 0$, define $U_n = \mathbb{A}^2_k - \{(0,0)\}$ with coordinates (x_n, y_n), for every pair of integers $0 < m < n$, define $U_{m,n} = D(x_m) \subset U_m$, define $U_{n,m} = D(x_n) \subset U_n$, define $\phi_{m,n} : U_{m,n} \to U_{n,m}$ to be $(a_m, b_m) \mapsto (a_m, b_m / a_m^{n-m})$, and define $\phi_{n,m} : U_{n,m} \to U_{m,n}$ to be $(a_n, b_n) \mapsto (a_n, a_n^{n-m} b_n)$.

(a) For every pair of integers $0 \leq m < n$, prove $\phi_{m,n}$ and $\phi_{n,m}$ are inverse isomorphisms.

(b) Prove the datum $((U_m), (U_{m,n}), (\phi_{m,n}))$ satisfies the hypotheses of the gluing lemma for varieties. Denote the associated variety by $(U, (\phi_m : U_m \to U))$.

(c) To prove that U is separated, it suffices to prove for every pair of integers $0 \leq m \leq n$, that $\phi_m(D(*_m)) \cap \phi_n(D(-_n))$ is affine and $\mathcal{O}_U(\phi_m(D(*_m)) \cap \phi_n(D(-_n)))$ is generated as a k-algebra by $\mathcal{O}_{U_m}(D(*_m))$ and $\mathcal{O}_{U_n}(D-_n)$, for $* = x, y$ and $- = x, y$. Check this, in particular, for $* = y, - = y$.

(d) Prove that all the open sets $\phi_m(D(x_m y_m)) \subset U$ are equal. This is an irreducible open subset that is dense in each $\phi_m(U_m)$, thus it is dense in U. Conclude that U is irreducible of dimension 2.

(e) Prove that U is not quasi-compact. **Hint:** Consider $\phi_m((0,1)) \in U$.

(f) You don't have to do this part, it is just motivational. For every integer $n \geq 0$, define $F_n : U_n \to \mathbb{A}^2_k$ by $F_n(a_n, b_n) = (a_n, a_n^n b_n)$. Prove that the morphisms F_n satisfy the gluing lemma for morphisms. Denote the corresponding morphism by $G : U \to \mathbb{A}^2_k$. Prove that G is an isomorphism over $\mathbb{A}^2_k - (0,0)$. What is the fiber of G over $(0,0) \in \mathbb{A}^2_k$?