18.725 PROBLEM SET 4

Due date: Friday, October 15 in lecture. Late work will be accepted only with a medical note or for another Institute-approved reason. You are strongly encouraged to work with others, but the final write-up should be entirely your own and based on your own understanding.

Read through all the problems. Write solutions to the "Required Problems", 1, 2, 3, and 4, together with 2 others of your choice to a total of 6 problems. One or two more optional problems may be added to the problem set early next week.

Required Problem 1: Let F be an element of $k[X_0, \dots, X_n]_e$. Prove the *Euler identity*,

$$e \cdot F(X_0, \dots, X_n) = X_0 \frac{\partial F}{\partial X_0} + \dots + X_n \frac{\partial F}{\partial X_n}.$$

Remark: This isn't a proof, but to see where this identity comes from, differentiate with respect to t both sides of the identity,

$$t^e F(X) = F(tX).$$

Required Problem 2: Let X_0, X_1, X_2 be homogeneous coordinates on $\mathbb{P}^2_k$. Let $(\mathbb{P}^2_k)^\vee$ be a copy of $\mathbb{P}^2_k$ with homogeneous coordinates Y_0, Y_1, Y_2. Denote by $(\mathbb{P}^2_k \times (\mathbb{P}^2_k)^\vee, \pi_1, \pi_2)$ a product of $(\mathbb{P}^2_k, (\mathbb{P}^2_k)^\vee)$. Define $\Lambda \subset \mathbb{P}^2_k \times (\mathbb{P}^2_k)^\vee$ to be,

$$\{([a_0, a_1, a_2], [b_0, b_1, b_2]) | a_0 b_0 + a_1 b_1 + a_2 b_2 = 0\}.$$

A *projective line* in $\mathbb{P}^2_k$ is $\mathbb{V}(s)$ for any nonzero $s \in k[X_0, X_1, X_2]_1$.

(a) Prove there is a bijection between $(\mathbb{P}^2_k)^\vee$ and the set of lines in $\mathbb{P}^2_k$ defined by $q \in (\mathbb{P}^2_k)^\vee \mapsto \pi_1(\Lambda \cap \pi_2^{-1}(q))$.

(b) Let $F \in k[X_0, X_1, X_2]_e$ be an irreducible polynomial. Denote $C = \mathbb{V}(F) \subset \mathbb{P}^2_k$. Let $p = [a_0, a_1, a_2]$ be an element of C. A line $L \subset \mathbb{P}^2_k$ is *tangent to* C *at* p if $p \in L$ and the restriction of F to L has a repeated root at p. Assuming $\mathrm{char}(k)$ does not divide e, prove the line L associated to $[b_0, b_1, b_2] \in (\mathbb{P}^2_k)^\vee$ is tangent to C at $[a_0, a_1, a_2]$ iff the following matrix has rank 1,

$$\begin{pmatrix} (\partial F)/(\partial X_0)(a_0, a_1, a_2) & (\partial F)/(\partial X_1)(a_0, a_1, a_2) & (\partial F)/(\partial X_2)(a_0, a_1, a_2) \\ b_0 & b_1 & b_2 \end{pmatrix}.$$

(**Hint:** After a change of coordinates, arrange that $(a_0, a_1, a_2) = (1, 0, 0)$ and $(b_0, b_1, b_2) = (0, 0, 1)$. Combine this with the Euler identity from Problem 1.)

(c) A line $L \subset \mathbb{P}^2_k$ is *tangent to* C if there exists $p \in L$ such that L is tangent to C at p. Using (b) and the universal closedness of $\mathbb{P}^2_k$, prove the following subset of $(\mathbb{P}^2_k)^\vee$ is Zariski closed,

$$\left\{q | \pi_1(\Lambda \cap \pi_2^{-1}(q)) \text{ is tangent to } C\right\}.$$

Required Problem 3: Let k be an algebraically closed field and let R be a finitely generated, reduced k-algebra. Define the *max spectrum of* R, $\mathrm{Spec}_{\max}(R)$, to be the set of k-algebra homomorphisms $\phi : R \to k$. For every element $r \in R$, there is a mapping $\tilde{r} : \text{Spec}_{\max}(R) \to \mathbb{A}^1_k = k$ by $\tilde{r}(\phi) = \phi(r)$. Define the *Zariski topology on Spec* $_{max}(R)$ to be the weakest topology such that $\tilde{r}$ is continuous (with respect to the Zariski topology on $\mathbb{A}^1_k$) for every $r \in R$. Denote by $\mathcal{F}$ the sheaf on $\text{Spec}_{\max}(R)$ of all continuous maps from open subsets to $\mathbb{A}^1_k$. Define the *structure sheaf of Spec* $_{max}(R)$, $\mathcal{O}$, to be the smallest subsheaf of $\mathcal{F}$ such that,

(i) for every nonempty open subset $U \subset \text{Spec}_{\max}(R)$, the constant mappings are in $\mathcal{O}(U)$,

(ii) for every open subset $U \subset \text{Spec}_{\max}(R)$ and every $g \in \mathcal{O}(U)$ that is everywhere nonzero, also $1/g \in \mathcal{O}(U)$, and

(iii) for every $r \in R$, $\tilde{r} \in \mathcal{O}(\text{Spec}_{\max}(R))$.

(a) Prove that a basis for the topology on $\text{Spec}_{\max}(R)$ is given by the *basic open affines*, $D(r) := \{\phi : R \to k | \phi(r) \neq 0\}$.

(b) Prove that for every open U, every continuous map $g : U \to \mathbb{A}^1_k$ and every point $\phi \in U$, there exists a neighborhood $p \in V \subset U$ such that $g|_V$ is in $\mathcal{O}(V)$ iff there exist $h, s \in R$ such that $\phi \in D(s) \subset U$ and $g|_{D(s)} = \tilde{h}/\tilde{r}$. Using Theorem 4.5, prove that for every $s \in R$, $\mathcal{O}(D(s)) \cong R[1/s]$.

(c) Prove that $(\text{Spec}_{\max}(R), \mathcal{O})$ is an affine variety. **Not to be written up:** What is the universal property of this affine variety?

Required Problem 4: Let $F : X \to Y$ be a regular morphism of affine algebraic sets.

(a) For every element $y \in Y$, denote by $\mathfrak{m}_y \subset k[Y]$ the corresponding maximal ideal. Prove there is a bijection between the elements of $F^{-1}(\{y\})$ and the maximal ideals of $k[X]/F^*(\mathfrak{m}_y)k[X]$.

(b) If F is a finite morphism, and if $F^{-1}(\{y\})$ is empty, prove there exists $g \in k[Y]$ such that $g(y) \neq 0$ and $F^*(g) = 0$, i.e., $F^*(g) \cdot k[X] = \{0\}$. (**Hint**: Apply Nakayama's lemma to the finitely-generated $k[Y]$-module $k[X]$.)

(c) If F is a finite morphism, conclude that $F(X) \subset Y$ is a closed subset: if $y \in Y - F(X)$, then there exists $g \in k[Y]$ such that $y \in D(g) \subset Y - F(X)$. **Not to be written up:** Combined with Corollary 14.19, conclude that finite morphisms of algebraic varieties are universally closed.

Problem 5 (a): Assume $\text{char}(k) \neq 2$. Prove the subset of $(\mathbb{P}^2_k)^\vee$ parametrizing lines tangent to $C = \mathbb{V}(X_0^2 + X_1^2 + X_2^2)$ is $\mathbb{V}(Y_0^2 + Y_1^2 + Y_2^2)$. For a "general" element $p \in \mathbb{P}^2_k$, how many tangent lines to C contain p?

(b) Let $F \in k[X_0, X_1, X_2]_e$ be an irreducible polynomial. Define $U = \mathbb{V}(F) - \mathbb{V}(\partial F/\partial X_0, \partial F/\partial X_1, \partial F/\partial X_2)$. Prove the following mapping $U \to (\mathbb{P}^2_k)^\vee$ is a regular morphism whose image is contained in the set of lines tangent to $\mathbb{V}(F)$ (this mapping is the *Gauss map*):

$$[p] \in U \mapsto [(\partial F/\partial X_0)(p), (\partial F/\partial X_1)(p), (\partial F/\partial X_2)(p)].$$

Problem 6: Let R be a finitely-generated k-algebra that is not necessarily reduced. Repeat the definition of $\text{Spec}_{\max}(R)$ and $\mathcal{O}^{\text{red}}$ as in Problem 3 (except that, for reasons that will become clear, the sheaf is denoted $\mathcal{O}^{\text{red}}$ instead of $\mathcal{O}$). Prove $(\text{Spec}_{\max}(R), \mathcal{O}^{\text{red}})$ is an affine variety, and identify the k-algebra $\mathcal{O}^{\text{red}}(\text{Spec}_{\max}(R))$.

Problem 7: Another proof of existence of sheafification Let X be a topological space and let $\mathcal{F}$ be a presheaf of sets. Define the *éspace étalè* as a *set* in Definition 10.8, $p : |\mathcal{F}| \to X$.

(a) Let $U \subset X$ be an open set, $p \in U$ an element and $f, g \in \mathcal{F}(U)$ elements whose images are equal in the stalk $\mathcal{F}_p$. Prove there exists an open neighborhood $p \in V \subset U$ such that $f|_V = g|_V$.

(b) For every open set $U \subset X$ and every $f \in \mathcal{F}(U)$, define $D(U, f) \subset |\mathcal{F}|$ to be the set of pairs (p, f_p) of an element $p \in U$ and the image f_p of f in $\mathcal{F}_p$. Prove these sets form the basis for a topology on $|\mathcal{F}|$, called the *natural topology.*

(c) For every open set $U \subset X$ and every $f \in \mathcal{F}(U)$, prove the induced set map $\tilde{f} : U \to |\mathcal{F}|$ is continuous with respect to the natural topology on $|\mathcal{F}|$.

(d) Denote by $\mathcal{F}^+$ the sheaf of sections of the continuous mapping $p : |\mathcal{F}| \to X$ as in Example 10.4(ii). By (c) there is a presheaf homomorphism $\phi : \mathcal{F} \to \mathcal{F}^+$. Prove this is a sheafification of $\mathcal{F}$.

Problem 8 Let $\mathcal{A}$ and $\mathcal{B}$ be categories. An *adjoint pair of functors* is a pair of functors (L, R), $L : \mathcal{A} \to \mathcal{B}$, $R : \mathcal{G} \to \mathcal{A}$, together with a rule associating to every object A of $\mathcal{A}$ and every object B of $\mathcal{B}$ a bijection,

$$\eta_{A,B} : \mathrm{Hom}_{\mathcal{B}}(L(A), B) \to \mathrm{Hom}_{\mathcal{A}}(A, R(B)),$$

which is a *natural bijection* in the sense that for every object A of $\mathcal{A}$, resp. every object B of $\mathcal{B}$, the induced transformation of functors $\mathcal{B} \to \mathrm{Sets}$,

$$\eta_{A,*} : \mathrm{Hom}_{\mathcal{B}}(L(A), *) \Rightarrow \mathrm{Hom}_{\mathcal{A}}(A, R(*)),$$

is a natural transformation, resp. the induced transformation of contravariant functors $\mathcal{A} \to \mathrm{Sets}$,

$$\eta(*, B) : \mathrm{Hom}_{\mathcal{B}}(L(*), B) \Rightarrow \mathrm{Hom}_{\mathcal{A}}(*, R(B)),$$

is a natural transformation.

(a) Let $\mathcal{A} = \mathrm{Sets}$ and let $\mathcal{B} = \mathrm{Groups}$, Rings, or $R -$ modules. Define $R : \mathcal{B} \to \mathcal{A}$ to be the functor that sends each object to its underlying set of elements. Prove there is a functor $L : \mathcal{A} \to \mathcal{B}$ and a natural bijection η so that (L, R) is an adjoint pair. **Hint:** For each $\mathcal{B}$, there is a notion of a *free object.*

(b) In each case above, prove that (L, R) has the additional property that a morphism $f : B \to B'$ in $\mathcal{B}$ is an isomorphism iff $R(f)$ is an isomorphism (this is not an axiom for an adjoint pair).

Problem 9: Let $\mathcal{A} = \mathrm{Sets}$, let $\mathcal{B}$ be a category, and let (L, R, η) be an adjoint pair such that for every morphism $f : B \to B'$ in $\mathcal{B}$, f is an isomorphism iff $R(f)$ is an isomorphism. Let X be a topological space, and let $\mathcal{F}$ be a presheaf of objects in $\mathcal{B}$ on X.

(a) Prove that $\mathcal{F}$ is a sheaf iff the presheaf of sets $R(\mathcal{F})$ on X is a sheaf.

(b) Prove that $\mathcal{F}$ satisfies Axiom (A) from Definition 10.1 iff $\mathcal{F}$ satisfies Axiom (A') from Remark 10.2.

Difficult Problem 10: Let $F : \mathbb{P}^1_k \to \mathbb{P}^3_k$ be the regular morphism $[a_0, a_1] \mapsto [a_0^3, a_0^2 a_1, a_0 a_1^2, a_1^3]$. Denote by $C \subset \mathbb{P}^3_k$ the image of F (which is a projective subvariety by Problem 10 from PS# 2). For every element $p = [b_0, b_1, b_2, b_3] \in \mathbb{P}^3_k - F(\mathbb{P}^1_k)$, define a morphism $G_p : C \to \mathbb{P}^5_k$ by

$$[c_0, c_1, c_2, c_3] \mapsto [b_1c_0 - b_0c_1, b_2c_0 - b_0c_2, b_3c_0 - b_0c_3, b_2c_1 - b_1c_2, b_3c_1 - b_1c_3, b_3c_2 - b_2c_3].$$

(a) Prove there exists a linear embedding $H : \mathbb{P}^2_k \subset \mathbb{P}^5_k$ whose image contains the image of G_p.

(b) With respect to your linear embedding, find the equation of the plane curve $C_p = H^{-1}(G_p(C))$ for $p = [1, 0, 0, 1]$. Write down all the elements $q \in C_p$ where there is *not* a unique tangent line to C_p at q.

(c) A *secant line to* C is a projective line in $\mathbb{P}^3$ that intersects C in at least 2 distinct points. How many secant lines to C contain p? **Not to be written up:** What if p is another (general) element of $\mathbb{P}^3_k$? How many secant lines to C contain p? Pay special attention if you go to Alexei Oblomkov's PUMA-GRASS lecture.

Problem 11: For every integer $n \in \mathbb{Z}$, define X_n to be a copy of the affine variety $\mathbb{V}(xy) \in \mathbb{A}^2$, define $X_{n,n+1} \subset X_n$ to be $D(x)$ and $X_{n,n-1} \subset X_n$ to be $D(y)$. Define $\phi_n : X_{n,n+1} \to X_{n+1,n}$ to be the regular morphism $(a, 0) \mapsto (0, 1/a)$. If $|m - n| > 1$, define $X_{m,n} = \emptyset$ and define $\phi_{m,n}$ to be the empty mapping.

(a) Prove that the collection $(\{X_n\}, \{X_{m,n}\}, \{\phi_{m,n}\})$ satisfy the axioms for Lemma 12.11, the Gluing Lemma for spaces with functions. Denote by X the associated space with functions.

(b) Prove that X is a connected algebraic variety that is *not* quasi-compact.