18.725 SOLUTIONS TO PROBLEM SET 2

Remark: Solutions are only given for some problems. For problems for which there is no solution given, if you did not write it up this week, you may write it up as one of the optional problems for next week.

Products: For every pair of objects X, Y in a category $\mathcal{C}$, a *product of* (X, Y) is a triple (U, π_1, π_2) of an object U, a morphism $\pi_1 : U \to X$ and a morphism $\pi_2 : U \to Y$ such that for every object T the following is a bijection,

$$(\pi_1, \pi_2) : \mathrm{Hom}_{\mathcal{C}}(T, U) \to \mathrm{Hom}_{\mathcal{C}}(T, X) \times \mathrm{Hom}_{\mathcal{C}}(T, Y), \quad f \mapsto (\pi_1 \circ f, \pi_2 \circ f).$$

Required Problem 1 Problem 3 from PS# 1 proves every pair in the category of affine algebraic sets has a product. Prove this affine algebraic set is also a product of the pair in the category of quasi-affine algebraic sets, i.e., the universal property holds for every quasi-affine algebraic set T. (**Hint:** Every quasi-affine algebraic set is a union of open affine sets. Combine with the gluing lemma and Problem 3(c).)

Solution: There are at least 2 solutions, one hinted at above. First is the short solution. The solution of Exercise 3 from PS 1 proves for every pair of affine algebraic sets, (X, Y), affine algebraic sets, there exists an affine algebraic set U and morphisms $\pi_1 : U \to X$, $\pi_2 : U \to Y$ such that for every reduced k-algebra A, the following set map is a bijection,

$$(\pi_1^*, \pi_2^*) : \mathrm{Hom}_{k-\mathrm{alg}}(k[U], A) \longrightarrow \mathrm{Hom}_{k-\mathrm{alg}}(k[X], A) \times \mathrm{Hom}_{k-\mathrm{alg}}(k[Y], A).$$

For every quasi-affine algebraic set T there is a commutative diagram of set maps,

$$\begin{array}{ccc} \mathrm{Hom}_{\text{Q-Aff}}(T, U) & \xrightarrow{(\pi_1 \circ -, \pi_2 \circ -)} & \mathrm{Hom}_{\text{Q-Aff}}(T, X) \times \mathrm{Hom}_{\text{Q-Aff}}(T, Y) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{k-\mathrm{alg}}(k[U], \mathcal{O}_T(T)) & \xrightarrow{(\pi_1^*, \pi_2^*)} & \mathrm{Hom}_{k-\mathrm{alg}}(k[X], \mathcal{O}_T(T)) \times \mathrm{Hom}_{k-\mathrm{alg}}(k[Y], \mathcal{O}_T(T)) \end{array}$$

By Prop. 4.8, the vertical arrows are bijections. Because $\mathcal{O}_T(T)$ is a reduced k-algebra, the bottom horizontal arrow is a bijection. Therefore the top horizontal arrow is a bijection.

Injectivity: Here is the second solution. Let T be a quasi-affine algebraic set. There is a collection of open subsets $T_1, \dots, T_r$ that are isomorphic to affine algebraic sets. Let $F, G : T \to U$ be morphisms such that $(\pi_1 \circ F, \pi_2 \circ F) = (\pi_1 \circ G, \pi_2 \circ G)$. For every $i = 1, \dots, r$, denote by $F_i, G_i : T_i \to U$ the restriction of F, resp. G to T_i. By restriction, $(\pi_1 \circ F_i, \pi_2 \circ F_i) = (\pi_1 \circ G_i, \pi_2 \circ G_i)$. Since T_i is affine, Exercise 3 from PS 1 proves $F_i = G_i$. By the uniqueness part of Prop. 4.10 (the gluing lemma), $F = G$.

Surjectivity: Let $F_X : T \to X$, $F_Y : T \to Y$ be regular morphisms. For every $i = 1, \dots, r$, denote the restrictions to T_i by $F_{X,i} : T_i \to X$, resp. $F_{Y,i} : T_i \to Y$. By Exercise 3 from PS 1, there exists a unique morphism $F_i : T \to U$ such that $(\pi_1 \circ F_i, \pi_2 \circ F_i) = (F_{X,i}, F_{Y,i})$. For every $1 \leq i, j \leq r$ and every point $p \in T_i \cap T_j$, there exists an open affine $T_{i,j,k} \subset T_i \cap T_j$ containing p. Again by Exercise 3 from

PS 1, the restriction of each F_i and F_j to this open affine set is the unique regular morphisms whose compositions with π_1, resp. π_2, is the restriction of F_X, resp. F_Y. Therefore the restrictions to $T_{i,j,k}$ of F_i and F_j are equal. By the gluing lemma again, the restrictions to $T_i \cap T_j$ of F_i and F_j are equal. So by the gluing lemma, there exists a unique morphism $F : T \to U$ whose restriction to T_i is F_i for every i. Again by the gluing lemma, $(\pi_1 \circ F, \pi_2 \circ F) = (F_X, F_Y)$.

Problem 2 Prove every pair in the category of quasi-affine algebraic sets has a product.

Solution: Let (X, Y) be a pair of quasi-affine algebraic sets. Denote by $\overline{X}$ and $\overline{Y}$ the Zariski closures of each. Let $(\overline{U}, \overline{\pi}_1, \overline{\pi}_2)$ be a product for $(\overline{X}, \overline{Y})$, which exists by Exercise 3 from PS 1 and the previous exercise. Because $\overline{\pi}_1$ and $\overline{\pi}_2$ are continuous, $U := \overline{\pi}_1^{-1}(X) \cap \overline{\pi}_2^{-1}(Y) \subset \overline{U}$ is an open subset, i.e., U is a quasi-affine algebraic set. Define $\pi_1 : U \to X$, $\pi_2 : U \to Y$ to be the regular morphisms obtained by restricting $\overline{\pi}_1$ and $\overline{\pi}_2$. The claim is that (U, π_1, π_2) is a product of (X, Y).

Injectivity: Let T be a quasi-affine algebraic set and let $F, G : T \to U$ be morphisms such that $(\pi_1 \circ F, \pi_2 \circ F) = (\pi_1 \circ G, \pi_2 \circ G)$. Denote by $\overline{F}, \overline{G} : T \to \overline{U}$ the morphisms obtained from F, resp. G, by composing with the inclusion. Then $(\overline{\pi}_1 \circ \overline{F}, \overline{\pi}_2 \circ \overline{F}) = (\overline{\pi}_1 \circ \overline{G}, \overline{\pi}_2 \circ \overline{G})$. By the uniqueness part of the universal property, $\overline{F} = \overline{G}$. Therefore $F = G$.

Surjectivity: Let $F_X : T \to X$, $F_Y : T \to Y$ be regular morphisms. Denote by $\overline{F}_X : T \to \overline{X}$ and $\overline{F}_Y : T \to \overline{Y}$ the morphisms obtained from F_X, resp. F_Y, by composing with the inclusions. By the existence part of the universal property, there exists a regular morphism $\overline{F} : T \to \overline{U}$ such that $(\overline{\pi}_1 \circ \overline{F}, \overline{\pi}_2 \circ \overline{F}) = (\overline{F}_X, \overline{F}_Y)$. Since the images of $\overline{F}_X$, resp. $\overline{F}_Y$, are contained in X, resp. Y, the image of $\overline{F}$ is contained in U. Denote by $F : T \to U$ the induced map. Because the composition with inclusion into $\overline{U}$ is regular, also F is a regular morphism (this is non-trivial, but easy). And $(\pi_1 \circ F, \pi_2 \circ F)$ equals (F_X, F_Y).

Fiber products: For every pair of morphisms $f : X \to Z$ and $g : Y \to Z$ in a category $\mathcal{C}$, a *fiber product of* (f, g) is a triple (U, g', f') of an object U and morphisms $g' : U \to X$, $f' : U \to Y$ such that,

(i) $f \circ g' = g \circ f'$, and
(ii) for every triple (V, g'', f'') satisfying $f \circ g'' = g \circ f''$ there exists a unique morphism $u : V \to U$ such that $g'' = g' \circ u$ and $f'' = f' \circ u$.

Let $\mathcal{C}$ be a category in which every pair (X, Y) has a product, denoted $(X \times Y, \pi_1, \pi_2)$ (this hypothesis holds in Problem 3), and for every pair of morphisms $f : U \to X$, $g : U \to Y$, denote by $f \times g : U \to X \times Y$ the unique morphism such that $\pi_1 \circ (f \times g) = f$, $\pi_2 \circ (f \times g) = g$; this is not standard notation, but will be less confusing for the following problem. For every object Z, the *diagonal morphism of* Z is $\mathrm{Id}_Z \times \mathrm{Id}_Z : Z \to Z \times Z$.

Required Problem 3 (a) Let (U, g', f') be a fiber product of (f, g). Denote by $h : U \to Z$ the morphism $f \circ g' = g \circ f'$. Prove $(U, g' \times f', h)$ is a fiber product of the pair of morphisms $(f \circ \pi_1) \times (g \circ \pi_2) : X \times Y \to Z \times Z$ and $\Delta_Z : Z \to Z \times Z$.

Solution: First of all,

$$((f \circ \pi_1) \times (g \circ \pi_2)) \circ (g', f') = (f \circ g') \times (g \circ f') = h \times h = \Delta_Z \circ h.$$

Injectivity: Let T be an object and let $F, G : T \to U$ be morphisms such that $((g' \times f') \circ F, h \circ F) = ((g' \times f') \circ G, h \circ G)$. Then in particular, $g' \circ F = \pi_1 \circ (g' \times f') \circ F = \pi_1 \circ (g' \times f') \circ G = g' \circ G$, and similarly $f' \circ F = f' \circ G$. By the uniqueness part of the fiber product, $F = G$.

Surjectivity: Let $a : T \to X \times Y$ and $b : T \to Z$ be morphisms such that $((f \circ \pi_1) \times (g \circ \pi_2)) \circ a = \Delta_Z \circ b$. Then,

$$f \circ \pi_1 \circ a = \pi_1 \circ ((f \circ \pi_1) \times (g \circ \pi_2)) \circ a = \pi_1 \circ \Delta_Z \circ b = b.$$

Similarly, $g \circ \pi_2 \circ a = b$. So, in particular, $f \circ (\pi_1 \circ a) = g \circ (\pi_2 \circ a)$. Therefore there exists a unique morphism $F : T \to U$ such that $(g' \circ F, f' \circ F) = (\pi_1 \circ a, \pi_2 \circ a)$. Therefore $(g' \times f') \circ F = (\pi_1 \circ a) \times (\pi_2 \circ a) = a$. Also,

$$h \circ F = \pi_1 \circ [\Delta_Z \circ h] \circ F = \pi_1 \circ [((f \circ \pi_1) \times (g \circ \pi_2)) \circ (g', f')] \circ F = \\ \pi_1 \circ ((f \circ \pi_1) \times (g \circ \pi_2)) \circ a = \pi_1 \circ \Delta_Z \circ b = b.$$

Therefore $F : T \to U$ is a morphism such that $((g' \times f') \circ F, h \circ F) = (a, b)$.

(b) Conversely, i.e., without assuming existence of a fiber product of (f, g), let (U, e, h) be a fiber product of the pair of morphisms $(f \circ \pi_1) \times (g \circ \pi_2) : X \times Y \to Z \times Z$ and $\Delta_Z : Z \to Z \times Z$. Define $g' = \pi_1 \circ e$ and $f' = \pi_2 \circ e$. Prove (U, g', f') is a fiber product of (f, g).

Solution: First of all,

$$f \circ g' = f \circ \pi_1 \circ e = \pi_1 \circ ((f \circ \pi_1) \times (g \circ \pi_2)) \circ e = \pi_1 \circ \Delta_Z \circ h = h,$$

and similarly,

$$g \circ f' = g \circ \pi_2 \circ e = \pi_2 \circ ((f \circ \pi_1) \times (g \circ \pi_2)) \circ e = \pi_2 \circ \Delta_Z \circ h = h.$$

Therefore, $f \circ g' = g \circ f'$.

Injectivity: Let T be an object and let $F, G : T \to U$ be morphisms such that $(f' \circ F, g' \circ F) = (f' \circ G, g' \circ G)$. By the uniqueness part of the product, $e \circ F = e \circ G$. Thus also,

$$h \circ F = \pi_1 \circ \Delta_Z \circ h \circ F = \pi_1 \circ ((f \circ \pi_1) \times (g \circ \pi_2)) \circ e \circ F = \pi_1 \circ ((f \circ \pi_1) \times (g \circ \pi_2)) \circ e \circ G = \cdots = h \circ G.$$

Therefore, by the uniqueness part of the fiber product, $F = G$.

Surjectivity: Let $F_X : T \to X$ and $F_Y : T \to Y$ be morphisms such that $f \circ F_X = g \circ F_Y$. Denote $F_Z = f \circ F_X = g \circ F_Y$. Then $F_X \times F_Y : T \to X \times Y$ and $F_Z : T \to Z$ are morphisms satisfying,

$$\pi_1 \circ \Delta_Z \circ F_Z = F_Z = f \circ F_X = \pi_1 \circ ((f \circ \pi_1) \times (g \circ \pi_2)) \circ (F_X \times F_Y).$$

Similarly $\pi_1 \circ \Delta_Z \circ F_Z = \pi_1 \circ ((f \circ \pi_1) \times (g \circ \pi_2)) \circ (F_X \times F_Y)$. By the uniqueness part of the product, $\Delta_Z \circ F_Z = ((f \circ \pi_1) \times (g \circ \pi_2)) \circ (F_X \times F_Y)$. By the existence part of the fiber product, there exists a morphism $F : T \to U$ such that $e \circ F = (F_X \times F_Y)$ and $h \circ F = F_Z$. Then,

$$g' \circ F = \pi_1 \circ e \circ F = \pi_1 \circ (F_X \times F_Y) = F_X,$$

and similarly $f' \circ F = F_Y$.

Coproducts: For every pair of objects X, Y of a category $\mathcal{C}$, a *coproduct of* (X, Y) is a triple (U, q_1, q_2) of an object U and a pair of morphisms $q_1 : X \to U$, $q_2 : Y \to U$ such that for every object T the following is a bijection,

$$(q_1, q_2) : \mathrm{Hom}_{\mathcal{C}}(U, T) \to \mathrm{Hom}_{\mathcal{C}}(X, T) \times \mathrm{Hom}_{\mathcal{C}}(Y, T), \quad f \mapsto (f \circ q_1, f \circ q_2).$$

Required Problem 4(a) Let $n \geq 0$ be an integer, let $U = \mathbb{V}(x_{n+1}(x_{n+1}-1)) \subset \mathbb{A}_k^{n+1}$, let $q_1 : \mathbb{A}_k^n \to U$ be $(a_1,\dots,a_n) \mapsto (a_1,\dots,a_n,0)$ and let $q_2 : \mathbb{A}_k^n \to U$ be $(a_1,\dots,a_n) \mapsto (a_1,\dots,a_n,1)$. Prove (U, q_1, q_2) is a coproduct of $(\mathbb{A}_k^n, \mathbb{A}_k^n)$ in the category of quasi-affine algebraic sets. (**Hint:** For every pair of regular functions f_1 and f_2 on $\mathbb{A}_k^n$, $f(x_1,\dots,x_n,x_{n+1}) = x_{n+1}f_2(x_1,\dots,x_n) + (1-x_{n+1})f_1(x_1,\dots,x_n)$ is a regular function on $\mathbb{A}_k^{n+1}$ such that $q_1^* f = f_1$ and $q_2^* f = f_2$.)

Solution: To better organize the solution, the main argument is stated as a lemma.

Lemma 0.1. (i) *A subset $V \subset U$ is open, resp. closed, iff the subsets $q_1^{-1}(V), q_2^{-1}(V) \subset \mathbb{A}_k^n$ are open, resp. closed. Therefore a subset $V \subset U$ is a quasi-affine algebraic subset of $\mathbb{A}_k^{n+1}$ iff the subsets $q_1^{-1}(V), q_2^{-1}(V) \subset \mathbb{A}_k^n$ is quasi-affine.*
(ii) *For every quasi-affine algebraic subset $V \subset U$ and every function g on V, g is regular iff $g \circ q_1$ is regular on $q_1^{-1}(U)$ and $g \circ q_2$ is regular on $q_2^{-1}(U)$.*
(iii) *For each integer $n \geq 0$, (U, q_1, q_2) is a coproduct of $(\mathbb{A}_k^n, \mathbb{A}_k^n)$.*

Proof. **(i)** Denote by $\mathbb{A}_k^n \sqcup \mathbb{A}_k^n$ the coproduct of $(\mathbb{A}_k^n, \mathbb{A}_k^n)$ in the category of topological spaces. Denote by $q_1 \sqcup q_2 : \mathbb{A}_k^n \sqcup \mathbb{A}_k^n \to U$ the continuous map determined by (q_1, q_2). This is a bijection of sets. To prove it is a homeomorphism, it suffices to prove it is open. Because the sets $D(s), s \in k[x_1,\dots,x_n]$ form a basis for the topology of $\mathbb{A}_k^n$, it suffices to prove $q_1(D(s))$ and $q_2(D(s))$ are both open for every s. Since $q_1(D(s)) = D((1-x_{n+1})s(x_1,\dots,x_n)) \cap U$ and $q_2(D(s)) = D(x_{n+1}s(x_1,\dots,x_n)) \cap U$, $q_1 \sqcup q_2$ is a homeomorphism. In particular, a subset $V \subset U$ is an open subset of a closed subset iff $q_1^{-1}(V), q_2^{-1}(V) \subset \mathbb{A}_k^n$ are open subsets of closed subsets.

(ii) It suffices to prove for every $x \in q_1^{-1}(V)$ and every $y \in q_2^{-1}(V)$, that g is regular at x and at y. Because $g \circ q_1$ is regular at x, there exist polynomials $h, s \in k[x_1,\dots,x_n]$ such that $s(x) \neq 0$ and the restriction of $g \circ q_1$ to $q_1^{-1}(V) \cap D(s)$ equals h/s. Denote $\widetilde{s} = (1-x_{n+1})s(x_1,\dots,x_n)$ and $\widetilde{h} = (1-x_{n+1})h(x_1,\dots,x_n)$. Then $D(\widetilde{s}) \cap V = q_1(D(s)) \cap V$, so it contains $q_1(x)$, and the restriction of g to $D(\widetilde{s}) \cap V$ equals $\widetilde{h}/\widetilde{s}$, i.e., g is regular at $q_1(x)$. A very similar argument proves g is regular at $q_2(y)$.

(iii) Injectivity of (q_1, q_2) is clear. Let T be a quasi-affine algebraic set and let $F_1, F_2 : \mathbb{A}_k^n \to T$ be regular morphisms. There is a unique set map $F : U \to T$ such that $F \circ q_1 = F_1$ and $F \circ q_2 = F_2$. The issue is whether F is regular. For every regular function g on T, $g \circ F \circ q_1 = g \circ F_1$ is regular because F_1 is regular, and $g \circ F \circ q_2 = g \circ F_2$ is regular because F_2 is regular. So by (ii), $g \circ F$ is regular. Therefore F is a regular morphism. □

(b) Assuming part (a), deduce every pair (X, Y) of quasi-affine algebraic sets has a coproduct (U, q_1, q_2). (**Hint:** Embed in a large affine variety and use (a).)

Solution: Most of the work is already done in the lemma (which is why the solution is organized this way). Let $X \subset \mathbb{A}_k^l$ and $Y \subset \mathbb{A}_k^m$ be quasi-affine algebraic subsets. Let n be an integer $n \geq l, m$. Define $i_1 : \mathbb{A}_X^l \to \mathbb{A}_k^n$, resp. $i_2 : \mathbb{A}_k^m \to \mathbb{A}_k^n$, to be the regular morphism $(a_1,\dots,a_l) \mapsto (a_1,\dots,a_l,0,\dots,0)$, resp. $(a_1,\dots,a_m) \mapsto (a_1,\dots,a_m,0,\dots,0)$. The image of i_1, resp. i_2, is the affine algebraic set $\mathbb{V}(x_{l+1},\dots,x_n)$, resp. $\mathbb{V}(x_{m+1},\dots,x_n)$. And the projection morphism $\pi_1 : \mathbb{A}_k^n \to \mathbb{A}_k^l$, resp. $\pi_2 : \mathbb{A}_k^n \to \mathbb{A}_k^m$, by $(a_1,\dots,a_n) \mapsto (a_1,\dots,a_l)$, resp. $(a_1,\dots,a_n) \mapsto (a_1,\dots,a_m)$, restricts on Image$(i_1)$ to an inverse of i_1, resp. restricts on Image(i_2) to an inverse of i_2. The upshot is that i_1 and i_2 are isomorphisms to affine algebraic subsets of $\mathbb{A}^n_k$. Therefore the restriction of i_1 to X, resp. of i_2 to Y, are isomorphisms to quasi-affine algebraic subsets of $\mathbb{A}^n_k$. Since $i_1 : X \to i_1(X)$ and $i_2 : Y \to i_2(Y)$ are isomorphisms, every coproduct (W, r_1, r_2) of $(i_1(X), i_2(Y))$ determines a coproduct $(W, r_1 \circ i_1, r_2 \circ i_2)$ of (X, Y). Hence, after replacing X and Y by $i_1(X)$ and $i_2(Y)$, assume X, Y are quasi-affine algebraic subsets of $\mathbb{A}^n_k$.

Let (U, q_1, q_2) be the coproduct of $(\mathbb{A}^n_k, \mathbb{A}^n_k)$ from part (a). By part (i) of the lemma, $W = q_1(X) \cup q_2(Y)$ is a quasi-affine algebraic subset of $\mathbb{A}^{n+1}_k$. Define $r_1 : X \to W$, resp. $r_2 : Y \to W$, to be the restriction of q_1 to X, resp. of q_2 to Y. These are regular morphisms. The claim is that (W, r_1, r_2) is a coproduct of (X, Y). For every quasi-affine algebraic set T, it is clear that the set map (q_1, q_2) is injective. It remains to prove it is surjective. Let $F_X : X \to T$ and $F_Y : Y \to T$ be regular morphisms. There is a unique set map $F : W \to T$ such that $F_X = F \circ r_1$ and $F_Y = F \circ r_2$; the issue is whether F is regular. For every regular function g on T, $g \circ F \circ r_1 = g \circ F_X$ is regular because F_X is regular, and $g \circ F \circ r_2 = g \circ F_Y$ is regular because F_Y is regular. Therefore, by part (ii) of the lemma, $g \circ F$ is a regular function on W, i.e., $F : W \to T$ is a regular morphism.

Some problems on irreducibility:

Required Problem 5(a) Prove every nonempty open subset of an irreducible topological space is dense.

Solution: Let U be a nonempty open subset of an irreducible topological space X. Denote by $\overline{U}$ the closure of U in X. Then $(X - U, \overline{U})$ is a decomposition of X. Because X is irreducible, one of these sets equals X. Since U is nonempty, $X - U \neq U$, therefore $\overline{U} = X$.

(b) Let $Y \subset X$ be a subset of a topological space, irreducible with the relative topology. Prove the closure of Y is also irreducible with the relative topology.

Solution: Denote by $\overline{Y}$ the closure of Y. Let $(\overline{Y}_1, \dots, \overline{Y}_r)$ be a finite decomposition of $\overline{Y}$. For each $i = 1, \dots, r$, denote $Y_i = \overline{Y}_i \cap Y$. Then $(Y_1, \dots, Y_r)$ is a finite decomposition of Y. Because Y is irreducible, there exists i such that $Y = Y_i$. Then $\overline{Y}_i$ is a closed subset of X containing Y, so $\overline{Y} \subset \overline{Y}_i$. Because also $\overline{Y}_i \subset \overline{Y}$, $\overline{Y}$ equals $\overline{Y}_i$, i.e., $\overline{Y}$ is irreducible.

(c) Prove the image of an irreducible topological space under a continuous map is irreducible with the relative topology from the target.

Solution: Let X be an irreducible topological space, and let $f : X \to Y$ be a continuous map of topological spaces. Let $(Z_1, \dots, Z_r)$ be a finite decomposition of $f(X)$ with the relative topology. Because f is continuous, for each $i = 1, \dots, r$, the subset $X_i := f^{-1}(Z_i) \subset X$ is closed. Therefore $(X_1, \dots, X_r)$ is a finite decomposition of X. Because X is irreducible, there exists i such that $X = X_i$, i.e., $f(X) \subset Z_i$. Since also $Z_i \subset f(X)$, $f(X)$ equals Z_i, i.e., $f(X)$ is irreducible.

Problem 6 Assuming Problem 5, prove the irreducible components of $\mathbb{V}(\langle x_1 - x_2x_3, x_1x_3 - x_2^2 \rangle) \subset \mathbb{A}^3_k$ are $V_1 = \{(0,0,a) | a \in \mathbb{A}^1_k\}$ and $V_2 = \{(b^3, b^2, b) | b \in \mathbb{A}^1_k\}$. This is the "affine hyperplane section $x_4 = 1$" of the example from lecture on 9/13.

Solution: Consider $f, g : \mathbb{A}^1_k \to \mathbb{A}^3_k$ by $f(a) = (0, 0, a)$ and $g(b) = (b^3, b^2, b)$. These are regular morphisms, hence continuous for the Zariski topologies. Because every 2 nonempty open subset of $\mathbb{A}^1_k$ intersect, $\mathbb{A}^1_k$ is irreducible. Therefore $V_1 = f(\mathbb{A}^1_k)$

and $V_2 = g(\mathbb{A}^1_k)$ are irreducible by (iii) or Problem 5. Also $V_1 = \mathbb{V}(x_1, x_2)$ and $V_2 = \mathbb{V}(x_1 - x_3^3, x_2 - x_3^2)$. So to prove (V_1, V_2) is an irreducible decomposition of V, it suffices to prove $V = V_1 \cup V_2$.

It is easy to see $V_1, V_2 \subset V$. Let (a_1, a_2, a_3) be an element of V. Assume first $a_1 = 0$. Then $a_2^2 = a_1 a_3 = 0$ so that $a_2 = 0$. Therefore $(a_1, a_2, a_3) = (0, 0, a_3)$, which is in V_1. Next assume $a_1 \neq 0$. Because $a_1 = a_2 a_3$, also $a_2, a_3 \neq 0$. Define $b = a_1/a_2$. Then $a_3 = a_1/a_2 = b$, $a_2 = a_2^2/a_2 = (a_1 a_3)/a_2 = a_3(a_1/a_2) = b^2$, and $a_1 = a_2(a_1/a_2) = b^2(b) = b^3$. So $(a_1, a_2, a_3) = (b^3, b^2, b)$, which is in V_2. Therefore $V = V_1 \cup V_2$.

Problem 7 Find the irreducible components of $\mathbb{V}(\langle x_1x_2, x_1x_3, x_2x_3\rangle) \subset \mathbb{A}^3_k$.

Solution: This is a special case of the next problem. The irreducible components are $V_1 \cup V_2 \cup V_3$, where $V_1 = \{(a, 0, 0)|a \in k\}$, $V_2 = \{(0, a, 0)|a \in k\}$, and $V_3 = \{(0, 0, a)|a \in k\}$.

Difficult Problem 8 For every integer $n \geq 1$ and every collection S of nonempty subsets of $\{1, \dots, n\}$, define $m(S) \subset S$ to be the collection of subsets of $\{1, \dots, n\}$ minimal among those in S, and define $S^\vee$ to be the collection of all nonempty subsets $A \subset \{1, \dots, n\}$ such that for every $B \in S$, $A \cap B \neq \emptyset$.

(a) Prove $m(S)^\vee = S^\vee$, $S \subset (S^\vee)^\vee$ and $m((S^\vee)^\vee) = m(S)$.

Solution: Since every set in S contains a set in $m(S)$, a subset $A \subset \{1, \dots, n\}$ intersects every set in S iff it intersects every set in $m(S)$, i.e., $S^\vee = m(S)^\vee$. Every set in S intersects every subset $A \subset \{1, \dots, n\}$ which intersects every set in S, i.e., $S \subset (S^\vee)^\vee$. In particular, $m(S) \subset (S^\vee)^\vee$. Let $B \subset \{1, \dots, n\}$ be a subset that contains no set in $m(S)$. Consider $A = \{1, \dots, n\} - B$. For every set C in $m(S)$, because $C \not\subset B$, $A \cap C \neq \emptyset$. Hence A is in $S^\vee$, and $A \cap C = \emptyset$. Therefore A is not in $(S^\vee)^\vee$. So every set in $(S^\vee)^\vee$ contains a set in $m(S)$, proving $m((S^\vee)^\vee) = m(S)$.

(b) Define $I_S \subset k[x_1, \dots, x_n]$ to be the ideal $\langle m_A | A \in S\rangle$, where $m_A = \prod_{i \in A} x_i$. Prove the set of irreducible components of $\mathbb{V}(I_S)$ is in bijection with $m(S^\vee)$.

Solution: For every set B in $m(S^\vee)$, define $I_B = \langle x_i | i \in B\rangle$ and $V_B = \mathbb{V}(I_B)$. For every $A \in S$, there exists $i \in A \cap B$ so that $m_A \in \langle x_i\rangle \subset I_B$. Therefore $I_S \subset I_B$, implying $V_B \subset \mathbb{V}(I_S)$. Of course V_B is isomorphic to an affine space $\mathbb{A}^m_k$, where $m = n - \text{card}(B)$. So each V_B is irreducible. Also, if B_1, B_2 are distinct elements of $m(S^\vee)$, there exists $i \in B_2 - B_1$. Let $p \in \mathbb{A}^n_k$ be the element whose only nonzero coordinate is the i^{th} coordinate, which is 1. Then $p \in V_{B_1} - V_{B_2}$ so that $V_{B_2} \not\subset V_{B_1}$. By symmetry $V_{B_1} \not\subset V_{B_2}$, therefore $(V_B | B \in m(S^\vee))$ is an indecomposable decomposition of $\cup_B V_B$.

Finally, suppose that $p \in \mathbb{V}(I_S)$. Let C be the set of elements $1 \leq i \leq n$ such that the i^{th} coordinate of p is zero. For every $A \in S$, because $m_A(p) = 0$, for at least one $i \in A$, $x_i(p) = 0$, i.e., $A \cap C \neq \emptyset$. Therefore $C \in S^\vee$. Let $B \in m(S^\vee)$ be a set contained in C. Then $p \in V_B$. Therefore $(V_B | B \in m(S^\vee))$ is the irreducible decomposition of $\mathbb{V}(I_S)$.

Images of some morphisms:

Problem 9 For every pair of integers $m, n \geq 0$, the *affine Segre mapping* $F : \mathbb{A}^m_k \times \mathbb{A}^n_k \to \mathbb{A}^{mn}_k$ is as follows. Let $x_1, \dots, x_m$ be coordinates on $\mathbb{A}^m_k$, let $y_1, \dots, y_n$ be coordinates on $\mathbb{A}^n_k$ and let $z_{i,j}, 1 \leq i \leq m, 1 \leq j \leq n$ be coordinates on $\mathbb{A}^{mn}_k$.

Then $F^*z_{i,j} = x_i y_j$. Find an ideal $I \subset k[z_{i,j}]$ such that $\mathbb{V}(I) = \text{Image}(F)$. (**Hint:** The generators of I are homogeneous degree 2 binomials.)

Solution: Denote by I the ideal,

$$I = \langle z_{i_1,j_1} z_{i_2,j_2} - z_{i_3,j_3} z_{i_4,j_4} | \{i_1, i_2\} = \{i_3, i_4\} \text{ and } \{j_1, j_2\} = \{j_3, j_4\} \rangle.$$

It is easy to see $\text{Image}(F) \subset \mathbb{V}(I)$, i.e., the pullback by F of each generator of I is zero. Let p be an element in $\mathbb{V}(I)$. If $p = 0$, then $p = F(0)$. Thus assume $p \neq 0$, i.e., there exists (i_0, j_0) such that $z_{i_0,j_0}(p) \neq 0$. For every $i = 1, \dots, m$, define $a_i = z_{i,j_0}(p)/z_{i_0,j_0}(p)$. For every $j = 1, \dots, n$, define $b_j = z_{i_0,j}(p)$. Define $q = (a_1, \dots, a_m)$ and $r = (b_1, \dots, b_n)$. For every $i = 1, \dots, m$ and $j = 1, \dots, n$,

$$z_{i,j}(p) z_{i_0,j_0}(p) = z_{i,j_0}(p) z_{i_0,j}(p), \text{ i.e., } z_{i,j}(p) = a_i b_j.$$

Thus $p = F(q, r)$, which is in $\text{Image}(F)$. So $\mathbb{V}(I) = \text{Image}(F)$. It is not necessary to prove this, and it is not obvious, but I is a radical ideal.

Problem 10 For every pair of integers $n, d \geq 0$, define $N = \binom{n+d}{d}$, and define the *affine Veronese mapping* $F : \mathbb{A}^n_k \to \mathbb{A}^N_k$ as follows. Let $x_1, \dots, x_n$ be coordinates on $\mathbb{A}^n_k$ and let $z_{i_1,\dots,i_n}$ be coordinates on $\mathbb{A}^N_k$ where $(i_1, \dots, i_n)$ runs through all n-tuples of nonnegative integers with $i_1 + \cdots + i_n = d$. Then $F^* z_{i_1,\dots,i_n} = x_1^{i_1} x_2^{i_2} \dots x_n^{i_n}$. Find an ideal $I \subset k[z_{i_1,\dots,i_n}]$ such that $\mathbb{V}(I) = \text{Image}(F)$. (**Hint:** The generators of I are homogeneous degree 2 binomials.)

Difficult Problem 11 For every integer $n \geq 2$, define $N = \binom{n}{2}$ and define $F : \mathbb{A}^{2n}_k \to \mathbb{A}^N_k$ as follows. Let $x_{1,1}, \dots, x_{1,n}, x_{2,1}, \dots, x_{2,n}$ be coordinates on $\mathbb{A}^{2n}_k$ and let $z_{i,j}, 1 \leq i < j \leq n$ be coordinates on $\mathbb{A}^N_k$. Then $F^* z_{i,j} = x_{1,i} x_{2,j} - x_{1,j} x_{2,i}$. The image of this morphism is the *affine cone over the Grassmannian* ***Grass***$(2, n)$. Find an ideal $I \subset k[z_{i,j}]$ such that $\mathbb{V}(I) = \text{Image}(F)$. (**Hint:** Interpret elements of $\mathbb{A}^{2n}_k$ as $2 \times n$ matrices; interpret elements of $\mathbb{A}^N_k$ as elements of the exterior square of the n-space, which also give anti-symmetric $n \times n$ matrices, and take Pfaffians of appropriate 4×4-submatrices of this $n \times n$-matrix. The generators are homogeneous degree 2 trinomials.)

Problem 12 Give an example of a regular morphism of affine varieties $F : V \to W$ whose image is not a quasi-affine algebraic set.

Solution: There are many solutions. Let $V \subset \mathbb{A}^3_k$ be $\mathbb{V}(\langle x(xz-1), y(xz-1), z(xz-1) \rangle)$, let $W = \mathbb{A}^2_k$ and let $F : V \to W$ be $F(a, b, c) = (a, b)$. Observe the irreducible components of V are $V_1 = \{(a, b, 1/a) | a \in k - \{0\}, b \in k\}$ and $V_2 = \{(0, 0, 0)\}$. Therefore $F(V) = \{(a, b) | a \in k - \{0\}, b \in k\} \cup \{(0, 0)\}$. Of course the Zariski closure of $F(V)$ is all of $W = \mathbb{A}^2_k$. So if $F(V)$ is quasi-affine, then $\mathbb{A}^2_k - V$ is a Zariski closed subset. But $\mathbb{A}^2_k - V = \{(0, b) | b \in k - \{0\}\}$. This is not Zariski closed; the Zariski closure of $\mathbb{V}(x) = \{(0, b) | b \in k\}$. Therefore $F(V)$ is not a quasi-affine algebraic subset of W.

Problem 13: Proposition 4.8 can fail if W is not affine Let $V = \mathbb{A}^2_k$, let $W = \mathbb{A}^2_k - \mathbb{V}(x_1, x_2)$ and let $i : W \to V$ be the inclusion. Prove that $i^* : \mathcal{O}_V(V) \to \mathcal{O}_W(W)$ is an isomorphism, but there is no inverse of i, i.e., Proposition 4.8 fails for V and W.

Solution: It is easy to see i^* is injective; the difficult part is proving i^* is surjective. Let g be a regular function on W. Chasing through the definition of regular function, there exists a collection of pairs of polynomials in $k[x, y]$, $(h_1, s_1), \dots, (h_r, s_r)$

such that $W \subset D(s_1) \cup \cdots \cup D(s_r)$ and such that the restriction of g to $W \cap D(s_i)$ equals h_i/s_i for each $i = 1, \ldots, r$. Throw out all pairs such that $s_i = 0$. Then each of the fractions $h_i/s_i \in k(x, y)$ is defined, and $h_i/s_i = h_j/s_j$ for every $1 \leq i < j \leq r$. Write this fraction as h/s where $h, s \in k[x, y]$ have no common irreducible factors: this makes sense because $k[x, y]$ is a unique factorization domain. The claim is that s is a constant. The proof is by contradiction.

By way of contradiction, assume s is not a constant. By the Nullstellensatz $\mathbb{V}(s) \neq \emptyset$. Because $\text{rad}\langle s\rangle \neq \langle x, y\rangle$, also $\mathbb{V}(s) \neq \{(0, 0)\}$. Hence there exists $p \in \mathbb{V}(s) - \{(0, 0)\}$ such that $s(p) = 0$. Because $p \in W$, there exists i such that $s_i(p) \neq 0$. Since $s_i h = s h_i$, s divides $s_i h$ in $k[x, y]$. Because no irreducible factor of s divides any irreducible factor of h, s divides s_i, i.e., $s_i = u_i s$ for some $u_i \in k[x, y]$. But then $s_i(p) = u_i(p)s(p) = 0$, which is a contradiction. Therefore s is a constant and $f/s \in k[x, y]$, i.e., g is in the image of i^*.

Very Difficult Problem 14 Prove there exists a quasi-affine algebraic set V such that $\mathcal{O}_V(V)$ is not a finitely-generated k-algebra. The examples I am aware of all have dimension ≥ 4. (**Warning:** This problem would be more appropriate at the end of 18.726. I mention it now because you can understand it, and it is a problem to keep in mind as the semester goes on.)

Problem 15 Prove the k-algebra $\mathcal{O}_V(V)$ of every quasi-affine algebraic set V is a *subalgebra* of a finitely-generated k-algebra.

Solution: Let $V \subset \mathbb{A}^n_k$ be a quasi-affine algebraic set. Denote by $\overline{V}$ the Zariski closure. Because a basis for the topology of $\overline{V}$ consists of basic open affines, there exist elements $s_1, \ldots, s_r \in k[\overline{V}]$ such that $V = D(s_1) \cup \cdots \cup D(s_r)$. As proved in lecture, $\mathcal{O}_{D(s)}(D(s)) = k[\overline{V}][x_{n+1}]/\langle x_{n+1}s - 1\rangle$. Consider the k-algebra homomorphism,

$$\phi : \mathcal{O}_V(V) \to \prod_{i=1}^{r} k[\overline{V}][x_{n+1}]/\langle x_{n+1}s - 1\rangle,$$

that sends a regular function g on V to $(g_1, \ldots, g_r)$, where g_i is the restriction of g to $D(s_i)$. By the gluing lemma, ϕ is an injective k-algebra homomorphism. Each factor $k[\overline{V}][x_{n+1}]\langle x_{n+1}s - 1\rangle$ is a finitely-generated k-algebra, and a finite product of finitely-generated k-algebras is a finitely-generated k-algebra (essentially as proved in Problem 4).

Problem 16, An open affine that is not a basic open affine, I Together with the next problem, this problem gives an open subset of an affine algebraic set, itself isomorphic to an affine algebraic set, but not a basic open affine $D(s)$. In both problems, assume $\text{char}(k) \neq 2$ and let i denote a solution of $x^2 + 1$ in k. Let $C \subset \mathbb{A}^2_k$ be the *affine nodal plane cubic*, $C = \mathbb{V}(y^2 - x^2(x - 1))$ Let $(a_0, b_0) \in C$ and define $F : D(x - a_0) \to \mathbb{A}^3_k$ by $F(a, b) = (a, b, (b + b_0)/(a - a_0))$.

(a) Prove there exists a regular morphism $G : C - \{(a_0, b_0)\} \to \mathbb{A}^3_k$ whose restriction to $D(x - a_0$ equals F. (**Hint:** Expand the defining equation of C in the coordinates $x - a_0$ and $y - b_0$.)

(b) Prove the image of G is an affine algebraic subset of $\mathbb{A}^3_k$.

(c) Prove the projection $\pi : \mathbb{A}^3_k \to \mathbb{A}^2_k$, $\pi(a, b, c) = (a, b)$ restricts on the image of G to an inverse morphism to G. Therefore $C - \{a_0, b_0\}$ is an open subset of C, itself isomorphic to an affine algebraic set.

Difficult Problem 17, An open affine that is not a basic open affine, II This problem continues Problem 16; again $\mathrm{char}(k) \neq 2$. Consider the morphism $H : \mathbb{A}^1_k \to C$ by $H(u) = (u^2+1, u(u^2+1))$. Let t be a coordinate on $\mathbb{A}^1_k$.

(a) Prove $H^* : k[C] \to k[t]$ maps $k[C]$ isomorphically to the subalgebra of functions $f(t) \in k[t]$ such that $f(i) = f(-i)$.

(b) For (b), (c) and (d), assume $(a_0, b_0) \in C - \{(0,0)\}$. Prove the ideal of $k[t]$ generated by $H^*(\langle x - a_0, y - b_0 \rangle)$ is the principal ideal $\langle a_0 t - b_0 \rangle$.

(c) If there is an element $s \in k[V]$ such that $\mathbb{V}(s) = \{(a_0, b_0)\}$, $H^*(s) = c(at - b)^n$ for some nonzero constant $c \in k$ and integer $n \geq 1$. (**Hint:** Consider the image of s in $k[V][1/xy] \cong k[t][1/(t^2+1)]$. Use this to express H^*s as $c(t^2+1)^r(at-b)^n$ for some $r \geq 0$, and then use that $s(0,0) \neq 0$.)

(d) Deduce that $(a_0 i - b_0)^n = (-a_0 i - b_0)^n$, because $c(at-b)^n$ is in the image of H^*. Therefore for every $(a_0, b_0) \in C - \{(0,0)\}$, if $(b_0 - ia_0)/(b_0 + ia_0)$ is not a root of unity, then $C - \{(a_0, b_0)\}$ is of the form $D(s)$ for *no* element $s \in k[V]$ (in fact these are equivalent conditions).