MAT 543 Problem Set 7

Homework Policy. Read through and carefully consider all of the following problems. Please write up and hand-in solutions to **five** of the problems.

Each student is encouraged to work with other students, but submitted problem sets must be in the student's own words and based on the student's own understanding. It is against university policy to copy answers from other students or from any other resource.

Textbook Problems.

Problem 1. Problem 15.1, p. 126, Forster.

Problem 2. Problem 15.2, p. 126, Forster.

Problem 3. Problem 15.3, p. 126, Forster.

Problem 4. Problem 15.4, p. 126, Forster.

Problem 5. Problem 16.1, p. 131, Forster.

Problem 6. Problem 16.2, p. 131, Forster.

Problem 7. Problem 16.3, p. 131, Forster.

Problem 8. Problem 16.4, p. 131, Forster.

Problem 9. Regarding Problem 1, prove that every complex-valued harmonic function is uniquely the sum of a holomorphic function and an antiholomorphic function, cf. the proof of Corollary 13.3. Conclude that the cokernel of

$$d'd'' : \mathcal{E}(X) \to \mathcal{E}^{(2)}(X),$$

is the direct sum of $H^1(X, \mathcal{O}_X)$ and its complex conjugate subspace in the complexification $H^1(X, \mathcal{H}_{\mathbb{C}})$ of the real vector space $H^1(X, \mathcal{H}_{\mathbb{R}})$ for the sheaf $\mathcal{H}_{\mathbb{R}}$ of real harmonic functions. What condition on a section in $\mathcal{E}^{(2)}(X)$ guarantees that the image in $H^1(X, \mathcal{H}_{\mathbb{C}})$ is contained in $H^1(X, \mathcal{O}_X)$?

Problem 10. Regarding Problem 2, the associated map of first cohomology groups,

$$c_1 : H^1(X, \mathcal{O}_X^*) \to H^1(X, \Omega_X^{1,0}),$$

is usually called the *first Chern class* map. Using the Leray covering of $\mathbb{P}^1$ from the proof of Theorem 13.5, compute explicitly the domain and target cohomology groups, as well as the first Chern class map.

Problem 11. Use Problem 3 to give a second proof that an Abelian differential of the second kind has a primitive if and only if its periods are all zero.

Problem 12. Read about higher sheaf cohomology (beyond H^1). Use the long exact associated to the short exact sequence of Problem 2 to obtain an exact sequence,

$$0 \to \mathrm{Coker}(c_1) \to H^2(X, \underline{\mathbb{C}}^*_X) \to H^2(X, \mathcal{O}^*_X) \to H^2(X, \Omega_X).$$

Since X is a Riemann surface, $H^2(X, \Omega_X)$ is zero. What does that imply about $H^1(X, \mathcal{O}^*_X)$? In algebraic geometry, the analogous theorem is called *Tsen's theorem*.