Jason Starr

MAT 322 Problem Set 8

Homework Policy. Please read through all the problems. Please write up solutions of the required problems. Please also read and attempt the extra problems, but please do not write up those solutions for grading. I will be happy to discuss the extra problems during office hours.

Each student is encouraged to work on problem sets with other students, but each submitted problem set must be in the student's own words and based on the student's own understanding. It is against university policy to copy answers from other students or from any other resource (such as a webpage).

Required Problems.

Problem 1. Denote L be the z-axis in $\mathbb{R}^3$, $L = \{(x,y,z) \in \mathbb{R}^3 : x = y = 0\}$. Denote by $U \subset \mathbb{R}^3$ the open complement of L. Define the following two functions,

$$f : U \to \mathbb{R}^2,\ f(x,y,z) = (x/\sqrt{x^2+y^2}, y/\sqrt{x^2+y^2}),$$

$$g : U \to \mathbb{R},\ g(x,y,z) = \|(x,y,z) - f(x,y,z)\|^2 = \left(x - \frac{x}{\sqrt{x^2+y^2}}\right)^2 + \left(y - \frac{y}{\sqrt{x^2+y^2}}\right)^2 + z^2.$$

(a) For every $p = (x,y,z) \in U$, compute the total derivative D_pf as a 2×3 matrix, and compute the total derivative D_pg as a 1×3 matrix.

(b) For every $p = (x,y,z)$, compute the rank, nullspace and column space of D_pf. Compute the image S of f.

(c) Repeat (b) for D_pg. In particular, identify the closed subset $S' \subset U$ where D_pg is the zero matrix.

Problem 2. For the following invertible 3×3 matrix A, find an orthogonal matrix Q and an upper triangular matrix R with positive entries on the diagonal such that A equals QR.

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 2 & 3 & 0 \\ -2 & 0 & 3 \end{bmatrix}.$$

Problem 3. For every $n \times n$ matrix A, prove that $\mathrm{Trace}(A^\dagger)$ equals $\mathrm{Trace}(A)$. Also, for every $n \times m$ matrix A and for every $m \times n$ matrix B, prove that $\mathrm{Trace}(AB)$ equals $\mathrm{Trace}(BA)$. Use this to prove that for every $n \times m$ matrix R, for every $m \times \ell$ matrix S, and for every $\ell \times n$ matrix T, $\text{Trace}(RST)$ equals $\text{Trace}(TRS)$. Show by example that, even when $\ell = m = n$, $\text{Trace}(RST)$ might not equal $\text{Trace}(RTS)$.

Problem 4. For every pair A, B of $n \times k$ matrices, define the **Hilbert-Schmidt inner product** of A and B by the following formula,

$$\langle A, B \rangle_{\text{HS}} := \text{Trace}(B^\dagger A).$$

Similarly, define the **Hilbert-Schmidt norm** of A by

$$\|A\|_{\text{HS}} = \sqrt{\langle A, A \rangle_{\text{HS}}} = \sqrt{\text{Trace}(A^\dagger A)}.$$

(a) Prove that the function defined above,

$$\langle \bullet, \bullet \rangle : \text{Mat}_{n\times k}(\mathbb{R}) \times \text{Mat}_{n\times k}(\mathbb{R}) \to \mathbb{R}, \ (A, B) \mapsto \text{Trace}(B^\dagger A),$$

is $\mathbb{R}$-bilinear.

(b) Use Problem 3 to prove that this function is also symmetric.

(c) For a matrix $A = [\mathbf{v}_1 | \dots | \mathbf{v}_k]$, prove that $\|A\|^2_{\text{HS}}$ equals $\|\mathbf{v}_1\|^2_{\mathbb{R}^n} + \cdots + \|\mathbf{v}_k\|^2_{\mathbb{R}^n}$. Conclude that the Hilbert-Schmidt inner product is, indeed, positive definite, and hence it is an inner product.

(d) Let B be an $n \times k$ matrix such that for every $n \times k$ matrix A, $\text{Trace}(B^\dagger A)$ equals 0. Prove that B is the zero matrix.

(e) Combine (d) and Problem 3 to prove the following: let A be an $n \times m$ matrix and let C be an $m \times n$ matrix. Then CA equals $\text{Id}_{m\times m}$ if and only if for every $m \times m$ matrix B, $\text{Trace}(ABC)$ equals $\text{Trace}(B)$. In particular, for $n = m$, for an $n \times n$ matrix Q, prove that the following $\mathbb{R}$-linear transformation

$$c_Q : \text{Mat}_{n\times n}(\mathbb{R}) \to \text{Mat}_{n\times n}(\mathbb{R}), \ B \mapsto Q^\dagger B Q,$$

is orthogogonal with respect to the Hilbert-Schmidt inner product if and only if Q is an orthogonal matrix.

Problem 5.(Problem 2, p. 193). Let $U \subset \mathbb{R}^k$ be an open subset. Let $g : U \to \mathbb{R}$ be a C^1 function. Define $h : U \to U \times \mathbb{R} \subset \mathbb{R}^{k+1}$ to be the graph of g, $h(x_1, \dots, x_k) = (x_1, \dots, x_k, g(x_1, \dots, x_k))$. Let $V \subset \mathbb{R}^{k+1}$ be an open subset containing $h(U)$, and let $f : V \to \mathbb{R}$ be a bounded continuous function. Compute $|v_{\mathbb{R}^{k+1},k}(D_{\mathbf{x}}h)|$ in terms of the k partial derivatives of g, and use this to write out $\int_{(U,h)} f$ as an integral $\int_U I$ for some explicit integrand I.

Problem 6.(Problem 1, p. 208) Let $r \in (0, 1)$ be a real number. For the function $g : U \to \mathbb{R}$ from Problem 1, prove that the subset $g^{-1}([-r, r]) \subset \mathbb{R}^3$ is a 3-dimensional manifold with boundary. You may either use the Implicit Function Theorem or explicit coordinate charts.
Extra Problems. p. 187, Exercises 1, 2, 3; p. 193, Exercise 3; p. 202, Exercises 1, 3, 6.