MAT 322 Problem Set 7

Homework Policy. Please read through all the problems. Please write up solutions of the required problems. Please also read and attempt the extra problems, but please do not write up those solutions for grading. I will be happy to discuss the extra problems during office hours.

Each student is encouraged to work on problem sets with other students, but each submitted problem set must be in the student's own words and based on the student's own understanding. It is against university policy to copy answers from other students or from any other resource (such as a webpage).

Required Problems.

The following exercises investigate various aspects of the connected components of the groups $\mathbf{GL}_n(\mathbb{R}) \subset \mathbf{Mat}_{n\times n}(\mathbb{R})$ and $\mathbf{O}_n(\mathbb{R})$, i.e., the set of all $n \times n$ matrices $A = [\mathbf{v}_1|\dots|\mathbf{v}_n]$ such that $\mathcal{B} = (\mathbf{v}_1, \dots, \mathbf{v}_n)$ is an ordered basis, resp. an orthonormal ordered basis. Recall, that $\mathbf{GL}_n(\mathbb{R})$ is an open subset of $\mathrm{Mat}_{n\times n}(\mathbb{R})$ and it has a connected component $\mathbf{GL}_n(\mathbb{R})^+$ consisting of all matrices with positive determinant. Similarly, $\mathbf{O}_n(\mathbb{R})$ contains a subset $\mathbf{SO}_n(\mathbb{R})$ consisting of matrices with determinant equal to $+1$. The Gram-Schmidt Theorem associates to every $A \in \mathbf{GL}_n(\mathbb{R})$ a unique orthogonal matrix $Q \in \mathbf{O}_n(\mathbb{R})$ and a unique upper triangular matrix U with positive entries on the diagonal such that $A = QU$. The rule $A \mapsto Q$ is a continuous function $\mathbf{GL}_n(\mathbb{R}) \to \mathbf{O}_n(\mathbb{R}) \subset \mathbf{GL}_n(\mathbb{R})$ that restricts as the identity on the subset $\mathbf{O}_n(\mathbb{R}) \subset \mathbf{GL}_n(\mathbb{R})$. Since $\det(A)$ equals $\det(Q)\det(U)$, and since $\det(U)$ is positive, $\mathbf{GL}_n(\mathbb{R})^+$ maps surjectively onto $\mathbf{SO}_n(\mathbb{R})$. Thus, also $\mathbf{SO}_n(\mathbb{R})$ is path connected.

Problem 1.(p. 177, Problem 5). **(a)** Find formulas $q_{i,j} = a_{i,j}\cos(\theta) + b_{i,j}\sin(\theta)$ for $1 \leq i, j \leq 2$ and some choice of real numbers $a_{i,j}$, $b_{i,j}$ such that the 2×2 matrix $Q(\theta) = (q_{i,j})_{1\leq i,j\leq 2}$ is a special orthogonal matrix, and every special orthogonal 2×2 matrix occurs for some choice of θ.

(b) Find a similar formula for all orthogonal 2×2 matrices that are not special orthogonal.

(c) Check directly in terms of your formula that the product of any two special orthogonal 2×2 matrices is special orthogonal, and the inverse of every special orthogonal 2×2 matrix is special orthogonal.

(d) Similarly, check directly that the product of any two orthogonal non-special 2×2 matrices is special orthogonal, the product of a special orthogonal and an orthogonal non-special 2×2 matrix is an orthogonal non-special 2×2 matrix, and the inverse of every orthogonal non-special 2×2 matrix is orthogonal non-special.

Problem 2. **(a)** For every integer $n \geq 1$, for every pair A, B of invertible $n \times n$ matrices with $\det(A) < 0$ and $\det(B) < 0$, prove that there exists a unique invertible $n \times n$ matrix C with $\det(C) > 0$ such that $B = CA$. Conclude that the continuous function $T_A : \mathbf{GL}_n(\mathbb{R})^+ \to \mathbf{GL}_n(\mathbb{R})^-$ by $T_A(C) = C \cdot A$ is one-to-one and onto. Thus, if $\mathbf{GL}_n(\mathbb{R})^+$ is connected, then also $\mathbf{GL}_n(\mathbb{R})^-$ is connected.

(b) For every integer $n \geq 1$, for every pair Q, R of orthogonal $n \times n$ matrices with $\det(Q) = \det(R) = -1$, prove that there exists a unique special orthogonal matrix S with $R = S \cdot Q$. Conclude that the continuous function $T_Q : \mathbf{SO}_n(\mathbb{R}) \to (\mathbf{O}_n(\mathbb{R}) \setminus \mathbf{SO}_n(\mathbb{R}))$ by $T_Q(S) = S \cdot Q$ is one-to-one and onto. Thus, if $\mathbf{SO}_n(\mathbb{R})$ is connected, then also $\mathbf{O}_n(\mathbb{R}) \setminus \mathbf{SO}_n(\mathbb{R})$ is connected.

Problem 3. Let V be an $\mathbb{R}$-vector space of finite dimension $n \geq 1$. For every ordered m-tuple $\mathcal{B} = (\mathbf{v}_1, \dots, \mathbf{v}_m)$ of elements of V, denote by $j_{\mathcal{B}}$ the following $\mathbb{R}$-linear transformation,

$$j_{\mathcal{B}} : \mathbb{R}^m \to V, \; j_{\mathcal{B}}(x_1\mathbf{e}_1 + \cdots + x_m\mathbf{e}_m) = x_1\mathbf{v}_1 + \cdots + x_m\mathbf{v}_m.$$

Similarly, for every $\mathbb{R}$-linear transformation $j : \mathbb{R}^m \to V$, denote by $\mathcal{B}_j$ the following ordered m-tuple of elements of V,

$$\mathcal{B}_j = (j(\mathbf{e}_1), \dots, j(\mathbf{e}_m)).$$

(a) Prove that these two operations define a bijection between the set $V^m = V \times \cdots \times V$ of ordered m-tuples $\mathcal{B}$ of elements of V and the set $\mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^m, V)$ of $\mathbb{R}$-linear transformations from $\mathbb{R}^m$ to V. Giving V^m the $\mathbb{R}$-vector space in which addition and scalar multiplication are defined componentwise, prove that this bijection is an $\mathbb{R}$-linear isomorphism.

(b) Prove that j is an isomorphism if and only if $\mathcal{B}$ is an ordered basis for V.

(c) Prove that for every pair $j : \mathbb{R}^n \to V$ and $j' : \mathbb{R}^n \to V$ of $\mathbb{R}$-linear isomorphisms, there exists a unique $\mathbb{R}$-linear isomorphism $T : \mathbb{R}^n \to \mathbb{R}^n$ such that j' equals $j \circ T$. Denoting by $\mathrm{Isom}_{\mathbb{R}}(\mathbb{R}^n, V) \subset \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^n, V)$ the open subset of $\mathbb{R}$-linear isomorphisms, conclude that the following map is a (linear) diffeomorphism of open subsets of finite dimensional vector spaces,

$$L_j : \mathbf{GL}_n(\mathbb{R}) \to \mathrm{Isom}_{\mathbb{R}}(\mathbb{R}^n, V), \; L_j(T) = j \circ T.$$

(d) In particular, since $\mathbf{GL}_n(\mathbb{R})$ has two connected components, $\mathbf{GL}_n(\mathbb{R})^+$ and $\mathbf{GL}_n(\mathbb{R}) \setminus \mathbf{GL}_n(\mathbb{R})^+$, prove that also $\mathrm{Isom}_{\mathbb{R}}(\mathbb{R}^n, V)$ has two connected components. Prove that precomposition with an element in $\mathbf{GL}_n(\mathbb{R})^+$ maps each connected component (diffeomorphically) back to itself, but precomposition by an element in $\mathbf{GL}_n(\mathbb{R}) \setminus \mathbf{GL}_n(\mathbb{R})^+$ permutes the two connected components.

Problem 4 This problem continues the previous problem. In this problem, you may assume all of the results from the previous problem. Define an **orientation** of V to be a surjective, continuous function,

$$\alpha : \mathrm{Isom}_{\mathbb{R}}(\mathbb{R}^n, V) \to \{-1, +1\},$$

which is necessarily constant on each connected component. For a given orientation α, $j \in \mathrm{Isom}_{\mathbb{R}}(\mathbb{R}^n, V)$ is called α-**orientation preserving** if $\alpha(j)$ equals $+1$, and j is α-**orientation reversing** if $\alpha(j)$ equals -1. For $V = \mathbb{R}^n$, the **standard orientation** is

$$a : \mathbf{GL}_n(\mathbb{R}) \to \{-1, +1\},\ a(T) := \frac{\det(T)}{|\det(T)|}.$$

A pair (V, α) of a vector space V of finite dimension $n \geq 1$ and an orientation α of V is called a **oriented vector space** (this notion does not make sense for the zero vector space, and we will not need this notion for infinite dimensional vector spaces).

(a) Prove that for every orientation α, also $-\alpha$ is an orientation, and the set of all orientations of V is $\{\alpha, -\alpha\}$, i.e., there are precisely two orientations of every n-dimensional vector space, $n \geq 1$.

(b) For $\mathbb{R}$-vector spaces V and W of dimension n, for every $\mathbb{R}$-linear isomorphism $k : V \to W$, prove that the following map is a (linear) diffeomorphism of open subsets of finite dimensional $\mathbb{R}$-vector spaces,

$$R_k : \mathrm{Isom}_{\mathbb{R}}(\mathbb{R}^n, V) \to \mathrm{Isom}_{\mathbb{R}}(\mathbb{R}^n, W),\ R_k(j) = k \circ j.$$

In particular, prove that for every orientation β of W, also $\beta \circ R_k$ is an orientation of V. For oriented vector spaces (V, α) and (W, β), for every $\mathbb{R}$-linear isomorphism $k \in \mathrm{Isom}_{\mathbb{R}}(V, W)$, the (α, β)-**orientation** of k equals $(\beta \circ R_k)/\alpha \in \{-1, +1\}$, i.e., k is (α, β)-**orientation preserving** if $\beta \circ R_k$ equals $+\alpha$, and it is (α, β)-**orientation reversing** if $\beta \circ R_k$ equals $-\alpha$. This is consistent with the previous use of "orientation preserving" and "orientation reversing" where we give $\mathbb{R}^n$ the standard orientation. Prove that $\mathrm{Isom}_{\mathbb{R}}(V, W)$ has two connected components, each (linearly) diffeomorphic to $\mathbf{GL}_n(\mathbb{R})$ given by the subset of (α, β)-orientation preserving isomorphisms and the subset of (α, β)-orientation reversing isomorphisms.

(c) For oriented vector spaces (U, α), (V, β), and (W, γ), for $\mathbb{R}$-linear isomorphisms $S : U \to V$ and $T : V \to W$, prove that the (α, γ)-orientation of $T \circ S$ equals the product of the (α, β)-orientation of S and the (β, γ)-orientation of T.

(d) For every $\mathbb{R}$-linear isomorphism $k : V \to W$, prove that the (α, β)-orientation of k equals the $(-\alpha, -\beta)$-orientation of k. In particular, $k : V \to V$ is (α, α)-orientation preserving if and only if k is $(-\alpha, -\alpha)$-orientation preserving, i.e., for an $\mathbb{R}$-linear isomorphism of a vector space V back to the same vector space V, the notion that V is (α, α)-orientation preserving is independent of the choice of orientation α of V. For this reason, there is a well-defined subset $\mathrm{Isom}_{\mathbb{R}}(V, V)^+ \subset \mathrm{Isom}_{\mathbb{R}}(V, V)$ of orientation-preserving $\mathbb{R}$-linear isomorphisms. This is the unique connected component that contains Id_V.

(e) Prove that standard orientation a is the unique orientation α of $\mathbb{R}^n$ having either of the following properties: (i) $\alpha(\mathrm{Id}_{\mathbb{R}^n})$ equals $+1$, (ii) $\alpha(T \circ S) = \alpha(T) \cdot \alpha(S)$ for every $S, T \in \mathbf{GL}_n(\mathbb{R})$. In particular, this means that $a : \mathbf{GL}_n(\mathbb{R}) \to \{-1, +1\}$ is a *group homomorphism* whose kernel equals $\mathbf{GL}_n(\mathbb{R})^+$.

Problem 5. Let V be an $\mathbb{R}$-vector space of finite dimension $n \geq 1$. Denote by $V^\vee$ the n-dimensional vector space $\mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$ of $\mathbb{R}$-linear transformations $\phi : V \to \mathbb{R}$; such $\mathbb{R}$-linear transformations are usually called *linear functionals*.

(a) For every ordered basis $\mathcal{B} = (\mathbf{v}_1, \dots, \mathbf{v}_n)$ of V, prove that there exists a unique ordered basis $\mathcal{B}^\vee = (\chi_1, \dots, \chi_n)$ of $V^\vee$ such that

$$\chi_i(\mathbf{v}_j) = \begin{cases} 1, & i = j, \\ 0, & i \neq j \end{cases}$$

The ordered basis $\mathcal{B}^\vee$ is called the **dual ordered basis** of $V^\vee$ associated with $\mathcal{B}$.

(b) For every $\mathbf{v} \in V$, define

$$e_\mathbf{v} : V^\vee \to \mathbb{R},\ e_\mathbf{v}(\phi) := \phi(\mathbf{v}).$$

Prove that $e_\mathbf{v}$ is $\mathbb{R}$-linear. Thus there is an induced set map,

$$e : V \to (V^\vee)^\vee,\ \mathbf{v} \mapsto e_\mathbf{v}.$$

Prove that $e_\mathbf{v}$ is $\mathbb{R}$-linear.

(c) For every nonzero $\mathbf{v} \in V$, recall that there exists a basis $\mathcal{B} = (\mathbf{v}_1, \dots, \mathbf{v}_n)$ of V with $\mathbf{v}_1 = \mathbf{v}$. For the dual ordered basis $\mathcal{B}^\vee = (\chi_1, \dots, \chi_n)$, check that $e_{\mathbf{v}_1}(\chi_1)$ equals 1. Thus, $e_\mathbf{v} = e_{\mathbf{v}_1}$ is nonzero. Conclude that e is an injective $\mathbb{R}$-linear transformation. Since all three of V, $V^\vee$, and $(V^\vee)^\vee$ have dimension n, use the Rank-Nullity Theorem to prove that e is an $\mathbb{R}$-linear isomorphism.

(c) Let $\mathcal{C} = (\chi_1, \dots, \chi_n)$ be an ordered basis for $V^\vee$. Let $\mathcal{C}^\vee$ be the dual ordered basis for $(V^\vee)^\vee$ associated with $\mathcal{C}$. Since e is an $\mathbb{R}$-linear isomorphism, there exists a unique ordered basis $\mathcal{B} = (\mathbf{v}_1, \dots, \mathbf{v}_n)$ for V such that $\mathcal{C}^\vee$ equals $(e_{\mathbf{v}_1}, \dots, e_{\mathbf{v}_n})$. Check that $\phi_i(\mathbf{v}_j)$ equals 1 if $i = j$, and otherwise equals 0. Conclude that $\mathcal{C}$ equals $\mathcal{B}^\vee$. Finally, conclude that the rule $\mathcal{B} \mapsto \mathcal{B}^\vee$ defines a bijection

$$\bullet^\vee : \mathbf{Isom}_\mathbb{R}(\mathbb{R}^n, V) \to \mathbf{Isom}_\mathbb{R}(\mathbb{R}^n, V^\vee),\ j_\mathcal{B} \mapsto j_{\mathcal{B}^\vee}.$$

(d) For $j \in \mathrm{Isom}_\mathbb{R}(\mathbb{R}^n, V)$ and for $j^\vee \in \mathrm{Isom}_\mathbb{R}(\mathbb{R}^n, V^\vee)$, for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, for $\mathbf{v} = j(\mathbf{x})$ and $\phi = j(\mathbf{y})$, prove that $\phi(\mathbf{v})$, i.e., $j^\vee(\mathbf{y})(j(\mathbf{x}))$, equals $\langle \mathbf{x}, \mathbf{y} \rangle_{\text{Eucl}}$. **Hint.** Since both sides of $j^\vee(\mathbf{y})(j(\mathbf{x}))$ and $\langle \mathbf{x}, \mathbf{y} \rangle$ are $\mathbb{R}$-bilinear in $\mathbf{x}$ and $\mathbf{y}$, reduce to checking the identity for $\mathbf{x}$ and $\mathbf{y}$ in the standard ordered basis $(\mathbf{e}_1, \dots, \mathbf{e}_n)$ for $\mathbb{R}^n$.

(e) Using the previous result, for every invertible matrix $A \in \mathbf{GL}_n(\mathbb{R})$, prove that $(j \circ A^{-1})^\vee$ equals $j^\vee \circ A^\dagger$. If A is in $\mathbf{GL}_n(\mathbb{R})^+$, then both A^{-1} and $A^\dagger$ are also in $\mathbf{GL}_n(\mathbb{R})^+$. Conclude that for every pair of elements j and k of $\mathbf{Isom}_\mathbb{R}(\mathbb{R}^n, V)$ that are in the same connected component, then also $j^\vee$ and $k^\vee$ are in the same connected component of $\mathbf{Isom}_\mathbb{R}(\mathbb{R}^n, V^\vee)$. Conclude that for every orientation α of V, there is a unique orientation $\alpha^\vee$ of $V^\vee$ such that for every α-orientation preserving j, also $j^\vee$ is $\alpha^\vee$-orientation preserving.

Problem 6. This problem extends the notion of *cross product* or *vector product* to every vector space of dimension $n \geq 1$ that has a specified orientation and a specified inner product. This operation is usually called the *Hodge dual* or *Hodge star*.

Let V be an $\mathbb{R}$-vector space of finite dimension $n \geq 1$, and let $\langle \bullet, \bullet \rangle : V \times V \to \mathbb{R}$ be an inner product on V. Denote by $\| \bullet \|$ the associated norm, i.e., $\|\mathbf{v}\|^2 = \langle \mathbf{v}, \mathbf{v} \rangle$.

(a) Prove that an ordered basis $\mathcal{B}$ of V is an orthonormal basis with respect to $\langle \bullet, \bullet \rangle$ if and only if the induced $\mathbb{R}$-linear isomorphism $j_{\mathcal{B}} : \mathbb{R}^n \to V$ is orthogonal in the sense that $\langle j_{\mathcal{B}}(\mathbf{x}), j_{\mathcal{B}}(\mathbf{y}) \rangle = \langle \mathbf{x}, \mathbf{y} \rangle_{\text{Eucl}}$ for every $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

(b) For every pair of ordered orthonormal bases $\mathcal{B}$ and $\mathcal{B}'$ of V, prove that the composite $j_{\mathcal{B}}^{-1} \circ j_{\mathcal{B}'} : \mathbb{R}^n \to \mathbb{R}$ is an orthogonal transformation Q of $(\mathbb{R}^n, \langle \bullet, \bullet \rangle_{\text{Eucl}})$. In particular, $\det(Q)$ equals $+1$ or -1. If both j and j' are α-orientation preserving for an orientation α of V, prove that $\det(Q)$ equals $+1$.

(c) For every $\mathbf{w} \in V$, define

$$i_{\mathbf{w}} : V \to \mathbb{R}, \ i_{\mathbf{w}}(\mathbf{v}) = \langle \mathbf{v}, \mathbf{w} \rangle.$$

Prove that $i_{\mathbf{w}}$ is an $\mathbb{R}$-linear transformation. Also prove that the induced set map

$$i = i_{\langle \bullet, \bullet \rangle} : V \to V^{\vee}, \ \mathbf{w} \mapsto i_{\mathbf{w}},$$

is an $\mathbb{R}$-linear transformation. Finally, prove that i is *symmetric*, i.e., $(i(\mathbf{w})(\mathbf{v})$ equals $(i(\mathbf{v}))(\mathbf{w})$ for every $\mathbf{v}, \mathbf{w} \in V$.

(d) Since $i_{\mathbf{w}}(\mathbf{w}) = \|\mathbf{w}\|^2$, prove that $i_{\mathbf{w}}$ equals 0 if and only if $\mathbf{w}$ equals 0. Conclude that $i : V \to V^{\vee}$ is an injective linear transformation. Since both V and $V^{\vee}$ are isomorphic to $\mathbb{R}^n$, use the Rank-Nullity Theorem to prove that i is an isomorphism of $\mathbb{R}$-vector spaces, i.e., i is also a surjective linear transformation.

(e) Prove that an ordered basis $\mathcal{B}$ of V is orthonormal if and only if the dual ordered basis $\mathcal{B}^{\vee}$ of $V^{\vee}$ equals $i(\mathcal{B})$. In particular, since one orthonormal basis exists (by the Gram-Schmidt Theorem), prove that for every orientation α of V, for the dual orientation $\alpha^{\vee}$ of V, the $\mathbb{R}$-linear isomorphism i is $(\alpha, \alpha^{\vee})$-orientation preserving.

(f) Fix an orientation α of V. Let $j : \mathbb{R}^n \to V$ be an $\mathbb{R}$-linear isomorphism that is both orthogonal and orientation preserving. Denote by $k : V \to \mathbb{R}^n$ the inverse of j; again this is orthogonal and orientation preserving. For every ordered $(n-1)$-tuple of vectors $\mathcal{A} = (\mathbf{v}_1, \dots, \mathbf{v}_{n-1})$ of V, define

$$\phi_{j,\mathcal{A}} : V \to \mathbb{R}, \ \mathbf{w} \mapsto \det(k(\mathbf{v}_1), \dots, k(\mathbf{v}_{n-1}), k(\mathbf{w})).$$

Since the determinant is n-multilinear, $\phi_{\mathcal{A}}$ is an $\mathbb{R}$-linear transformation, i.e., an element of $V^{\vee}$. Check that for every special orthogonal matrix $Q \in \mathbf{SO}_n(\mathbb{R})$, $(j \circ Q)^{-1} = Q^{\dagger} \circ k$, so that

$$\phi_{j \circ Q, \mathcal{A}}(\mathbf{w}) = \det(Q^{\dagger}) \phi_{j,\mathcal{A}}(\mathbf{w}) = +1 \cdot \phi_{j,\mathcal{A}}(\mathbf{w}).$$

Conclude that, for a fixed inner product $\langle \bullet, \bullet \rangle$ and orientation α on V, $\phi_{j,\mathcal{A}}$ does not depend on the choice of the orthogonal, orientation preserving isomorphism j.

(g) Since $\phi_{\mathcal{A}}$ is an element of $\mathbb{R}^{\vee}$, and since i is an $\mathbb{R}$-linear isomorphism, conclude that there exists a unique element $*\mathcal{A} \in V$ such that $i_{*\mathcal{A}} = \phi_{\mathcal{A}}$, i.e., $\langle *\mathcal{A}, \mathbf{w} \rangle$ equals $\det(k(\mathcal{A}), k(\mathbf{w}))$ for every orthogonal, orientation preserving isomorphism $k : V \to \mathbb{R}^n$. Check that reversing the orientation of V multiplies $*\mathcal{A}$ by -1.

Problem 7 This problem computes the relation between the Hodge star operator from the last problem and the volume in $\mathbb{R}^n$ of an $(n-1)$-dimensional parallelepiped. As above, let V be an $\mathbb{R}$-vector space of finite dimension $n \geq 1$. Let $\langle \bullet, \bullet \rangle$ be an inner product on V. Let α be an orientation of V.

(a) By the Cauchy-Schwarz inequality, $|i_{\mathbf{w}}(\mathbf{v})| \leq \|\mathbf{v}\| \cdot \|\mathbf{w}\|$. Thus, if $\mathbf{v} \in B_1^{\|\bullet\|}(0) \subset V$, i.e., if $\|\mathbf{v}\| < 1$, then $|i_{\mathbf{w}}(\mathbf{v})| < \|\mathbf{w}\|$. Prove that $\|\mathbf{w}\|$ equals $\sup\{|i_{\mathbf{w}}(\mathbf{v})| : \mathbf{v} \in B_1^{\|\bullet\|}(0)\}$.

(b) For every ordered $(n-1)$-tuple $\mathcal{A} = (\mathbf{v}_1, \dots, \mathbf{v}_{n-1})$ of elements of V, prove that

$$\| * \mathcal{A}\| = \sup\{|\det(k(\mathbf{v}_1), \dots, k(\mathbf{v}_{n-1}), k(\mathbf{w}))| : \mathbf{w} \in B_1^{\|\bullet\|}(0)\}.$$

(c) For the standard Euclidean inner product on $\mathbb{R}^n$, for a linearly independent set $(\mathbf{v}_1, \dots, \mathbf{v}_{n-1})$ of elements of $\mathbb{R}^n$, check that

$$|v_{\mathbb{R}^n, n-1}(\mathbf{v}_1, \dots, \mathbf{v}_{n-1})| = \sup\{|\det(\mathbf{v}_1, \dots, \mathbf{v}_{n-1}, \mathbf{w})| : \mathbf{w} \in B_1^{\|\bullet\|}(0)\}$$

(d) For the standard Euclidean inner product and the standard orientation on $\mathbb{R}^n$, for $\mathbf{v}_i = x_{1,i}\mathbf{e}_1 + \cdots + x_{n,i}\mathbf{e}_n$, prove that the j-coordinate of $*\mathcal{A}$ equals $\det(\mathbf{v}_1, \dots, \mathbf{v}_{n-1}, \mathbf{e}_j)$, which is $(-1)^{n+1-j}\det(A_j)$, where A_j is the $(n-1) \times (n-1)$-matrix obtained from $[\mathbf{v}_1|\dots|\mathbf{v}_{n-1}]$ by removing the j^{th} row. In particular, for $n = 2$, check that

$$* \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix},$$

the vector obtained from $\mathbf{v}_1$ by rotation by $\pi/2$ in the counterclockwise direction. Similarly, for $n = 3$, check that the Hodge star operator is the usual cross product / vector product,

$$* \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{bmatrix} = (x_2y_3 - x_3y_2)\mathbf{e}_1 + (x_3y_1 - x_1y_3)\mathbf{e}_2 + (x_1y_2 - x_2y_1)\mathbf{e}_3 = \det \begin{bmatrix} x_1 & y_1 & \mathbf{e}_1 \\ x_2 & y_2 & \mathbf{e}_2 \\ x_3 & y_3 & \mathbf{e}_3 \end{bmatrix},$$

where the last determinant is just formal.

Problem 8. This final problem confirms an unproved assertion announced in lecture: for every pair of integers (n, k) with $1 \leq k < n$, every $\mathbb{R}$-linear isomorphism $T : \mathbb{R}^n \to \mathbb{R}^n$ that preserves volumes of k-dimensional parallelepipeds is orthogonal. As above, let V be a vector space of dimension $n \geq 1$ with a specified inner product $\langle \bullet, \bullet \rangle$ and a specified orientation α. Fix an orthogonal, orientation preserving $\mathbb{R}$-linear isomorphism $j : \mathbb{R}^n \to V$ and its inverse $k : V \to \mathbb{R}^b$.

(a) For every unit vector $\mathbf{w}$ in V, use the Gram-Schmidt Theorem to prove that there exists an orthonormal basis $(\mathbf{v}_1, \dots, \mathbf{v}_{n-1}, \mathbf{w})$. For $\mathcal{A} = (\mathbf{v}_1, \dots, \mathbf{v}_{n-1})$, prove that $*\mathcal{A}$ equals $+\mathbf{w}$ or $-\mathbf{w}$ (the sign depends on the orientation on V). In particular, up to replacing $\mathbf{v}_1$ by $\lambda\mathbf{v}_1$ for a nonzero scalar λ, prove that every vector $\mathbf{w}$ in V (whether or not it is a unit vector) is of the form $*\mathcal{A}$ for some choice of $\mathcal{A}$.

(b) Let $T : V \to V$ be an $\mathbb{R}$-linear isomorphism. For every ordered $(n-1)$-tuple $\mathcal{A} = (\mathbf{v}_1, \dots, \mathbf{v}_{n-1})$ and for every $\mathbf{w} \in V$, check the equality of the following $n \times n$-matrices,

$$[k(T(\mathbf{v}_1))|\dots|k(T(\mathbf{v}_{n-1}))|k(T(\mathbf{w}))] = (k \circ T \circ k^{-1}) \cdot [k(\mathbf{v}_1)|\dots|k(\mathbf{v}_{n-1})|k(\mathbf{w})].$$

Conclude that $\phi_{T\mathcal{A}}(T(\mathbf{w}))$ equals $\det(T)\phi_{\mathcal{A}}(\mathbf{w})$.

(c) Use (b) to prove that $T^\dagger(*(T\mathcal{A}))$ equals $\det(T)\cdot *\mathcal{A}$ as elements in V. In particular, $\|T^\dagger(*(T\mathcal{A}))\|$ equals $|\det(T)|\| *\mathcal{A}\|$.

(d) Let $r \neq 0$ be a real number. Assume that for every $\mathcal{A}$, $|v_{\mathbb{R}^n,n-1}(T\mathcal{A})|$ equals $|r||v_{\mathbb{R}^n,n-1}(\mathcal{A})|$. Conclude that $\|T^\dagger(*(T\mathcal{A}))\|$ equals $(|\det(T)|/|r|)\| *(T\mathcal{A})\|$. For every ordered $(n-1)$-tuple $\mathcal{C} = (\mathbf{w}_1, \dots, \mathbf{w}_{n-1})$, setting $\mathcal{A} = T^{-1}\mathcal{C}$ so that $T\mathcal{A} = \mathcal{C}$, conclude that $\|T^\dagger(*\mathcal{C})\|$ equals $(|\det(T)|/|r|)\| *\mathcal{C}\|$. For the operator $Q = (|r|/|\det(T)|) \cdot T^\dagger$ conclude that $\|Q(*\mathcal{C})\|$ equals $\| *\mathcal{C}\|$ for every $\mathcal{C}$. Finally, combine this with (a) to conclude that Q is orthogonal.

(e) From the previous part, conclude that $T^\dagger = (|\det(T)|/|r|) \cdot Q$, so that T equals $(T^\dagger)^\dagger = (|\det(T)|/|r|) \cdot Q^\dagger$. Since Q is orthogonal, $Q^\dagger$ is also orthogonal. Thus, T equals $s \cdot Q^\dagger$ for a nonzero real scalar s and an orthogonal transformation $Q^\dagger$. Finally, since $Q^\dagger$ also preserves $v_{\mathbb{R}^n,n-1}$, conclude that $|s|$ equals 1. Therefore T is itself orthogonal.

(f) Now let k and n be integers such that $1 \leq k < n$. Let $T : V \to V$ be an $\mathbb{R}$-linear isomorphism that preserves k-volumes. Since $n > k \geq 1$, for every vector $\mathbf{v} \in V$, there exists a $(k+1)$-dimensional subspace $U \subset V$ that contains $\mathbf{v}$. Choose an orthogonal $\mathbb{R}$-linear isomorphism $j : \mathbb{R}^{k+1} \circ U$, and choose an orthogonal $\mathbb{R}$-linear isomorphism $j' : T(U) \to \mathbb{R}^{k+1}$. Conclude that the composition $j' \circ T|_U \circ j : \mathbb{R}^{k+1} \to \mathbb{R}^{k+1}$ is an $\mathbb{R}$-linear isomorphism that preserves all k-volumes. From (e), conclude that $j' \circ T|_U \circ j$ is orthogonal. Since j and j' are orthogonal, conclude that $T|_U : U \to T(U)$ is orthogonal. In particular $\|T(\mathbf{v})\|$ equals $\|\mathbf{v}\|$. Since this holds for every $\mathbf{v} \in V$, T is orthogonal.