MAT 322 Problem Set 6

Homework Policy. Please read through all the problems. Please write up solutions of the required problems. Please also read and attempt the extra problems, but please do not write up those solutions for grading. I will be happy to discuss the extra problems during office hours.

Each student is encouraged to work on problem sets with other students, but each submitted problem set must be in the student's own words and based on the student's own understanding. It is against university policy to copy answers from other students or from any other resource (such as a webpage).

Required Problems.

Problem 1.(p. 121, Problem 8) Let $A \subset \mathbb{R}^m$ and $B \subset \mathbb{R}^n$ be bounded, closed rectangles. Thus $A \times B \subset \mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^{m+n}$ is a bounded, closed rectangle. Let $S \subset A \times B$ be a compact, rectifiable set. For every $\mathbf{x} \in A$, define $S_{\mathbf{x}} = \{\mathbf{y} \in B : (\mathbf{x}, \mathbf{y}) \in S\}$. Also define $f(\mathbf{x}) = \underline{\int}_B 1_{S_{\mathbf{x}}} \in \mathbb{R}_{\geq 0}$. Prove that whenever $S_{\mathbf{x}}$ is rectifiable, then $f(\mathbf{x})$ equals $v_{\mathbb{R}^n}(S_{\mathbf{x}})$. Also prove that f is integrable and $\int_A f$ equals $v_{\mathbb{R}^{m+n}}(S)$.

Problem 2.(p. 132, Problem 1) For $\mathbb{R} = \mathbb{R}_{<0} \sqcup \{0\} \sqcup \mathbb{R}_{>0}$, define $f : \mathbb{R} \to \mathbb{R}$ to be constant equal to -1 on $\mathbb{R}_{<0}$, to have value 0 on 0, and to be constant equal to $+1$ on $\mathbb{R}_{>0}$. Prove that there exists a sequence of bounded, closed intervals $(C_n)_{n \in \mathbb{N}}$ of $\mathbb{R}$ with each $C_n \subset \mathrm{Int}(C_{n+1}) \subset C_{n+1}$ such that $\cup_{n \in \mathbb{N}} C_n$ equals $\mathbb{R}$ and such that every $\int_{C_n} f$ equals 0. Conclude that $\lim_{n \to \infty} \int_{C_n} f$ equals 0. Nonetheless, prove that the extended (improper) integral $\int_{\mathbb{R}} f$ is not defined.

Problem 3.(p. 143, Problem 1) For every integer $n \geq 0$, define $f_n : \mathbb{R} \to \mathbb{R}$ as follows,

$$f_n(x) = \begin{cases} \frac{1}{x^n} e^{-1/x}, & x > 0, \\ 0, & x \leq 0 \end{cases}$$

Prove that every $f_n(x)$ is continuous at 0, hence continuous everywhere. Find real numbers a and b such that $f_n(x) = \int_0^x a f_{n+1}(t) + b f_{n+2}(t) dt$. Use the Fundamental Theorem of Calculus to conclude that $f_n(x)$ is differentiable everywhere, including at $x = 0$. Conclude that every $f_n(x)$ is of class C^∞.

Problem 4.(p. 151, Problem 4) Let $u > 0$ be a real number. For the everywhere nonnegative function $f_u : \mathbb{R} \to \mathbb{R}$ by $f_u(t) = e^{-ut^2}$, denote by I_u the extended (improper) integral

$$I_u = \int_{\mathbb{R}}^* f_u = \sup_{A > 0} \int_{t=-A}^{A} e^{-ut^2} dt,$$

where possibly this supremum is undefined if the integrals are unbounded. Use Fubini's Theorem to relate this to the extended integral

$$J_u = \int^*_{(x,y)\in\mathbb{R}^2} f_u(x)f_u(y) = \int^*_{(x,y)\in\mathbb{R}^2} e^{-u(x^2+y^2)}.$$

Use the polar coordinates change of variables from Exercise 5, p. 54, and the Change of Variables Theorem to rewrite J_u as an integral over $(r,\theta) \in (0,\infty)\times(0,2\pi)$. For the new integrand, use Fubini's Theorem to evaluate the extended integral. Compute I_u as a function of u.

Not to be turned in. What do you get from the equation

$$I^n_u = \int^*_{(x_1,\dots,x_n)\in\mathbb{R}^n} e^{-u(x_1^2+\dots+x_n^2)}$$

if you differentiate m times with respect to u?

Problem 5.(p. 168 , Problem 6) For every integer n, denote by $\|\bullet\|_{\mathbb{R}^n,2}$ the standard Euclidean norm on $\mathbb{R}^n$. For every real $u>0$, denote by $B^n_2(u)$ the ball of radius u in $\mathbb{R}^n$ centered at 0 with respect to $\|\bullet\|_{\mathbb{R}^n,2}$.

(a) Prove that there exists real $\lambda_n > 0$ such that $v_{\mathbb{R}^n}(B^n_2(u))$ equals $\lambda_n u^n$ for every real $u>0$.

(b) Compute λ_1 and λ_2.

(c) Use an iterated integral to obtain a formula for λ_{n+2} in terms of λ_n.

(d) For every n even, respectively for every n odd, derive a formula for λ_n.

Problem 6. Repeat Problem 5 with respect to the ℓ_1-norm,

$$\|(x_1,\dots,x_n)\|_{\mathbb{R}^n,1} = |x_1| + \dots + |x_n|.$$

Extra Problems.

pp. 120-121, Exercises 2, 3, 4, 5, 7; pp. 132-133, Exercises 2, 3, 4, 5, 6; p. 151, Exercises 3, 6; pp. 167-168, Exercise 1.