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MAT 322 Problem Set 5

Homework Policy. Please read through all the problems. Please write up solutions of the required problems. Please also read and attempt the extra problems, but please do not write up those solutions for grading. I will be happy to discuss the extra problems during office hours.

Each student is encouraged to work on problem sets with other students, but each submitted problem set must be in the student's own words and based on the student's own understanding. It is against university policy to copy answers from other students or from any other resource (such as a webpage).

Required Problems.

Problem 1.(p. 97, Problem 1) Give an example of a subset $A \subset [0,1]$ that has measure zero, yet the closure of A and the boundary of A equal all of $[0,1]$. On the other hand, prove that if A has measure zero, then the interior of A is empty.

Problem 2. For every $\epsilon > 0$, prove that there exists a countable collection $(I_n)_{n\in\mathbb{N}}$ of open subintervals $I_n \subset (0,1)$ such that the following series is convergent and bounded above by ϵ,

$$\sum_{n=1}^{\infty} v(I_n) < \epsilon$$

yet the open subset $U = \cup_{n\in\mathbb{N}} I_n$ is a dense subset of $(0,1)$.

Problem 3. (p. 97, Problem 3) For every integer $n \geq 1$, prove that the subset $\mathbb{R}^{n-1} \times \{0\} \subset \mathbb{R}^n$ has measure zero as a subset of $\mathbb{R}^n$.

Problem 4.(p. 103, Problem 4) Let A be an open subset of $\mathbb{R}^2$, and let $f : A \to \mathbb{R}$ be a C^2 function. Prove that for every rectangle $Q \subset A$,

$$\int_Q D_2 D_1 f \text{ equals } \int_Q D_1 D_2 f.$$

Use this to prove that D_2D_1f is identically equal to D_1D_2f on A. Do not simply cite the result from Section 6.

Problem 5. For real numbers $0 < a < b$ and $0 < \alpha < \beta$. compute the integral

$$\int_{(x,y)\in[a,b]\times[\alpha,\beta]} \frac{dxdy}{(x+y)^2}.$$

Problem 6. For every integer $n \geq 1$, define $f_n : [0,1] \to [0,1]$ to be the function such that $f(p/2^n) = 1$ for every integer $p = 0, \dots, 2^n - 1$, and otherwise $f(x) = 0$. Prove that every f_n is integrable with integral equal to 0. Prove that for every $x \in \mathbb{R}$, $f_1(x) \leq f_2(x) \leq \cdots \leq 1$ so that the function $f(x) = \sup\{f_n(x) : n \in \mathbb{N}\}$ is well-defined. Compute $f(x)$ and prove that f is not (Riemann) integrable. (This is one of the main defects of the Riemann integral compared to other theories of integration.)

Extra Problems.

p. 97, Exercises 2, 4, 6, 7, 9; p. 103, Exercise 2.