Topological Methods in the Quest for Periodic Orbits

Lecture Notes

MM613 2016-2

v0 [1]

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To the cycles of life

unimaginable in variety
stunning surprises

Preface

The present text originates from lecture notes written during the graduate course "MM613 Métodos Topológicos da Mecânica Hamiltoniana" held from august to november 2016 at UNICAMP. The manuscript has then been extended in order to serve as accompanying text for an advanced mini-course during the 31$^{\text{st}}$ *Colóquio Brasileiro de Matemática*, IMPA, Rio de Janeiro, in august 2017.

Scope

We aim to present some steps in the history of the problem of detecting closed orbits in Hamiltonian dynamics. This not only relates to symplectic geometry, but also to an odd cousin, called contact geometry and leading to Reeb dynamics. Ultimately we'd like to introduce the reader to Rabinowitz-Floer homology, an active area of contemporary research.

When we started to write these lecture notes we aimed in the introduction "*The following text is meant to provide an introductory overview, throwing in some details occasionally, preferably such which are usually omitted.*" Obviously we failed: In the end our text contains quite a lot of details and, as it turned out, basically all of them can be found somewhere in the literature..

Content

There are two parts, Hamiltonian dynamics and Reeb dynamics, each one coming with, maybe largely motivated by, a famous conjecture: The Arnol′d conjecture on existence of 1-periodic Hamiltonian trajectories and the Weinstein conjecture concerning closed characteristics, that is images of periodic Reeb trajectories of whatever period but on the same energy level.

Part one recalls basics of symplectic geometry, in particular, we review the Conley-Zehnder index from various angles. Then we present the construction of Floer homology, rather detailed, as analogous steps are used in the construction of Rabinowitz-Floer homology. Floer homology was deviced to prove the Arnol′d conjecture. Part two recalls basics of contact geometry and reviews the construction of Rabinowitz-Floer homology. The Weinstein conjecture is reconfirmed for certain classes of hypersurfaces in exact symplectic manifolds.

It goes without saying that the references simply reflect the knowledge, not to say ignorance, of the author. They are not meant to be exhaustive. Certainly many more people contributed to the many research fields, and all their facetes, touched upon in these notes.

Audience

The intended audience are graduate students. Recommendable background includes manifolds and basics of differential geometry and functional analysis.

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Campinas, *Joa Weber*
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Chapter 1

Introduction

The quest for closed orbits of dynamical systems - for instance periodic geodesics or periodic trajectories of particles in a magnetic field - dates back to the foundational work by Hamilton [Ham35] and Jacobi [Jac09] around 1840 and by Poincaré [Poi95] around 1900, followed by work, among many others, by Lusternik-Schnirelmann [LS30] in the 1920s, Kolmogorov-Arnol′d-Moser [Kol54, Arn63, Mos62] around the 1960s, and Rabinowitz [Rab79, Rab86] and Conley-Zehnder [CZ83] around the early 1980s. Floer's approach [Flo89] to infinite dimensional Morse theory in the second half of the 1980s, combining the Conley-Zehnder approach with Gromov's J-holomorphic curves introduced in his 1985 landmark paper [Gro85], marked a breakthrough in the efforts to prove the *Arnol′d conjecture*: The number of 1-periodic trajectories of a Hamiltonian vector field on a closed symplectic manifold M is bounded below by the Lusternik-Schnirelmann category of M or, in the non-degenerate case, by the sum of the Betti numbers of M. At about the same time Hofer entered the stage and together with Wysocki, Zehnder, Eliashberg, among others, contactized the symplectic world, eventually leading to the (occasionally so-called) theory of everything [EGH00]: Symplectic Field Theory – SFT.

Departing from Poincaré's last geometric theorem

We briefly sketch how Poincaré's last geometric theorem inspired the Arnol′d conjecture. For many more facets and further related results along these developments see the excellent presentations [HZ11, Ch. 6] and [Arn78, App. 9]. The following result was announced by Poincaré [Poi12] shortly before his death in 1912 and proved by Birkhoff [Bir13] shortly thereafter.

Theorem 1.0.1 (Poincaré-Birkhoff). *Every area and orientation preserving homeomorphism h of an annulus $A := \mathbb{S}^1 \times [a, b]$ rotating the two boundaries in opposite directions*[1] *possesses at least 2 fixed points in the interior.*

[1] This so-called **twist condition** excludes rotations (they have no fixed points in general).

Exercise 1.0.2. Show that h in the Poincaré-Birkhoff Theorem 1.0.1 is homotopic to the identity. [Hint: Identify each of the two boundary components of the annulus A to a point to obtain a space homeomorphic to $\mathbb{S}^2$ equipped with an induced homeomorphism $\tilde{h}$. Apply the Hopf degree theorem.[2]]

Lefschetz fixed point theory, introduced in 1926 [Lef26], cf. [Hir76, Ch.5 §2 Excs.] or [GP74, Ch.3 §4], guarantees existence of a fixed point for a continuous map $h : X \to X$ on a compact topological space X whenever a certain integer L_h, called the Lefschetz number, is non-zero. Key properties concerning applications are, firstly, that L_h is a homotopy invariant and, secondly, if X is a closed manifold then L_{id} is the Euler characteristic $\chi(X)$.

For the Poincaré-Birkhoff Theorem 1.0.1 Lefschetz theory fails, as $\chi(A) = 0$. A direct proof of the existence of one fixed point of h is given in the beautyful presentation [MS98, §8.2] where, furthermore, existence of infinitely many periodic points[3] of h is proved whenever the boundary twist is 'sufficiently strong'.

Exercise 1.0.3. Show that any continuous map $f : \mathbb{S}^2 \to \mathbb{S}^2$ homotopic to the identity, in symbols $f \sim \mathrm{id}$, has at least one fixed point. This result is sharp even for homeomorphisms: Find a homeomorphism of $\mathbb{S}^2$ with exactly one fixed point. [Hint: Consider the Riemann sphere $\mathbb{R}^2 \cup \{\infty\}$ and translations on $\mathbb{R}^2$.]

Remark 1.0.4. By [Nik74,Sim74] one gets back to at least two guaranteed fixed points, if one requires a homeomorphism $f \sim \mathrm{id}$ on $\mathbb{S}^2$ to preserve, in addition, a regular measure. So any diffeomorphism f of $\mathbb{S}^2$ leaving an area form ω invariant, that is $f^*\omega = \omega$, admits at least two[4] fixed points; cf. Section 5.4.3.

In dimension *two*, but *not* in higher dimension, the diffeomorphisms of a surface that preserve an area form are the **symplectomorphisms** of the form.

Arriving at the Arnol′d conjecture

In [Arn78, App. 9] Arnol′d suggested to glue together two copies of the annulus in the Poincaré-Birkhoff Theorem 1.0.1 along their boundaries each of which equipped with the same area and orientation preserving map h which, in addition, is now assumed to be a diffeomorphism and not too far C^1-away from the identity. This results in the 2-torus $\mathbb{T}^2$ equipped with an area and orientation preserving diffeomorphism, say $\tilde{h}$, which is C^1-close to id and by the twist condition satisfies a condition illustratively called "preservation of center of mass". Note that Lefschetz theory does not predict any fixed point for $\tilde{h}$ since $\chi(\mathbb{T}^2) = 0$. However, due to the additional C^1-close-to-id condition, the fixed points of $\tilde{h}$ correspond precisely to the critical points of a function F on

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[2] **Hopf degree theorem:** Two maps of a closed connected oriented n-dimensional manifold Q into $\mathbb{S}^n$ are homotopic if and only if they have the same degree. See e.g. [GP74, Ch.3 §6] or [Hir76, Ch.5 Thm. 1.10].

[3] A **periodic point** x of h is a fixed point of one of the iterates of h, that is $h^k(x) := (h \circ \dots h)(x) = x$ for some $k \in \mathbb{Z}$.

[4] Note that $\deg f = 1 = \deg \mathrm{id}$ and apply the Hopf degree theorem.

$\mathbb{T}^2$ called the generating function of $\tilde{h}$. The number of critical points of F is bounded below by the **Lusternik-Schnirelmann category**

$$|\mathrm{Crit} F| \geq \mathrm{cat}(\mathbb{T}^2) = 3 > \mathrm{cup}_{\mathbb{R}}(\mathbb{T}^2) = 2,$$

more modestly, by the **cuplength** plus one, or via Morse theory by the **sum of the Betti numbers** $\mathrm{SB}(\mathbb{T}^2) = 4$ in the **non-degenerate case**, that is in case all fixed points of h, equivalently all critical points of F, are non-degenerate. See e.g. [Web] for basics on Lusternik-Schnirelmann and Morse theory. So the number of fixed points of $\tilde{h}$ is at least three. But this number is even by symmetry of the construction (the fixed points come in pairs). Consequently $\tilde{h}$ has at least four fixed points. Hence h has at least two and this reconfirms the Poincaré-Birkhoff Theorem 1.0.1 for diffeomorphisms and under the *additional assumption* of h being C^1-close to id.
Hence one might conjecture, as Arnol′d did in [Arn76], that the torus result should be true as well without the C^1-close-to-id condition and, furthermore, not only for "doublings" $\tilde{h}$ of h. It is important to observe that $h \sim \mathrm{id}$ leads to the fact that $\tilde{h}$ is a Hamiltonian diffeomorphism[5] for the area form, that is it is the time-1-map of the flow generated by the Hamiltonian vector field X_H for some function $H : \mathbb{S}^1 \times \mathbb{T}^2 \to \mathbb{T}^2$. Fixed points of $\tilde{h}$ are then in bijection with 1-periodic orbits of X_H.

Arnol′d conjecture. Suppose (M, ω) is a closed symplectic manifold and $H : \mathbb{R} \times M \to \mathbb{R}$ is a smooth time-1 periodic function $H_t(x) := H(t, x) = H_{t+1}(x)$, denoted $H : \mathbb{S}^1 \times M \to \mathbb{R}$. Consider the time-dependent Hamiltonian equation

$$\dot{z} = X_{H_t}(z), \quad z : \mathbb{R} \to M,$$

and the set $\mathcal{P}_0(H) = \mathcal{P}_0(H; M, \omega)$ of all contractible[6] 1-periodic[7] solutions. The Arnol′d conjecture states that the number $|\mathcal{P}_0(H)|$ of contractible 1-periodic solutions is bounded below by the least number of critical points that a function on M must have, that is by the infimum $\mathrm{Crit}(M)$ over all functions $f : M \to \mathbb{R}$ of the number $\mathrm{Crit} f$ of critical points. The commonly addressed weaker forms of the Arnol′d conjecture are suggested by Lusternik-Schnirelmann and Morse theory, respectively. They state that

$$|\mathcal{P}_0(H)| \geq \mathrm{cup}_{\mathbb{R}}(M) + 1 \tag{1.0.1}$$

in general and that

$$|\mathcal{P}_0(H)| \geq \mathrm{SB}(M) \tag{1.0.2}$$

in case all contractible 1-periodic solutions are non-degenerate.

[5] Some authors use the terminology $\boldsymbol{\tilde{h}}$ **is homologous to the identity**.

[6] Multiplying X_H by a small constant implies that all 1-periodic solutions are very short, hence contractible; see Proposition 2.3.16. Firstly, this inspires the conjecture that it is the contractible solutions which are related to the topology of M. Secondly, this has the consequence that any Floer complex on a component of the free loop space that consists of non-contractible loops is chain homotopy equivalent to the trivial (no generators) complex.

[7] Concerning period *one* see Remark 2.3.14.

As we tried to stress, the Arnol′d conjecture for $\mathbb{T}^2$ is the *differentiable* generalization of the Poincaré-Birkhoff Theorem 1.0.1. Are there *topological* generalizations, that is topological analogues of the Arnol′d conjecture, as well? There are – in dimension *two* – and these are extremely far reaching indeed; see discussion towards the end of §6.1 in [HZ11]. For instance, they led to the affirmative solution [Fra92, Ban93] of the longstanding open question if all Riemannian 2-spheres carry infinitely many geometrically distinct periodic geodesics.

Floer homology – period one

Cornerstones in the confirmation of the Arnol′d conjecture were the solution by Conley and Zehnder [CZ83] for Hamiltonians H_t on the standard torus $(\mathbb{T}^n, \omega_0)$ and the solution by Floer [Flo88, Flo89] for ω-aspherical (and other) closed symplectic manifolds (M, ω); see e.g. [Sal99a] or [HZ11] for detailed accounts of further contributions. Floer's seminal contribution was to develop a meaningful Morse (homology) theory for the **symplectic action functional**

$$\mathcal{A}_H : \mathcal{L}_0 M \to \mathbb{R}, \quad z \mapsto \int_{\mathbb{D}} \bar{z}^*\omega - \int_0^1 H_t(z(t))\, dt, \tag{1.0.3}$$

on the component $\mathcal{L}_0 M$ of the **free loop space** $\mathcal{L}M = C^\infty(\mathbb{S}^1, M)$ that consists of contractible smooth 1-periodic loops $z : \mathbb{S}^1 = \partial\mathbb{D} \to M$, where $\bar{z} : \mathbb{D} \to M$ is any smooth extension of z. Floer mastered the obstructions presented by

- *infinite* Morse index of the critical points $z \in \mathrm{Crit}\mathcal{A}_H = \mathcal{P}_0(H)$ (which by definition are the generators of the Floer chain groups), cf. Ex. 3.2.15;
- the fact that the formal *downward* gradient equation for the L^2-gradient

  $$\operatorname{grad} \mathcal{A}_H(z) = -J_t(z)\big(\dot{z} - X_{H_t}(z)\big)$$

  does not generate a flow on loop space, not even a semi-flow; see Remark 3.2.8. Here J_t is a family of ω-compatible almost complex structures.

By definition and for generic H the Floer chain group $\mathrm{CF}_*(M, \omega, H)$ is the free abelian group generated by $\mathcal{P}_0(H)$ and graded by the canonical Conley-Zehnder index assuming that the first Chern class $c_1(M)$ vanishes. Roughly speaking, the Floer boundary operator counts downward flow lines and the Floer isomorphism

$$\mathrm{HF}_{n-\ell}(M, \omega; H, J) \cong \mathrm{H}_\ell(M)$$

to singular homology of M proves the Arnol′d conjecture (1.0.2) for *closed* symplectic manifolds that are ω-aspherical; see Definition 3.0.9. Floer homology of the closed manifold M of dimension $2n$ is restricted to degrees in $[-n, n]$.

Floer homology of cotangent bundles. Floer homology of non-compact symplectic manifolds can be highly different if it can be defined. For instance, consider a cotangent bundle T^*Q over a closed orientable manifold Q equipped with the canonical symplectic structure $\omega_{\mathrm{can}} = d\lambda_{\mathrm{can}} =$ "$dp \wedge dq$" on T^*Q. Pick a Riemannian metric g on Q; it is convenient to identify $T^*Q \cong TQ$ via g. Now consider a **mechanical Hamiltonian**

$$H_V(q,v) = \frac{1}{2}|v|^2 + V_t(q), \qquad q \in Q,\ v \in T_qQ, \tag{1.0.4}$$

of the form *kinetic* plus *potential energy* where the **potential** $V(t,q) =: V_t(q)$ is a smooth function on $\mathbb{S}^1 \times Q$ and $|v|^2$ abbreviates $g_q(v,v) =: \langle\cdot,\cdot\rangle$; see Section 2.4.1. For these data the action functional (1.0.3) takes on the form

$$\mathcal{A}_V : \mathcal{L}TQ \to \mathbb{R}, \quad z = (q,v) \mapsto \int_0^1 \langle v(t), \dot{q}(t)\rangle - H_{V_t}(q(t), v(t))\, dt, \tag{1.0.5}$$

which makes sense on arbitrary loops, not just contractible ones. Its critical points are of the form $z_x = (x, \dot{x})$ where x is a **perturbed 1-periodic geodesic**, that is an element of the set

$$\mathcal{P}(V) := \{x \in \mathcal{L}Q \mid -\nabla_t \dot{x} - \nabla V_t(x) = 0\}. \tag{1.0.6}$$

By the Morse index theorem the Morse index of a periodic geodesic is finite; still true after perturbation by a zero order term. In [Web02] it is shown that for generic V the canonical Conley-Zehnder index is well defined and equal to

$$\mu^{\mathrm{CZ}}(z_x) = \mathrm{ind}_{\mathcal{S}_V}(x) \in \mathbb{N}_0 \tag{1.0.7}$$

the **Morse index**; cf. (1.0.15): The number of negative eigenvalues, counted with multiplicities, of the Hessian at a critical point x of the classical action functional given by $\mathcal{S}_V(\gamma) = \int_0^1 \frac{1}{2}|\dot{\gamma}|^2 - V_t(\gamma)\, dt$ for $\gamma \in \mathcal{L}Q$. The upshot is that the **Floer homology of the cotangent bundle**, graded by μ^{CZ}, is naturally isomorphic to singular integral homology of the free loop space: That is

$$\mathrm{HF}_*(\mathcal{A}_V) := \mathrm{HF}_*(T^*Q, \omega_{\mathrm{can}}; H_V, \bar{J}_g) \cong \mathrm{H}_*(\mathcal{L}Q),$$

at least if the orientable manifold Q carries a spin structure or, equivalently, if the first and second Stiefel-Whitney classes of Q are both trivial; cf. Section 3.5. If Q is not simply connected, there is a separate isomorphism for each component $\mathcal{L}_\alpha Q$ of the free loop space. If Q is not orientable, choose $\mathbb{Z}_2$ coefficients.

Weinstein conjecture

Given a symplectic manifold (M,ω), consider an autonomous Hamiltonian $F : M \to \mathbb{R}$, also called an energy function. In this case the Hamiltonian flow ϕ^F generated by the Hamiltonian vector field X_F is energy preserving: Energy level sets $F^{-1}(c)$ are invariant under ϕ^F. It is a natural question if there exists a Hamiltonian flow trajectory that closes up in finite time T on a given, say closed, *regular* level set $\Sigma := F^{-1}(c)$. Observe that by regularity there are no zeroes of X_F or, equivalently, no constant flow trajectories. Restricting the non-degenerate 2-form ω to the odd-dimensional submanifold Σ yields the so-called **characteristic line bundle**

$$\mathcal{L}_\Sigma := \ker \omega|_\Sigma \to \Sigma$$

which even comes with a non-vanishing section, namely X_F. Therefore flow lines of X_F are integral curves of the distribution $\mathcal{L}_\Sigma$ and those that close up are called the **closed characteristics** P of the energy surface, in symbols $TP = \mathcal{L}_\Sigma|_P$.

On $\mathbb{R}^{2n}$ equipped with the canonical symplectic form $\omega_{\text{can}} = "dp \wedge dq"$ existence of a closed characteristic was confirmed on convex and star-shaped Σ by, respectively, Weinstein [Wei78] and Rabinowitz [Rab78]. Weinstein then isolated key *geometric* features of these hypersurfaces, and of the slightly more general class treated by Rabinowitz in [Rab79], thereby coining the notion of *contact type* hypersurfaces in [Wei79] and formulating the famous[8]

Weinstein conjecture. *A closed hypersurface of contact type with trivial first real cohomology carries a closed characteristic.*

Rabinowitz-Floer homology – free period fixed energy

For about three decades the potential of the variational setup used by Rabinowitz in his breakthrough result [Rab78], cf. [Rab79], went widely unnoticed. Given an autonomous Hamiltonian system $(V, \omega, F : M \to \mathbb{R})$, his idea was to incorporate a Lagrange multiplier τ into the standard action functional (1.0.3) whose presence causes that the critical points are periodic Hamiltonian trajectories of *whatever period* and constrained to a *fixed energy* level surface, namely $\Sigma := F^{-1}(0)$. Only around 2007 the Rabinowitz action functional

$$\mathcal{A}^F : \mathcal{L}V \times \mathbb{R}, \quad (z, \tau) \mapsto \int_{\mathbb{S}^1} z^*\lambda - \tau \int_0^1 F(z(t))\, dt,$$

on certain exact symplectic manifolds $(V, \omega = d\lambda)$, namely convex ones, was brought to new, if not spectacular, honours by Cieliebak and Frauenfelder in their landmark construction [CF09] of a Floer type homology theory: Rabinowitz-Floer homology $\mathrm{RFH}(\Sigma, V) := \mathrm{HF}(\mathcal{A}^F)$ associated certain closed hypersurfaces $\Sigma = F^{-1}(0)$, for instance such of restricted contact type that bound a closed submanifold-with-boundary $M \subset V$, by picking any regular F.

The power of their theory is shown by the fact that Rabinowitz-Floer homology of the archetype example of the unit bundle $\Sigma = S^*Q$ in the cotangent bundle $(V, \lambda) = (T^*Q, \lambda_{\text{can}})$ over a closed Riemannian manifold Q, not only represents the homology of the loop space of Q, but *simultaneously* its cohomology.

Symplectic and contact topology

For an overview of the development of symplectic and contact topology, starting with Lagrange's 1808 formulation of classical mechanics and culminating in the moduli space techniques initiated by Gromov [Gro85] and Floer [Flo86,Flo89] in the mid 1980's we recommend the article [Nel16]. The article also explains the origin of the adjective *symplectic* as the greek version of the originally advocated latin adjective *complex*. The latter was abandoned as it was already used in the prominent notion of complex number; see also the wiktionary entry 'symplectic'.

[8] For more background and context we recommend the fine recent survey in [Hut10].

Notation and conventions

Symbol	Terminology	Remark
$\mathbb{R}^*$, $\mathbb{Z}^*$	non-zero reals, integers	$\mathbb{R}\setminus\{0\}$, $\mathbb{Z}\setminus\{0\}$
N	manifold (mf)	modelled on $\mathbb{R}^k$
N	manifold-with-boundary	modelled on $\mathbb{R}^{k-1}\times\{x_k\geq 0\}$
Q	closed manifold	compact, no boundary, dim n
(M,ω)	symplectic mf/mf-w-bdy	$\dim M=2n$
(W,α)	contact mf/mf-w-bdy	$\dim W=2n-1$
F, ϕ^F	autonomous Ham., flow	$\dot\phi_t=X_F\circ\phi_t$, $\phi_0=\mathrm{id}$
	generates 1-param. group:	$\phi_{t+s}=\phi_s\circ\phi_s$
H_t, ψ^H	non-auton. Hamiltonian, flow	$\dot\psi_t=X_{H_t}\circ\psi_t$, $\psi_0=\mathrm{id}$
	is not a 1-param. group:	$\psi_t=\psi_{t,0}$; see (2.3.19)
	Hamiltonian path/loop	Remark 2.3.13
$\mathcal{P}(H)$	1-periodic trajectories of X_H	$\mathcal{P}(H)^*$, non-constant ones
$\mathfrak{c}$	flow line, integral curve, orbit	embedded submf $\mathfrak{c}\hookrightarrow N$
	of vector field X	s.t. X is tangent to $\mathfrak{c}$
	closed orbit	orbit $\mathfrak{c}\cong\mathbb{S}^1$ or $\mathfrak{c}=\{\mathsf{pt}\}$
$\mathfrak{c}$, P	closed characteristic of X	integral curve diffeom. to $\mathbb{S}^1$
u	trajectory (parametr. orbit)	$u:\mathbb{R}\to\mathrm{mf}$, $u'=\dots$
z	periodic trajectory	$z:\mathbb{R}/\tau\mathbb{Z}\to M$, $\dot z=X(z)$
	(parametrized closed orbit)	
γ, z	loop (periodic path)	$\gamma:\mathbb{R}/\tau\mathbb{Z}\to M$, Def. 2.3.4

Notation 1.0.5. Unless mentioned otherwise, the following conventions apply throughout. All quantities, including homotopies and paths, are supposed to be C^∞ smooth. By a **manifold** we mean a **differentiable manifold**[9] N modeled locally on open subsets of $\mathbb{R}^n$. For **manifold-with-boundary** N replace $\mathbb{R}^n$ by its closed upper half space. The boundary ∂N might be empty though. A **closed manifold**, usually denoted by $Q=Q^n$, is a compact manifold (hence no boundary). The empty set $\emptyset$ generates the trivial group $\{0\}$. It is often convenient to set $\inf\emptyset:=\infty$. Vector spaces are real. Neighborhoods are open. Given a map $\gamma:\mathbb{R}\to N$, a path, denote time shift and uniform speed change by

$$\gamma_{(T)}:=\gamma(T+\cdot),\qquad \gamma^\mu:=\gamma(\mu\cdot),$$

whereas subindex $\gamma_\tau:[0,\tau]\to N$, $\tau\in\mathrm{Per}(\gamma)\setminus\{0\}$, denotes a divisor part, see (2.3.11), and simultaneously the induced loop $\gamma_\tau:\mathbb{R}/\tau\mathbb{Z}\to N$, but subindex $u_s(\cdot):=u(s,\cdot)$ also denotes the operation of freezing a variable.

[9] The topological Hausdorff space N comes with countably many (second axiom of countability) homeomorphisms $\varphi_i:N\supset U_i\to\mathbb{R}^n$, called local coordinate charts, such that all transition maps $\varphi_j\circ{\varphi_i}^{-1}:\mathbb{R}^n\to\mathbb{R}^n$ are diffeomorphisms.

Linear space. On $\mathbb{R}^{2n}$ there are two natural structures, the **euclidean metric** $\langle v,w\rangle_0 := \sum_{j=1}^{2n} v_j w_j$ and the **standard almost complex structure**

$$J_0 := \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}; \qquad \bar{J}_0 := -J_0 = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}.$$

The matrizes $\pm J_0$ represent multiplication by $\pm i$ under the natural isomorphism

$$\mathbb{R}^{2n} \xrightarrow{\simeq} \mathbb{C}^n, \quad z = (x,y) = (x_1,\dots,x_n,y_1,\dots,y_n) \mapsto x + iy.$$

We shall use this isomorphism freely whenever convenient, even writing $\mathbb{R}^{2n} = \mathbb{C}^n$ as real vector spaces and $J_0 = i$. Furthermore, given the coordinates $z = (x,y) \in \mathbb{R}^{2n}$ it is natural to combine them in the form

$$\omega_0 := \sum_j dx_j \wedge dy_j$$

called the **standard symplectic form**. While the 2-form ω_0 is exact for several choices of primitives,[10] such as for instance $\sum_j x_j dy_j$, the natural **radial vector field** $Y_0(z) = z = \sum_j \left(x_j \partial_{x_j} + y_j \partial_{y_j}\right)$ is compatible with the ω_0-primitive

$$\lambda_0 := \frac{1}{2} \sum_{j=1}^{n} \left(x_j dy_j - y_j dx_j\right), \qquad d\lambda_0 = \omega_0 =: dx \wedge dy, \tag{1.0.8}$$

in the sense that $i_{Y_0}\omega_0 := \omega_0(Y_0,\cdot) = \lambda_0$.[11] Hence $L_{Y_0}\omega_0 = di_{Y_0}\omega_0 = \omega_0$. These identities play a crucial role in the history of the Weinstein Conjecture 4.1.9 and the development of the notion of contact type hypersurfaces.

On the other hand, on cotangent bundles, say $T^*Q \ni (q,p)$, there is a canonical globally defined 1-form, the **Liouville form** λ_{can}, see (2.4.27), the **canonical symplectic form** $\omega_{\mathrm{can}} := d\lambda_{\mathrm{can}}$, and the canonical **fiberwise radial vector field** Y_{rad}, see (4.5.13). In natural cotangent bundle coordinates $(q_1,\dots,q_n,p^1,\dots,p^n)$ these structures are of the form $Y_{\mathrm{rad}} = \sum_j p^j \partial_{p^j}$ and

$$\lambda_{\mathrm{can}} := \sum_{j=1}^{n} p^j \, dq_j =: p\, dq, \quad \omega_{\mathrm{can}} := + \, d\lambda_{\mathrm{can}} = \sum_{j=1}^{n} dp^j \wedge dq_j =: dp \wedge dq.$$

Note that $\omega_{\mathrm{can}}(Y_{\mathrm{can}},\cdot) = \lambda_{\mathrm{can}}$ where $Y_{\mathrm{can}} := 2Y_{\mathrm{rad}}$ is the **canonical Liouville vector field**. Of course, these definitions make sense on $\mathbb{R}^{2n} \simeq T^*\mathbb{R}^n$. For better readability we often use the notation $(q_1,\dots,q_n,p_1,\dots,p_n)$.

The two natural symplectic structures ω_0 and ω_{can} on $\mathbb{R}^{2n}$ are compatible with $J_0 = i$ and $\bar{J}_0 = -i$, respectively, in the sense that the two compositions

$$\omega_0(\cdot, J_0 \cdot) = \langle \cdot,\cdot\rangle_0, \qquad \omega_{\mathrm{can}}(\cdot, \bar{J}_0 \cdot) = \langle \cdot,\cdot\rangle_0, \tag{1.0.9}$$

both reproduce the euclidean metric.

[10] A differential form λ is called a **primitive** of ω if its exterior derivative $d\lambda$ is ω.

[11] We define the **wedge product** by $dx_j \wedge dy_j := \frac{1}{2}\left(dx_j \otimes dy_j - dy_j \otimes dx_j\right)$ as in [GP74].

Remark 1.0.6 (Canonical normalization of Conley-Zehnder index)**.** In Hamiltonian dynamics of classical physical Hamiltonians on cotangent bundles, e.g. on $\mathbb{R}^{2n} \simeq T^*\mathbb{R}^n$, the second choice in (1.0.9) is natural since the dynamics is governed by **Hamilton's equations** [Ham35, Eq. (A.)] given by

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} \partial_p H \\ -\partial_q H \end{pmatrix} = \bar{J}_0 \nabla H \tag{1.0.10}$$

and exhibiting most prominently $\bar{J}_0$. For the most basic physical system, the harmonic oscillator on $\mathbb{R}^2$, which is also most basic mathematically in the sense that in place of ∇H one has the identity $\mathbb{1}$, the system is linear and given by

$$\dot{\zeta} = \bar{J}_0 \zeta, \qquad \zeta(t) = e^{\bar{J}_0 t} = e^{-it}, \quad t \in [0,1].$$

Hence it is natural to favorize $\bar{J}_0$ and the finite path $t \mapsto e^{\bar{J}_0 t}$ concerning sign conventions and normalize the index function by associating the value 1 to the *distinct symplectic path* e^{-it} in $\mathrm{Sp}(2)$, as we do in (1.0.11).[12] However, probably since e^{it} is also rather distinct in the sense that it represents the mathematically positive sense of rotation (counter-clockwise), just like $i = J_0$, basically all of the original papers on the Conley-Zehnder index use the normalization

$$\mu_{\mathrm{CZ}}(\{e^{it}\}_{t\in[0,1]}) = 1.$$

This is the **standard normalization** or the **counter-clockwise normalization** and μ_{CZ} the (standard) Conley-Zehnder index. For compatibility with the literature our review in Section 2.1 of the various variants of Maslov-type indices and the various constructions of each of them uses the standard normalization.

A simple method to deal with the need, when dealing with ω_{can}, for a Conley-Zehnder index normalized clockwise is to introduce a new name and notation. We call the Conley-Zehnder index based on the **canonical normalization**

$$\boxed{\mu^{\mathrm{CZ}}(\{e^{-it}\}_{t\in[0,1]}) = 1.} \tag{1.0.11}$$

the **canonical Conley-Zehnder index**, denoted by μ^{CZ} for distinction. It is just the negative $\mu^{\mathrm{CZ}} = -\mu_{\mathrm{CZ}}$ of the standard Conley-Zehnder index.

By $\overline{\mathbb{R}^{2n}} \times \mathbb{R}^{2n}$ we denote the vector space $\mathbb{R}^{2n} \times \mathbb{R}^{2n}$ equipped with the almost complex structure $-J_0 \oplus J_0$ and the symplectic form $-\omega_0 \oplus \omega_0$.

Manifolds. Suppose M is a manifold. Let $\mathbb{S}^1 \subset \mathbb{C} \cong \mathbb{R}^2$ be the unit circle. We slightly abuse notation to express periodicity in time $t \in \mathbb{R}/\mathbb{Z}$. We usually write

$$H : \mathbb{S}^1 \times M \to \mathbb{R} \quad \text{or} \quad H : \mathbb{R}/\mathbb{Z} \times M \to \mathbb{R}$$

to denote a function $H : \mathbb{R} \times M \to \mathbb{R}$ with $H_{t+1} = H_t \; \forall t$ where $H_t := H(t, \cdot)$. Given a symplectic form ω on M, the identities of 1-forms

$$\boxed{dH_t(\cdot) = -\omega(X_{H_t}, \cdot)}, \qquad X_t := X_{H_t} = X^{\omega}_{H_t}, \tag{1.0.12}$$

[12] As a rotation e^{-it} is mathematically negatively oriented (counter-clockwise is positive).

one for each $t \in \mathbb{S}^1$, determines the family of **Hamiltonian vector fields** X_t.[13] The set $\mathcal{P}_0(H)$ of contractible 1-periodic Hamiltonian trajectories is precisely the set of critical points of the **perturbed symplectic action functional**

$$\mathcal{A}_H = \mathcal{A}_H^\omega : C^\infty_{\mathrm{contr}}(\mathbb{S}^1, M) \to \mathbb{R}, \quad z \mapsto + \int_{\mathbb{D}} \bar{z}^*\omega - \int_0^1 H_t(z(t))\, dt.$$

Here $\bar{z} : \mathbb{D} \to M$ denotes an extension[14] of the contractible loop $z : \mathbb{S}^1 \to M$ and the two signs arise as follows. The sign " $+$ " is due to the requirement that on cotangent bundles (convention $\omega_{\mathrm{can}} := + \, d\lambda_{\mathrm{can}} = dp \wedge dq$) the first integral should reduce to $\int_{\mathbb{S}^1} \lambda_{\mathrm{can}}$. Since the critical points of $\mathcal{A}_H$ should be orbits of X_H, as opposed to $-X_H$, the sign choice " $-$ " in (1.0.12) dictates the second sign " $-$ " in $\mathcal{A}_H$.

Suppose J_t is a family of **almost complex structures** on TM, that is each J_t is a section of the endomorphism bundle $\mathrm{End}(TM) \to M$ with ${J_t}^2 = -\mathbb{1}$. Assume, in addition, that each J_t is ***ω*-compatible**[15] in the sense that

$$\boxed{g_{J_t} := \langle \cdot, \cdot \rangle_t := \omega(\cdot, J_t \cdot)}$$

defines a Riemannian metric on M, one for each t. Such a triple (ω, J_t, g_{J_t}) is called a **compatible triple** and for *such* there is the identity

$$X^\omega_{H_t} = + \, J_t \circ \nabla^{g_{J_t}} H_t, \tag{1.0.13}$$

one for each $t \in \mathbb{S}^1$. Two compatible triples in $\mathbb{R}^{2n}$ are shown in (1.0.9).

Exercise 1.0.7. Suppose J is an ω-compatible almost complex structure. Show that both, the associated Riemannian metric g_J and ω itself, are J-invariant, that is $g_J(J\cdot, J\cdot) = g_J(\cdot,\cdot)$ and $\omega(J\cdot, J\cdot) = \omega(\cdot,\cdot)$.

Cotangent bundles. Consider a cotangent bundle $(T^*Q, \omega_{\mathrm{can}} := + \, d\lambda_{\mathrm{can}})$ over a closed Riemannian manifold (Q, g) of dimension n. Use g to identify T^*Q with TQ via the inverse of the isomorphism $v \mapsto g(v, \cdot)$, again denoted by g. By exactness of ω_{can} there is no need to restrict to contractible loops. Just define[16]

$$\mathcal{A}_H^{\lambda_{\mathrm{can}}} : \mathcal{L}TQ \to \mathbb{R}, \quad z = (q, v) \mapsto + \int_0^1 g\left(v(t), \dot{q}(t)\right) - H_t(q(t), v(t))\, dt.$$

For Hamiltonians H_V of the form *kinetic plus potential energy* for some potential $V_{t+1} = V_t : Q \to \mathbb{R}$, see (1.0.4), the critical points of $\mathcal{A}_V := \mathcal{A}_{H_V}^{\lambda_{\mathrm{can}}}$ are precisely

---

[13] In our previous papers [Web02, SW06] we used twice opposite signs, firstly for ω_{can} and secondly in (1.0.9). Hence the Hamiltonian vector field there and here is the same.

[14] To avoid that $\mathcal{A}_H(z)$ depends on the extension $\bar{z}$, suppose that ω vanishes over $\pi_2(M)$.

[15] In the euclidean case the convention $g_{J_0} := \omega_0(\cdot, J_0 \cdot)$ leads to the euclidean metric, so the opposite convention $g'_{J_0} := \omega_0(J_0 \cdot, \cdot)$ is negative definite and therefore not an inner product.

[16] Here all signs are dictated: By physics (integrand should be $p\, dq - H\, dt$) as well as by mathematics (the cousin $\mathcal{S}_V$ of $\mathcal{A}$, given by (3.5.53), is bounded below which suggests that the downward gradient flow encodes homology and the upward flow cohomology).

of the form $z_x := (x, \dot{x})$ with $x \in \mathcal{P}(V) = \{-\nabla_t \dot{x} - \nabla V(x) = 0\}$. Hence x is a (perturbed) 1-periodic geodesic in the Riemannian manifold (Q, g) and as such admits a Morse index $\mathrm{ind}_{\mathcal{S}_V}(x) \in \mathbb{N}_0$ and a nullity $\mathrm{null}_{\mathcal{S}_V}(x) \in \mathbb{N}_0$.

Suppose the nullity of $x \in \mathcal{P}(V)$ is zero and the vector bundle $x^*TQ \to \mathbb{S}^1$ is trivial; pick an orthonormal trivialization. Then the linearized Hamiltonian flow along x provides a finite path $\Psi_x : [0,1] \to \mathrm{Sp}(2n)$ of symplectic matrices that starts at $\mathbb{1}_{2n}$ and whose endpoint does not admit 1 in its spectrum (by the nullity assumption). Thus Ψ_x has a well defined canonical Conley-Zehnder index $\mu^{\mathrm{CZ}}(\Psi_x)$. Recall from (1.0.11) that μ^{CZ} is based on the canonical $\bar{J}_0$ (clockwise) normalization and equal to $-\mu_{\mathrm{CZ}}$. In other words, compared to our previous papers [Web02, SW06] we use the opposite `(signature)` axiom:

`(signature)`$_{\texttt{can}}$ If $S = S^T \in \mathbb{R}^{2n \times 2n}$ is a symmetric matrix of norm $\|S\| < 2\pi$, then

$$\mu^{\mathrm{CZ}}\left(\{[0,1] \ni t \mapsto e^{t\bar{J}_0 S}\right) = \frac{1}{2}\mathrm{sign}\,(S) := \frac{n^+(S) - n^-(S)}{2}. \qquad (1.0.14)$$

Since $\mu_{\mathrm{CZ}}(\Psi_x)$ does not depend on the choice of trivialization, one defines $\mu^{\mathrm{CZ}}(z_x) := \mu^{\mathrm{CZ}}(\Psi_x)$. The relation to the Morse index is

$$\mu^{\mathrm{CZ}}(z_x) = + \,\mathrm{ind}_{\mathcal{S}_V}(x), \qquad (1.0.15)$$

as shown in [Web02].[17] If $x^*TM \to \mathbb{S}^1$ is not orientable a correction term enters.

Remark 1.0.8 (Homology or cohomology?)**.** As the energy functional $\mathcal{S}_V : \mathcal{L}Q \to \mathbb{R}$ given by (3.5.53) is bounded below, the *downward* gradient direction is the right choice to construct Morse homology, whereas counting upwards naturally suits cohomology. The functional is Morse for generic V and the critical points are given by $\mathrm{Crit}\,\mathcal{S}_V = \mathcal{P}(V)$. For each of them there is a finite Morse index $\mathrm{ind}_{\mathcal{S}_V}(x)$ and the index function $\mathrm{ind}_{\mathcal{S}_V}$ decreases along connecting *downward* gradient flow lines under the Morse-Smale condition. Thus $\mathrm{ind}_{\mathcal{S}_V}$ is a natural grading of Morse homology $\mathrm{HM}_*(\mathcal{L}Q, \mathcal{S}_V)$.

Because the critical points of $\mathcal{S}_V$ coincide with those of $\mathcal{A}_V$ under the correspondence $x \mapsto z_x = (x, \dot{x})$ and there is the identity (1.0.15) of indices and the identity $\mathcal{S}_V(x) = \mathcal{A}_V(z_x)$ of functionals, it is natural to use the canonical Conley-Zehnder index μ^{CZ} and the downward gradient of $\mathcal{A}_V$ to construct Floer homology of cotangent bundles.

[17] The identity $\mu_{\mathrm{CZ}} = -\mathrm{ind}_{\mathcal{S}_V}$ in [Web02] uses the anti-clockwise normalization of μ_{CZ}.

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Part I

Hamiltonian dynamics

Chapter 2

Symplectic geometry

Consider a manifold M of finite dimension. A **Riemannian metric** g on M is a family g_x of symmetric non-degenerate bilinear forms on T_xM that varies smoothly in x. To define a **symplectic form** ω on M replace symmetric by skew-symmetric[1] – consequently the dimension is necessarily even, say $2n$ – and impose, in addition, the *global* condition that the non-degenerate differential 2-form ω is closed ($d\omega = 0$). Symplectic manifolds are orientable since the $2n$-form $\omega^{\wedge n}$ nowhere vanishes by non-degeneracy of ω, in other words $\omega^{\wedge n}$ is a volume form. Thus, if the manifold M is *closed*, then the differential form ω cannot be exact by Stoke's theorem. Thus the global condition $d\omega = 0$ implies that $[\omega] \neq 0$. Hence the second real cohomology of a closed symplectic manifold is necessarily non-trivial. For existence of symplectic structures see e.g. [Gom01, Sal13].

In contrast to Riemannian geometry there are *no local invariants* in symplectic geometry: By Darboux's Theorem a symplectic manifold looks locally like the prototype symplectic vector space $(\mathbb{R}^{2n}, \omega_0)$. In contrast to Riemannian geometry[2] the global theory is rich, already for the space $\mathrm{Sp}(2n)$ of *linear* symplectic transformations of $(\mathbb{R}^{2n}, \omega_0)$. In Chapter 2 we follow mainly [MS98].

Exercise 2.0.9. Show that only one of the unit spheres $\mathbb{S}^k \subset \mathbb{R}^{k+1}$, $k \in \mathbb{N}$, carries a symplectic form. Which of the tori $\mathbb{T}^k := (\mathbb{S}^1)^{\times k}$ carry a symplectic form? How about the real projective plane $\mathbb{R}\mathrm{P}^2$? And, in contrast, the $\mathbb{C}\mathrm{P}^k$'s?

Exercise 2.0.10 (The unit 2-sphere $\mathbb{S}^2 \subset \mathbb{R}^3$)**.** Show that

$$\omega_p(x, y) := \langle p, x \times y \rangle, \qquad p \in \mathbb{S}^2, \quad x, y \in (\mathbb{R}p)^\perp,$$

defines a symplectic form on $\mathbb{S}^2$ and that the unit tangent bundle $T^1\mathbb{S}^2$ is diffeomorphic to $\mathrm{SO}(3)$. [Hint: The three columns of any matrix $\boldsymbol{a} \in \mathrm{SO}(3)$ are of the form $p, v, p \times v$ where $p \perp v$ are unit vectors.]

Exercise 2.0.11. Show that ω on $\mathbb{S}^2$ defined above is in cylindrical coordinates given by $\omega_{\mathrm{cyl}} = d\theta \wedge dz$, for $(\theta, z) \in [0, 2\pi) \times (-1, 1)$, and in spherical coordinates by $\omega_{\mathrm{sph}} = (\sin \varphi)\, d\theta \wedge d\varphi$, for $(\theta, \varphi) \in [0, 2\pi) \times (0, \pi)$.

[1] Such non-degenerate skew-symmetric ω_x is called a **symplectic bilinear form** on T_xM.

[2] The space of Riemannian metrics is convex, hence contractible, thus topologically trivial.

2.1 Linear theory

The **symplectic linear group** $\mathrm{Sp}(2n)$ consists of all real $2n \times 2n$ matrices Ψ that preserve the standard symplectic structure $\omega_0 = "dx \wedge dy"$, that is

$$\omega_0 = \Psi^* \omega_0 := \omega_0(\Psi \cdot, \Psi \cdot). \tag{2.1.1}$$

Observe that this identity implies that $\det \Psi = 1$.

Exercise 2.1.1. Show that (2.1.1) is equivalent to

$$\Psi^T J_0 \Psi = J_0$$

where Ψ^T is the transposed matrix. [Hint: Compatibility with euclidean metric.]

Consider the group $\mathrm{GL}(2n, \mathbb{R})$ of invertible real $2n \times 2n$ matrices. The orthogonal group $\mathrm{O}(2n) \subset \mathrm{GL}(2n)$ is the subgroup of those matrices that preserve the euclidean metric. The linear map $\mathbb{R}^{2n} \to \mathbb{C}^n$, $z = (x, y) \mapsto x + iy$, is an isomorphism of vector spaces that identifies J_0 and the imaginary unit i. Under this identification $X + iY \in \mathrm{GL}(n, \mathbb{C})$ corresponds to

$$\begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} \in \mathrm{GL}(2n, \mathbb{R}).$$

Similarly the unitary group $\mathrm{U}(n)$ is a subgroup of $\mathrm{GL}(2n, \mathbb{R})$, in fact of $\mathrm{Sp}(2n)$.

Exercise 2.1.2. Show that the identities of real $n \times n$ matrices

$$X^T Y = Y^T X, \qquad X^T X + Y^T Y = \mathbb{1},$$

are precisely the condition that $X + iY \in \mathrm{U}(n)$.

Exercise 2.1.3. Prove that the intersection of any two of $\mathrm{O}(2n)$, $\mathrm{Sp}(2n)$, and $\mathrm{GL}(n, \mathbb{C})$ is precisely $\mathrm{U}(n)$ as indicated by Figure 2.1.

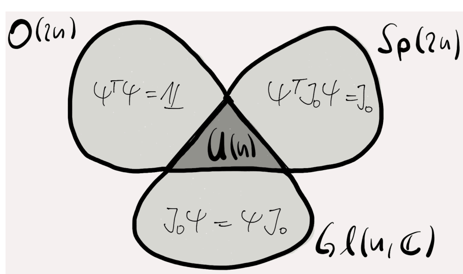

Figure 2.1: Relation among four classical matrix Lie groups

The **eigenvalues of a symplectic matrix** occur either as pairs $\lambda, \lambda^{-1} \in \mathbb{R} \setminus \{0\}$ or $\lambda, \bar{\lambda} \in \mathbb{S}^1$ or as complex quadruples

$$\lambda, \lambda^{-1}, \bar{\lambda}, \bar{\lambda}^{-1}.$$

In particular, both 1 and -1 occur with even multiplicity.

2.1.1 Topology of $\mathrm{Sp}(2)$

Major topological properties of $\mathrm{Sp}(2n)$, such as the fundamental group being $\mathbb{Z}$ or the existence of certain cycles, can nicely be visualized using the Gel′fand-Lidskiĭ [GL58] homeomorphism between $\mathrm{Sp}(2)$ and the open solid 2-torus in $\mathbb{R}^3$. It is a diffeomorphism away from the center circle $\mathrm{U}(1)$; for details see also [Web99, App. D]. Consider the sets $\mathcal{C}_\pm$ of all symplectic matrices which have ± 1 in their spectrum.

The set $\mathcal{C} := \mathcal{C}_+$ *is called the* ***Maslov cycle*** *of* $\mathrm{Sp}(2n)$.

It consists of disjoint subsets which are submanifolds, called strata. For $n = 1$ there are only two strata one of which contains only one element, namely the identity matrix $\mathrm{E} = \mathbb{1}$; see Figure 2.2 which shows $\mathcal{C}_\pm$ in the Gel′fand-Lidskiĭ parametrization of $\mathrm{Sp}(2)$, namely, as the open solid 2-torus in $\mathbb{R}^3$.

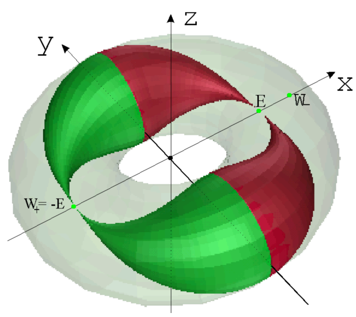

Figure 2.2: Subsets $\mathcal{C}_-$ and $\mathcal{C}_+$ of $\mathrm{Sp}(2) = \mathbb{S}^1 \times \operatorname{int}\mathbb{D}$. Notation $\mathrm{E} = \mathbb{1}$

For the spectrum of the elements Ψ of $\mathrm{Sp}(2)$ there are three possibilities:

(pos. hyp.) real positive pairs $\lambda, \lambda^{-1} > 0$; those Ψ with $\lambda = 1$ are $\mathcal{C}_+$;

(neg. hyp.) real negative pairs $\lambda, \lambda^{-1} < 0$; those Ψ with $\lambda = -1$ are $\mathcal{C}_-$;

(elliptic) complex pairs $\lambda, \bar\lambda \in \mathbb{S}^1 \setminus \{-1, +1\}$; those enclosed by $\mathcal{C}_- \cup \mathcal{C}_+$.

Exercise 2.1.4 (Eigenvalues of first and second kind)**.** Note that the set enclosed by $\mathcal{C}_- \cup \mathcal{C}_+$ has *two* connected components. What distinguishes them?[3] Suppose $\lambda, \bar\lambda \in \mathbb{S}^1 \setminus \{-1, +1\}$ are eigenvalues of $\Psi \in \mathrm{Sp}(2n)$. Thus $\lambda \neq \bar\lambda$ and the eigenvectors $\xi_\lambda, \xi_{\bar\lambda} = \overline{\xi_\lambda} \in \mathbb{C}^n \setminus \mathbb{R}^n$ are linearly independent. Show that $\omega_0\left(\overline{\xi_\lambda}, \xi_\lambda\right) \in i\mathbb{R} \setminus \{0\}$. If the imaginary part of this quantity is positive, then λ is called **of the first kind**, otherwise **of the second kind**. Show that one of $\lambda, \bar\lambda$ is of the first kind and the other one of the second kind.

For $\mathrm{Sp}(2n)$, $2n \geq 4$, where the eigenvalues can be quadruples the notion of eigenvalues of the first and second kind becomes important concerning *stability properties of Hamiltonian trajectories*: Two pairs of eigenvalues on $\mathbb{S}^1$ can meet and leave $\mathbb{S}^1$, if and only if, eigenvalues of different kind meet.

[3] Those Ψ whose eigenvalue of the first kind lies in the upper half plane, e.g. J_0, lie in the same component, those where this location is the lower half plane lie in the other component.

2.1.2 Maslov index μ

The map

$$h : [0,1] \times \mathrm{Sp}(2n) \to \mathrm{Sp}(2n), \quad (t, \Psi) \mapsto (\Psi\Psi^T)^{-t/2}\Psi, \tag{2.1.2}$$

is a strong deformation retraction of $\mathrm{Sp}(2n)$ onto $\mathrm{Sp}(2n) \cap \mathrm{GL}(n, \mathbb{C}) \simeq \mathrm{U}(n)$; cf. Figure 2.1. So the quotient space $\mathrm{Sp}(2n)/\mathrm{U}(n)$ is contractible. It is well known that the determinant map $\det : \mathrm{U}(n) \to \mathbb{S}^1$ induces an isomorphism of fundamental groups. Consequently the fundamental group of $\mathrm{Sp}(2n)$ is given by the integers. Define a map $\rho : \mathrm{Sp}(2n) \to \mathrm{Sp}(2n) \cap \mathrm{GL}(n, \mathbb{C}) \to \mathbb{S}^1$ by

$$\rho : \Psi \mapsto h(\Psi, 1) = \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} \mapsto \det\big(\underbrace{X + iY}_{\in \mathrm{U}(n)}\big). \tag{2.1.3}$$

Maslov index – degree

An explicit isomorphism $[\mu] : \pi_1(\mathrm{Sp}(2n)) \to \mathbb{Z}$ is realized by the **Maslov index** μ which assigns to every loop $\Phi : \mathbb{S}^1 \to \mathrm{Sp}(2n)$ of symplectic matrices the integer

$$\mu(\Phi) := \deg\left(\mathbb{S}^1 \xrightarrow{\Phi} \mathrm{Sp}(2n) \xrightarrow{\rho} \mathbb{S}^1\right).$$

Exercise 2.1.5. Show that the Maslov index satisfies the following axioms:

`(homotopy)` Two loops in $\mathrm{Sp}(2n)$ are homotopic iff they have the same Maslov index.

`(product)` For any two loops $\Phi_1, \Phi_2 : \mathbb{S}^1 \to \mathrm{Sp}(2n)$ we have

$$\mu(\Phi_1\Phi_2) = \mu(\Phi_1) + \mu(\Phi_2).$$

Consequently $\mu(\mathbb{1}) = 0$, hence $\mu(\Phi^{-1}) = -\mu(\Phi)$ where $\Phi^{-1}(t) := \Phi(t)^{-1}$.

`(direct sum)` If $n = n' + n''$, then $\mu(\Phi' \oplus \Phi'') = \mu(\Phi') + \mu(\Phi'')$.

`(normalization)` The loop $\Phi : \mathbb{R}/\mathbb{Z} \to \mathrm{U}(1)$, $t \mapsto e^{i2\pi t}$, has Maslov index 1.

Show that these axioms determine μ uniquely. [Hints: The problem reduces to loops in $\mathrm{U}(n)$ by (2.1.2). On diagonal matrizes and products of matrizes det behaves nicely. A complex matrix $\Phi(t) \in \mathrm{U}(n)$ is triangularizable via conjugation by a unitary matrix, continuously in t. Diagonal elements are loops $\mathbb{S}^1 \to \mathrm{U}(1)$.]

Maslov index – intersection number with Maslov cycle

Looking at the Maslov cycle $\mathcal{C}$ in Figure 2.3, alternatively at the Robbin-Salamon cycle $\overline{\mathrm{Sp}}_1$,[4] suggests that the Maslov index $\mu(\Phi)$ should be half the intersection number with either cycle of generic, that is transverse, loops $\Phi : \mathbb{S}^1 \to \mathrm{Sp}(2n)$.[5]

---

[4] Write $\Psi \in \mathrm{Sp}(2n)$ in the form of a block matrix $\Psi = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with four $n \times n$ matrices and consider the function $\chi : \mathrm{Sp}(2n) \to \mathbb{R}$, $\Psi \mapsto \det B$. By definition the **Robbin-Salamon cycle** $\overline{\mathrm{Sp}}_1$ is the pre-image $\chi^{-1}(0)$, i.e. $\overline{\mathrm{Sp}}_1$ consists of all Ψ with $\det B$=0. Actually $\mathrm{Sp}(2n)$ is partitioned by the submanifolds $\mathrm{Sp}_k(2n)$ of codimension $k(k+1)/2$ which consist of those Ψ with rank $B = n - k$ and $\overline{\mathrm{Sp}}_1$ is the complement of the codimension zero stratum $\mathrm{Sp}_0(2n)$.

[5] This is indeed the case: The **Robbin-Salamon index** μ_{RS} of a generic loop is the intersection number with $\mathrm{Sp}_1(2n)$; see definition in [RS93, §4]. The equality $\mu = \frac{1}{2}\mu_{\mathrm{RS}}$

2.1.3 Conley-Zehnder index μ_{CZ} of symplectic path

Consider the map

$$\mathrm{Sp}(2n) \to \mathbb{R}, \quad \Psi \mapsto \det(\Psi - \mathbb{1}) \tag{2.1.4}$$

and note that the pre-image of 0 is precisely the Maslov cycle $\mathcal{C}$. Let Sp^*, Sp^*_+, and Sp^*_- be the subsets of $\mathrm{Sp}(2n)$ on which this map is, respectively, different from zero, positive, and negative. Thus we obtain the partitions

$$\mathrm{Sp}(2n) = \mathrm{Sp}^*_+ \,\dot\cup\, \mathcal{C} \,\dot\cup\, \mathrm{Sp}^*_-, , \qquad \mathrm{Sp}^* = \mathrm{Sp}^*_+ \,\dot\cup\, \mathrm{Sp}^*_-.$$

Geometrically the **Conley-Zehnder index** of admissible paths Ψ, namely

$$\mu_{\mathrm{CZ}} : \mathcal{SP}^*(2n) := \{\Psi : [0,1] \xrightarrow{C^0} \mathrm{Sp}(2n) \mid \Psi(0) = \mathbb{1},\ \Psi(1) \in \mathrm{Sp}^*\} \to \mathbb{Z}$$

can be defined as intersection number with the Maslov cycle $\mathcal{C}$ of generic paths in $\mathrm{Sp}(2n)$ starting at the identity and ending away from the Maslov cycle. Here generic not only means transverse to the codimension one stratum, but also in the sense that the paths depart from $\mathbb{1}$ immediately into Sp^*_-. The need for the latter condition can be read off from Figure 2.4 easily.

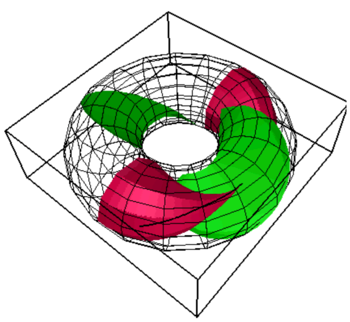

Figure 2.3: Cycles $\mathcal{C}$, $\mathrm{Sp}_1(2)$

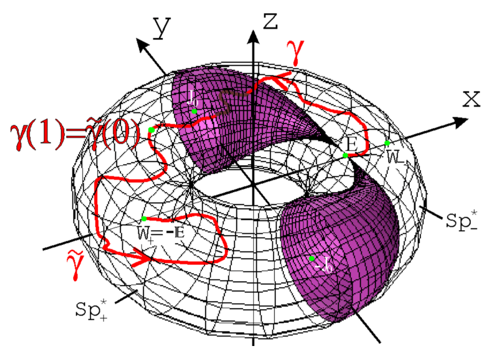

Figure 2.4: CZ-index 1 of path $\Psi = \gamma$

Exercise 2.1.6. Following the original definition by Conley and Zehnder [CZ84, §1], pick a path $\Psi \in \mathcal{SP}^*(2n)$. Then its endpoint lies in one of the two *connected* open sets Sp^*_+ or Sp^*_-. Show that these sets contain, respectively, the matrices

$$W_+ := -\mathbb{1}, \qquad W_- := \operatorname{diag}\left(2, -1, \dots, -1, \tfrac{1}{2}, -1, \dots, -1\right).$$

Extend Ψ from its endpoint to the corresponding matrix inside the component $\mathrm{Sp}^*_\pm$ of the endpoint. Consider the extended path $\tilde\Psi : [0,2] \to \mathrm{Sp}(2n)$. Define

$$\mu_{\mathrm{CZ}}(\Psi) := \deg\left(\rho^2 \circ \tilde\Psi\right)$$

follows either by the formula in [RS93, Rmk. 5.3] or by the fact that $\frac{1}{2}\mu_{\mathrm{RS}}$ satisfies, by [RS93, Thm. 4.1], the first three axioms for μ in Exercise 2.1.5. To confirm `(normalization)` calculate $\mu_{\mathrm{RS}}(t \mapsto e^{i2\pi t}) = 2$. Hint: $B(t) = -\sin(2\pi t)$, intersection form above [RS93, Thm. 4.1].]

and show that $\rho \circ \Psi(0) = 1 \in \mathbb{S}^1$ and that

$$\rho(W_+) = \det(-\mathbb{1}) = (-1)^n, \quad \rho(W_-) = \det(1, -1, \dots, -1) = (-1)^{n-1}.$$

Clearly taking the square of ρ yields $+1$ in either case. Hence the path $\rho^2 \circ \tilde{\Psi} : [0, 2] \to \mathbb{S}^1$ closes up at time 1 and therefore taking the degree makes sense.

The Conley-Zehnder index $\mu_{\mathrm{CZ}} : \mathcal{SP}^*(2n) \to \mathbb{Z}$ satisfies certain axioms, similar to those of the Maslov index μ in Exercise 2.1.5.

Theorem 2.1.7 (Conley-Zehnder index)**.** *For $\Psi \in \mathcal{SP}^*(2n)$ the following holds.*

`(homotopy*)` *The Conley-Zehnder index is constant on the components of $\mathcal{SP}^*(2n)$.*

`(loop*)` $\mu_{\mathrm{CZ}}(\Phi\Psi) = 2\mu(\Phi) + \mu_{\mathrm{CZ}}(\Psi)$ *for any loop* $\Phi : \mathbb{S}^1 \to \mathrm{Sp}(2n)$.

`(signature*)` *If $S = S^T \in \mathbb{R}^{2n\times 2n}$ is a symmetric matrix of norm $\|S\| < 2\pi$, then*

$$\mu_{\mathrm{CZ}}\left(t \mapsto e^{tJ_0S}\right) = \frac{1}{2}\mathrm{sign}\,(S) := \frac{n^+(S) - n^-(S)}{2}$$

where $\mathrm{sign}\,(S)$ *is the* ***signature*** *of S and $n^\pm(S)$ is the number of positive/negative eigenvalues of S.*

`(direct sum)` *If $n = n' + n''$, then $\mu(\Psi' \oplus \Psi'') = \mu(\Psi') + \mu(\Psi'')$.*

`(naturality)` $\mu_{\mathrm{CZ}}(\Theta\Psi\Theta^{-1}) = \mu_{\mathrm{CZ}}(\Psi)$ *for any path* $\Theta : [0, 1] \to \mathrm{Sp}(2n)$.

`(determinant)` $(-1)^{n-\mu_{\mathrm{CZ}}(\Psi)} = \mathrm{sign}\,\det\,(\mathbb{1} - \Psi(1))$.

`(inverse)` $\mu_{\mathrm{CZ}}(\Psi^{-1}) = \mu_{\mathrm{CZ}}(\Psi^T) = -\mu_{\mathrm{CZ}}(\Psi)$.

The `(signature)` axiom normalizes μ_{CZ}. The $*$-axioms determine μ_{CZ} uniquely; see e.g. [Sal99a, §2.4].

Remark 2.1.8 (Canonical Maslov and Conley-Zehnder indices)**.** Clockwise rotation appears naturally in Hamiltonian dynamics, cf. (1.0.10), since $\bar{J}_0 := -J_0$ is compatible with $\omega_{\mathrm{can}} = dp \wedge dq$, not J_0. Thus it is natural and convenient to introduce versions of the Maslov and Conley-Zehnder indices normalized clockwise and denoted by μ_{can} and μ^{CZ}, respectively, namely

`(normalization)`$_{\texttt{can}}$

$$\mu_{\mathrm{can}}(\{e^{-i2\pi t}\}_{t\in[0,1]}) := 1, \qquad \mu^{\mathrm{CZ}}(\{e^{-it}\}_{t\in[0,1]}) := 1. \tag{2.1.5}$$

These indices are just the negatives of the standard anti-clockwise normalized indices, that is $\mu_{\mathrm{can}} = -\mu$ and $\mu^{\mathrm{CZ}} = -\mu_{\mathrm{CZ}}$. They satisfy corresponding versions of the previously stated axioms, e.g. `(loop)` becomes `(loop)`$_{\texttt{can}}$ $\mu^{\mathrm{CZ}}(\Phi\Psi) = 2\mu_{\mathrm{can}}(\Phi) + \mu^{\mathrm{CZ}}(\Psi)$ and `(signature)`$_{\texttt{can}}$ is displayed in (1.0.14).

In the present text we use for Floer, and also Rabinowitz-Floer, homology the canonical (clockwise) version μ^{CZ} of the Conley-Zehnder index, because on cotangent bundles (see Section 3.5) these theories relate canonically to the classical action functional $\mathcal{S}_V$ which requires no choices at all to establish Morse homology; see Remark 1.0.8.

The reason why we introduced here in great detail the standard (counter-clockwise) version $\mu_{\rm CZ}$ is better comparability with the literature. It spares the reader continuously translating between the normalizations. So while we explain $\mu_{\rm CZ}$, the reader can conveniently consult the literature for details of proofs, and once everything is established for $\mu_{\rm CZ}$ we simply note that $\mu^{\rm CZ}(\Psi) = -\mu_{\rm CZ}(\Psi)$.

Symmetric matrizes are rather intimately tied to symplectic geometry:

Exercise 2.1.9. Show that for symmetric $S = S^T \in \mathbb{R}^{2n\times 2n}$ the matrizes e^{tJ_0S} and $e^{t\bar{J}_0S}$ are elements of $\mathrm{Sp}(2n)$ whenever $t \in \mathbb{R}$.

Exercise 2.1.10 (Symmetric matrizes)**.** More generally, given a smooth path $[0,1] \ni t \mapsto S(t) = S(t)^T \in \mathbb{R}^{2n\times 2n}$ of symmetric matrizes, show that the path of matrizes $\Psi : [0,1] \to \mathbb{R}^{2n\times 2n}$ determined by the initial value problem

$$\frac{d}{dt}\Psi(t) = J_0 S(t)\Psi(t), \qquad \Psi(0) = \mathbb{1}, \tag{2.1.6}$$

takes values in $\mathrm{Sp}(2n)$. Note that $\Psi \in \mathcal{SP}^*(2n)$ iff $\det(\mathbb{1} - \Psi(1)) \neq 0$. Vice versa, given a symplectic path Ψ, show that the family of matrizes defined by

$$S(t) := -J_0\dot{\Psi}(t)\Psi(t)^{-1} \tag{2.1.7}$$

is symmetric.

Exercise 2.1.11. a) The map (2.1.4) provides a natural co-orientation[6] of the Maslov cycle $\mathcal{C}$. Does this co-orientation serve to define the Maslov index μ as intersection number with $\mathcal{C}$? (For simplicity suppose $n = 1$.) [Hint: Given this co-orientation, calculate μ for any generic loop winding around 'the hole' once.]
b) Is the situation better for the Robbin-Salamon cycle $\overline{Sp}_1$? Suppose $n = 1$ and co-orient $\overline{Sp}_1 = \mathrm{Sp}_1$ by the increasing direction of the function χ in an earlier footnote. Show that the intersection number with Sp_1 of the loop $\mathbb{R}/\mathbb{Z} \ni t \mapsto e^{i2\pi t} \in \mathrm{U}(1) \subset \mathrm{Sp}(2)$ is -1 at $t = 0$ and $+1$ at $t = 1/2$.
c) For $n = 1$ consider the **parity** $\nu(B, D)$ of $\Psi \in \mathrm{Sp}_1$ defined in [RS93, Rmk. 4.5]. Check that for $t \mapsto e^{i2\pi t}$ the parity is -1 at $t = 0$ and $+1$ at $t = 1/2$. Check that the intersection number of loops with Sp_1 co-oriented by *χ-co-orientation times parity* recovers the Maslov index μ.

By now several alternative descriptions of the Conley-Zehnder index have been found, for instance, the interpretation as intersection number with the Maslov cycle of a symplectic path, even with arbitrary endpoints, has been defined by Robbin and Salamon [RS93]; see Section 2.1.5. In case $n = 1$ there is a description of $\mu_{\rm CZ}$ in terms of winding numbers which we discuss right below.

For further details concerning Maslov, Conley-Zehnder, and other indices see e.g. [Arn67, CZ84, RS93, Gut14] and [Sal99a].

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[6] A **co-orientation** is an orientation of the normal bundle (to the top-diml. stratum).

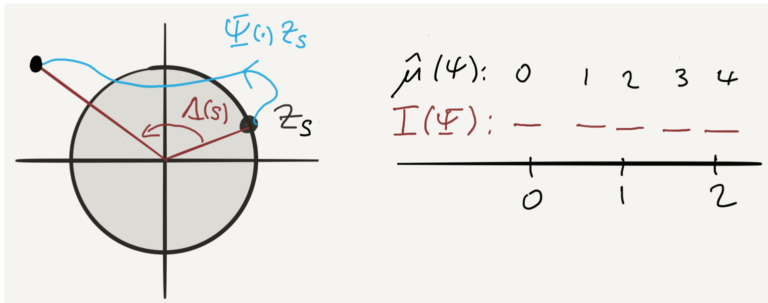

Figure 2.5: Winding number $\Delta(s)$ of point $z_s = e^{i2\pi s} \in \mathbb{S}^1$ under path Ψ

Winding number descriptions of μ_{CZ} in the case $n = 1$

For the following geometric and analytic construction we recommend the presentations in [HWZ03, §8] and [HMSa15, §2]. It is convenient to naturally identify $\mathbb{R}^2$ with $\mathbb{C}$ and J_0 with i.

Geometric description (winding intervals [HK99, §3]). A path $\Psi : [0,1] \to \mathrm{Sp}(2)$ with $\Psi(0) = \mathbb{1}$ uniquely determines via the identity

$$\Psi(t) z_s = r(t,s) e^{i\theta(t,s)}, \qquad z_s := e^{i2\pi s} \in \mathbb{S}^1,$$

two continuous functions r and θ. Note that $r > 0$ and $\theta(0,s) = 2\pi s$. Define the winding number of the point $z_s := e^{i2\pi s} \in \mathbb{S}^1$ under the symplectic path Ψ, i.e. the change in argument of $[0,1] \ni t \mapsto \Psi(t) z_s \in \mathbb{C} \setminus \{0\}$, see Figure 2.5, by

$$\Delta(s) := \frac{\theta(1,s) - 2\pi s}{2\pi} \in \mathbb{R}.$$

The **winding interval of the symplectic path** Ψ is the union

$$I(\Psi) := \{\Delta(s) \mid s \in [0,1]\}$$

of the winding numbers under Ψ of the elements of $\mathbb{S}^1$. The interval $I(\Psi)$ is compact, its boundary is disjoint from the integers iff $\Psi(1) \notin \mathcal{C}$, that is iff $\Psi \in \mathcal{SP}^*(2)$, and most importantly its length $|I(\Psi)| < 1/2$ is less then $1/2$. Thus, for $\Psi \in \mathcal{SP}^*(2)$, the winding interval either lies between two consecutive integers or contains precisely one of them in its interior. Thus one can define

$$\mu'(\Psi) := \begin{cases} 2k & , \text{ if } k \in I(\Psi), \\ 2k+1 & , \text{ if } I(\Psi) \subset (k, k+1), \end{cases}$$

for some integer $k \in \mathbb{Z}$. One verifies the $*$-axioms in Theorem 2.1.7 to get that $\mu' = \mu_{\mathrm{CZ}}$ is the Conley-Zehnder index itself.

Observe that the winding number $\Delta(s)$ is an integer k iff $\Psi(1) z_s = \lambda z_s$ is a positive multiple of z_s. But the latter means that λ is a positive eigenvalue of $\Psi(1)$. Thus $k \in I(\Psi)$ which shows that positive hyperbolic paths are of *even* Conley-Zehnder index. Similar considerations show that negative hyperbolic and elliptic paths both have *odd* Conley-Zehnder indices.

Analytic description (eigenvalue winding numbers, [HWZ95, §3]). The integer $\mu'(\Psi)$ can be characterized in terms of the spectral properties of the unbounded self-adjoint differential operator on L^2 with dense domain $W^{1,2}$, namely

$$L_S := -J_0 \frac{d}{dt} - S(t) : L^2(\mathbb{S}^1, \mathbb{R}^2) \supset W^{1,2} \to L^2$$

where the family S of symmetric matrices corresponds to Ψ via (2.1.7). Here we assume that the symplectic path Ψ is defined on $\mathbb{R}$ and satisfies $\Psi(t+1) = \Psi(t)\Psi(1)$. This extra condition corresponds to periodicity $S(t+1) = S(t)$.

The spectrum $\sigma(L_S)$ of the operator L_S consists, by compactness of the resolvent, of countably many isolated real eigenvalues of finite multiplicity accumulating precisely at $\pm\infty$. Suppose that $v : \mathbb{S}^1 \to \mathbb{R}^2$ is eigenfunction associated to an eigenvalue λ. Note that $v : \mathbb{S}^1 \to \mathbb{R}^2$ cannot have any zero. Thus we can write $v(t) = \rho(t)e^{i\vartheta(t)}$ and define its winding number by $\mathrm{wind}(v) := \frac{\vartheta(1)-\vartheta(0)}{2\pi}$. This integer only depends on the eigenvalue λ, but not on the choice of eigenvector. So it is denoted by $\mathrm{wind}(\lambda)$ and called the **winding number of the eigenvalue $\boldsymbol{\lambda}$**. For each integer k there are precisely two eigenvalues (counted with mulitplicities) whose winding number is k. If there is only one such eigenvalue, its multiplicity is 2. Moreover, if $\lambda_1 \le \lambda_2$, then $\mathrm{wind}(\lambda_1) \le \mathrm{wind}(\lambda_2)$. Let $\lambda_-(S) < 0$ be the largest negative eigenvalue and $\lambda_+(S) \in \mathbb{N}_0$ the next larger one. Define the maximal winding number among the negative eigenvalues of the operator L_S and its parity by

$$\alpha(S) := \mathrm{wind}(\lambda_-) \in \mathbb{Z}, \qquad p(S) := \begin{cases} 0 & \text{, if } \mathrm{wind}(\lambda_-) = \mathrm{wind}(\lambda_+), \\ 1 & \text{, if } \mathrm{wind}(\lambda_-) < \mathrm{wind}(\lambda_+). \end{cases}$$

Theorem 2.1.12. *If $\Psi \in \mathcal{SP}^*(2)$, then $2\alpha(S) + p(S) = \mu'(\Psi)$.*

2.1.4 Lagrangian subspaces

A **symplectic vector space** (V, ω) is a real vector space with a non-degenerate skew-symmetric bilinear form. So $\dim V = 2n$ is necessarily even.

Exercise 2.1.13. Show that, firstly, each symplectic vector space admits a **symplectic basis**, that is vectors $u_1, \dots, u_n, v_1 \dots, v_n$ such that

$$\omega(u_j, u_k) = \omega(v_j, v_k) = 0, \quad \omega(u_j, v_k) = \delta_{jk},$$

and, secondly, there is a **linear symplectomorphism** – a vector space isomorphism preserving the symplectic forms – to $(\mathbb{R}^{2n}, \omega_0)$.

The **symplectic complement** of a vector subspace $W \subset V$ is defined by

$$W^\omega := \ker \omega := \{v \in V \mid \omega(v, w) = 0 \ \forall w \in W\}.$$

In contrast to the orthogonal complement, the symplectic complement is not necessarily disjoint to V, but V and V^ω are still of complementary dimension (as non-degeneracy is imposed in both worlds) and $(W^\omega)^\omega = W$. Thus the maximal dimension of $W \cap W^\omega$ is $n = \frac{1}{2}\dim V$. Such W, that is those with $W = W^\omega$, are called **Lagrangian subspaces**. Equivalently these are characterized as the n dimensional subspaces restricted to which ω vanishes identically.

A subspace $W \subset V$ is called **isotropic** if $W \subset W^\omega$, in other words, if ω vanishes on W, and **coisotropic** if $W^\omega \subset W$.

Exercise 2.1.14 (Graphs of symmetric matrizes are Lagrangian)**.** Show that

$$\Gamma_S := \{(x, Sx) \mid x \in \mathbb{R}^n\}$$

is a Lagrangian subspace of $(\mathbb{R}^{2n}, \omega_0)$ if and only if $S = S^T \in \mathbb{R}^{n\times n}$ is symmetric.

Exercise 2.1.15 (Natural structures on $W \oplus W^*$)**.** Let W be a real vector space and W^* its dual space. Show that on $W \oplus W^*$ a symplectic form Ω_0 is naturally given by $((v,\eta),(\tilde{v},\tilde{\eta})) \mapsto \tilde{\eta}(v) = \eta(\tilde{v})$. Show that both summands of $W \oplus W^*$ are Lagrangian. Now pick, in addition, an **inner product** on W, that is a non-degenerate symmetric bilinear form g on W. This provides a natural isomorphism $W \to W^*$, $v \mapsto g(v,\cdot)$, again denoted by g, which naturally leads to the inner product $g^* = g(g^{-1}\cdot, g^{-1}\cdot)$ on W^*. Moreover, on $W \oplus W^*$ one obtains an inner product $G_g = g \oplus g^*$ and an almost complex structure $\bar{J}_g$; cf. (2.4.28). Show their compatibility in the sense that $\Omega_0(\cdot, \bar{J}_g\cdot) = G_g$.

Exercise 2.1.16. Show that the graph Γ_Ψ of a linear symplectomorphism $\Psi : V \to V$ is a Lagrangian subspace of the cartesian product $V \times V$ equipped with the symplectic form $(-\omega)\oplus\omega$ that sends $((v,w),(v',w'))$ to $-\omega(v,v')+\omega(w,w')$. Note that the **diagonal subspace** $\Delta := \{(v,v) \mid v \in V\}$ is Lagrangian.

2.1.5 Robbin-Salamon index – degenerate endpoints

A symplectic path $\Psi : [0,1] \to \mathrm{Sp}(2n)$ gives rise to the family of symmetric matrizes $S(t)$ given by (2.1.7) for which, in turn, it is a solution to the ODE (2.1.6). A number $t \in [0,1]$ is called a **crossing** if $\det(\mathbb{1} - \Psi(t)) = 0$ or, equivalently, if 1 is eigenvalue of $\Psi(t)$. In other words, if $\Psi(t)$ hits the Maslov cycle $\mathcal{C}$: The eigenspace $\mathrm{Eig}_1\Psi(t) = \ker(\mathbb{1} - \Psi(t)) \neq 0$ must be non-trivial.

At a crossing t the quadradic form given by

$$\Gamma(\Psi, t) : \mathrm{Eig}_1\Psi(t) \to \mathbb{R}, \quad \xi_0 \mapsto \omega_0(\xi_0, \dot{\Psi}(t)\xi_0) = \langle \xi_0, S(t)\xi_0\rangle,$$

is called the **crossing form**. A crossing is called **regular** if the crossing form is non-degenerate. Regular crossings are isolated. If all crossings are regular the **Robbin-Salamon index** $\mu_{\mathrm{RS}}(\Psi)$ was introduced in [RS93], although here we repeat the presentation given in [Sal99a, § 2.4], as the sum over all crossings t of the signatures of the crossing forms where crossings at the boundary points $t = 0, 1$ are counted with the factor $\frac{1}{2}$ only. For the particular paths $\Psi \in \mathcal{SP}^*(2n)$

the Robbin-Salamon index

$$\mu_{\mathrm{RS}}(\Psi) := \frac{1}{2}\mathrm{sign}\, S(0) + \sum_t \mathrm{sign}\, \Gamma(\Psi, t) = \mu_{\mathrm{CZ}}(\Psi)$$

reproduces the Conley-Zehnder index.

Exercise 2.1.17. a) Check the identity in the definition of Γ. b) The factor $\frac{1}{2}$ at the endpoints is introduced in order to make $\mu_{\mathrm{RS}}(\Psi)$ invariant under homotopies with fixed endpoints. To see what happens homotop the path γ in Figure 2.4 to $\Psi(t) = e^{i\pi t}$ and calculate the crossing forms at $t = 0$ in both cases.

In [RS93] an index for a rather more general class of paths is constructed: A *relative* index $\mu_{\mathrm{RS}}(\Lambda, \Lambda')$ for *pairs* of paths of *Lagrangian subspaces* of a symplectic vector space (V, ω); cf. Exercise 2.1.14. Here crossings are nontrivial intersections $\Lambda(t) \cap \Lambda'(t) \neq \{0\}$. The Conley-Zehnder index on $\mathcal{SP}^*(2n)$ is recovered by choosing the symplectic vector space $(\mathbb{R}^{2n} \times \mathbb{R}^{2n}, -\omega_0 \oplus \omega_0)$ and the Lagrangian path given by the graphs $\Gamma_{\Psi(t)}$ of Ψ relative to the constant path given by the diagonal Δ. Indeed $\mu_{\mathrm{CZ}}(\Psi) = \mu_{\mathrm{RS}}(\Gamma_\Psi, \Delta)$ by [RS93, Rmk. 5.4]. Note that $\Gamma_{\Psi(t)} \cap \Delta \simeq \mathrm{Eig}_1 \Psi(\mathrm{t})$.

2.2 Symplectic vector bundles

Suppose $E \to N$ is a vector bundle of real rank $2n$ over a manifold-with-boundary of dimension k; where $\partial N = \emptyset$ is not excluded. A **symplectic vector bundle** is a pair (E, ω) where ω is a family of symplectic bilinear forms ω_x, one on each fiber E_x. Similarly a **complex vector bundle** is a pair (E, J) where J is a family of *complex structures* J_x *on the fibers* E_x, that is $J_x^2 = -\mathrm{Id}_{E_x}$. Existence of a deformation retraction, such as h in (2.1.2), of $\mathrm{Sp}(2n)$ onto $\mathrm{U}(n)$ has the consequence that any symplectic vector bundle (E, ω) is fiberwise homotopic, thus isomorphic as a vector bundle,[7] to a complex vector bundle (E, J_ω) called the **underlying complex vector bundle**.[8] A **Hermitian vector bundle** (E, ω, J, g_J) is a symplectic and complex vector bundle (E, ω, J) such that J is ω-compatible, that is $g_J := \omega(\cdot, J\cdot)$ is a Riemannian bundle metric on E.

Proposition 2.2.1. *Two symplectic vector bundles* (E_1, ω_1) *and* (E_2, ω_2) *are isomorphic if and only if their underlying complex bundles are isomorphic.*

Two proofs are given in [MS98, §2.6], one based on the deformation retraction (2.1.2), the other on constructing a homotopy equivalence between $\mathcal{J}(V, \omega)$, the space of ω-compatible complex structures on a symplectic vector space, and the convex, thus contractible, non-empty space of all inner products on V.

A **trivialization** of a bundle E is an isomorphism to the trivial bundle which preserves the structure under consideration. A **unitary trivialization**

[7] A **vector bundle isomorphism** is a diffeomorphism between the total spaces whose fiber restrictions are vector space isomorphisms.

[8] The complex structure J_ω, but not its isomorphism class, depends on h.

of a Hermitian vector bundle E is a smooth map

$$\Phi : N \times \mathbb{R}^{2n} \to E, \quad (x, \xi) \mapsto \Phi(x, \xi) =: \Phi(x)\xi, \tag{2.2.8}$$

which maps fibers linearly isomorphic to fibers, that is Φ^{-1} is a vector bundle isomorphism to the trivial bundle, and simultaneously identifies the compatible triple ω, J, g_J on E with the standard compatible triple $\omega_0, J_0, \langle\cdot,\cdot\rangle_0$ on $\mathbb{R}^{2n}$.

Proposition 2.2.2. *A Hermitian vector bundle $E \to \Sigma$ over a compact Riemann surface Σ with* non-empty *boundary $\partial\Sigma$ admits a unitary trivialization.*

The idea is to prove in a first step that for any path $\gamma : [0,1] \to \Sigma$ the pull-back bundle $\gamma^* T\Sigma \to [0,1]$ can be unitarily trivialized even if one fixes in advance unitary isomorphisms $\Phi_0 : \mathbb{R}^{2n} \to E_{\gamma(0)}$ and $\Phi_1 : \mathbb{R}^{2n} \to E_{\gamma(1)}$ over the two endpoints of γ. To see this construct unitary frames over small subintervalls of $[0,1]$ starting with a unitary basis of $E_{\gamma(t)}$ at some t, extend to a small intervall via parallel transport, say with respect to some Riemannian connection on E, and then exposed to the Gram-Schmidt process over $\mathbb{C}$. The coupling of the resulting unitary trivializations over the subintervals is based on the fact that the Lie group $\mathrm{U}(n)$ is connected. In the second step one uses a parametrized version of step one to deal with the case that Σ is diffeomorphic to the unit disk $\mathbb{D} \subset \mathbb{R}^2$. [9] Step three is to prove the general case by an induction that starts at step two and whose induction step is again by a parametrized version of step one, this time for the disk with two open disks removed from its interior (called a **pair of pants**).

2.2.1 Compatible almost complex structures

Given a symplectic manifold (M, ω), consider an endomorphism J of TM with $J^2 = -\mathbb{1}$. Such J is called an **almost complex structure** on M.[10] If, in addition, the expression

$$g_J(\cdot,\cdot) := \omega(\cdot, J\cdot)$$

defines a Riemannian metric on M, then J is called an **ω-compatible almost complex structure** on M. The space $\mathcal{J}(M, \omega)$ of all such J is non-empty and contractible by [MS98, Prop. 2.63].

Exercise 2.2.3. Pick $J \in \mathcal{J}(M, \omega)$ and let ∇ be the Levi-Civita connection associated to g_J. Suppose ξ is a smooth vector field on M, show (i) and (ii):[11]

(i) J preserves g_J and $(\nabla_\xi J)J + J(\nabla_\xi J) = 0$;

(ii) J and $(\nabla_\xi J)$ are anti-symmetric with respect to g_J;

(iii) $J(\nabla_{J\xi} J) = \nabla_\xi J$.

[9] Trivialize along rays starting at the origin: Extend a chosen frame sitting at the origin simultaneously along all rays, say by parallel transport, along an interval $[0, \varepsilon]$. Now apply Gram-Schmidt to the family of frames and repeat the process on $[\varepsilon/2, 3\varepsilon/2]$, and so on.

[10] Such J is called a **complex structure** or an **integrable complex structure** on M, if it arises from an atlas of M consisting of complex differentiable coordinate charts to $(\mathbb{C}^n, i)$.

[11] Part (iii) is less trivial; see [Sal99b, Le. 3.18].

2.2.2 First Chern class

Up to isomorphism, symplectic[12] vector bundles E over manifolds N are classified by a family $c_k(E) \in \mathrm{H}^{2k}(N)$ of integral cohomology classes of N called **Chern classes**. If $N = \Sigma$ is a closed orientable Riemannian surface, the first Chern class is uniquely determined by the **first Chern number** which is the integer obtained by evaluating the first Chern class on the fundamental cycle Σ. Thus, slightly abusing notation, in case $E \to \Sigma$ we shall denote the first Chern number by $c_1(E) \in \mathbb{Z}$. We cite again from [MS98].

Theorem 2.2.4. *There exists a unique functor c_1, called the **first Chern number**, which assigns an integer $c_1(E) \in \mathbb{Z}$ to every symplectic vector bundle E over a closed oriented Riemann surface Σ and satisfies the following axioms.*

`(naturality)` *Two symplectic vector bundles E and E' over Σ are isomorphic iff they have the same rank and the same Chern number.*

`(functoriality)` *For any smooth map $\varphi : \Sigma' \to \Sigma$ of oriented Riemann surfaces and any symplectic vector bundle E it holds $c_1(\varphi^* E) = \deg(\varphi) \cdot c_1(E)$.*

`(additivity)` *For any two symplectic vector bundles $E_1 \to \Sigma$ and $E_2 \to \Sigma$*

$$c_1(E_1 \oplus E_2) = c_1(E_1 \otimes E_2) = c_1(E_1) + c_1(E_2).$$

`(normalization)` *The Chern number of Σ is $c_1(\Sigma) := c_1(T\Sigma) = 2 - 2g$ where g is the genus.*

The proof is constructive, based on the Maslov index μ for symplectic loops: Pick a splitting $\Sigma = \Sigma_1 \cup_C \Sigma_2$ such that $\partial\Sigma_1 = C = -\partial\Sigma_2$ as *oriented* manifolds. So the union $C = S^1 \dot\cup \ldots \dot\cup S^1$ of, say ℓ, embedded 1-spheres is oriented as the boundary of Σ_1, say by the outward-normal-first convention. Given a symplectic vector bundle E over Σ, pick unitary[13] trivializations

$$\Sigma_i \times \mathbb{R}^{2n} \to E_i, \quad (x, \xi) \mapsto \Phi_i(x)\xi, \quad i = 1, 2 \tag{2.2.9}$$

and consider the **overlap map** $\Psi : C \to \mathrm{Sp}(2n)$ defined by $x \mapsto \Phi_1(x)^{-1}\Phi_2(x)$.

Exercise 2.2.5 (First Chern number)**.** Prove uniqueness in Theorem 2.2.4. Show that the first Chern number of $E \to \Sigma$ is the degree of the composition

$$c_1(E) = \deg\Big(C \xrightarrow{\Psi} \mathrm{Sp}(2n) \xrightarrow[(2.1.3)]{\rho} \mathbb{S}^1\Big) = \sum_{j=1}^{\ell} \mu(\gamma_j)$$

by verifying for $\deg(\rho \circ \Psi)$ the four axioms for the first Chern number.
(The second identity for the Maslov index μ is obvious: Just pick an orientation preserving parametrization $\gamma_j : \mathbb{S}^1 \to S^1$ for each connected component of C.)
[Hint: Show, or even just assume, first that $\deg(\rho \circ \Psi)$ is independent of the choice of, firstly, trivialization and, secondly, splitting. Use these two facts, whose proofs rely heavily on Lemma 2.2.6 below, to verify the four axioms.]

[12] equivalently, complex vector bundles, by Proposition 2.2.1
[13] a **symplectic trivialization** (just required to identify ω_0 with ω) is already fine

Lemma 2.2.6. *Let Σ be a compact oriented Riemann surface with* non-empty *boundary. A smooth map $\Psi : \partial\Sigma \to \mathrm{Sp}(2n)$ extends to Σ iff $\deg(\rho \circ \Psi) = 0$.*

Exercise 2.2.7 (Obstruction to triviality)**.** Use the axioms to show that the first Chern number $c_1(E)$ vanishes iff the symplectic vector bundle is **trivial**, that is isomorphic to the trivial Hermitian bundle $\Sigma \times (\mathbb{R}^{2n}, \omega_0, J_0, \langle\cdot,\cdot\rangle_0)$.

Exercise 2.2.8 (First Chern class)**.** Suppose E is a symplectic vector bundle over any manifold N. Observe that the first Chern number assigns an integer $c_1(f^*E)$ to every smooth map $f : \Sigma \to N$ defined on a given closed oriented Riemannian surface. Use the axioms to show that this integer depends only on the homology class of f and so the first Chern number generalizes to an integral cohomology class $c_1(E) \in \mathrm{H}^2(N)$ called the **first Chern class** of E.

The **first Chern class of a symplectic manifold**, denoted by $c_1(M, \omega)$ or just by $c_1(M)$, is the first Chern class of the tangent bundle $E = TM$.

Exercise 2.2.9 (Splitting Lemma[14])**.** Every symplectic vector bundle E over a closed oriented Riemannian surface Σ decomposes as a direct sum of rank-2 symplectic vector bundles.
[Hint: View E as complex vector bundle with $\mathbb{C}$-dual E^*, so $c_1(E) = -c_1(E^*) = -c_1(\Lambda^n E^*) = c_1((\Lambda^n E^*)^*)$; cf. [GH78, p.414]. Remember `(naturality)`.]

Exercise 2.2.10 (Lagrangian subbundle)**.** Suppose $E \to \Sigma$ is a symplectic vector bundle over a closed oriented Riemannian surface. If E admits a Lagrangian subbundle L, then the first Chern number $c_1(E) = 0$ vanishes. (Consequently the vector bundle E is unitarily trivial by Exercise 2.2.7.)
[Hint: The unitary trivializations (2.2.9) identify Lagrangian subspaces. Modify them so that each Lagrangian in L gets identified with the horizontal Lagrangian $\mathbb{R}^n \times 0$. Then the overlap map Ψ will be of the form (2.1.3) with $Y = 0$, so the determinant is real and the degree therefore zero.]

2.3 Hamiltonian trajectories

2.3.1 Loops – continuous, immersed, simple

Suppose N is a manifold.

Paths, periods, loops – continuous case

Definition 2.3.1 (Paths and curves, closed, simple, constant)**.** A **path** is a *continuous* map of the form $\gamma : \mathbb{R} \to N$ and its image $\mathfrak{c} \subset N$ is called a **curve** in N. A **finite path** is a continuous map $\alpha : [a, b] \to N$ defined on a compact interval. Its image $\mathfrak{c}$ is a **compact curve**. Note: A finite path is not a path.

A finite path $\alpha : [a, b] \to N$ is called **closed** if $\alpha(a) = \alpha(b)$ and its image is called a **closed curve**. A path β, finite or not, is called a **simple** if it does not

[14] There is a far more general theory behind called *splitting principle*; see e.g. [BT82, §21].

admit self-intersections $\beta(t) = \beta(s)$ at times $t \neq s$ in the interior of the domain. The image of a simple path is called a **simple curve**. A **constant path** is a path whose image is a point, its image is a **constant curve**. The image of a **point path** $\alpha : [a, a] \to N$ is called a **point curve**.

Of course, a constant path is not simple – unless it is a point path, of course..

Definition 2.3.2 (Paths, periodic and non-periodic)**.** Consider a path $\gamma : \mathbb{R} \to N$. If there is a real $\tau \neq 0$ such that $\gamma(\tau + \cdot) = \gamma(\cdot)$, then γ is called a **periodic path** and τ **a period** of γ. One also says that the path γ is $\boldsymbol{\tau}$**-periodic**. If there is no such $\tau \neq 0$, then γ is called **non-periodic**. By definition $\tau = 0$ is considered a period of any path, called the **trivial period**. Let $\operatorname{Per}(\gamma)$ be the **set of all periods** of γ, including the trivial period 0.

Observe that $\operatorname{Per}(\gamma) = \{0\}$ iff γ is a non-periodic path and $\operatorname{Per}(\gamma) = \mathbb{R}$ iff γ is a constant path. Do not confuse *closed finite path* with *periodic path* – the domains $[a, b]$ and $\mathbb{R}$ are different. However, a closed finite path $\alpha : [0, b] \to N$ naturally comes with an **associated** $\boldsymbol{b}$**-periodic path**

$$\alpha^{\#} : \mathbb{R} \to N, \quad t \mapsto \alpha(t \bmod b), \qquad t \bmod 0 := 0.$$

Vice versa, a non-constant τ-periodic loop $\gamma : \mathbb{R} \to N$ is the infinite concatenation of the closed finite paths $\alpha_k = \gamma| : [k\tau, (k+1)\tau] \to N$, $k \in \mathbb{Z}$.

Exercise 2.3.3. Given a path $\gamma : \mathbb{R} \to N$, show that $\operatorname{Per}(\gamma)$ is a closed subgroup of $(\mathbb{R}, +)$. Under the convention $\inf \emptyset = \infty$ define the **minimal period**

$$\tau_\gamma := \inf\{\tau \in \operatorname{Per}(\gamma) \mid \tau > 0\} \in [0, \infty].$$

Show that the period group $\operatorname{Per}(\gamma)$ of a path comes in three flavors, namely

$$\operatorname{Per}(\gamma) = \begin{cases} \{0\} & \text{, non-periodic path } (\tau_\gamma = \infty), \\ \mathbb{R} & \text{, constant path } (\tau_\gamma = 0), \\ \tau_\gamma \mathbb{Z} & \text{, periodic path of } \textbf{prime period } \tau_\gamma > 0. \end{cases} \tag{2.3.10}$$

[Hint: Consult [PP09, Prop. 1.3.1] if you get stuck.] If the period groups of two paths are equal, will they in general become equal after suitable time shift?

Definition 2.3.4 (Divisor parts and corresponding loops)**.** A **divisor part** of a periodic path $\gamma : \mathbb{R} \to N$ is a closed finite path of the form

$$\gamma_\tau : [0, |\tau|] \to N, \quad t \mapsto \begin{cases} \gamma(t) & , \tau > 0, \\ \gamma(0) & , \tau = 0, \\ \hat{\gamma}(t) := \gamma(-t) & , \tau < 0, \end{cases} \tag{2.3.11}$$

one for each period $\tau \in \operatorname{Per}(\gamma)$. The corresponding map on the quotient[15]

$$\gamma_\tau : \mathbb{R}/\tau\mathbb{Z} \to N, \quad [t] \mapsto \begin{cases} \gamma(t) & , \tau > 0, \\ \gamma(0) & , \tau = 0, \\ \hat{\gamma}(t) = \gamma(-t) & , \tau < 0, \end{cases} \tag{2.3.12}$$

[15] By u_τ we also denote "freezing the variable τ", but application context should be different.

is called the **loop associated to the period** $\tau \in \mathrm{Per}(\gamma) = \tau_\gamma\mathbb{Z}$ of the path γ. A **loop** is a map of the form (2.3.12). (We often simply write γ.) For a non-constant periodic path divisor part and loop associated to τ_γ are denoted by

$$\gamma_{\mathrm{prime}} : [0, \tau_\gamma] \to N, \quad \gamma_{\mathrm{prime}} : \mathbb{R}/\tau_\gamma\mathbb{Z} \to N, \qquad t \mapsto \gamma(t), \tag{2.3.13}$$

and called the **prime part** and the **prime loop** of the path $\gamma : \mathbb{R} \to N$, respectively. It is also useful to call the loop γ_{prime} the **prime loop** of any loop γ_τ with $\tau \in \mathrm{Per}(\gamma)$. Observe that $\gamma_{k\tau_\gamma} : \mathbb{R}/k\tau_\gamma\mathbb{Z} \to N$ is a k-fold cover of γ_{prime}. A **simple loop** is an injective prime loop γ_{prime}, equivalently, the finite path $\gamma_{\mathrm{prime}} : [0, \tau_\gamma] \to N$ must be simple.

Exercise 2.3.5. Find a path whose prime part is not simple. Show that a simple loop is a homeomorphism from a circle to its image.

Do not confuse *prime period* of a path with the **time of first return**, namely $\gamma(T) = \gamma(0)$ but $\gamma(t) \neq \gamma(0)$ at earlier times $t \in (0, T)$; just think about a figure eight with $\gamma(0)$ being the crossing point. For trajectories of smooth *autonomous* vector fields both notions coincide, prime parts are automatically simple, and prime loops are circle embeddings; cf. Exercise 2.3.19! For a periodic *immersion* $\gamma : \mathbb{R} \to N$, thus non-constant, an associated loop is simple iff it is an embedding.

Remark 2.3.6 (Negative periods)**.** Consider a path $\gamma : \mathbb{R} \to N$. Note that for negative periods τ divisor parts (2.3.11) and associated loops (2.3.12) *run backwards*, more precisely they follow the **time reversed path**

$$\hat{\gamma}(t) := \gamma(-t), \quad t \in \mathbb{R}, \qquad \hat{\tau} := -\tau.$$

Certainly γ is τ-periodic iff $\hat{\gamma}$ is $\hat{\tau}$-periodic and $\mathrm{Per}(\hat{\gamma}) = \mathrm{Per}(\gamma)$.

Definition 2.3.7 (Concatenation of finite paths and loops)**.** (i) Consider two **consecutive** finite paths, that is $\alpha : [a, b] \to N$ and $\beta : [b, c] \to N$ such that α ends at the point at which β begins.[16] The **concatenation** of two consecutive finite paths is defined by following first α and then β, notation

$$\beta\#\alpha : [a, c] \to N.$$

In particular, closed finite paths can be self-concatenated. Suppose $\alpha : [0, b]$ with $b > 0$ is a closed finite path and $k \in \mathbb{Z}$ is an integer. Consider the closed finite path given by the ***k*-fold concatenation** $\alpha\#\ldots\#\alpha : [0, |k|b] \to N$, traversed backwards in case $k < 0$; cf. (2.3.11). Denote by

$$\alpha^{\#k} : \mathbb{R}/kb\mathbb{Z} \to N \tag{2.3.14}$$

the **associated *kb*-periodic loop**; mind convention (2.3.12) if $k \leq 0$. To $k = 0$ associate the point path $[0, 0] \to \alpha(0)$ and the constant loop $\alpha^{\#0} \equiv \alpha(0)$.

[16] If the domains are $[a, b]$ and $[c, d]$ replace β by $\tilde{\beta} : [b, b + d - c] \ni t \mapsto \beta(t - b + c)$.

(ii) In case γ_τ is a loop with period $\tau \neq 0$, use in (2.3.14) the closed finite path α given by the divisor part $\gamma_\tau : [0, |\tau|] \to N$, cf. (2.3.11), to get the loop

$$\gamma_\tau^{\# k} : \mathbb{R}/k\tau\mathbb{Z} \to N$$

of period $k\tau$. It is a ***k*-fold cover** of the τ-periodic loop γ_τ. In particular $\gamma_{\tau_\gamma}^{\# k} = \gamma_{\mathrm{prime}}^{\# k} : \mathbb{R}/k\tau_\gamma\mathbb{Z} \to N$ is a k-fold cover of the prime loop of the loop γ_τ.

Definition 2.3.8 (Time shift and uniform change of speed)**.** Certainly if a path $\gamma : \mathbb{R} \to N$ is τ-periodic, then so is any **time shifted** path

$$\gamma_{(T)} := \gamma(T + \cdot), \qquad T \in \mathbb{R}, \qquad \mathrm{Per}(\gamma_{(T)}) = \mathrm{Per}(\gamma) = \tau_\gamma \mathbb{Z}.$$

The operation **uniform change of speed** applied to a loop γ, namely

$$\gamma^\mu := \gamma(\mu \cdot), \quad \mu \in \mathbb{R}, \qquad \text{where } \gamma^0 \equiv \gamma(0), \tag{2.3.15}$$

changes the prime period by $\tau_{\gamma^\mu} = \frac{1}{\mu} \cdot \tau_\gamma$ for $\mu \in \mathbb{R} \setminus \{0\}$ and $\tau_{\gamma^0} = 0$.

Remark 2.3.9 (In/compatibilities)**.** The operation of k-fold self-concatenation $\gamma_\tau^{\# k}$ of a τ-periodic loop γ_τ is compatible with ODEs and also preserves periods in the sense that τ is still a period after the operation. Period preservation also holds true for uniform *integer* speed changes γ^k, but these do in general not map ODE solutions to solutions for $k \neq 1$.

Remark 2.3.10 (Loops and periods – immersed case)**.** An **immersion** is a smooth map whose differential is injective at every point. Starting from Definition 2.3.1 redo all definitions and constructions replacing *continuous path* by *immersed path* and investigate if and how things change.

Remark 2.3.11 (Loops and periods – embedded case)**.** Given only immersed paths $\gamma : \mathbb{R} \to N$, a loop is an *embedding* iff it is simple, in which case it is prime (but prime is not sufficient). Investigate if and how the previous constructions change on the space of embedded loops. A crucial observation is that, given a non-constant periodic trajectory of an *autonomous* smooth vector field on N, an associated loop is embedded iff it is prime, as there are no self-intersections.

2.3.2 Hamiltonian flows

Throughout let (M, ω) be a symplectic manifold.

Autonomous Hamiltonians *F*

Given a function $F : M \to \mathbb{R}$, by non-degeneracy of ω the identity of 1-forms

$$dF = -i_{X_F}\omega := -\omega(X_F, \cdot) \tag{2.3.16}$$

determines a vector field $X_F = X_F^\omega$ on M, called the **Hamiltonian vector field** associated to H or the **symplectic gradient** of H. The function F is called the

Hamiltonian of the dynamical system (M, X_F), it is also called **autonomous** since it does not depend on time. For ω-compatible almost complex structures J the Hamiltonian vector field is given by

$$X_F = J\nabla F \tag{2.3.17}$$

where the **gradient** ∇F is taken with respect to the induced Riemannian metric, that is ∇F is determined by $dF = g_J(\nabla F, \cdot)$. We denote the flow generated by the Hamiltonian vector field of an *autonomous* Hamiltonian by $\phi = \{\phi_t\}$, alternatively by $\phi^F = \{\phi_t^F\}$, as opposed to the greek letter $\psi = \{\psi_t\}$ used in case of non-autonomous Hamiltonians which are usually denoted by $H = H_t$.

An **energy level** is a pre-image $F^{-1}(c) \subset M$ of an autonomous Hamiltonian. It is called an **energy surface** if c is a regular value of F, notation $S = F^{-1}(c)$. In particular, energy surfaces S contain no singularities, that is zeroes, of X_F and by the regular value theorem they are smooth codimension one submanifolds of M. Most importantly, the Hamiltonian flow preserves its energy levels:

$$\frac{d}{dt}F(\phi_t^F p) = 0, \qquad F : M \to \mathbb{R}, \tag{2.3.18}$$

for every initial condition $p \in M$.

Exercise 2.3.12. Show that (2.3.16) determines X_F uniquely. Prove that X_F is tangent to energy surfaces and, slightly more general, prove (2.3.18).

Remark 2.3.13 (Closed orbit versus image of closed finite trajectory)**.**

a) Suppose the flow of X_F is complete, that is $\phi = \{\phi_t\}_{t\in\mathbb{R}}$. The solution path $z(t) := \phi_t z_0$, $t \in \mathbb{R}$, of $\dot z = X_F(z)$ with $z(0) = z_0$ is called a **Hamiltonian path** or a **flow trajectory**. It is either an immersed line $z : \mathbb{R} \looparrowright M$ (embedded: $z(\mathbb{R}) \cong \mathbb{R}$, or self-tangent: $z(\mathbb{R}) \cong \mathbb{S}^1$, but not self-transverse) or it is constant: $z(\mathbb{R}) = \{\mathsf{pt}\}$. The image $\mathfrak{c}$ of a flow trajectory $z : \mathbb{R} \to M$ is called a **flow line** or an **orbit** in M. In case the solution path forms a loop $z : \mathbb{R}/\tau\mathbb{Z} \to M$ we call it a **Hamiltonian loop** and its image $\mathfrak{c} = z(\mathbb{R})$ a **closed orbit**.
Thus a closed orbit is either an *embedded* circle or a point, in fact, with respect to any *autonomous* vector field X,

b) For a non-constant trajectory z of X_F the prime period τ_z as a loop is the **time of first return** $z(T) = z(0)$ and $z(t) \neq z(0)$ at all earlier times $t \in (0, T)$. Thus the loop prime period τ_z is also called the trajectory prime period.

c) For a time-dependent vector field X_t on a manifold N the non-constant trajectories are still immersions, but now they can have self-crossings: A trajectory $\gamma : \mathbb{R} \to M$ might close up at some time $\gamma(t_*) = \gamma(0)$ without being t_*-periodic, even having a corner; think about the figure eight curve. In this case $\gamma : [0, t_*] \to N$ is just a closed finite path that solves the equation, but its image has nothing to do with an orbit.[17] In general there is no first return condition $z(T) = z(0)$ – even when requiring equality $\dot z(T) = \dot z(0)$ of first or higher order derivatives – that can guarantee that γ is a loop, let alone that T determines

[17] An orbit is the image of a *trajectory*, but the domain of such is $\mathbb{R}$.

the loop prime period. However, if γ is a periodic trajectory, then its image is either an immersed circle or a point. The vector field being autonomous, or not, the image of a periodic trajectory is called a **closed orbit**.

d) The running-at-k-fold-speed operation z^k, see (2.3.15), that produces new loops from a non-constant trajectory z does not produce new trajectories in general, as z^k might not satisfy the ODE any more.

Remark 2.3.14. Given $F : M \to \mathbb{R}$, note that z is a τ-periodic trajectory of F iff $z^{1/\tau} = z(\cdot/\tau)$ is a 1-periodic trajectory of $\tau^{-1}F$. So while changing speed maps loops to loops, it does not map F-trajectory to F-trajectory, in general.

Remark 2.3.15 (Multiple cover problem – variable period)**.** This doesn't refer to change of speed, but to path concatenation: Given a τ-periodic trajectory $z : \mathbb{R} \to M$, consider that same map instead of on $[0, \tau]$ on the larger domain $[0, k\tau]$, $k \in \mathbb{N}$, to get a periodic *trajectory* k times covering z – *same speed* but k-fold time.

Proposition 2.3.16 (C^1 and C^2 small Hamiltonians, [HZ11, §6.1])**.** *Suppose M is a closed symplectic manifold. Sufficiently C^1 small Hamiltonians $H : \mathbb{S}^1 \times M \to \mathbb{R}$ do not admit* non-contractible $\underline{1}$*-periodic trajectories. Sufficiently C^2 small autonomous Hamiltonians $F : M \to \mathbb{R}$ do not admit $\underline{1}$-periodic trajectories at all – except the constant ones sitting at the critical points.*

Idea of proof. Pick an ω-compatible almost complex structure to conclude that the length of a periodic trajectory z of period one is small if the Hamiltonian is C^1 small, autonomous or not. Indeed

$$\text{length}(z) = \int_0^1 |\dot{z}(t)|\, dt = \int_0^1 |\nabla H_t(z(t))|\, dt.$$

But a short loop z in a compact manifold is contractible and its image is covered by a Darboux chart. For autonomous $F : M \to \mathbb{R}$ the argument on page 185 in [HZ11] shows that $\dot{z} = 0$ whenever the Hessian of F is sufficiently small. □

Non-autonomous Hamiltonians *H*

A time dependent Hamiltonian $H : \mathbb{R} \times M \to \mathbb{R}$, notation $H_t(x) := H(t, x)$, generates a time dependent Hamiltonian vector field $X_t := X_{H_t}$ by considering (2.3.16) for each time t. One obtains a family $\psi_t = \psi_t^H$ of symplectomorphisms[18] on M, called the **Hamiltonian flow** generated by H, via

$$\frac{d}{dt}\psi_{t,0} = X_t \circ \psi_{t,0}, \qquad \psi_{0,0} = \text{id}, \qquad \psi_t := \psi_{t,0}. \tag{2.3.19}$$

The family[19] $\psi = \{\psi_t\}$ is called a **complete flow** if it exists for all $t \in \mathbb{R}$. Important examples are autonomous Hamiltonians F and periodic in time Hamiltonians $H_{t+1} \equiv H_t$, both on closed manifolds. A **Hamiltonian trajectory**,

[18] A **symplectomorphism** is a diffeomorphism preserving the symplectic form: $\psi^*\omega = \omega$.

[19] If X_t depends on time, it is wise to keep track of the initial time t_0. As indicated in (2.3.19) we shall always use $t_0 = 0$. The notation $\psi_{t,0}$ helps to remember that $\psi_{t+s,0}$ is in general not a composition of $\psi_{s,0}$ and $\psi_{t,0}$. To obtain the composition law $\psi_{t,s}\psi_{s,r} = \psi_{t,r}$ one would have to allow for variable initial times, not just $t_0 = 0$. For simplicity $\psi_{t,0} =: \psi_t$.

is a path of the form $z(t) = \psi_t p$ with $p \in M$. In case z is a loop we call it a **Hamiltonian loop**. In either case z satisfies the **Hamiltonian equation**

$$\dot{z}(t) = X_t(z(t)), \qquad z(0) = p.$$

Hamiltonian flows, autonomous or not, preserve the symplectic form. By definition[20] of the **Lie derivative** and **Cartan's formula** one gets

$$\frac{d}{dt}\psi_t^*\omega =: \psi_t^*\left(\mathcal{L}_{X_t}\omega\right) = \psi_t^*\left(i_{X_t}d\omega + di_{X_t}\omega\right). \tag{2.3.20}$$

This shows that the family of diffeomorphisms ψ_t generated by the family of vector fields X_t preserves ω, that is $\psi_t^*\omega = \omega$, if and only if the 1-form $i_{X_t}\omega$ is closed.[21] This holds, for instance, if X_t is Hamiltonian ($di_{X_t}\omega = ddH = 0$).

Periodic trajectories and their loop types

Remark 2.3.17 (Loops, periodic trajectories, closed characteristics)**.**

Topology. A non-constant loop $\gamma : \mathbb{R}/\tau\mathbb{Z} \to N$ is a *simple*, thus *prime*, loop if it admits no self-intersections, in symbols $\gamma^{-1}(\gamma(t)) = \{t\}\ \forall t$. Given two non-constant loops γ and $\tilde{\gamma}$, if $\tilde{\gamma} = \gamma(k\cdot)$ for some integer k, one says that γ is ***k*-fold covered** by $\tilde{\gamma}$, or a **multiply covered loop** in case $|k| > 1$, in symbols $\gamma^k = \tilde{\gamma}$. Two loops γ and $\tilde{\gamma}$ are called **geometrically distinct** if their images are not equal as sets. Otherwise, they are **geometrically equivalent**, in symbols $\gamma \sim \tilde{\gamma}$. Geometrically equivalent loops, although having the same image set, certainly can be very different as maps. For instance, subloops of a figure-eight can be traversed a different number of times or in a different order.

Analysis. Whereas any loop in a manifold N is a periodic trajectory of some periodic vector field X_t, only the rather restricted class of *embedded* loops arises as (prime parts of) periodic trajectories of autonomous vector fields; see Exercise 2.3.18. By (2.3.10) the prime part of a periodic trajectory of an autonomous vector field X is either constant, an embedded loop or self-concatenations of such (same speed but k-fold life time). Moreover, in the autonomous case two periodic trajectories $z, \tilde{z}$ are geometrically distinct iff their images are disjoint, they are geometrically equivalent iff one k-fold covers the other one.

Geometry. Suppose X is an autonomous vector field on a manifold N. Let $\mathcal{P}_{\rm all}(X)$ be the set of loop trajectories, whatever (finite) period, and $\mathcal{P}^*_{\rm all}(X)$ the non-constant ones. The set of equivalence classes

$$\mathcal{C}(X) := \mathcal{P}^*_{\rm all}(X)/\sim \tag{2.3.21}$$

represents the geometrically distinct non-constant closed orbits of X. Different elements are disjoint embedded circles tangent to X. Representatives y, z of the same element of $\mathcal{C}(X)$ are multiple covers of a common simple trajectory x. In other words, the set $\mathcal{C}(X)$ corresponds to the integral submanifolds of the

[20] Alternatively defining $\mathcal{L}_X$ axiomatically, the definition becomes Thm. 2.2.24 in [AM78].

[21] Such vector fields are called **symplectic**, generalizing the Hamiltonian ones.

vector field X which are diffeomorphic to $\mathbb{S}^1$. These are called the **closed characteristics** of the (autonomous) vector field X.

In Chapters 4 and 5 we will deal with the following special case: The manifold is a closed regular level set $S := F^{-1}(c)$ of an autonomous Hamiltonian F on a symplectic manifold (M, ω) and $X = X_F$ is the Hamiltonian vector field. Note that in this case there are no zeroes of X_F, equivalently of dF, on $F^{-1}(c)$ by regularity of the value c. Furthermore, whenever S is a regular level set of both Hamiltonians F and K, then $X_F = f X_K$ along S for some non-vanishing function f on S; cf. Exercise 4.1.8. Thus we use the notation

$$\mathcal{C}(S) = \mathcal{C}(S, \omega) := \mathcal{C}(X_F|_S) \tag{2.3.22}$$

for the set of closed characteristics on the closed regular level set $S \subset (M, \omega)$.

Exercise 2.3.18 (Loops are generated by vector fields)**.**
a) A loop z in N is the trajectory of some periodic vector field $X_{t+\tau} = X_t$.
b) An embedded loop z in N is a trajectory of some autonomous vector field X.
[Hint: First case $N = \mathbb{R}^k$, graph of z in $[0,1] \times \mathbb{R}^k$, cutoff functions.]

Exercise 2.3.19 (Periodic geodesics are self-transverse, but not self-tangent)**.** Let N be a Riemannian manifold with Levi-Civita connection ∇. Let $\gamma : \mathbb{R} \to N$ be a non-trivial **periodic geodesic**, that is γ is non-constant periodic and satisfies $\nabla_t \dot\gamma = 0$. There are precisely two options. Such γ is

1) either multiply covered or

2) self-transverse and $\gamma_{\mathrm{prime}} : \mathbb{R}/\tau_\gamma\mathbb{Z} \to N$ is called **prime closed**.

Self-transverse means that if two arcs of γ meet in M they intersect transversely, so there is just a finite number of intersection points by compactness of $\mathbb{S}^1$.

a) Show that the two options are characterized by the two possibilities whether the set $\mathcal{T}$ of times t_0 such that $\gamma(t_0)$ has more than one pre-image[22] under γ is an infinite set (thus equal to $\mathbb{S}^1$ itself) or a finite set.

b) Why are there no self-intersections of trajectories of autonomous vector fields, but for geodesics they can appear?
[Hint: Either γ admits a self-tangency (mind: ODE of 2^{nd} order), or not.]

2.3.3 Conley-Zehnder index of periodic trajectories

Given a symplectic manifold (M, ω), consider a 1-periodic family of Hamiltonians $H_{t+1} = H_t : M \to \mathbb{R}$ with Hamiltonian flow $\psi_t = \psi_{t,0}$. Let $\mathcal{P}(H)$ be the **set of** 1**-periodic trajectories**.

Exercise 2.3.20. Check that $\mathcal{P}(H) \to \operatorname{Fix}\psi_1$, $z \mapsto z(0)$, provides a bijection between the set of 1-periodic trajectories and the set of fixed points of the time-1-map corresponding to initial time zero. [Hint: Recall that ψ_1 abbreviates $\psi_{1,0}$.]

A 1-periodic trajectory z is called **non-degenerate** if 1 is not an eigenvalue of the linearized time-1-map, that is

$$\det\left(d\psi_1(p) - \mathbb{1}\right) \neq 0, \qquad p := z(0). \tag{2.3.23}$$

[22] In symbols $|\gamma^{-1}(\gamma(t_0))| \geq 2$. Such $\gamma(t_0)$ is called a **multiple** or a **double** $(= 2)$ **point**.

Exercise 2.3.21. Show that condition (2.3.23) implies that p is an isolated fixed point of ψ_1. Vice versa, would isolatedness imply (2.3.23)?
[Hint: Condition (2.3.23) means that the graph of ψ_1 in the product manifold $M \times M$ is transverse at p to the **diagonal** $\Delta := \{(p,p) \mid p \in M\}$.]

Exercise 2.3.22 (Finite set)**.** If the manifold M is closed and all 1-periodic trajectories are non-degenerate, then the set $\operatorname{Fix}\psi_1$, hence $\mathcal{P}(H)$, is a finite set.

In addition to non-degeneracy, suppose the loop trajectory $z : \mathbb{S}^1 = \partial\mathbb{D} \to M$ is contractible. Fix an extension of z, namely a smooth map $v : \mathbb{D} \to M$ that coincides with z on $\partial\mathbb{D}$. Moreover, pick an auxiliary ω-compatible almost complex structure $J \in \mathcal{J}(M,\omega)$, so the Hermitian vector bundle (E, ω, J, g_J) with $E = v^*TM \to \mathbb{D}$ admits a unitary trivialization Φ_v by Proposition 2.2.2, that is Φ_v identifies the compatible triples $(\omega_0, J_0, \langle\cdot,\cdot\rangle_0)$ and (ω, J, g_J) where J_0 rotates counter-clockwise and corresponds to i. Restriction to the boundary $\mathbb{S}^1$ provides a unitary trivialization, say Φ_z, of the pull-back bundle $z^*TM \to \mathbb{S}^1$. These choices provide a symplectic path $\Psi_{z,v} \in \mathcal{SP}^*(2n)$ defined by

$$\begin{tikzcd} \mathbb{R}^{2n} \arrow[r, dashed, "\Psi_{z,v}(t)"] & \mathbb{R}^{2n} \\ T_{z(0)}M \arrow[u, "\Phi_v(z(0))^{-1}"] \arrow[r, "d\psi_t(z(0))"] & T_{z(t)}M \arrow[u, "\Phi_v(z(t))"'] \end{tikzcd} \tag{2.3.24}$$

The **standard** and the **canonical Conley-Zehnder indices of the non-degenerate 1-periodic trajectory** z are defined and related by

$$\underline{\mu^{\mathrm{CZ}}(z) := \mu^{\mathrm{CZ}}(\Psi_{z,v})} = -\mu_{\mathrm{CZ}}(\Psi_{z,v}) =: -\mu_{\mathrm{CZ}}(z). \tag{2.3.25}$$

In general these indices depend on the spanning disk v, unless $c_1(M)|_{\pi_2(M)} = 0$. Sometimes it is useful to denote $\mu^{\mathrm{CZ}}(z)$ by $\mu^{\mathrm{CZ}}(z;H)$ or even by $\mu^{\mathrm{CZ}}(z;H,\omega)$.

Exercise 2.3.23. Show that $\Psi_{z,v}(t) \in \mathrm{Sp}(2n)$ and that $\Psi_{z,v}(1) \in \mathrm{Sp}^*$. Show that $\mu^{\mathrm{CZ}}(\Psi_{z,v})$ does not depend on the particular unitary trivialization Φ_z. Show that $\mu^{\mathrm{CZ}}(z)$ is independent of the choice of v if $c_1(M)$ vanishes on $\pi_2(M)$.

Exercise 2.3.24 (Critical points are periodic trajectories)**.** Suppose z_0 is a non-degenerate critical point of a time independent function $H : M \to \mathbb{R}$. Then the constant periodic trajectory $t \mapsto z_0$ is non-degenerate and the canonical Conley-Zehnder index and the Morse index of z_0, see Section 3.1.1, are related by

$$\begin{aligned} \mu^{\mathrm{CZ}}(z_0) :&= \mu^{\mathrm{CZ}}\left(\Psi_{z_0} : t \mapsto e^{-tJ_0St}\right) = \frac{1}{2}\operatorname{sign}(S) \\ &= n - \operatorname{ind}_H(z_0) = \operatorname{ind}_{-H}(z_0) - n \end{aligned} \tag{2.3.26}$$

whenever $\|S\| < 2\pi$ and where $\dim M = 2n$. To obtain the first displayed formula, pick an ω-compatible almost complex structure J, the induced metric g_J, and an orthonormal basis of eigenvectors of the Hessian of H at z_0 and denote the corresponding Hessian matrix by S.[23] Apply the axiom $\texttt{(signature)}_{\texttt{can}}$.

[23] One has $X_H^\omega = J\nabla H$ for (M,ω,J,g_J), but $X_K^{\omega_0} = -J_0\nabla K$ for $(\mathbb{R}^{2n},\omega_0,J_0,\langle\cdot,\cdot\rangle_0$.

2.4 Cotangent bundles

Cotangent bundles are the phase spaces in the Hamiltonian formulation of classical mechanics. Given the tremendous success of the theory in physics, not to mention daily life, one wouldn't risk much predicting that these bundles should have a distinct position in the mathematical world as well. Indeed

Cotangent bundles $\pi : T^*N \to N$ over a manifold

(S1) admit a **canonical symplectic form** and a canonical 1-form given by[24]

$$\omega_{\mathrm{can}} = d\lambda_{\mathrm{can}} = \text{"}dp \wedge dq\text{"}, \qquad \begin{aligned}\lambda_{\mathrm{can}}(z) : T_zT^*N &\to \mathbb{R},\\ \zeta &\mapsto z \circ d\pi(z)\zeta\end{aligned}$$

The **Liouville form** λ_{can} is characterized by the property that

$$\sigma^*\lambda_{\mathrm{can}} = \sigma \tag{2.4.27}$$

for every 1-form $\sigma \in \Omega^1(N)$, so one calls λ_{can} the **tautological 1-form**;

(S2) admit a canonical Lagrangian subbundle of the tangent bundle, namely[25]

$$V := \ker d\pi \subset T(T^*N);$$

(S3) admit *along the zero section* $\mathcal{O}_N = \mathfrak{o}(N)$ a natural Lagrangian splitting[26]

$$\begin{aligned} T_{\mathcal{O}_N}T^*N = \overbrace{\operatorname{im} d\mathfrak{o}}^{=:H} \oplus \overbrace{\ker d\pi}^{=:V} &\xrightarrow{\cong} TN \oplus T^*N.\\ (h, v) &\mapsto (w_h, \theta_v) \end{aligned}$$

The isomorphisms[27] $w : \operatorname{im} d\mathfrak{o} \to TN$ and $\theta : \ker d\pi \to T^*M$ are determined in terms of the inclusion $\mathfrak{o} : N \hookrightarrow T^*N$, $q \mapsto (q, 0)$, by

$$d\mathfrak{o}(w_h) = h, \qquad \theta_v(\cdot) := \omega_{\mathrm{can}}(v, d\mathfrak{o}\cdot),$$

and $\mathfrak{o}^*\omega_{\mathrm{can}} = (w, \theta)^*\Omega_{\mathrm{can}}$ where

$$\Omega_{\mathrm{can}}(w_h \oplus \theta_v, w_{h'} \oplus \theta_{v'}) := \theta_v(w_{h'}) - \theta_{v'}(w_h);$$

(S4) have trivial first Chern class in the sense that $c_1(T_QT^*N) = 0$ for every oriented closed[28] submanifold Q. If $Q = N$, then $c_1(TQ \oplus T^*Q) = 0$.[29]

---

[24] Local coordinates $\varphi : U \to \mathbb{R}^n$ on N induce the diffeomorphism $T^*\varphi : T^*U \to \varphi(U) \times \mathbb{R}^n$ taking $z = (q, p)$ to $(x, y) := (\varphi(q), (d\varphi(q)^{-1})^*p) \in \mathbb{R}^{2n}$ and identifying ω_{can} with $\omega_{\mathrm{can}}(\mathbb{R}^{2n})$.

[25] The previous coordinates identify each V_z symplectically with the Lagrangian $0 \times \mathbb{R}^n$.

[26] $T_{\mathcal{O}_N}T^*N = H \oplus V$ is a direct sum: Linearize the composition $\pi \circ \mathfrak{o} = \mathrm{id} : N \to T^*N \to N$ of injection and surjection to get $H \cap V = \{0\}$. So $H + V = T_{\mathcal{O}_N}T^*N$ since ranks add up.

[27] It suffices to show either surjectivity or injectivity (equal dimension of domain/codomain). As $d\mathfrak{o}$ is injective it is an isomorphism $d\mathfrak{o} : TN \to \operatorname{im} d\mathfrak{o}$ onto its image with inverse w. Assuming $\theta_v \equiv 0$ with $v \in V$ means $v \in H^\omega$. So $v \in V \cap H = \{0\}$, as $H = H^\omega$ is Lagrangian: The restriction $\mathfrak{o}^*\omega_{\mathrm{can}} = d\mathfrak{o}^*\lambda_{\mathrm{can}} = d\mathfrak{o}$ is zero as $\mathfrak{o} \in \Omega^1(N)$ is the zero section.

[28] The argument involves Poincaré duality. Closedness: *Push the obstruction to infinity.*

[29] If $Q = N = \Sigma$ use Exercise 2.2.10. In general, let $\tilde{E}^*$ be the $\mathbb{C}$-dual of the vector bundle $\mathbb{C}^n \hookrightarrow \tilde{E} = T_QT^*N \to Q$. Then $c_1(\tilde{E}) = -c_1(\tilde{E}^*) = -c_1(\Lambda^n\tilde{E}^*) = 0 \in \mathrm{H}^2(Q)$ by [GH78, p.414] and as the restriction of $\omega_{\mathrm{can}}^{\wedge n}$ to Q is a non-vanishing section; cf. [Web99, App. B.1.7].

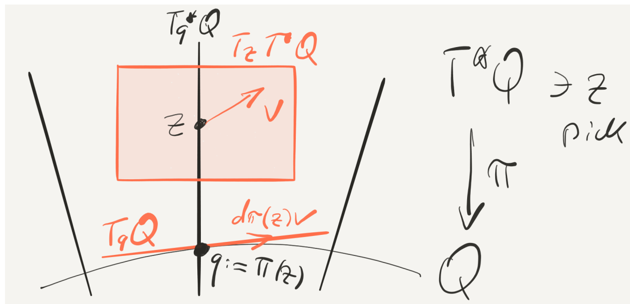

Figure 2.6: Liouville 1-form $\lambda_{\mathrm{can}} \in \Omega^1(T^*Q)$ on cotangent bundle $\pi : T^*Q \to Q$

To get in all the other structures introduced in earlier sections is rather simple:

Pick a Riemannian metric g on N to obtain

(H1) a *global* Lagrangian splitting $T(T^*N) \cong \pi^* (TN \oplus T^*N)$ defined pointwise by the isomorphism which takes the derivative of a curve $t \mapsto z(t) = (x(t), y(t))$ in T^*N to the pair of derivatives, namely

$$T_{z(t)}T^*N \xrightarrow{\cong} T_{x(t)}N \oplus T^*_{x(t)}N, \quad \dot{z}(t) \mapsto (\dot{x}(t), \nabla_t y(t)) .$$

The isomorphism takes ω_{can} to Ω_{can} extending the one in (S3);

(H2) a canonical Hermitian structure $(\omega_{\mathrm{can}}, \bar{J}_g, G_g)$ on $E = T(T^*N)$ defined pointwise for $z = (x, y)$ on $T_xN \oplus T^*_xN$ by

$$\bar{J}_g := \begin{pmatrix} 0 & g^{-1} \\ -g & 0 \end{pmatrix}, \qquad G_g := \begin{pmatrix} g & 0 \\ 0 & g^* \end{pmatrix}. \tag{2.4.28}$$

Concerning notation see Exercise 2.1.15. Throughout we denote by g not only the Riemannian metric on N, but also the induced isomorphism

$$g : TN \to T^*N, \quad v \mapsto g(v, \cdot).$$

Exercise 2.4.1. Prove properties (S1–S4) and (H1–H2).[30]

Exercise 2.4.2. Any diffeomorphism $\psi : N \to N$ of a manifold lifts to a symplectomorphism of the cotangent bundle

$$\begin{tikzcd} (T^*N, \omega_{\mathrm{can}}) \arrow[r, dashed, "\Psi := T^*\psi", "\cong"'] \arrow[d, "\pi"'] & (T^*N, \omega_{\mathrm{can}}) \arrow[d, "\pi"] \\ N \arrow[r, "\psi", "\cong"'] & N \end{tikzcd}$$

defined by $\Psi(q, p) = \big(\psi(q), (d\psi(q)^{-1})^* p\big)$. Prove that $\Psi^*\lambda_{\mathrm{can}} = \lambda_{\mathrm{can}}$. What can one say if (N, g) is a Riemannian manifold and ψ is an **isometry** $(\psi^* g = g)$?

[30] For details of the proof of (H1–H2) see e.g. [Web99, App. B.1.2–B.1.4].

Exercise 2.4.3. Suppose the vector field $Y : N \to TN$ generates a 1-parameter group of diffeomorphisms $\psi_t : N \to N$. Consider the corresponding group of symplectomorphisms Ψ_t of $(T^*N, \omega_{\mathrm{can}})$ and denote by $X : T^*N \to TT^*N$ the generating vector field, that is

$$X(q,p) = \tfrac{d}{dt}\Big|_{t=0} \Psi_t(q,p) = \left(Y(q),\, \tfrac{d}{dt}\Big|_{t=0} (d\psi_t(q)^{-1})^* p\right).$$

Show that X is the Hamiltonian vector field of the function $H(q,p) := p \circ Y(q)$. [Hint: Observe that $0 = \mathcal{L}_X \lambda_{\mathrm{can}} = \omega_{\mathrm{can}}(X,\cdot) + d\left(\lambda_{\mathrm{can}}(X)\right)$ by the previous exercise and Cartan's formula and that $\lambda_{\mathrm{can}}(X(z)) := z \circ (d\pi(z)X(z))$.]

2.4.1 Electromagnetic flows – twisted cotangent bundles

Classical mechanics describes the motion of a particle of unit mass $m = 1$ and unit charge $e = 1$ located at time t at position $\gamma(t)$ in a configuration space, a manifold Q. The **electromagnetic system**[31] is described by the following three structures on Q: A Riemannian metric g (providing kinetic energy), a smooth one form θ (the **magnetic potential**), and a smooth function V (the **electric potential**).

Exercise 2.4.4 (Reformulating Maxwell's equations)**.** Let us reformulate (some of) Maxwell's equations on $\mathbb{R}^3$ in terms of quantities which are at home on manifolds – differential forms (see e.g. [BT82, War83]). Consider $\mathbb{R}^3$ with its natural orientation and the euclidean metric g_0. One-forms $\theta \in \Omega^1 = \Omega^1(\mathbb{R}^3)$ are in bijection with vector fields $\boldsymbol{A} \in \mathcal{X} = \mathcal{X}(\mathbb{R}^3)$, just identify components:

$$\Omega^1 \ni \theta = \theta_1 dx_1 + \theta_2 dx_2 + \theta_3 dx_3 \overset{g_0^{-1}}{\mapsto} \theta_1 \partial_{x_1} + \theta_2 \partial_{x_2} + \theta_3 \partial_{x_3} =: \boldsymbol{A} \in \mathcal{X}.$$

Let us indicate the isomorphism $\Omega^1 \simeq \mathcal{X}$ by writing $dx_j \mapsto \partial_{x_j}$. Consider the isomorphism $* : \Omega \to \Omega$ which relates the degree of the differential forms by $\Omega^j \to \Omega^{3-j}$ and is determined on the elements of the natural bases by

$$\begin{aligned}
&\Omega^0 \to \Omega^3, && 1 \mapsto dx_1 \wedge dx_2 \wedge dx_3,\\
&\Omega^1 \to \Omega^2, && dx_1 \mapsto dx_2 \wedge dx_3,\\
&\Omega^2 \to \Omega^1, && dx_1 \wedge dx_2 \mapsto dx_3,\\
&\Omega^3 \to \Omega^0, && dx_1 \wedge dx_2 \wedge dx_3 \mapsto 1,
\end{aligned}$$

and cyclic permutations. Note that $** = \mathbb{1}$.[32] With $\nabla = (\partial_{x_1}, \partial_{x_2}, \partial_{x_3})$ check[33]

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[31] **Maxwell's equations** for a **magnetic field** $\boldsymbol{B}$ and an **electric field** $\boldsymbol{E}$ in $\mathbb{R}^3$ are

$$\boldsymbol{B} = \mathrm{rot}\, \boldsymbol{A} = \nabla \times \boldsymbol{A}, \qquad \boldsymbol{E} = -\nabla V - \dot{\boldsymbol{A}}.$$

Here $\boldsymbol{A}$ is the **magnetic vector potential** and V the **electric scalar potential**. The **Lorentz force law** of a particle of mass m and charge e at position $\boldsymbol{r}(t)$ is (cf. [AKN06, §1.1.2])

$$m\ddot{\boldsymbol{r}} = e\left(\boldsymbol{E} + \dot{\boldsymbol{r}} \times \boldsymbol{B}\right).$$

[32] More generally, on odd dimensional space $** = \mathbb{1}$, but on even dimensional space $** = \pm\mathbb{1}$ on forms of even/odd degree.

[33] To $\boldsymbol{B} = B_1\partial_{x_1} + B_2\partial_{x_2} + B_3\partial_{x_3}$ corresponds $\sigma = B_1 dx_2 \wedge dx_3 + B_2 dx_3 \wedge dx_2 + B_3 dx_1 \wedge dx_2$.

$\mathcal{X}$	$\xrightarrow{g_0}$	Ω^1	$\xleftrightarrow{*}$	Ω^2	Remark
$\boldsymbol{A}$		θ			magnetic potential
$\nabla \times \boldsymbol{A}$		$*d\theta$		$d\theta$	exact magnetic field
$\boldsymbol{B}$		$*\sigma$		σ	magnetic field, requires
					$\operatorname{div} \boldsymbol{B} = 0$ ($\Leftrightarrow d\sigma = 0$)
$\boldsymbol{V}$		ν			velocity $\boldsymbol{V} = \dot{\boldsymbol{r}}$
$\boldsymbol{V} \times \boldsymbol{B}$		$*(\nu \wedge *\sigma)$		$\nu \wedge *\sigma$	
		$= -i_{\boldsymbol{V}}\sigma$			explicit calculation
$\boldsymbol{Y}(\boldsymbol{V})$					Lorentz force
					$\boldsymbol{Y}_{\boldsymbol{r}} \dot{\boldsymbol{r}} := \dot{\boldsymbol{r}} \times \boldsymbol{B}_{\boldsymbol{r}}$

This shows that the Lorentz force $\boldsymbol{Y}_{\boldsymbol{r}}\, \dot{\boldsymbol{r}} := \dot{\boldsymbol{r}} \times \boldsymbol{B}_{\boldsymbol{r}}$ experienced by a particle of unit mass and unit charge is determined in terms of the closed 2-form σ encoding the magnetic field by the identity

$$i_{\boldsymbol{Y}(\boldsymbol{V})} g_0 = -i_{\boldsymbol{V}}\sigma. \tag{2.4.29}$$

Concerning differential forms in electrodynamics see e.g. [Des81, Bot85, WR14].

Exercise 2.4.5 (Twisted symplectic structures $\omega_\sigma = \omega_{\rm can} + \pi^*\sigma$)**.** Suppose σ is a closed 2-form on the closed manifold Q. Denote by $\pi : T^*Q \to Q$ the bundle projection. Show that $\omega_{\rm can} + \pi^*\sigma$ is a symplectic form on T^*Q.
Now fix a Riemannian metric g on Q and check that the identity

$$g_q(Y_q v, \cdot) = -\sigma_q(v, \cdot)$$

pointwise at $q \in Q$ and $v \in T_qQ$ determines a fiber preserving anti-symmetric vector bundle map $Y : TQ \to TQ$, i.e. fiberwise the map $Y_q : T_qQ \to T_qQ$ is linear and anti-symmetric. The map Y is the **Lorentz force** associated to the magnetic field σ; see e.g. [CMP04]. We chose the minus sign in order to match the classical scenario (2.4.29) in $\mathbb{R}^3$; cf. [BRCF05].

Exercise 2.4.6 (Twisted geodesic flow)**.** A curve γ solves the Euler-Lagrange equations (2.4.30) associated to the Lagrangian $L_\theta : TQ \to \mathbb{R}$ given by

$$L_\theta(q, v) = T(q, v) + \theta(q)v - V(q), \qquad T(q, v) := \tfrac{1}{2}|v|^2 := \tfrac{1}{2} g_q(v, v),$$

where T is called **kinetic energy**, if and only if (cf. [Gin96, §2]) the pair $(\gamma, g\dot\gamma)$ is an integral curve of the Hamiltonian vector field $X_H^{\omega_{d\theta}}$ associated to the Hamiltonian H and the twisted symplectic structure $\omega_{d\theta}$ on T^*Q given by

$$H(q, p) = T(q, g^{-1}p) + V(q), \qquad \omega_{d\theta} := \omega_{\rm can} + \pi^* d\theta.$$

The flow of $X_H^{\omega_{d\theta}}$ is called **twisted geodesic flow**, its flow lines **twisted geodesics**. Note: In the Hamiltonian formulation there is no need that the magnetic field $\sigma = d\theta$ is exact, any closed 2-form on Q will do by Exercise 2.4.5.

Concerning existence of periodic electromagnetic trajectories we recommend the (older) survey [Gin96] and the comments [Gin01]. For more recent results see e.g. [Mer11] and references therein.

Lagrangian and Hamiltonian formalism

The **Lagrangian formulation** of the dynamics is as follows. Given two points $q_0, q_1 \in Q$, the motion of the particle is a curve $\gamma : [t_0, t_1] \to M$ with $\gamma(t_i) = q_i$ extremizing the **classical action** functional

$$\mathcal{S}(\gamma) = \int_{t_0}^{t_1} L(\gamma(t), \dot\gamma(t))\, dt.$$

Here the **electromagnetic Lagrangian** $L : TQ \to \mathbb{R}$ of the system is given by

$$\begin{aligned} L(q,v) = L_\theta(q,v) :&= \frac{1}{2}\, m\, g_q(v,v) + e\,(\theta_q v - V(q)) \\ &= \frac{1}{2}\, \lvert v\rvert^2 + \theta v - V. \end{aligned}$$

The extremals (critical points) γ are the solutions of the **Euler-Lagrange equations** which in local coordinates can be written as

$$\frac{d}{dt}\partial_v L(\gamma(t), \dot\gamma(t)) = \partial_q L(\gamma(t), \dot\gamma(t)). \tag{2.4.30}$$

For the physics behind we recommend [FLS64]. For the variational theory in the more general setting of **Tonelli Lagrangians**[34] see e.g. [Maz12] or [Abb13].

The **Hamiltonian description** of the system replaces the Lagrangian L_θ by its Legendre transform $H_\theta : T^*Q \to \mathbb{R}$ called the **electromagnetic Hamiltonian** of the system and given by

$$\begin{aligned} H_\theta(q,p) &= (\partial_v L(q,v))v - L(q,v) \\ &= \frac{1}{2}\, \lvert v\rvert^2 + V \\ &= \frac{1}{2}\, \lvert p-\theta\rvert^2 + V \end{aligned}$$

where we substituted v according to $p := \partial_v L_\theta(q,v) = g_q v + \theta_q$. The dynamics of the particle on T^*Q is then determined by the Hamiltonian vector field $X_{H_\theta}^{\omega_{\mathrm{can}}}$.

Alternatively, the dynamics of the same particle is described by the Hamiltonian vector field $X_H^{\omega_{d\theta}}$ associated to the standard *non-magnetic* Hamiltonian

$$H(q,p) = \frac{1}{2}\, \lvert p\rvert^2 + V(q), \qquad \omega_{d\theta} := \omega_{\mathrm{can}} + \pi^* d\theta = d(\lambda_{\mathrm{can}} + \pi^*\theta),$$

and a *magnetically* **twisted symplectic structure**. This alternative description works not only for exact magnetic fields $\sigma = d\theta$, but for any closed 2-form σ on Q; cf. [Gin96, Thm. 2.1 (ii)].

Exercise 2.4.7. Hamiltonian dynamics of $(H_\theta, \omega_{\mathrm{can}})$ and $(H, \omega_{d\theta})$ coincides.

For further details of the relation of symplectic geometry and classical mechanics see e.g. [Arn78] or [AG01, Ch. 3] or [AKN06, §1, §4].

[34] A Lagrangian $L : TQ \to \mathbb{R}$ is **Tonelli** if it is fiberwise uniformly convex and superlinear.

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Chapter 3

Fixed period – Floer homology

Towards Floer homology

Consider a symplectic manifold (M, ω). Given an autonomous Hamiltonian $F : M \to \mathbb{R}$, it is an interesting but rather challenging problem to investigate the set "of geometrically distinct closed orbits" of the Hamiltonian vector field on a given regular energy level $S = F^{-1}(E)$. A little thought reveals that this cannot be a set of Hamiltonian loops $\mathbb{R}/\tau\mathbb{Z} \to M$, but rather it should be a set of equivalence classes of such or, in geometric terms, of their images – embedded circles tangent to the Hamiltonian vector field X_F. These circles form the set $\mathcal{C}(S, \omega)$ of *closed characteristics* of X_F on S; cf. (2.3.22).

Back to Hamiltonian loops (parametrized closed orbits). Let us simplify the problem, firstly, by searching for loop trajectories $\mathbb{R}/\tau\mathbb{Z} \to M$ of X_F without taking any equivalence classes at all, and secondly, by just focussing on the plain existence problem. Let us break this further down into smaller pieces.

Period one. The problem reduces to detect Hamiltonian loops of period *one*: A τ-periodic trajectory z of F on the level set $F^{-1}(E)$ corresponds to the 1-periodic trajectory $z^{1/\tau}(t) := z(t/\tau)$ of $\tau^{-1}F$ on the level set $(\tau^{-1}F)^{-1}(\tau^{-1}E)$ $(= F^{-1}(E))$; indeed $\tau^{-1}X_F = X_{\tau^{-1}F}$. So there is a little price to pay: One looses the freedom to fix an energy value E. (The notion of energy value/level is lost anyway as soon as one allows time-dependent Hamiltonians.) So let us then investigate the set $\mathcal{P}(F)$ of 1-periodic Hamiltonian loops z of X_F. Two loops $z, \tilde{z} \in \mathcal{P}(F)$ are geometrically distinct if their images $z(\mathbb{S}^1) \cap \tilde{z}(\mathbb{S}^1) = \emptyset$ are disjoint, otherwise they are equivalent $z \sim \tilde{z}$; cf. Remark 2.3.17.

There are two problems with autonomous Hamiltonians F concerning *non-constant* Hamiltonian loops z. Suppose you found such z of period 1.

(multiple covers) How can you find out a Hamiltonian loop $z : \mathbb{R}/\mathbb{Z} \to M$ does not multiply cover another one $\tilde{z}$? In other words, is 1 the prime period of $z : \mathbb{R} \to M$?

(degeneracy) A non-constant Hamiltonian loop z provides an $\mathbb{S}^1$ family via time-shift

$$\mathbb{S}^1 \ni \tau \mapsto z_{(\tau)}(\cdot) := z(\cdot + \tau).$$

The multiple cover problem makes it difficult to decide if a newly detected periodic trajectory, say in the form of a critical point of a functional on loop space, is geometrically different from known ones. The fact that the elements of $\mathcal{P}(F)$ are never isolated obstructs reformulating the problem in terms of Morse theory, a powerful tool to analyze sets of critical points in terms of topology.

Non-autonomous 1*-periodic Hamiltonians and their* 1*-periodic trajectories.* Both problems disappear if we allow time-dependent Hamiltonians $H : \mathbb{S}^1 \times M \to \mathbb{R}$ and direct our attention to the set $\mathcal{P}(H)$ of 1-periodic Hamiltonian loops with respect to $X_{t+1} = X_t := X_{H_t}$; see (2.3.19). Is there a lower bound for the cardinality $|\mathcal{P}(H)|$ uniformly in H? in fact, this problem reduces to study $\mathcal{P}_0(H)$, the **set of contractible 1-periodic Hamiltonian loops**:

Contractible 1*-periodic Hamiltonian loops and closedness of* M. For the special case of C^2 small autonomous Hamiltonians all 1-periodic trajectories are constant by Proposition 2.3.16, so there are no non-contractible ones. But a C^2 small Hamiltonian is obtained by multiplying any given $H : M \to \mathbb{R}$ by a small constant $\varepsilon > 0$. Really? Correct, at least, if M is closed. So from now on we assume closedness of M – conveniently guaranteeing completeness of flows – and aim for a lower bound for the cardinality of the set $\mathcal{P}_0(H)$, uniformly in H.

Assumption 3.0.8. Throughout Chapter 3 we assume, unless said differently, that the symplectic manifold (M, ω) is closed and symplectically aspherical, see (3.0.1), and we study the set $\mathcal{P}_0(H)$ for Hamiltonians $H : \mathbb{S}^1 \times M \to \mathbb{R}$.

Arriving at Floer's ideas. It was well known before Floer that the contractible 1-periodic Hamiltonian loops are precisely the critical points of a (possibly multi-valued) functional $\mathcal{A}_H$, but at the time it seemed that this functional was "certainly not suitable for an existence proof. [Mos76, (1.5)]". However, this changed with the success of Rabinowitz [Rab78] in applying minimax methods to detect critical values of $\mathcal{A}_H$ and Floer' s idea [Flo88] to rather differently overcome the obstruction presented by infinite Morse index and coindex, namely by looking at a *relative* index between critical points – which is finite! Thereby Floer discovered **relative Morse theory** and successfully reformulated the problem. Floer's insights also included departing from looking at the L^2 gradient equation formally as an ODE on the loop space, but instead noticing that it represents a well posed PDE Fredholm problem for maps from the cylinder $\mathbb{R} \times \mathbb{S}^1$ into the manifold M itself whenever suitably compactified by imposing non-degenerate Hamiltonian loops $z^\mp$ to sit at $\pm\infty$. Assuming transversality, the dimension of the associated moduli space is finite, as it is the Fredholm index which itself is given by the spectral flow, the relative Morse index, along a flow cylinder. Non-degeneracy amounts to $\mathcal{A}_H$ being a Morse functional, this holds true for generic H, and in good cases (say $\mathrm{I}_{c_1} = 0$) the relative index becomes the difference of an absolute index associated to the non-degenerate Hamiltonian loops $z^\mp$ – the Conley-Zehnder index of Section 2.1.3. The fundamental results of Floer's construction are the lower bounds (1.0.1) and (1.0.2) proving the Arnol′d conjecture [Flo89] in many cases; in general see [FO99, LT98].

Back to autonomous Hamiltonians: Closed characteristics. Given an autonomous Hamiltonian $F : M \to \mathbb{R}$, a natural approach to analyze the set $\mathcal{C}(X_F)$ of closed characteristics of X_F, see (2.3.21), would certainly be – in view of Floer's estimate $|\mathcal{P}_0(H)| \geq \mathrm{SB}(M)$ – to focus on $\mathcal{P}_0(F)$ in a first step: Approximate F in an appropriate topology by non-degenerate, thus non-autonomous, Hamiltonians $H_\nu \to F$. By Floer's estimate every set $\mathcal{P}_0(H_\nu)$ is non-empty, so picking one element z_ν for every ν provides a sequence of Hamiltonian loops. Now one can try to extract a convergent subsequence using the Arzelà-Ascoli Theorem 3.2.10 and show that the limit loop, say z, satisfies the equation $\dot{z} = X_F(z)$ of the limit Hamiltonian. There are two problems.

I. Firstly, the Hamiltonian limit loop z, in fact already some or all of the z_ν, could be constant. This is excluded if the energy hypersurface $S = F^{-1}(c)$ of z is known to be regular, that is if c is a regular value of F. This brings us to action filtered Floer homology; cf. Section 3.4.5.

II. The second problem are multiplicities. If you get two, or more, limit solutions $z, \tilde{z}$ this way and suppose you even already know that they are different and non-constant elements of $\mathcal{P}_0(F)$, say by having information about the action values $\mathcal{A}_{H_\nu}(z_\nu)$ and those of the $\tilde{z}_\nu$'s. Even then, how would one decide whether z and $\tilde{z}$ are geometrically distinct or whether one multiply covers the other one? It helps looking at the particular Hamiltonian loops on a cotangent bundle over a closed Riemannian manifold Q which correspond to geodesics $\gamma : \mathbb{S}^1 \to Q$ in the base manifold and remembering Bott's analysis [Bot56] of how the Morse index changes under iterations $\gamma^k(\cdot) := \gamma(k\cdot) : \mathbb{S}^1 \to Q$. For Hamiltonian loops the corresponding index formulae have been pioneered by Long [Lon02]. For recent tremendous success of studying iterations, in a slightly different direction though, see Ginzburg's proof [Gin10] of the Conley conjecture. See [GG15] for a recent survey about existence of infinitely many simple periodic trajectories.

Cotangent bundles and loop spaces. If one gives up the compactness requirement for the symplectic manifold and looks at the natural class of cotangent bundles $(T^*Q, \omega_{\mathrm{can}})$ over closed, say orientable spin, Riemannian manifolds (Q, g), equipped with physical Hamiltonians H of the form kinetic plus potential energy and a natural almost complex structure $\bar{J}_g$, then Floer homology is totally different: It does not represent singular homology of T^*M, but it is naturally isomorphic to singular homology of the free loop space $\mathcal{L}Q$; cf. Section 3.5. If Q is not simply connected, then there is one isomorphisms for each component of $\mathcal{L}Q$, so here Floer homology even detects non-contractible periodic trajectories.

Outline of Chapter 3

Consider a Hamiltonian $H : \mathbb{S}^1 \times M \to \mathbb{R}$. Motivated by the Morse complex Floer's program, see [Flo89], is to use the symplectic action functional

$$\mathcal{A}_H : \mathcal{L}_0 M = C^\infty_{\mathrm{contr}}(\mathbb{S}^1, M) \to \mathbb{R}, \quad z \mapsto \int_{\mathbb{D}} \bar{z}^* \omega - \int_0^1 H_t(z(t))\, dt,$$

as a Morse function to construct a Morse type chain complex.

In Section 3.1 we briefly recall the usual construction of the Morse complex associated to a Morse function $f : Q \to \mathbb{R}$ on a closed Riemannian manifold of dimension n by using the critical points as generators, the Morse index as grading, and counting downward gradient flow lines to define a boundary operator. We also recall the geometric realization of the Morse cochain complex using the same generators and grading, but counting upward flow lines.

Section 3.2 is devoted to a detailed study of the action functional $\mathcal{A}_H : \mathcal{L}_0 M \to \mathbb{R}$ starting with a list of serious deficiency and explaining sign conventions. Then we calculate the differential and, with the help of a family J_t of ω-compatible almost complex structures, also the L^2 gradient of $\mathcal{A}_H$. This shows that the critical points are precisely given by the set

$$\mathrm{Crit}\mathcal{A}_H = \{z : \mathbb{R} \to M \mid \dot z = X_{H_t}(z),\, 1 \in \mathrm{Per}(z),\, z \sim \mathsf{pt}\} =: \mathcal{P}_0(H).$$

of 1-periodic contractible Hamiltonian loops. We calculate the Hessian operator A_z of $\mathcal{A}_H$ at a critical point z to define non-degeneracy of critical points. Next we insert an excursion to Baire's category theorem trying to separate the surrounding and easily confusable notions of "residual" and "second category" subsets. The purpose is to detail the informal notion of genericity in theorems like the one asserting that $\mathcal{A}_H$ is Morse for generic H ("transversality on loops"). Given Section 3.2, we define, for generic H, the Floer chain group $\mathrm{CF}_*(H)$ as the $\mathbb{Z}_2$ vector space generated by the finite set $\mathcal{P}_0(H)$ of contractible 1-periodic trajectories and graded by the canonical Conley-Zehnder index μ^{CZ} in (2.3.25):

$$\mathrm{CF}_k(H) = \mathrm{CF}_k(M, \omega, H) := \bigoplus_{\substack{z\in\mathcal{P}_0(H)\\ \mu^{\mathrm{CZ}}(z)=k}} \mathbb{Z}_2 z.$$

Section 3.3 introduces the substitute for the non-existent downward gradient flow, namely solutions $u : \mathbb{R} \times \mathbb{S}^1 \to M$ to Floer's elliptic PDE[1]

$$0 = \partial_s u - J_t(u)\Big(\partial_t u - X_{H_t}(u)\Big) = \partial_s u - J_t(u)\partial_t u - \nabla H_t(u)$$

called **Floer cylinders** or **Floer trajectories**. A Floer trajectory defines what we informally call a "flow line" or "integral curve" in the loop space, namely the image set $\{u(s, \cdot) \mid s \in \mathbb{R}\} \subset \mathcal{L}M$. Any two trajectories producing the same flow line differ by composition with time-shift $s \mapsto \sigma + s$. It is useful to switch view points using the correspondence "cylinder in M" $\leftrightarrow$ "path in $\mathcal{L}M$", namely

$$u : \mathbb{R} \times \mathbb{S}^1 \to M,\ (s,t) \mapsto u(s,t) \quad \leftrightarrow \quad u : \mathbb{R} \to \mathcal{L}M,\ s \mapsto u_s(\cdot) := u(s, \cdot)$$

Back to Floer trajectories u. Imposing as asymptotic boundary conditions at $\pm\infty \times \mathbb{S}^1$ Hamiltonian loops $z^\mp$ we call u a **connecting trajectory** from z^-

[1] Gromov's [Gro85] J-holomorphic curve equation is $\partial_s u + J(u)\partial_t u = 0$, now add a lower order perturbation ∇H. Here we have a $\bar J$-holomorphic curve equation where $\bar J := -J$. Alternatively, time reflection $\tilde u(s,t) := u(-s,t)$ relates the solutions u of the displayed equation to solutions of the perturbed J-holomorphic curve equation $\partial_s \tilde u + J_t(\tilde u)\partial_t \tilde u + \nabla H_t(\tilde u) = 0$.

to z^+. Let $\mathcal{M}(z^-, z^+)$ be the space of all of them. At this point we insert Section 3.3.2 on relevant elements of Fredholm theory needed to analyze under which conditions the spaces $\mathcal{M}(z^-, z^+)$ are manifolds and to calculate their dimensions. Since $\mathcal{A}_H$ is Morse all asymptotic boundary conditions are non-degenerate. This causes the operators D_u defined by linearizing Floer's equation at any connecting trajectory u to be Fredholm. For the manifold property of $\mathcal{M}(z^-, z^+)$ and the dimension formula $\mu^{\mathrm{CZ}}(z^-) - \mu^{\mathrm{CZ}}(z^+)$ one needs to make sure that D_u is onto, often referred to as "transversality on cylinders". But this can be achieved for generic H again. The necessary perturbation not only preserves the Morse property, but even the set of critical points. The machinery to deal with transversality issues, be it non-degeneracy of the critical points of $\mathcal{A}_H$ or surjectivity of the Fredholm operators, goes under the name Thom-Smale transversality theory and will be discussed in detail following [Sal99b]. In Section 3.3, also in 3.4, we essentially follow [Sal99a].
Given Section 3.3, we define, for generic (H, J), Floer's boundary operator as the mod two count of flow lines connecting Hamiltonian loops of index difference 1: On basis elements $x \in \mathcal{P}_0(H)$ of canonical Conley-Zehnder index k, set

$$\partial x = \partial^{\mathrm{F}}(M, \omega, H, J)x := \sum_{\substack{y \in \mathcal{P}_0(H) \\ \mu^{\mathrm{CZ}}(y) = k-1}} \#_2(m_{xy})\, y$$

where $\#_2(m_{xy})$ is the number modulo two of flow lines connecting x and y.

Section 3.4 is the heart of Chapter 3. First the property $\partial^2 = 0$ is shown. The resulting chain complex is denoted by $\mathrm{CF}(H) := (\mathrm{CF}_*(H), \partial)$. Its homology is a graded $\mathbb{Z}_2$ vector space denoted by $\mathrm{HF}_*(H) = \mathrm{HF}_*(M, \omega, H; J)$ and called **Floer homology**. Next continuation isomorphisms are constructed which naturally identify Floer homology under change of Hamiltonian. Then we discuss two methods of constructing a natural isomorphisms to singular homology of the manifold M itself, namely by choosing for H a C^2 small Morse function or, alternatively, by studying "spiked disks". Section 3.4 concludes with a brief account of action filtered Floer homology.

In Section 3.5 we have a glimpse at Floer homology for cotangent bundles, as opposed to compact symplectic manifolds.

Preliminaries

As a line in $\mathcal{L}M$ is a cylinder in M, the formal L^2 gradient equation for $\mathcal{A}_H$ on $\mathcal{L}M$ corresponds to a PDE in M. Thus carrying out the program of constructing a Morse complex relies heavily on non-linear functional analysis. So it is useful to impose in a first step conditions in order to "facilitate" the analysis and look for generalizations subsequently. Closedness of M we already mentioned.

(C1) The symplectic manifold (M, ω) is closed.

(C2) Evaluating ω and $c_1(M)$ on $\pi_2(M)$ is identically zero (cf. (3.0.2))

$$\mathrm{I}_\omega = 0 = \mathrm{I}_{c_1}. \tag{3.0.1}$$

Under these conditions we sketch the construction of the Floer complex in Sections 3.2–3.4; for details, including history, see [Sal99a], or [MS04].

In Section 3.5 the closedness condition (C1) gets dropped and we consider cotangent bundles T^*Q equipped with the canonical symplectic form $\omega_{\mathrm{can}} = d\lambda_{\mathrm{can}}$ over closed base manifolds Q. Here both conditions in (C2) are satisfied automatically, the lack of compactness (C1) will be compensated by restricting to a class of Hamiltonians that grow fiberwise sufficiently fast, such as physical Hamiltonians of the form kinetic plus potential energy.

Definition 3.0.9. A symplectic manifold (M, ω) is called **symplectically aspherical**, or **ω-aspherical** for short, if the homomorphism

$$\mathrm{I}_\omega : \pi_2(M) \to \mathbb{R}, \quad [v] \mapsto [\omega]([v]) := \int_{\mathbb{S}^2} v^*\omega, \tag{3.0.2}$$

vanishes for every class and each smooth representative $v : \mathbb{S}^1 \to M$. We denote by $\mathrm{I}_{c_1} : \pi_2(M) \to \mathbb{Z}$ the corresponding evaluation homomorphism for the first Chern class $c_1 \in \mathrm{H}^2(M; \mathbb{Z})$ of the (homotopic) complex vector bundles $TM \to M$ associated to any family J_t of ω-compatible almost complex structures.

Exercise 3.0.10. Show that I_ω is well defined and a homomorphism of groups.

Example 3.0.11 (Condition (C2), thus $\pi_1(M) \neq 0$)**.** Since M is closed $\mathrm{H}^2(M)$ is non-trivial, indeed $[\omega] \neq 0$. Hence the condition $\mathrm{I}_\omega = 0$ causes $\pi_1(M) \neq 0$ via the Hurewicz homomorphism; see [HZ11, p.228].

(tori) Since any torus $\mathbb{T}^\ell$ is **aspherical**, in fact $\pi_k(\mathbb{T}^\ell) = 0$ for $k \geq 2$, any symplectic form ω on $\mathbb{T}^{2n}$ satisfies (C2). Tori were treated in [CZ83].

3.1 Toy model

The Morse complex goes back to the work of Thom, Smale, and Milnor in the 40s, 50s, and 60s, respectively, and was rediscovered in an influential paper of Witten in 1982. It has been studied since by many people. The standard reference is the 1993 monograph [Sch93] by Schwarz. For more on the history and references after 1993 see also our recent lecture notes manuscript [Web] which covers in detail the dynamical systems approach from [Web93]; cf. [Web06b].

3.1.1 Morse homology

Given a closed manifold Q of dimension n, one can utilize gradient dynamical systems to recover the integral singular homology $\mathrm{H}_*(Q) := \mathrm{H}_*(Q; \mathbb{Z})$. Among all smooth functions on Q there is an open and dense subset consisting of **Morse functions** $f : Q \to \mathbb{R}$, that is all critical points x are **non-degenerate** in the sense that all eigenvalues of the Hessian symmetric bilinear form $\mathrm{Hess}_x f$ on T_xQ are non-zero. The number of negative eigenvalues, counted with multiplicities, is called the **Morse index** of x denoted by $\mathrm{ind}_f(x)$. The **negative space** associated to the critical point x is the subspace $E_x \subset T_xQ$ spanned by all eigenvectors associated to negative eigenvalues. Non-degenerate critical points are isolated, so by compactness of Q they form a finite set $\mathrm{Crit} f$.

An **oriented critical point** o_x, also called an **orientation of a critical point** and alternatively denoted by $\langle x\rangle$, is a critical point x together with a choice of orientation of its negative space E_x. For each $k \in \mathbb{Z}$, let the **Morse chain group** $\mathrm{CM}_k(f)$ be the abelian group generated by the oriented critical points o_x of Morse index k and subject to the relations $o_x + \bar{o}_x = 0$ where $\bar{o}_x$ is the opposite orientation of o_x.[2] Let us denote by $[x]$ the equivalence class of an oriented critical point under the relation $o_x + \bar{o}_x = 0$.[3]

To define a boundary operator on $\mathrm{CM}_*(f)$ pick a Riemannian metric g on Q and consider the corresponding **downward gradient flow** on Q, i.e. the 1-parameter group of diffeomorphisms $\varphi = \{\varphi_t\}_{t\in\mathbb{R}}$ determined by

$$\frac{d}{dt}\varphi_t = -\nabla f \circ \varphi_t, \qquad \varphi_0 = \mathrm{id}. \tag{3.1.3}$$

By non-degeneracy of $x \in \mathrm{Crit}_k f$ the **un/stable manifolds** $W^{u/s}(x) := \{q \in \mathbb{Q} \mid \varphi_t q \to x, \text{ as } t \to -/+\infty\}$ are embedded submanifolds of Q of dimension/codimension k; see e.g. [Web15]. Slightly perturbing the Morse function f outside a small neighborhood of its critical points leads to a function with the same critical points, still Morse and still denoted by f, but whose flow satisfies in addition the **Morse-Smale condition**:[4] Namely, any intersection

$$M_{xy} := W^u(x) \pitchfork W^s(y), \qquad \dim M_{xy} = \mathrm{ind}_f(x) - \mathrm{ind}_f(y), \tag{3.1.4}$$

of an unstable and a stable manifold is cut out transversely, hence a manifold – the **connecting manifold** of x and y. The spaces of **connecting flow lines**

$$m_{xy} := M_{xy} \pitchfork f^{-1}(r), \qquad \dim m_{xy} = \mathrm{ind}_f(x) - \mathrm{ind}_f(y) - 1,$$

where $r \in (f(y), f(x))$ is any choice of a regular value of f, are not only manifolds, but are what is called *compact up to broken trajectories*; cf. Figure 3.1. Consequently in case of index difference one the m_{xy} are finite sets whose elements u represent isolated flow lines running from x to y. Given such u and an orientation o_x of E_x, one can define a **push-forward orientation** $u_* o_x$ of E_y that respects orientation reversal, that is $u_*\bar{o}_x = \overline{u_* o_x}$. Thus $u_*[x] := [u_* o_x]$ is well defined on the generators $[x]$ of the quotient group $\mathrm{CM}_k(f)$. The **Morse boundary operator** is then defined on the generators by

$$\partial_k = \partial_k(f, g) : \mathrm{CM}_k(f) \to \mathrm{CM}_{k-1}(f), \quad [x] \mapsto \sum_{y\in\mathrm{Crit}_{k-1}} \sum_{u\in m_{xy}} u_*[x]$$

and extended to the whole group by linearity. That $\partial^2 = 0$ boils down to the fact that $\partial^2[x]$ is a sum over all 1-fold broken flow lines (u, v) where u

[2] By convention the empty set $\emptyset$ generates the trivial group $\{0\}$; see Notation 1.0.5.

[3] A choice $\langle \mathrm{Crit}_k f\rangle$ of an orientation for all critical points of index k is a basis of $\mathrm{CM}_k(f)$.

[4] If the Morse-Smale condition (3.1.4) holds true, we call $h := (f, g)$ a **Morse-Smale pair**.

is a flow line from x to some y and v is one from *that same* y to some z as indicated by Figure 3.1. As also indicated by the figure such broken orbits correspond precisely to the ends of a 1-dimensional manifold-with-boundary. In other words, these broken orbits appear in pairs and, moreover, one partner provides the opposite coefficient $v_* u_* [x] = -\tilde{v}_* \tilde{u}_* [x]$ in front of $[z]$ as the other one. So in sum each partner pair contributes zero, but $\partial^2 [x]$ is precisely a sum of partner pair contributions. For details of the facts above/below see e.g. [Web].

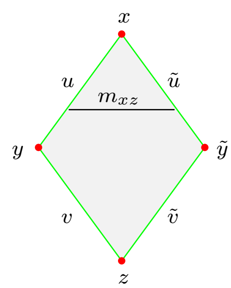


Figure 3.1: Partner pair property $(u, v) \sim (\tilde{u}, \tilde{v})$ leads to $\partial^2 = 0$

The corresponding homology groups $\mathrm{HM}_k(Q; h)$ are actually independent of the Morse-Smale pair $h = (f, g)$ as one shows, for instance,[5] by choosing a generic homotopy f_t between two Morse functions f^α and f^β and similarly g_t among the Riemannian metrics. Counting flow lines of the time-dependent gradient equation – just replace $h = (f, g)$ in (3.1.3) by the time-dependent pair $h_{\alpha\beta} = (f_t, g_t)$ – provides a chain complex homomorphisms $\psi_k^{\beta\alpha}(h_{\alpha\beta})$. The induced maps on homology $\Psi_k^{\beta\alpha} : \mathrm{HM}_k(f^\alpha, g^\alpha) \to \mathrm{HM}_k(f^\beta, g^\beta)$ are called **continuation maps**. They do not depend on the choice of homotopy $h_{\alpha\beta}$. On the chain level the continuation maps have the trivial, but important, property that the constant homotopy, denoted by h_α, induces the identity map, that is $\psi_k^{\alpha\alpha}(h_\alpha) = \mathbb{1}$. Looking at homotopies of homotopies not only shows that the $\Psi_k^{\beta\alpha}$ are independent of $h_{\alpha\beta}$, but also provides the crucial relations

$$\Psi_k^{\gamma\beta} \Psi_k^{\beta\alpha} = \Psi_k^{\gamma\alpha}, \qquad \Psi_k^{\alpha\alpha} = \mathbb{1}.$$

A rather nice way to construct a natural isomorphism

$$\Psi^h : \mathrm{HM}_k(Q; h) \xrightarrow{\cong} \mathrm{H}_k(Q) \tag{3.1.5}$$

to singular homology of Q is via the Abbondandolo-Majer filtration [AM06].[6]

Remark 3.1.1 ($\mathbb{Z}_2$ coefficients)**.** The k^{th} Morse chain group with $\mathbb{Z}_2$ coefficients is the $\mathbb{Z}_2$ vector space $\mathrm{CM}_k(f; \mathbb{Z}_2)$ whose canonical basis $\mathrm{Crit}_k f$ are the critical points of Morse index k. The Morse boundary operator is defined by

$$\partial_k x := \sum_{y \in \mathrm{Crit}_{k-1} f} \#_2(m_{xy})\, y \tag{3.1.6}$$

[5] Alternatively, use Poźniak cones; see [Web].

[6] If $f(x) = k$ for $x \in \mathrm{Crit}_k f$, then $\{f < k + \frac{1}{2}\}$ leads to such filtration; cf. [Mil65, Thm. 7.4].

on the basis elements $x \in \mathrm{Crit}_k f$; here $\#_2$ denotes 'number of elements mod 2'.

The $\mathbb{Z}_2$ Morse boundary operator counts modulo two
downward flow trajectories between critical points of index difference 1.

Exercise 3.1.2 (Closed orientable surfaces)**.** Calculate the $\mathbb{Z}_2$ Morse homology of your favorite closed orientable surface. [Hint: Embedd in $\mathbb{R}^3$; height function.]

Exercise 3.1.3 (Real projective plane $\mathbb{R}\mathrm{P}^2$)**.** Find a Morse function on $\mathbb{R}\mathrm{P}^2$ with exactly three critical points. Find the $\mathbb{Z}_2$ Morse complex and homology. [Hint: Think of $\mathbb{R}\mathrm{P}^2$ as unit disk $\mathbb{D} \subset \mathbb{R}^2$ modulo opposite boundary points.]

3.1.2 Morse cohomology

By definition **cohomology** arises from homology by *dualization*: Any chain complex $\mathrm{C} = (\mathrm{C}_*, \partial_*)$ comes naturally with a cochain complex $\mathrm{C}^\# = (\mathrm{C}^*, \delta^*)$, the **dual complex of C**: It consists of the dual spaces $\mathrm{C}^k := \mathrm{C}_k^\#$ and transposed maps $\delta^k := \partial_{k+1}^\#$. The cohomology $\mathrm{H}^*(\mathrm{C}^\#)$ of the cochain complex $\mathrm{C}^\#$ is called the **cohomology of C** and denoted by $\mathrm{H}^*(\mathrm{C})$.

Morse cohomology

For a Morse-Smale pair $h = (f, g)$ the **Morse cochain groups** are defined by

$$\mathrm{CM}^k(f) := \mathrm{CM}_k^\#(f) := \mathrm{Hom}(\mathrm{CM}_k(f), \mathbb{Z})$$

for any k and the associated **Morse coboundary operators** $\delta^k(h)$ by

$$\begin{tikzcd}
\mathrm{CM}^{k+1}(f) \arrow[r, equal] & \Big(\mathrm{CM}_{k+1}(f) \to \mathbb{Z}\Big) \arrow[d, "\partial_{k+1}"] \\
\mathrm{CM}^k(f) \arrow[u, "\delta^k_\alpha := (\partial_{k+1})^\#"] \arrow[r, equal] & \Big(\mathrm{CM}_k(f) \longrightarrow \mathbb{Z}\Big).
\end{tikzcd}$$

The transposed map acts by $\delta_\alpha^k(\gamma) = \gamma \circ \partial_{k+1}$, of course. The quotient space

$$\mathrm{HM}^k(Q; h) := \frac{\ker \delta_\alpha^k}{\mathrm{im}\, \delta_\alpha^{k-1}}$$

is called the $\boldsymbol{k}^{\mathbf{th}}$ **Morse cohomology** of Q with $\mathbb{Z}$ coefficients.

From now on we restrict to $\mathbb{Z}_2$ coefficients for simplicity of the presentation. Since $\mathbb{Z}_2$ is a field the Kronecker duality theorem implies that the homomorphism induced on cohomology $[\psi^{\beta\alpha}(h_{\alpha\beta})^\#]$ by the transpose is the transpose of the homology continuation isomorphism $\Psi^{\beta\alpha}$. Consequently the transposes

$$(\Psi^{\beta\alpha})^\# = [\psi^{\beta\alpha}(h_{\alpha\beta})]^\# = [\psi^{\beta\alpha}(h_{\alpha\beta})^\#] : \mathrm{HM}^*(Q; h^\beta; \mathbb{Z}_2) \to \mathrm{HM}^*(Q; h^\alpha; \mathbb{Z}_2)$$

are isomorphisms and satisfy the identities

$$(\Psi^{\beta\alpha})^\#(\Psi^{\gamma\beta})^\# = (\Psi^{\gamma\alpha})^\#, \qquad (\Psi^{\alpha\alpha})^\# = \mathbb{1}. \tag{3.1.7}$$

Geometric realization

For simplicity we restrict to $\mathbb{Z}_2$ coefficients, so there are no orientations involved and so $\mathcal{B}_f := \mathrm{Crit} f$ is a **canonical basis** of $\mathrm{CM}_*(f;\mathbb{Z}_2)$. By compactness of Q the dimension of $\mathrm{CM}_*(f;\mathbb{Z}_2)$, thus of its dual space $\mathrm{CM}^*(f;\mathbb{Z}_2)$, is finite. So the **dual basis** of $\mathcal{B}_f$ exists. It is given by the set $\mathcal{B}_f^\# = \mathrm{Crit}^\# f := \{\eta^x \mid x \in \mathrm{Crit} f\}$ of Dirac δ-functionals; denoted by η^x for distinction from the coboundary operator δ. Each functional is determined by its values

$$\eta^x : \mathrm{CM}_*(f;\mathbb{Z}_2) \to \mathbb{Z}_2, \quad y \mapsto \begin{cases} 1 & , y = x, \\ 0 & , \text{else}, \end{cases} \tag{3.1.8}$$

on the basis elements $y \in \mathrm{Crit} f$. Since $\mathrm{CM}^*(f;\mathbb{Z}_2)$ is of finite dimension any element ω can indeed be written as a linear combination of the η^x's, that is

$$\omega = \sum_{x \in \mathrm{Crit} f} \omega_x \eta^x, \qquad \omega_x := \omega(x) \in \mathbb{Z}_2. \tag{3.1.9}$$

The dual basis, thus $\mathrm{CM}^*(f;\mathbb{Z}_2)$, inherits the Morse index grading of f, i.e.

$$|\eta^x| := |x| := \mathrm{ind}_f(x).$$

To geometrically identify the action of the coboundary operator $\delta^k := \partial_{k+1}^\#$ on a cochain $\omega \in \mathrm{CM}^k(f;\mathbb{Z}_2)$ observe that

$$(\delta^k\omega)_x = (\delta^k\omega)(x) = \omega(\partial_{k+1}x) = \sum_{y \in \mathrm{Crit}_k^{f(x)} f} \#_2(m_{xy})\, \omega_y$$

for every $x \in \mathrm{Crit}_{k+1} f$. Here we used definition (3.1.6) of ∂_{k+1}. Thus by (3.1.9)

$$\delta^k\omega = \sum_{x \in \mathrm{Crit}_{k+1} f} \left(\delta^k\omega\right)_x \eta^x = \sum_{x \in \mathrm{Crit}_{k+1} f} \Bigg(\sum_{y \in \mathrm{Crit}_k^{f(x)} f} \#_2(m_{xy})\, \omega_y \Bigg) \eta^x$$

for every cochain $\omega \in \mathrm{CM}^k(f;\mathbb{Z}_2)$. In particular, we obtain that

$$\delta^k \eta^y = \sum_{x \in \mathrm{Crit}_{k+1} f} \#_2(m_{xy})\, \eta^x \tag{3.1.10}$$

for every basis element $\eta^y \in \mathrm{Crit}_k^\# f$. But this means the following.

The $\mathbb{Z}_2$ Morse coboundary operator counts modulo two
upward flow trajectories between critical points of index difference 1.

3.2 Symplectic action $\mathcal{A}_H$ – period one

Suppose (M,ω) is a *closed* symplectic manifold. The by H **perturbed symplectic action functional** on the space $\mathcal{L}_0M$ of contractible smooth loops $z:\mathbb{S}^1\to M$ is defined by

$$\mathcal{A}_H:\mathcal{L}_0M\to\mathbb{R},\quad z\mapsto\int_{\mathbb{D}}\bar{z}^*\omega-\int_0^1 H_t(z(t))\,dt, \tag{3.2.11}$$

where $\bar{z}=v:\mathbb{D}\to M$ is a **spanning disk**, i.e. a smooth extension of $z=v|_{\partial\mathbb{D}}$. Some remarks are in order. The symplectic action functional

- is **not well defined**, unless M is ω-aspherical (dA_H makes sense though);
- is **not bounded** below, neither above.[7] Unfortunately, common variational techniques build on at least semi-boundedness, say from below.

  One circumvents this problem by restricting attention to those L^2 (not $W^{1,2}$) gradient trajectories $\mathbb{R}\to\mathcal{L}_0M$ along which the action remains bounded; see Remark 3.3.3. It is the set $\mathcal{M}$ of these – called the set of **finite energy trajectories** of the L^2 gradient $\operatorname{grad}\mathcal{A}_H$ – that carries the complete homology information of M whenever $\mathcal{A}_H$ is Morse. This brings in, through the back door, another common assumption in variational theory: Although in general $\mathcal{A}_H$ is not Palais-Smale with respect to the $W^{1,2}$ gradient (cf. [Hof85, VI.1] and [HZ11, § 3.3]), it is sufficient that the **Palais-Smale condition** holds *on* $\mathcal{M}$;
- has critical points of **infinite Morse index**; see Example 3.2.15 for $H=0$. Unfortunately, therefore the symplectic action functional will not admit fundamental Morse theoretical tools such as the cell attachment theorem: The unit sphere in an infinite dimensional Hilbert space is contractible![8]

  One circumvents this problem by looking at the change of Morse index, also called **relative Morse index** or **spectral flow**, along a trajectory between two critical points.

Remark 3.2.1 (Signs in $\mathcal{A}_H$ – see Notation 1.0.5 for a detailed discussion)**.**
Closed manifolds. At the level of closed symplectic manifolds the sign choices in (3.2.11) are not relevant. Changing the sign of ω is equivalent by (1.0.12) to changing the sign of H. But changing the sign of H, more precisely replacing $H=H_t$ by $\hat{H}=\hat{H}_t:=-H_{-t}$, results in $\mathrm{HF}_k(H)\simeq\mathrm{HF}^{-k}(\hat{H})$ induced by natural identification of the two chain complexes. Together with continuation $\mathrm{HF}^{-k}(\hat{H})\simeq\mathrm{HF}^{-k}(H)$ and the natural isomorphisms to singular (co)homology such change of sign induces nothing but the Poincaré duality isomorphisms $\mathrm{H}_{k+n}(M)\simeq\mathrm{H}^{2n-(k+n)}(M)$ of the closed symplectic, so orientable, manifold M.
Cotangent bundles. Motivated by classical mechanics one would like to have as integrand $p\,dq-H\,dt$; this is the case for convention (3.2.11) with $\omega_{\mathrm{can}}=d\lambda_{\mathrm{can}}$.

[7] Figure out how term one behaves under replacing *loops* z by $z^k(t):=z(kt)$ und v by v^k.
[8] See Kakutani [Kak43] or apply Kuiper's theorem [Kui65].

Remark 3.2.2 (Palais-Smale condition)**.** Suppose f is a C^1 function on a Banach manifold $\mathcal{B}$ equipped with a Riemannian metric, see [Pal66], and ∇f denotes the gradient. A sequence $z_i \in \mathcal{B}$ along which f is bounded and ∇f converges to zero is called a **Palais-Smale sequence**. One says that the **Palais-Smale condition holds on a subset** $U \subset \mathcal{B}$ if every Palais-Smale sequence in U admits a subsequence converging to a critical point. For a detailed account of the Palais-Smale condition and its history see the survey [MW10].

Remark 3.2.3 (Non-exact cases)**.** Exactness of ω facilitates the definition of action functionals on loops, but it can be dropped on the cost of either

- restricting to contractible loops and spanning in disks, cf. (3.2.11), or
- fixing one reference loop in each component of loop space and spanning in cylinders.

The thereby potentially arising multi-valuedness of the action can then

- either be ruled out by requiring ω to be symplectically aspherical (case of spanning disks) or symplectically atoroidal (case of spanning cylinders),[9]
- or be accepted and dealt with by constructing chain complexes with coefficients in Novikov rings; cf. [HS95].

3.2.1 Critical points and L^2 gradient

The critical points of the action functional $\mathcal{A}_H$ are the 1-periodic trajectories of X_H by formula (3.2.12) for the differential of $\mathcal{A}_H$ at any[10] loop z.

Remark 3.2.4 (Paying dynamics to get compactness)**.** To turn the differential into a gradient one needs to pick a Riemannian metric on the loop space. Looking at the differential suggests the $W^{1,2}$ topology (absolutely continuous loops with square integrable derivatives), but for this choice desirable compactness properties fail, as mentioned above. It was Floer's insight that taking L^2 gradient instead provides sufficient compactness on relevant parts of loop space; see Section 3.4.1. The price to pay will be that the L^2 gradient does not generate a flow on the whole loop space – but it does on relevant parts. The relevant part actually consists of the loops represented by the space $\mathcal{M}$ of finite energy trajectories; cf. (3.3.27). It is this space that carries the homology of M.

Pick a 1-periodic family J_t of ω-compatible almost complex structures and let $g_{J_t} = \langle\cdot,\cdot\rangle_t$ be the associated family of Riemannian metrics on M. Define the $\boldsymbol{L^2}$ **inner product** on the loop space at any loop z, contractible or not, by

$$\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_{0,2} : T_z\mathcal{L}M \times T_z\mathcal{L}M \to \mathbb{R}, \quad (\xi,\eta) \mapsto \int_0^1 \langle\xi(t),\eta(t)\rangle_t \, dt,$$

where ξ and η are smooth vector fields along the loop z.

[9] **Symplectically** or $\boldsymbol{\omega}$**-atoroidal** means that $\int_{\mathbb{T}^2} v^*\omega = 0$ for every smooth map $v : \mathbb{T}^2 \to M$; see [Mer11, §2.3] for sufficient conditions in the cotangent bundle case $M = T^*Q$.

[10] The differential is well defined at any loop, contractible or not.

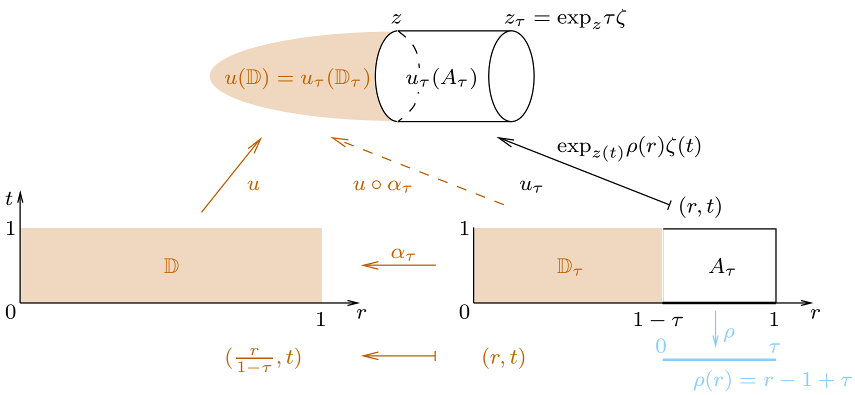


Figure 3.2: Derivative $d\mathcal{A}_H(z)\zeta := \frac{d}{d\tau}\big|_0 \mathcal{A}_H(\exp_z \tau\zeta)$ and spanning disks u_τ

Exercise 3.2.5. a) Show that $\mathcal{A}_H$ is well defined, if $\mathrm{I}_\omega = 0$. Recall that the identity $dH_t = -\omega(X_{H_t}, \cdot)$ determines the vector field X_{H_t}. Prove that

$$
\begin{aligned}
d\mathcal{A}_H(z)\zeta &= \int_0^1 \omega\left(\zeta, \dot z - X_{H_t}(z)\right) dt \\
&= \int_0^1 \big\langle \zeta, -J_t(z)\big(\dot z - X_{H_t}(z)\big)\big\rangle_t \, dt \\
&= \int_0^1 \langle \zeta, -J_t(z)\dot z - \nabla H_t(z)\rangle_t \, dt \\
&= \big\langle \zeta, \underbrace{-J_t(z)\dot z - \nabla H(z)}_{=:\, \operatorname{grad} \mathcal{A}_H(z)} \big\rangle
\end{aligned}
\tag{3.2.12}
$$

for every smooth vector field ζ along a contractible loop z in M; see also the hint to Exercise 3.3.2.[11]

b) For general I_ω show that, although $\mathcal{A}_H$ is not well defined, its linearization does not depend on the choice of spanning disk. Think about spanning disks

[11] Hint: To see identity one, pick an auxiliary Riemannian metric on M with corresponding Levi-Civita connection ∇ and exponential map exp. Given z and ζ, pick the families of loops z_τ and spanning disks u_τ in Figure 3.2. (In the figure we have identified the closed unit disk in $\mathbb{R}^2$ minus the origin with the cylinder $(0,1] \times \mathbb{S}^1$ which we denote by $\mathbb{D}$! This abuses notation, but might facilitate reading.) In the calculation use that the integral is additive under the domain decomposition $\mathbb{D} = \mathbb{D}_\tau \cup A_\tau$ to obtain a sum of three terms: One of them vanishes, as $\int_{\mathbb{D}_\tau} u_\tau^*\omega = \int_{\mathbb{D}} u^*\omega$ is constant in τ, and one of them leads easily to X_{H_t}. The third one is

$$
\tfrac{d}{d\tau}\Big|_0 \int_{A_\tau} u_\tau^*\omega = \int_0^1 \underbrace{\tfrac{d}{d\tau}\big|_0 \textstyle\int_{1-\tau}^1 F(\tau,r,t)\, dr}_{-F(\tau, r=1-\tau, t)|_0 \frac{d}{d\tau}\big|_0 (1-\tau)} \, dt = \int_0^1 \underbrace{F(\tau=0, r=1, t)}_{\omega(\zeta,\dot z)} dt
$$

for $F = \omega\big(E_2(z,\rho\zeta)\zeta, E_1(z,\rho\zeta)\dot z + E_2(z,\rho\zeta)\rho\nabla_t\zeta\big)$ and $E_i(z(t),\zeta(t)) : T_{z(t)}M \to T_{\exp_{z(t)} \zeta(t)}M$ for $i = 1,2$ denoting the covariant partial derivatives of the exponential map ($E_i(z,0) = \mathbb{1}$ as shown e.g. in the section *Analytic setup near hyperbolic singularities* in [Web, App.]).

as cylinders connecting the periodic trajectory z with some fixed constant loop $z_0(t) \equiv p \in M$. Extend the definition of $\mathcal{A}_H$ to components of the free loop space other than that of the contractible loops.

Lemma 3.2.6 (Compactness of critical set)**.** *The set of critical points*

$$\mathrm{Crit}\mathcal{A}_H = \{z \in \mathcal{L}_0 M \mid \dot{z} = X_{H_t}(z) = J_t(z)\nabla H_t(z)\} =: \mathcal{P}_0(H)$$

is compact with respect to the C^1 topology.

Proof. Pick a sequence z_i of critical points and consider the sequence $w_i = (z_i(0), \dot{z}_i(0))$ in the tangent bundle TM. Pick a Riemannian metric on M. As $|\dot{z}_i(0)| = |X_{H_0} \circ z_i(0)| \leq \max_{\mathbb{S}^1 \times M} |X_H|$, the sequence lives in a compact subset of TM. Thus there is a subsequence, still denoted by w_i, converging to an element $w = (p, v) \in TM$. But the $z_i(0)$ are fixed points of the time-1-map ψ_1 of the Hamiltonian flow (2.3.19) and so is the limit p by continuity of ψ_1. Hence the periodic trajectories z_i converge in C^1 to the periodic trajectory $z(t) := \psi_t p$. □

Exercise 3.2.7. Use the Arzelà-Ascoli Theorem 3.2.10 to prove Lemma 3.2.6.

The L^2 gradient grad $\mathcal{A}_H$ as section of a Hilbert space bundle

Concerning analysis one prefers to work in Banach or even Hilbert spaces. As the gradient (3.2.12) of $\mathcal{A}_H$ involves *one* derivative of the loop, the most natural space to consider is the space

$$\Lambda M := W^{1,2}_{\mathrm{contr}}(\mathbb{S}^1, M)$$

of contractible absolutely continuous loops $z : \mathbb{S}^1 \to M$ with square integrable derivative.[12] Although not a linear space, it is a Hilbert manifold, modeled locally at a loop z on the Hilbert space

$$W^{1,2}(\mathbb{S}^1, z^*TM) = T_z \Lambda M$$

of absolutely continuous vector fields along z with square integrable derivative. A standard reference for the **geometry of manifolds of maps** is [Elĭ67]. If $z \in \Lambda M$, then grad $\mathcal{A}_H(z)$ is an L^2 integrable vector field along z, that is

$$\operatorname{grad} \mathcal{A}_H(z) \in L^2(\mathbb{S}^1, z^*TM) =: \mathcal{E}_z.$$

Remark 3.2.8 (No flow)**.** To put it differently, the L^2 gradient grad $\mathcal{A}_H$ of the action functional is not a tangent vector field to the Hilbert manifold $W^{1,2}(\mathbb{S}^1, M)$, nor to any $W^{k,2}$, due to the *loss of a derivative.*[13] The initial value problem is not well posed and grad $\mathcal{A}_H$ does not generate a flow. The reason is that, by regularity, the loops of which a solution cylinder is composed are smooth, so the flow cannot pass any of the many non-smooth elements $z \in \Lambda M$.

[12] An absolutely continuous map $\mathbb{S}^1 \to M$ admits a derivative almost everywhere.

[13] Given $u \in W^{k,2}$, then grad $\mathcal{A}_H(u)$ lies in $W^{k-1,2}$. So grad $\mathcal{A}_H$ is not a vector field on $W^{k,2}$, i.e. a section of $TW^{k,2}$, so the formal equation $\frac{d}{ds}u(s) = -\operatorname{grad} \mathcal{A}_H(u(s))$ isn't an ODE.

The union of all the Hilbert spaces $\mathcal{E}_z = L^2(\mathbb{S}^1, z^*TM)$ forms a Hilbert space bundle $\mathcal{E}$ over ΛM. A section is given by the L^2 gradient $\operatorname{grad} \mathcal{A}_H$. Figure 3.3 illustrates the gradient section and indicates the natural splitting

$$T_z\mathcal{E}M \cong T_z\Lambda M \oplus \mathcal{E}_z = W^{1,2}(\mathbb{S}^1, z^*TM) \oplus L^2(\mathbb{S}^1, z^*TM)$$

of the tangent bundle $T\mathcal{E}$ along the zero section of $\mathcal{E}$, denoted still by ΛM.

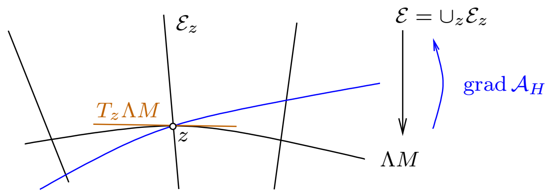


Figure 3.3: Hilbert bundle $\mathcal{E} \to \Lambda M$ over loop space and L^2 gradient section

Lemma 3.2.9 (Regularity)**.** *Any zero $z \in \Lambda M$ of* $\operatorname{grad} \mathcal{A}_H$ *is C^∞ smooth.*

Proof. By the Sobolev embedding theorem $W^{1,2}(\mathbb{S}^1) \hookrightarrow C^0(\mathbb{S}^1)$ we get $z \in C^0$. By assumption z admits a weak derivative of class L^2, say y, and $y = X_{H_t}(z)$ almost everywhere. But the RHS, hence y, is of class C^0. Thus the weak derivative is actually the ordinary derivative $\dot z = y \in C^0$. Hence $\dot z = X_{H_t}(z) \in C^0$, therefore $z \in C^1$. But then $\dot z = X_{H_t}(z) \in C^1$, hence $z \in C^2$, and so on. □

3.2.2 Arzelà-Ascoli – convergent subsequences

Theorem 3.2.10 (Arzelà-Ascoli Theorem)**.** *Suppose $(\mathcal{X}, d)$ is a compact metric space and $C(\mathcal{X})$ is the Banach space of continuous functions on $\mathcal{X}$ equipped with the sup norm. Then the following is true. A subset $\mathcal{F}$ of $C(\mathcal{X})$ is precompact if and only if the family $\mathcal{F}$ is **equicontinuous**[14] and **pointwise bounded**[15].*

Proof. [Rud91, Thm. A.5]. □

The theorem generalizes to functions taking values in a metric space.

Exercise 3.2.11. Suppose $(\mathcal{X}, d)$ is a metric space and $L > 0$ is a constant.
a) Show that any family $\mathcal{F}$ of Lipschitz continuous functions on $\mathcal{X}$ with Lipschitz constant L is equicontinuous.
b) Show that any family of differentiable functions on a closed manifold Q whose derivative is bounded by L is equicontinuous.

Further examples of equicontinuous families are provided by α-Hölder continuous functions. In practice one often encounters families of weakly differentiable functions on a compact manifold Q that are uniformly bounded in some Sobolev space $W^{k,p}(Q)$. If $\alpha = k - \frac{n}{p} > 0$ where $n = \dim Q$ then these functions are α-Hölder continuous by the Sobolev embedding theorem which applies by compactness of Q.

[14] $\forall \varepsilon > 0\ \exists \delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $d(x, y) < \delta$ and $f \in \mathcal{F}$.
[15] $\sup_{f \in \mathcal{F}} |f(x)| < \infty$ for every $x \in \mathcal{X}$.

3.2.3 Hessian

A Hessian is usually the second derivative of a function or, more generally, of a section of a vector bundle at a point in the domain. However, this is in general only well defined at a critical point, respectively a zero. To extend the concept to general points one chooses a connection or, equivalently, a family of horizontal subspaces. Furthermore, it is often convenient to express the Hessian bilinear form via an inner product as a linear operator, the Hessian operator.

Our setting is the following. Given a symplectic manifold (M,ω) and a Hamiltonian $H:\mathbb{S}^1\times M\to\mathbb{R}$, pick a family $J_t=J_{t+1}$ of ω-compatible almost complex structures and denote by g_t the associated family of Riemannian metrics on M. At each time t consider the corresponding Levi-Civita connection ∇^t with exponential map $\exp^t$ and **parallel transport** $\mathcal{T}_p^t(v):T_pM\to T_{\exp_p^t}M$ along the curve $[0,1]\ni\tau\mapsto\exp_p^t\tau v$.[16] Given a vector field ζ along an arbitrary loop z,[17] set

$$\exp_z\zeta:\mathbb{S}^1\to M,\quad t\mapsto\exp^t_{z(t)}\zeta(t),$$

to obtain a loop in M homotopic to z through $\tau\mapsto z_\tau:=\exp_z\tau\zeta$.

Now consider the map between Banach spaces defined near the origin by

$$f_z:T_z\Lambda M\to\mathcal{E}_z,\quad\zeta\mapsto\mathcal{T}_z(\zeta)^{-1}\operatorname{grad}\mathcal{A}_H(\exp_z\zeta).\tag{3.2.13}$$

Since a $W^{1,2}$ vector field is in particular of class L^2, there is the natural inclusion $T_z\Lambda M\subset\mathcal{E}_z$ which suggests to view this linearization as an unbounded operator with dense domain. Taking the derivative at the origin, that is $Df_z(0)\zeta=\frac{d}{d\tau}\big|_{\tau=0}f_z(\tau\zeta)$ defines the **covariant Hessian operator** of $\mathcal{A}_H$, namely

$$\begin{aligned}A_z:=Df_z(0):L^2_z\supset W^{1,2}_z&\to L^2_z=L^2(\mathbb{S}^1,z^*TM)\\ \zeta&\mapsto -J_t(z)\nabla_t\zeta-(\nabla_\zeta J)(z)\dot z-\nabla_\zeta\nabla H_t(z),\end{aligned}\tag{3.2.14}$$

at any loop z.

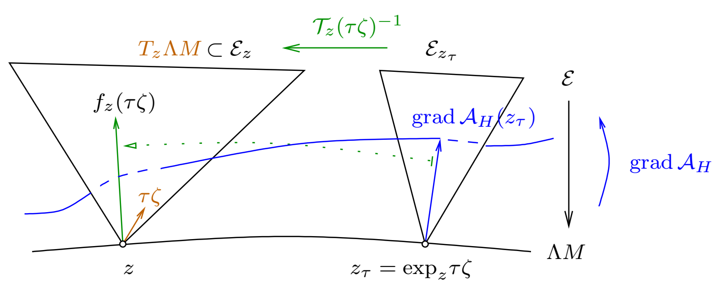


Figure 3.4: The Hessian operator $A_z\zeta=\frac{d}{d\tau}\big|_{\tau=0}f_z(\tau\zeta)$ at any loop z

[16] To easy notation we write $\nabla=\nabla^t$ and $\exp=exp^t$ and so on. But we keep indicating time dependence of quantities which involve perturbation at some stage, such as H_t and J_t.

[17] We assume tacitly that $\|\zeta(t)\|$ is smaller than the injectivity radius of (M,g_t) at $z(t)$.

Exercise 3.2.12. Check that $Df_z(0)\zeta$ is indeed given by (3.2.14). Show that A_z is symmetric and even self-adjoint with compact resolvent. Conclude that the eigenvalues of A_z are real and converge to $\pm\infty$.
[Hint: For symmetry use Exercise 2.2.3. Self-adjointness is a regularity problem. Concerning compact resolvent compare discussion in [Web02, §2.3].]

The eigenvalues of A_z are real by self-adjointness. A critical point z of the symplectic action $\mathcal{A}_H$ is called **non-degenerate** if zero is not an eigenvalue of the Hessian A_z. After picking a trivialization below we shall say more about the spectrum. A **Morse function** is a function all of whose critical points are non-degenerate. A Hamiltonian H is a **regular Hamiltonian** if $\mathcal{A}_H$ is Morse.

Exercise 3.2.13. Show that non-degeneracy of $z \in \mathrm{Crit}\mathcal{A}_H$ as a critical point coincides with non-degeneracy (2.3.23) of z as a 1-periodic trajectory.

Lemma 3.2.14. *A non-degenerate critical point $z \in \Lambda M$ of $\mathcal{A}_H$ is isolated.*

Proof v1. The critical points of $\mathcal{A}_H$ near z are in bijection with the zeroes near the origin of the map f_z given by (3.2.13) and z corresponds to the origin. But no point other than the origin gets mapped to zero, because f_z is a local diffeomorphism near the origin by the inverse function theorem. The latter applies since the linearization $Df_z(0) = A_z$ is a bijection: It is injective by the non-degeneracy assumption, hence surjective by self-adjointness. □

Proof v2. Recall the bijection $z \mapsto z(0) =: p$ from Exercise 2.3.20. Note that $\zeta(t) := d\psi_t(p)\zeta_0$ already lies in the 'kernel' of the differential equation (3.2.14) for any $\zeta_0 \in T_{z(0)}M$. But $\zeta(t)$ must close up at time one in order to lie in the kernel of A_z. This happens precisely if ζ_0 is eigenvector of $d\varphi_1(z(0))$ associated to the eigenvalue 1. Thus there is an isomorphism

$$\ker A_z \cong \mathrm{Eig}_1 d\psi_1(z(0)), \quad \zeta \mapsto \zeta(0). \tag{3.2.15}$$

Now recall Exercise 2.3.21. □

The Hessian with respect to a unitary trivialization

Given a loop z, pick a unitary trivialization (2.2.8) of the symplectic vector bundle $z^*TM \to \mathbb{S}^1$, namely a smooth family Φ of vector space isomorphisms $\Phi(t) : \mathbb{R}^{2n} \to T_{z(t)}M$ intertwining the Hermitian triples $\omega_0, J_0, \langle\cdot,\cdot\rangle_0$ and ω, J, g_J; cf. (1.0.9). Conjugation transforms the Hessian A_z into the unbounded linear operator on $L^2 = L^2(\mathbb{S}^1, \mathbb{R}^{2n})$ with dense domain $W^{1,2}$ given by

$$\begin{aligned} A(z) := \Phi^{-1} A_z \Phi : L^2 \supset W^{1,2} &\to L^2 = L^2(\mathbb{S}^1, \mathbb{R}^{2n}) \\ \zeta &\mapsto -J_0\dot\zeta - S_t\zeta \end{aligned} \tag{3.2.16}$$

where S_t is a 1-periodic family of symmetric matrizes (Exercise 3.2.12), namely

$$S_t v = \Phi^{-1}\bigl(J_t(z)\,(\nabla_t\Phi)\,v + (\nabla_{\Phi v}J)(z)\,\dot z + \nabla_{\Phi v}\nabla H_t(z)\bigr), \quad v \in \mathbb{R}^{2n}. \tag{3.2.17}$$

Example 3.2.15 (Infinite Morse index)**.** The operator $-i\frac{d}{dt}$ on $C^\infty(\mathbb{S}^1, \mathbb{C}^n)$ has eigenvectors $\zeta_k = e^{-i2\pi kt}z$ and eigenvalues $\lambda_k = 2\pi k$, $k \in \mathbb{Z}$, $z \in \mathbb{C} \setminus \{0\}$.

3.2.4 Baire's category theorem – genericity

Since the notions surrounding Baire's category theorem can be slightly confusing we enlist them for definiteness in more detail than needed here. However, all you should take with you in Section 3.2.4 is Baire's category Theorem 3.2.18 part (C) and the first application in Theorem 3.2.22 ($\mathcal{A}_H$ Morse for generic H).

A subset A of a topological space $\mathcal{T}$ is called **nowhere dense** if its closure $\bar{A}$ has empty interior, that is if there is no neighborhood U on which A is dense: For each non-empty open set U in $\mathcal{T}$ there is a non-empty open subset V contained in U and disjoint from A.[18] Any countable union of nowhere dense subsets is called a **meager subset** of $\mathcal{T}$.[19] All other subsets, that is all non-meager subsets, are said to be of the **second category in the sense of Baire**. These are, of course, somewhere dense. As first readings we recommend the Wikipedia article Meagre set, the online handout The Baire category theorem and its consequences, the Blog The Baire category theorem and its Banach space consequences, and §1 of the book [Oxt80].

Exercise 3.2.16. Show that the set $\mathcal{M}$ of meager subsets of $\mathcal{T}$ is a σ-ideal of subsets: Subsets of a meager set are meager, countable unions of meager sets are meager. [Hint: Countable unions of countable unions are countable.]

How about the complements of meager sets? Let's call them **residuals**. Actually they are called **residual subsets** or **comeager subsets**. Let $\mathcal{R}$ be the set of residual subsets. Can the complement R of a meager set be meager? Or is it always non-meager, i.e. of the second category?

Exercise 3.2.17. Suppose U and R are subsets of a topological space $\mathcal{T}$, show:

(i) U open and dense $\Leftrightarrow$ complement U^{C} closed and nowhere dense;

(ii) R residual $:\Leftrightarrow$ R^{C} meager $\Leftrightarrow$ $R \supset$ countable intersect. of open dense sets.

(iii) Countable intersections of residuals are residuals.

[Hint: (i) Interior $\operatorname{int} U = \overline{U^{\mathrm{C}}}^{\mathrm{C}}$. (ii) '$\Rightarrow$' Families de Morgan [Kel55, Thm. 0.3]. '$\Leftarrow$' Suffices to show: Complement R^{C} is contained in a meager set.]

Complete metric spaces

Theorem 3.2.18 (Baire category theorem)**.** *In a complete metric space $\mathcal{X}$*

(A) *meager sets (countable unions of nowhere dense sets) have empty interior;*

(B) *the complement of any meager set is dense, that is residual sets are dense;*

(C) *countable intersections of dense open sets, hence residuals, are dense.*

Exercise 3.2.19. Show that the three assertions (A), (B), (C) are equivalent. [Hint: Take complements.]

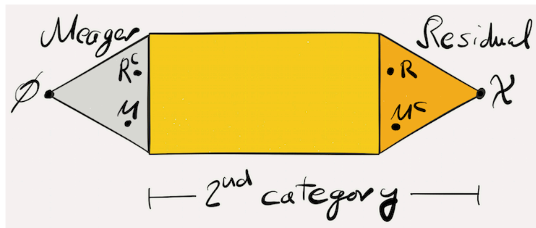


Figure 3.5: Partition of power set $\mathcal{P}(\mathcal{X})$ of non-empty *complete* metric space $\mathcal{X}$

By the exercise it suffices to prove part (C) of the theorem.[20] In applications one often gets away with the following weak form of the Baire theorem (just replace 'dense' by 'non-empty').

Corollary 3.2.20 (Baire category theorem – weak form)**.** *For a non-empty complete metric space $\mathcal{X}$ the following is true.*

(b) *One cannot write $\mathcal{X}$ as a countable union of nowhere dense sets. (A non-empty complete metric space is non-meager in itself.)*

($\widetilde{\text{b}}$) *If $\mathcal{X}$ is written as a countable union of closed sets, then at least one of them has non-empty interior.*

(c) *Countable intersections of dense open sets are non-empty.*

So in a non-empty complete metric space, any set with non-empty interior is of the second category (non-meager) by (A). Moreover, the complement R of a meager set M cannot be meager: Otherwise $\mathcal{X} = M \cup M^{\mathrm{C}}$ contradicting (b). Thus $\mathcal{R} \cap \mathcal{M} = \{M^{\mathrm{C}} \mid M \in \mathcal{M}\} \cap \mathcal{M} = \emptyset$ which answers the introductory questions and is illustrated by Figure 3.5.

Note that the properties *nowhere dense*, *dense*, and *somewhere dense* do not correspond to the sets $\mathcal{M}$, $\mathcal{R}$, and $\mathcal{P}(\mathcal{X}) \setminus (\mathcal{M} \cup \mathcal{R})$. While all elements of $\mathcal{R}$ are dense subsets of $\mathcal{X}$ and all nowhere dense subsets are located in $\mathcal{M}$, it is possible that even dense subsets are elements of $\mathcal{M}$, e.g. $M = \mathbb{Q} \subset \mathcal{X} = \mathbb{R}$.

Remark 3.2.21 (Warning)**.** Obviously, not all subsets of the second category are dense. For example, the non-dense subset $A = [-1, 1] \subset \mathcal{X} = \mathbb{R}$ is of the second category: It is not meager since its complement is not dense. In view of this, the in the literature not uncommon wording *"every set of the second category is dense by Baire's category theorem"* is rather misleading, often based on defining the sets of the second category as those that contain countable intersections of open dense sets, that is on confusing second category and residual.

---

[18] Equivalently, the closure $\bar{A}$ of a nowhere dense set has a dense complement. Equivalently, the complement A^{C} of a nowhere dense set is a set with dense *interior*. (Not every dense set has a nowhere dense complement.)

[19] A meager subset, although a union of nowhere dense sets, can be dense: Consider $\mathbb{Q} \subset \mathbb{R}$.

[20] For a proof see e.g. [Rud91, Thm. 2.2] or [Kel55, Thm. 6.34] or [Oxt80, Thm. 9.1]. See also [KN76, §9] and for references to the original papers see [RS80, Notes to Ch. III].

It is common to call a property P **generic** if it holds for 'typical' examples. More precisely, in our context a generic property is one that is true for the elements of the set $\mathcal{R}$ of residual subsets of a complete metric space $\mathcal{X}$. In this case the property is shared by the elements of a dense set. In other words, by a small perturbation of an arbitrary pick one can get the desired property.[21]

For closed manifolds Q the set of smooth functions equipped with the metric

$$d(f,g) := \sum_{k=0}^{\infty} \frac{1}{2^k} \frac{\|f-g\|_{C^k}}{1+\|f-g\|_{C^k}} \tag{3.2.18}$$

is a complete metric space denoted by $C^\infty(Q)$; see e.g. [Rud91, 1.46] and [Con85, IV.2]. The C^k norm is the sum of the C^0 norms of all partial derivatives up to order k where $\|f\|_{C^0} := \sup_Q |f|$.

Non-degeneracy is a generic property

Theorem 3.2.22 ($\mathcal{A}_H$ Morse for generic H)**.** *Suppose (M,ω) is a symplectic manifold. There exists a residual subset $\mathcal{H}_{\rm reg} \subset C^\infty(\mathbb{S}^1 \times M)$ such that the symplectic action functional $\mathcal{A}_H : \mathcal{L}_0 M \to \mathbb{R}$ is Morse whenever $H \in \mathcal{H}_{\rm reg}$. If M is closed, then $\mathcal{H}_{\rm reg}$ is open and dense in C^∞.*

The proof will be given at the end of Section 3.3.4 on Thom-Smale transversality. It will serve to illustrate abstract transversality theory in a simple setting.

Definition 3.2.23 (Non-degenerate case, Morse-regular Hamiltonians)**.** The terminology **non-degenerate case** refers to the situation $H \in \mathcal{H}_{\rm reg}$, that is all 1-periodic trajectories $z \in \mathcal{P}_0(H) = \mathrm{Crit}\mathcal{A}_H$ are non-degenerate, that is $\mathcal{A}_H$ is Morse. Let us call the elements $H \in \mathcal{H}_{\rm reg}$ **M(orse)-regular Hamiltonians**.

Proposition 3.2.24 (Finite set)**.** *Given a closed symplectic manifold (M,ω) and a M-regular Hamiltonian $H \in \mathcal{H}_{\rm reg}$, then $\mathrm{Crit}\mathcal{A}_H = \mathcal{P}_0(H)$ is a finite set.*

Proof. v1. $\mathrm{Crit}\mathcal{A}_H$ is compact (Lemma 3.2.6) and discrete (Lemma 3.2.14).
v2. Exercise 2.3.22. □

3.3 Downward gradient equation

Throughout (M,ω) is a closed symplectic manifold. To emphasize time dependence we denote time 1-periodic Hamiltonians $H \in C^\infty(\mathbb{S}^1 \times M)$ by H_t. Let

$$J = \{J_t = J_{t+1}\} \subset \mathcal{J}(M,\omega) \tag{3.3.19}$$

be a 1-periodic family of ω-compatible almost complex structures with associated 1-periodic families of Riemannian metrics $g_J = \{g_{J_t} =: g_t\}$ and Levi-Civita connections $\nabla = \{\nabla(g_t) =: \nabla^t\}$. Let $\mathcal{L}_0 M = C^\infty_{\rm contr}(\mathbb{S}^1, M)$ be the space of free

[21] Saying *"pick a generic element x of $\mathcal{X}$"* actually means that one chooses $x \in \mathcal{R}$.

contractible loops in M and $\mathcal{P}_0(H) = \mathrm{Crit}\mathcal{A}_H$ the set of 1-periodic *contractible* trajectories of the Hamiltonian flow ψ given by (2.3.19).

As we indicated in Remark 3.2.8 the initial value problem of the L^2 gradient $\mathrm{grad}\,\mathcal{A}_H$ is ill-posed on the loop space, no matter which Hilbert or Banach completion one takes into consideration; see also Remark 3.2.4. The way out is to interpret a curve $\mathbb{R} \to \mathcal{L}M$ in the loop space asymptotic to critical points $z^\mp \in \mathrm{Crit}\mathcal{A}_H$ as a cylinder in M and the formal downward gradient equation on $\mathcal{L}M$ as a PDE for the cylinder $\mathbb{R} \times \mathbb{S}^1 \to M$ in the manifold with asymptotic boundary conditions given by the two 1-periodic trajectories $z^\mp$. The key property that makes the analysis work is non-degeneracy of the critical points $z^\mp$: This assumption leads to a Fredholm problem, hence to solution spaces of *finite* dimension. In Section 3.3 we follow mainly [Sal99a, FHS95] and [HZ11, §6.5].

Throughout Section 3.3 we fix J as in (3.3.19) and pick H Morse-regular.

3.3.1 Connecting trajectories

Fix J as in (3.3.19). Pick $H \in \mathcal{H}_{\mathrm{reg}}$, that is the functional $\mathcal{A}_H$ is Morse, so its critical points are isolated. A smooth map $u : \mathbb{R} \times \mathbb{S}^1 \to M$, $(s,t) \mapsto u(s,t)$, is called a **trajectory** or a **Floer cylinder** if it satisfies the perturbed non-linear Cauchy-Riemann type elliptic PDE, also called **Floer's equation**, given by

$$\mathcal{F}(u) := \partial_s u + \mathrm{grad}\,\mathcal{A}_H(u_s) = \partial_s u - J_t(u)\partial_t u - \nabla H_t(u) = 0. \tag{3.3.20}$$

Here u_s denotes the loop $u(s,\cdot)$, for the L^2 gradient $\mathrm{grad}\,\mathcal{A}_H(u_s)$ see (3.2.12). Floer's equation generalizes three theories, as indicated in Figure 3.6.

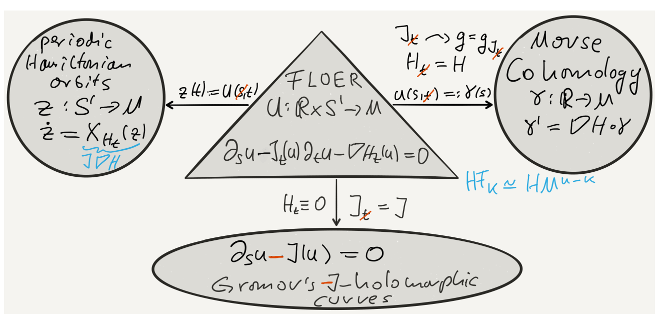


Figure 3.6: Floer's equation interpolates between three theories

For $H \in \mathcal{H}_{\mathrm{reg}}$ and $z^\mp \in \mathrm{Crit}\mathcal{A}_H$ the **set of connecting trajectories** or the **connecting manifold**[22] or the **moduli space** of connecting solutions

$$\mathcal{M}(z^-, z^+) = \mathcal{M}(z^-, z^+; H, J) \tag{3.3.21}$$

consists of all Floer cylinders u with asymptotic limits

$$\lim_{s\to\mp\infty} u(s,t) = z^\mp(t) \tag{3.3.22}$$

[22] Although at this stage $\mathcal{M}(z^-, z^+)$ is not yet a manifold.

where the convergence is uniform in $t \in \mathbb{S}^1$, in other words in $C^0(\mathbb{S}^1)$.

Remark 3.3.1 (Asymptotic convergence: C^0 versus $W^{1,2}$)**.** Only asking $C^0(\mathbb{S}^1)$ convergence in the boundary condition of a connecting trajectory u may come as a surprise, given that the natural domain of the symplectic action functional $\mathcal{A}_H$ is the Banach manifold of $W^{1,2}$ loops in M. For instance, one would expect that the energy of a connecting trajectory is the difference of the action values of the two asymptotic boundary conditions whenever the asymptotic convergence happens with respect to the functional's natural topology, in this case $W^{1,2}$. However, Lemma 3.3.2 below shows that C^0 asymptotic convergence (3.3.22) is already sufficient to enforce finite energy of u. But is this really that surprising?[23] Indeed, together with non-degeneracy of the boundary conditions $z^\mp$, Theorem 3.3.5 below guarantees even exponential convergence – based on just a C^0 convergence assumption, but not to forget that u solves an elliptic PDE..

Energy

A useful notion concerning gradient type equations is the **energy** of arbitrary paths, that is arbitrary cylinders in M in our case. It is defined by

$$E(u) := \frac{1}{2}\int_0^1 \int_{-\infty}^{\infty} \left(|\partial_s u|^2 + |J_t(u)\partial_t u + \nabla H_t(u)|^2 \right) ds\, dt \geq 0 \tag{3.3.23}$$

for *any* smooth cylinder $u : \mathbb{R} \times \mathbb{S}^1 \to M$ and where the integrand is evaluated at (s, t), of course. Note the following.

- The energy $E(u) \geq 0$ vanishes precisely *on the periodic trajectories*, that is the **constant trajectories** $u(s, \cdot) \equiv z \in \mathcal{P}_0(H)$;
- The energy of a *trajectory* is the square of the L^2 norm, namely
$$E(u) = \|\partial_s u\|_2^2; \tag{3.3.24}$$
- Among all smooth cylinders $w : \mathbb{R} \times \mathbb{S}^1 \to M$ subject to the same asymptotic boundary conditions (3.3.22), whether $z^\mp \in \mathrm{Crit}\mathcal{A}_H$ are non-degenerate or not, it is precisely the Floer cylinders u that *minimize the energy* with $E(u) = \mathcal{A}_H(z^-) - \mathcal{A}_H(z^+)$; see Exercise 3.3.4.

Lemma 3.3.2 (Connecting trajectories are of finite energy)**.** *Given a connecting trajectory $u \in \mathcal{M}(z^-, z^+)$, non-degeneracy of $z^\mp$ is actually not needed, then*[24]

$$\begin{aligned} E(u) &= \int_{\mathbb{R}\times\mathbb{S}^1} u^*\omega - \int_0^1 H_t(z^-(t))\, dt + \int_0^1 H_t(z^+(t))\, dt \\ &= \mathcal{A}_H(z^-) - \mathcal{A}_H(z^+). \end{aligned} \tag{3.3.25}$$

[23] As the L^2 gradient does not generate a flow, the cylinder substitutes are already special to start with. Secondly, being solutions to a PDE, as opposed to an ODE, they are rather exotic, hence rare, creatures. But from exceptional objects one may expect exceptional behavior.

[24] By definition $\int_{\mathbb{R}\times\mathbb{S}^1} v^*\omega := \int_0^1 \int_{-\infty}^{\infty} \omega(\partial_s u, \partial_t u)\, dsdt$, mind that the order of s, t in $\mathbb{R}\times\mathbb{S}^1$ must be the same as of ∂_s, ∂_t when inserted into $u^*\omega$.

Proof. By (3.3.24) and the gradient nature of the Floer equation we obtain that

$$\begin{aligned}
E(u) &= \int_{-\infty}^{\infty} \langle \partial_s u_s, \partial_s u_s \rangle_{L^2(\mathbb{S}^1)}\, ds \\
&= \lim_{T\to\infty} \underbrace{\left(\mathcal{A}_H(u_{-T}) - \mathcal{A}_H(u_T)\right)}_{\le \mathcal{A}_H(z^-)-\mathcal{A}_H(z^+)} \\
&= \lim_{T\to\infty} \left(\int_{[-T,T]\times\mathbb{S}^1} u^*\omega - \int_0^1 H\circ u_{-T}\, dt + \int_0^1 H\circ u_T\, dt \right).
\end{aligned}$$

To obtain the final step we fixed an extension $\bar{v} : \mathbb{D} \to M$ of $z^- : \mathbb{S}^1 \to M$. Then we chose the extensions of the loops u_{-T} and u_T required by the definition of $\mathcal{A}_H$ by simply connecting these loops along the cylinder u to z^- and its spanning disk $\bar{v}$. In the difference $\mathcal{A}_H(u_{-T}) - \mathcal{A}_H(u_T)$ each of the integrals $\int_{\mathbb{D}} \bar{v}^*\omega$ and $\int_{(-\infty,-T]\times\mathbb{S}^1} u^*\omega$ appears twice, but with opposite signs. Because the difference is uniformly bounded from above, the limit as $T \to \infty$ exists and is given by the RHS of the first identity in (3.3.25); here C^0 convergence (3.3.22) enters. The second identity in (3.3.25) holds by the earlier argument: Given an extension $\bar{v} : \mathbb{D} \to M$ of the loop z^- choose the natural extension $u\#v$ of z^+. $\square$

Remark 3.3.3 (Circumventing non-boundedness of $\mathcal{A}_H$ – finite energy)**.** To construct a Morse complex one needs that the trajectories used to define the boundary operator have precisely one critical point sitting asymptotically at each of the two ends. In our case, suppose $u : \mathbb{R} \times \mathbb{S}^1 \to M$ is a trajectory, how can we guarantee existence of asymptotic limits $z^\mp \in \mathrm{Crit}\mathcal{A}_H$? Well, if they exist, the energy identity (3.3.26) shows that u is of *finite energy*. Indeed it turns out, see Theorem 3.3.5, that finite energy of a trajectory u is sufficient to enforce existence of asymptotic limits $z^\mp \in \mathrm{Crit}\mathcal{A}_H$ – under the assumption that our functional $\mathcal{A}_H$ is Morse. This is why we pick $H \in \mathcal{H}_{\mathrm{reg}}$ in Section 3.3.

Exercise 3.3.4. Any smooth cylinder $w : \mathbb{R}\times\mathbb{S}^1 \to M$ subject to the asymptotic boundary conditions (3.3.22), whether $z^\mp \in \mathrm{Crit}\mathcal{A}_H$ are non-degenerate or not, satisfies the identity

$$\begin{aligned}
E(u) = \frac{1}{2}\int_0^1 \int_{-\infty}^{\infty} &\left|\partial_s u - J_t(u)\partial_t u - \nabla H_t(u)\right|^2 ds\, dt \\
&+ \mathcal{A}(z^-) - \mathcal{A}(z^+).
\end{aligned} \tag{3.3.26}$$

[Hint: Start at the integral term in (3.3.26). Permute the integrals, so the integral over t becomes the $L^2(\mathbb{S}^1)$ inner product of $\partial_s u_s + \operatorname{grad} \mathcal{A}_H(u_s)$ with itself. Use the gradient nature of the Floer equation to end up with $E(u) + \lim_{T\to\infty}(\mathcal{A}_H(u_T) - \mathcal{A}_H(u_{-T}))$ which is equal to $E(u) + \mathcal{A}_H(z^+) - \mathcal{A}_H(z^-)$, as shown in the proof of Lemma 3.3.2.]

Finite energy trajectories

Consider the **set of finite energy trajectories**

$$\mathcal{M} := \{\text{solutions } u : \mathbb{R}\times\mathbb{S}^1 \to M \text{ of Floer's equation (3.3.20)} \mid E(u) < \infty\}.$$

If $\mathcal{A}_H$ is Morse, then by Theorem 3.3.5 below every finite energy trajectory is connecting and this non-trivial fact contributes the inclusion $\subset$ in the identity

$$\mathcal{M} = \bigcup_{z^\mp \in \mathcal{P}_0(H)} \mathcal{M}(z^-, z^+). \tag{3.3.27}$$

The other inclusion $\supset$ already holds without the Morse assumption by (3.3.25). In the non-degenerate case, still assuming that ω and $c_1(M)$ vanish over $\pi_2(M)$, counting with appropriate signs the 1-dimensional components appearing on the RHS of (3.3.27) defines the Floer boundary operator. In the special case of an autonomous C^2 small Morse function H, see Proposition 2.3.16, and autonomous J, the count defines the Morse boundary operator and the RHS of (3.3.27) is naturally homeomorphic to M itself. Consequently Floer homology represents the singular integral co/homology of M.[25] These remarks show the significance of

Theorem 3.3.5. *Let $u : \mathbb{R} \times \mathbb{S}^1 \to M$ be a Floer cylinder in the non-degenerate case,[26] then the following are equivalent.*

`(finite energy)` $E(u) < \infty$.

`(asymp.limits)` *There exist periodic trajectories $z^\mp \in \mathcal{P}_0(H)$ which are the C^0 limits (3.3.22) of the loops u_s and $\partial_s u_s(t) \to 0$, as $s \mp \infty$, again uniformly in t.*

`(exp.decay)` *There exist constants $\delta, c > 0$ such that*

$$|\partial_s u(s,t)| \le c e^{-\delta |s|}$$

at every point (s,t) of the cylinder $\mathbb{R} \times \mathbb{S}^1$.

Outline of proof. The proof is non-trivial, for details we recommend [Sal99a]. Roughly speaking, `(finite energy)`, namely by (3.3.24) finiteness of the integral $\|\partial_s u\|_2^2 = E(u) < \infty$ over the whole cylinder $\mathbb{R} \times \mathbb{S}^1$, enforces via additivity of the integral with respect to disjoint union of the domain that the integrals over annuli $(T, T-1) \times \mathbb{S}^1$ must converge to zero, as $T \to \mp\infty$. But a mean value inequality for $e = |\partial_s u(s,t)|^2$ based on a differential inequality of the form $\Delta e := {\partial_s}^2 + {\partial_t}^2 \ge -c_1 - c_2 e^2$ provides a pointwise estimate of $|\partial_s u(s,t)|$ in terms if the L^2 norm over some annulus A_s which does not depend on s. Thus $|\partial_s u_s| = |\partial_t u_s - X_{H_t}(u_s)|$ converges to zero, as $s \to \mp\infty$, uniformly in t. From this one already concludes existence of a sequence $s_k \to \infty$ such that u_{s_k} converges uniformly to some periodic trajectory z^+. But by non-degeneracy all periodic trajectories are isolated which implies that any sequence diverging to $+\infty$ leads to the same limit; similarly for $s \to -\infty$. One has confirmed `(asymptotic limits)`. But `(asymptotic limits)` implies `(finite energy)` by Lemma 3.3.2.

[25] As M is closed and automatically orientable, Poincaré duality identifies $\mathrm{H}^{2n-k} \cong \mathrm{H}_k$.

[26] That is u satisfies the Floer equation (3.3.20) on the closed manifold M and $H \in \mathcal{H}_{\mathrm{reg}}$.

Obviously `(exponential decay)` immediately leads to `(finite energy)` by (3.3.24) and explicit integration. Conversely, how `(finite energy)` leads to `(exponential decay)` is hard to illustrate, have a look at [Sal99a, §2.7]. A key observation is that $\xi_s(t) = \partial_s u_s(t)$ lies in the kernel of the trivialization $D = \frac{d}{ds} + A(s) = \partial_s - \bar{J}_0\partial_t - S(s,t) - C(s,t)$ of the linearized operator D_u, see (3.3.31) and (3.3.35), and that for kernel elements the function $f(s) := \frac{1}{2}\int_0^1 |\xi_s(t)|^2\,dt$ satisfies a differential inequality $f''(s) \geq \delta^2 f(s)$ for $|s|$ sufficiently large; this is based on invertibility of the Hessian operators $A(s)$ at $\mp\infty$, hence near $\mp\infty$. It is here where non-degeneracy of the asymptotic boundary conditions $z^{\mp}$ enters. This way one arrives at an L^2 version of the desired estimate, namely $f(s) \leq c' e^{-\delta|s|}$. Application of the operator $\partial_s + \bar{J}_0\partial_t$ to $D\xi = 0$ leads to a differential inequality $\Delta|\xi_s(t)|^2 \geq -c''|\xi_s(t)|^2$, thus to a mean value inequality for $|\xi_s(t)|^2$ which together with the formerly obtained L^2 estimate establishes `(exponential decay)`. □

3.3.2 Fredholm theory

For convenience of the reader we enlist some basic notions and tools of Fredholm theory. Concerning details we highly recommend [Sal99b].

Suppose throughout that X, Y, Z are Banach spaces. Denote by $\mathcal{L}(X,Y)$ the **Banach space of bounded linear operators** $T : X \to Y$ equipped with the **operator norm** $\|T\| := \sup_{\|x\|=1}\{\|Tx\| \colon x \in X\}$. A bounded linear operator $D : X \to Y$ is a **Fredholm operator** if it has a closed range and finite dimensional kernel and cokernel. The latter is given by the quotient space $\operatorname{coker} D = Y/\operatorname{im} D$ and inherits the Banach space structure of Y by closedness. The map defined on the space of Fredholm operators $\mathcal{F}(X,Y)$ by

$$\operatorname{index} D := \dim\ker D - \dim\operatorname{coker} D$$

is called the **Fredholm index** of D.

Exercise 3.3.6 (Stability properties)**.** a) Show that the subset $\mathcal{F}(X,Y) \subset \mathcal{L}(X,Y)$ is open with respect to the operator norm and the index is **locally constant**, that is constant on each component.

b) A **compact operator** is a (bounded) linear operator $K : X \to Y$ which takes bounded sets to **precompact sets**, that is sets of compact closure. Show that the sum $D + K$ of a Fredholm operator D and a compact operator K is Fredholm and of the same index.

Theorem 3.3.7 (semi-Fredholm estimate)**.** *Given $D \in \mathcal{L}(X,Y)$ and a compact operator $K : X \to Z$, suppose there is a constant $c > 0$ such that*

$$\|x\|_X \leq c\left(\|Dx\|_Y + \|Kx\|_Z\right) \tag{3.3.28}$$

for every $x \in X$. Then D has closed range and finite dimensional kernel.

Exercise 3.3.8. Prove the previous theorem. Use it to show openness $\mathcal{F} \subset \mathcal{L}$. [Hint: Concerning finite dimensionality it suffices to show that the unit ball in

$\ker D$ is compact. Use finite dimension together with the Hahn-Banach theorem to reduce the proof of closed range to the case in which D is injective; choose a complement X_1 of $\ker D$ and replace X by X_1. Concerning openness use again finiteness of $k = \dim \ker D$ to define an augmentation $(D, K_0) : X \to Y \oplus \mathbb{R}^k$ of D which is injective and has closed range Z. Apply the open mapping theorem, cf.proof of Lemma 3.3.9, to conclude that the inverse of the bounded linear bijection $(D, K_0) : X \to Z$ is continuous.]

Lemma 3.3.9. *For a bounded linear operator $D : X \to Y$ are equivalent:*

- *D is an injection with closed range*
- *There is a constant $c > 0$ such that*

$$\|x\|_X \le c \, \|Dx\|_Y \tag{3.3.29}$$

for every $x \in X$.

Proof. '$\Rightarrow$' By the open mapping theorem[27] *the inverse of a bounded linear bijection between Banach spaces is continuous.* Now pick $c := \|(\tilde{D})^{-1}\|_{\mathcal{L}(\operatorname{im} D, X)}$ where $\tilde{D} : X \to \operatorname{im} D$, $x \mapsto Dx$. '$\Leftarrow$' By contradiction the inequality itself shows that $\ker D = \{0\}$. By Theorem 3.3.7 the range of D is closed. □

Definition 3.3.10. A **complement**, often called **topological complement**, of a closed linear subspace $X_0 \subset X$ is a closed linear subspace $Z \subset X$ such that $X_0 \oplus Z = X$.[28] If X_0 admits a complement it is a **complemented subspace**.

Examples of complemented subspaces are finite dimensional subspaces and closed subspaces of finite codimension. In a Hilbert space every closed subspace is complemented. We recommend the book by Brezis [Bre83, II.4].

Definition 3.3.11. A **right inverse** of a surjective bounded linear operator $D : X \to Y$ is a bounded linear operator $T : Y \to X$ such that $DT = 1\!\!1_Y$.

Exercise 3.3.12. Given a surjective bounded linear operator $D : X \to Y$, then

$$D \text{ admits a right inverse } T \quad \Leftrightarrow \quad \ker D \text{ is complemented.}$$

[Hint: '$\Rightarrow$' A natural try is $Z := \operatorname{im} T$. Use $DT = 1\!\!1_Y$ to derive the injectivity estimate (3.3.29) for T and to conclude $\operatorname{im} T \cap \ker D = \{0\}$. Writing $x = x - TDx + TDx$ shows that $X = (\ker D) + Z$. '$\Leftarrow$' Note that the restriction $D|_Z : Z \to Y$ to the complement is a bounded bijection.]

The previous exercise shows that any surjective Fredholm operator admits a right inverse. This generalizes, see part ii), as follows; for details see e.g. [Web02].

[27] **The open mapping theorem.** A bounded linear surjection between Banach spaces is open. A map is called **open** if it maps open sets to open sets.

[28] Here $\oplus$ denotes the **internal direct sum** of two closed subspaces which by definition means $X_0 + Z = X$ and $X_0 \cap Z = \{0\}$.

Exercise 3.3.13. Let $D : X \to Y$ be Fredholm and $Z \in \mathcal{L}(Z, Y)$. It holds that:

i) The range of the bounded operator

$$D \oplus L : X \oplus Z \to Y, \quad (x, z) \mapsto Dx + Lz,$$

is closed with finite dimensional complement. How about $\dim \ker(D \oplus L)$? Give an example in which $\ker(D \oplus L)$ is infinite dimensional.

ii) If $D \oplus L$ is surjective, then $\ker(D \oplus L)$ admits a complement, thus a right inverse. Moreover, the projection $P : \ker(D \oplus L) \to Z$, $(x, z) \mapsto z$, to the second component is a Fredholm operator with

$$\ker P \simeq \ker D, \qquad \operatorname{coker} P \simeq \operatorname{coker} D,$$

thus $\operatorname{index} P = \operatorname{index} D$.

Remark 3.3.14 (Non-linear Fredholm theory)**.** A continuously differentiable map $f : X \to Y$ is a **Fredholm map** if the linearization $df(x) : X \to Y$ is a Fredholm operator for every $x \in X$. In this case $\operatorname{index}(f) := \operatorname{index} df(x)$ is the **Fredholm index of the Fredholm map** f.

For any map $f : X \to Y$ of class C^ℓ with $\ell \geq 1$, Fredholm or not, an element $y \in Y$ is called a **regular value** of f if the linear operator $D = df(x) : X \to Y$ is onto and admits a *right inverse* for every element $x \in f^{-1}(y)$ in the pre-image of y.[29] Then by the implicit function theorem, see e.g. [MS04, Thm. A.3.3], the **regular level set**

$$\mathcal{M} := f^{-1}(y) \subset X \tag{3.3.30}$$

is a C^ℓ Banach manifold whose tangent spaces are given by the kernels, that is

$$T_x\mathcal{M} = \ker df(x).$$

This result is called the **regular value theorem**. If f is even a Fredholm map, then $\mathcal{M}$ is finite dimensional and $\dim \mathcal{M} = \operatorname{index}(f)$.

Exercise 3.3.15. The Fredholm index of a Fredholm map is well defined.

3.3.3 Connecting manifolds

Consider a connecting manifold $\mathcal{M}(z^-, z^+; H, J)$ as defined by (3.3.21). Recall that $H \in \mathcal{H}_{\mathrm{reg}}$ is Morse-regular. In this section we show that it is a smooth manifold for generic Hamiltonian H and its dimension is the difference of the canonical Conley-Zehnder indices of $z^\mp$.

To start with denote the left hand side of Floer's equation (3.3.20) by $\mathcal{F}_H(u)$, often denoted by $\bar\partial_{H,J}$ to emphasize its Cauchy-Riemann type nature. Let

$$D_u = D\mathcal{F}_H(u) : W^{1,p}_u := W^{1,p}(\mathbb{R} \times \mathbb{S}^1, u^*TM) \to L^p_u$$

[29] Any y outside the image of f, i.e. with empty pre-image $f^{-1}(y) = \emptyset$, is a regular value.

denote linearization at a zero u. Here $p > 2$ is a constant; see Remark 3.3.21. As outlined below, see (3.3.32), the linearized operator is of the form

$$\begin{aligned} D_u\zeta &= \nabla_s\zeta - J_t(u)\nabla_t\zeta - \nabla_\zeta\nabla H_t(u) - (\nabla_\zeta J)(u)\partial_t u \\ &= \left(\tfrac{D}{ds} + A_{u_s}\right)\zeta \end{aligned} \tag{3.3.31}$$

for every smooth compactly supported vector field ζ along u. Here A_{u_s} denotes the covariant Hessian operator (3.2.14) of $\mathcal{A}_H$ based at a loop $u_s := u(s,\cdot)$. Actually formula (3.3.31) not only makes sense for $p > 2$, let us allow $p > 1$. The Sobolev spaces L^p_u and $W^{1,p}_u$ are the closures of the vector space of smooth compactly supported vector fields ζ along u with respect to the Sobolev norms

$$\|\zeta\|_p := \left(\int_{-\infty}^\infty\int_0^1 |\zeta|^p\right)^{\frac{1}{p}}, \quad \|\zeta\|_{1,p} := \left(\int_{-\infty}^\infty\int_0^1 |\zeta|^p + |\nabla_s\zeta|^p + |\nabla_t\zeta|^p\right)^{\frac{1}{p}}.$$

Definition 3.3.16. Abbreviate $\mathcal{H} := C^\infty(\mathbb{S}^1 \times M)$. The elements of the set

$$\mathcal{H}_{\rm reg}(J) := \{H \in \mathcal{H} \mid D\mathcal{F}_H(u) \text{ onto } \forall u \in \mathcal{M}(z^-, z^+; H, J)\ \forall z^\mp \in \mathcal{P}_0(H)\}$$

are called **MS-regular Hamiltonians** or **Morse-Smale Hamiltonians**.

Exercise 3.3.17 (Constant connecting trajectories, MS-regular ⇒ M-regular)**.** Suppose $H \in \mathcal{H}_{\rm reg}(J)$. Show that every Hamiltonian loop $z \in \mathcal{P}_0(H)$ is nondegenerate, hence $\mathcal{A}_H$ is Morse. This shows that $\mathcal{H}_{\rm reg}(J) \subset \mathcal{H}_{\rm reg}$.
[Hint: Consider the constant trajectory $u_s \equiv z$.]

Exercise 3.3.18 (Morse-Smale condition)**.** As indicated in Figure 3.6, if u, H, J are independent of t, then the Floer equation recovers the gradient flow of ∇H on the closed Riemannian manifold (M, g_J). Show that in this case the elements of $\mathcal{H}_{\rm reg}(J)$ are precisely those Morse functions on M which satisfy the **Morse-Smale condition**, i.e. all stable and unstable manifolds intersect transversely.

The significance of MS-regular Hamiltonians $H \in \mathcal{H}_{\rm reg}(J)$ lies in the fact that their connecting manifolds are smooth manifolds, even of finite dimension, as a consequence of the regular value theorem; cf. Remark 3.3.14. However, to apply that theorem, two assumptions need to be verified: Firstly, for zero to be a regular value, the kernel of D_u needs to be complemented; cf. Exercise 3.3.12. Secondly, to get to finite dimension of $\mathcal{M}(z^-, z^+; H, J)$ the kernel has to be finite dimensional. Fredholm operators satisfy both criteria.

Thus, modulo proving that D_u is a Fredholm operator and calculating its Fredholm index, the regular value theorem provides part (ii) of

Theorem 3.3.19 (Connecting manifolds)**.** *Given a closed symplectic manifold (M, ω) and a family of ω-compatible almost complex structures $J_t = J_{t+1}$, then*

(i) *the set $\mathcal{H}_{\rm reg}(J)$ of regular Hamiltonians is a residual of $\mathcal{H} := C^\infty(\mathbb{S}^1 \times M)$;*

(ii) *for any* $H \in \mathcal{H}_{\mathrm{reg}}(J)$ *any* $\mathcal{M}(z^-, z^+) := \mathcal{M}(z^-, z^+; H, J)$ *is a smooth manifold. The dimension of the component of* u *is given by*[30]

$$\dim \mathcal{M}(z^-, z^+)_u = \operatorname{index} D\mathcal{F}_H(u) = \mu^{\mathrm{CZ}}(z^-) - \mu^{\mathrm{CZ}}(z^+)$$

where μ^{CZ} *is the canonical Conley-Zehnder index normalized by (2.1.5).*

For the proof of part (i) we refer to [FHS95, §5]. It utilizes a tool called Thom-Smale transversality theory, see Section 3.3.4 below. It is crucial that the linearized operator $D\mathcal{F}_H(u)$ is already Fredholm to start with, but this holds true precisely for Morse-regular Hamiltonians. This explains one reason for our standing assumption $H \in \mathcal{H}_{\mathrm{reg}}$. Concerning part (ii) we shall sketch below the proof of the Fredholm property of $D_u = D\mathcal{F}_H(u)$ and the calculation of its Fredholm index denoted by index D_u.

Remark 3.3.20 (Critical points unaffected by MS-perturbation)**.** By [FHS95, Thm. 5.1 (ii)] one can C^∞ approximate a given Morse-regular H by MS-regular Hamiltonians H^ν which C^2 agree with H along its (finitely many) 1-periodic trajectories. Thus $\mathrm{Crit}\mathcal{A}_H \subset \mathrm{Crit}_{H^\nu}$. In [FHS95, Rmk. 5.2 (ii)] it is pointed out that it is an *open problem* whether it is sufficient to perturb H outside some open neighborhood $U \subset M$ of the images of the 1-periodic trajectories of H. This would guarantee equality $\mathrm{Crit}\mathcal{A}_H = \mathrm{Crit}\mathcal{A}_{H^\nu}$ for large ν. However, the authors point out that it is possible to perturb the family J outside such neighborhood U to achieve MS-regularity for the given M-regular Hamiltonian H itself, just with respect to a perturbed 1-periodic family of ω-compatible structures.

Linearization at general cylinders u

When it comes to gluing, in Section 3.4.2, it is necessary to linearize $\mathcal{F} = \mathcal{F}_H$ given by (3.3.20) not only at zeroes of $\mathcal{F}_H$, but at more general smooth cylinders $u : \mathbb{R} \times \mathbb{S}^1 \to M$. The procedure is completely analogous to the definition of the covariant Hessian (3.2.14), just replace the map in (3.2.13) by the map

$$f_u : L^p_u \supset W^{1,p}_u \to L^p_u, \quad \zeta \mapsto \mathcal{T}_u(\zeta)^{-1} \mathcal{F}_H(\exp_u \zeta), \tag{3.3.32}$$

given any cylinder $u \in \mathcal{B}^{1,p}(z^-, z^+)$. Here $\mathcal{B} = \mathcal{B}^{1,p}(z^-, z^+)$ denotes the Banach manifold which, roughly speaking, consists of all continuous cylinders $u : \mathbb{R} \times \mathbb{S}^1 \to M$ which are locally of Sobolev class[31] $W^{1,p}$ and converge asymptotically in a suitable way to the given periodic trajectories $z^\mp$. Convergence of the elements u of $\mathcal{B}$ must be such that the tangent space $T_u\mathcal{B}$ coincides with the space $W^{1,p}_u := W^{1,p}(\mathbb{R} \times \mathbb{S}^1, u^*TM)$ of $W^{1,p}$ vector fields along u; see [FHS95, Thm. 5.1] for details. Note that $\mathcal{F}_H(u) \in L^p_u$ is a vector field along u of class L^p. All these spaces fit together in the form of a Banach space bundle $\mathcal{E} \to \mathcal{B}$ whose fiber over $u \in \mathcal{B}$ is $\mathcal{E}_u = L^p_u$; see Figure 3.3 for a similar case. Now $\mathcal{F}_H$

[30] Do not miss our standing assumptions $\mathrm{I}_{c_1} = 0 = \mathrm{I}_\omega$.

[31] Here it is crucial to choose $p > 2$, because in this case $W^{1,p}$ implies continuity, so one can work with local coordinate charts on M to analyze u.

is a section of $\mathcal{E}$ whose regularity depends on the regularity of the function H; which is smooth here. To show that

$$D\mathcal{F}_H(u)\zeta := \left.\frac{d}{d\tau}\right|_0 f_u(\tau\zeta) \tag{3.3.33}$$

is equal to the operator D_u displayed in (3.3.31) one can either utilize local coordinates on M or work with global notions.[32] Details of both possibilities can be found in [Web99, App. A] in the slightly different but related context of Section 3.5.

Remark 3.3.21 (The condition $p > 2$)**.** Roughly speaking, maps on a 2-dimensional domain of Sobolev class $W^{k,p}$ are continuous, so one can localize, and well behaved with respect to relevant compositions and products whenever $kp > 2$; for details see in [MS04] the paragraphs prior to Prop. 3.1.9 and App. B, see also the Blog L^p or not L^p, that is the question.
For instance, the non-linear Fredholm theory, see Remark 3.3.14, requires to equip the domain $\mathcal{B}$ of the section $\mathcal{F} = \mathcal{F}_H$ in (3.3.20) with the structure of a differentiable Banach manifold. Choosing $\mathcal{B} = \mathcal{B}^{1,p}(z^-, z^+)$ with $p > 2$, as we did in the previous remark, does the job (and is the usually selected option), but leads into the realm of L^p estimates which are much harder to obtain than L^2 estimates. Another option would be to choose $k = 2$ and $p > \frac{2}{2}$, of course $p = 2$, and view $\mathcal{F}_H$ as a section of the Hilbert space bundle $\mathcal{E}^{1,2} \to \mathcal{B}^{2,2}(z^-, z^+)$. Unfortunately, this choice brings in higher derivatives.

The Fredholm operator D_u

Consider two (non-degenerate) critical points $z^\mp \in \mathrm{Crit}\mathcal{A}_H$ and suppose that $u \in \mathcal{M}(z^-, z^+; H, J)$ is a connecting trajectory.

Theorem 3.3.22 (Fredholm operator)**.** *By non-degeneracy of $z^\mp$ the linear operator $D_u : W^{1,p}_u \to L^p_u$ given by (3.3.31) is Fredholm for $1 < p < \infty$ and*

$$\operatorname{index} D_u = \mu^{\mathrm{CZ}}(z^-) - \mu^{\mathrm{CZ}}(z^+). \tag{3.3.34}$$

To prove the theorem it is convenient to represent D_u by an operator D acting on vector fields that take values in $\mathbb{R}^{2n}$.

Trivialization

The connecting trajectory $u \in \mathcal{M}(z^-, z^+; H, J)$ extends continuously from the open cylinder $Z = \mathbb{R} \times \mathbb{S}^1$ to its compactification $\overline{Z} = (\mathbb{R} \cup \{\mp\infty\}) \times \mathbb{S}^1$, because there are the two periodic trajectories $z^\mp$ sitting at the ends by the asymptotic C^0 limit condition (3.3.22). Thus we have in fact a hermitian vector bundle $\bar{u}^*TM$ over the compact cylinder-with-boundary $\overline{Z}$. Pick a canonical unitary trivialization $\Phi(s,t) : \mathbb{R}^{2n} \to T_{\bar{u}(s,t)}M$ according to Proposition 2.2.2. In fact,

[32] We use the notation $D\mathcal{F}_H(u)\zeta$ of a Fréchet derivative, but define it by Gâteaux derivative $\frac{D}{d\tau}|_{\tau=0}\, \mathcal{F}_H(\exp_u \tau\zeta)$. See e.g. [LV03, § 3.1] for these notions and when they coincide.

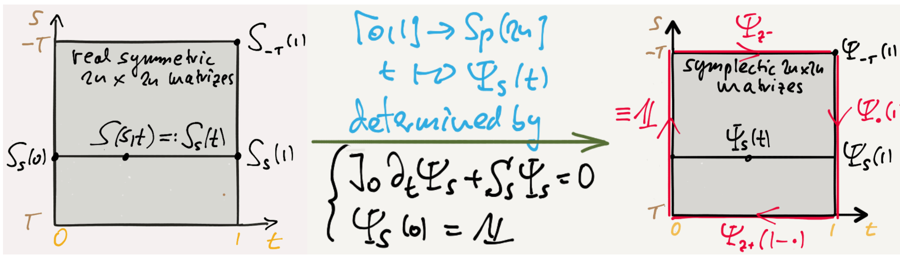


Figure 3.7: Rectangles worth of matrices: Symmetric $\longrightarrow$ symplectic

for ease of notation, let us right away agree to omit any 'bars' from now on completely. Proceed as in (3.2.16) and represent $D_u : W^{1,p}_u \to L^p_u$ by the operator

$$D = \tfrac{d}{ds} + A(s) = \partial_s - J_0\partial_t - S(s,t) + \cancel{C(s,t)} : W^{1,p} \to L^p \tag{3.3.35}$$

where $W^{1,p} := W^{1,p}(\mathbb{R}\times\mathbb{S}^1, \mathbb{R}^{2n})$ and $L^p := L^p(\mathbb{R}\times\mathbb{S}^1,\mathbb{R}^{2n})$. Here the family of symmetric matrizes $S(s,t) = S(s,t)^T \in \mathbb{R}^{2n\times 2n}$ is given by replacing in (3.2.17) the loop z by the family of loops $s \mapsto u_s$ and $\Phi(t)$ by $\Phi(s,t)$. The only new element is the matrix family $C(s,t) = \Phi(s,t)^{-1}\left(\nabla_s \Phi(s,t)\right)$ which converges to zero, as $s \mp \infty$, uniformly in t, because ∇_s actually stands for $\nabla_{\partial_s u}$ and $\partial_s u$ vanishes asymptotically by `(asymptotic limits)`.[33] The bad news is that in general C is not symmetric, thereby destroying self-adjointness of $A(s)$. The good news is that C only amounts to a compact perturbation of D (cf. [RS95, Le. 3.18]), so Fredholm property and index, if any, will not change by Exercise 3.3.6 if we simply ignore C (while investigating *these two* properties).

From now on we set $C = 0$. For fixed $s \in \mathbb{R}$ the path of symmetric matrizes $S_s : [0,1] \to \mathbb{R}^{2n\times 2n}$, $t \mapsto S(s,t)$, determines a symplectic path $t \mapsto \Psi_s(t)$ by

$$\dot\Psi_s = J_0 S_s \Psi_s = \bar J_0(-S_s)\Psi_s, \qquad \Psi_s(0) = \mathbb{1},$$

see Figure 3.7 and Exercise 2.1.10. The asymptotic limit matrizes $S^\mp(t) := \lim_{s\to\mp\infty} S(s,t)$ are given by (3.2.17) with $z = z^\mp$ and their corresponding symplectic paths $\Psi^\mp$ lie in $\mathcal{SP}^*(2n)$: Indeed the paths $\Psi^\mp$ coincide with the paths $\Psi_{z^\mp,v^\mp}$ in (2.3.24), because they satisfy the same ODE and initial condition, but the latter lie in $\mathcal{SP}^*(2n)$ by non-degeneracy of $z^\pm$; cf. Exercises 2.3.23 and 3.2.13. Thus $\Psi^\mp$ do admit Conley-Zehnder indices for which we shall choose the canonical normalization (2.1.5), of course. By the uniform limit condition (3.3.22) these indices are actually already shared by the paths $\Psi_{\mp T} \in \mathcal{SP}^*(2n)$ whenever $T > 0$ is sufficiently large. For simplicity we set

$$\Psi_{z^\mp} := \Psi_{\mp T}. \tag{3.3.36}$$

Fredholm property

To prove that $D : W^{1,p} \to L^p$ is a Fredholm operator one first establishes for D the semi-Fredholm estimate (3.3.28) with some suitable compact operator, say

[33] In fact, for any cylinder $u \in \mathcal{B}^{1,p}(z^-, z^+)$ asymptotic convergence $\partial_s u \to 0$ holds.

the operator $K : W^{1,p} \to W^{1,p}_T \hookrightarrow L^p_T$ given by restriction (continuous) followed by inclusion (compact by [RS95, Le. 3.8]). Here $T > 1$ is a large constant and $W^{1,p}_T$ denotes $W^{1,p}([-T,T] \times \mathbb{S}^1, \mathbb{R}^{2n})$. The semi-Fredholm estimate can be established for any $1 < p < \infty$.[34] This is non-trivial, we recommend the excellent presentation in [Sal99a, § 2.3], Via the transformation $s \mapsto -s$, one derives a semi-Fredholm estimate for the **formal adjoint operator**[35] $D^* = -\frac{d}{ds} + A(s)$ acting on $W^{1,q} \to L^q$ where $\frac{1}{p} + \frac{1}{q} = 1$. Consequently both D and D^* have finite dimensional kernel and closed range. Using elliptic regularity theory one shows that if η lies in the annihilator of the image of D, that is

$$\eta \in \mathrm{Ann}(\mathrm{im}\, D) := \{\eta \in (L^p)^* = L^q \mid \langle \eta, D\xi \rangle = 0 \ \forall \xi \in W^{1q}\},$$

then $\eta \in W^{1,q}$ and $D^*\eta = 0$; cf. [Sal99a, Pf. of Thm. 2.2]. Thus one gets

$$\begin{aligned} W^{1,q} \supset \ker D^* = \mathrm{Ann}(\mathrm{im}\, D) \cong \mathrm{coker}\, D &:= \frac{L^p}{\mathrm{im}\, D} \\ \eta &\mapsto [\eta] \end{aligned} \tag{3.3.37}$$

and similarly one gets the analogous isomorphism

$$W^{1,p} \supset \ker D = \mathrm{Ann}(\mathrm{im}\, D^*) \cong \mathrm{coker}\, D^* := \frac{L^q}{\mathrm{im}\, D^*}. \tag{3.3.38}$$

This shows that both D and D^* are Fredholm operators.

Fredholm index

Let's get back to the rectangle $[0, 1] \times [-T, T] \ni (t, s)$ worth of symplectic matrizes $\Psi_s(t) \in \mathrm{Sp}(2n)$ introduced prior to (3.3.36) and illustrated by Figure 3.7. Let Γ denote the loop of symplectic matrizes obtained by cycling along the rectangle's boundary once. In [RS93] Robbin and Salamon introduced a Conley-Zehnder type index for rather general symplectic paths in the sense that there are no restrictions on initial and endpoint; see Section 2.1.5. Among the most useful features of the Robbin-Salamon index μ_{RS} is that it is additive under concatenations of paths. Furthermore, constant paths have index zero and paths homotopic with fixed endpoints share the same index. Moreover, the Robbin-Salamon index coincides with the Conley Zehnder index μ_{CZ} $(= -\mu^{\mathrm{CZ}})$ on the set $\mathcal{SP}^*(2n)$ of admissible paths; see Section 2.1.3. Thus

$$0 = \mu_{\mathrm{RS}}(\Gamma) = \underbrace{\mu_{\mathrm{RS}}(\mathbb{1})}_{0} + \underbrace{\mu_{\mathrm{RS}}(\Psi_{z^-})}_{-\mu^{\mathrm{CZ}}(\Psi_{z^-})} + \mu_{\mathrm{RS}}(\Psi_\cdot(1)) + \underbrace{\mu_{\mathrm{RS}}(\Psi_{z^+}(1-\cdot))}_{\mu^{\mathrm{CZ}}(\Psi_{z^+})}. \tag{3.3.39}$$

So to conclude the proof of the index formula (3.3.34) it remains to identify the yet anonymous term in the sum with the Fredholm index of D_u. We relate the unkown term in an intermediate step to another quantity called spectral flow.

[34] Here we do *linear* Fredholm theory. The necessity for $p > 2$ only arises later on when we use *non-linear* Fredholm theory, cf. Remark 3.3.21, to show the manifold property of the spaces of connecting flow lines. Of course, the non-linear application is based on the present linear findings.

[35] defined by the identity $\langle D^*\eta, \xi \rangle = \langle \eta, D\xi \rangle$ on $C^\infty_0 \subset W^{1,p}$ (compactly supported maps)

The Robbin-Salamon index counts with multiplicities the intersections (called crossings) of a symplectic path $s \mapsto \Psi(s)$ with the Maslov cycle $\mathcal{C}$ in the symplectic linear group $\mathrm{Sp}(2n)$. Let us repeat the definition given in [Sal99a, § 2.4] that starts from $\Phi' = J_0 S \Phi$. Slightly perturbing the path, if necessary, leads to finitely many *regular crossings* s_i, that is crossings at which the following quadratic form is non-degenerate. Suppose that Ψ has only regular crossings. The multiplicity at a regular crossing s is measured by the signature of the quadratic form

$$\Gamma(\Psi, s) : \ker\left(\mathbb{1} - \Psi(s)\right) \to \mathbb{R}, \quad \zeta_0 \mapsto \omega_0\left(\zeta_0, \Psi'(s)\zeta_0\right) = \langle \zeta_0, S(s)\zeta_0 \rangle_0,$$

called the *crossing form of the symplectic path* at s. The Robbin-Salamon index, in case there are no crossings at the endpoints, is the sum of signatures

$$\mu_{\mathrm{RS}}(\Psi) := \sum_{s_i} \operatorname{sign} \Gamma(\Psi, s_i)$$

over all crossings. As long as endpoints are fixed, the definition does not depend on perturbations, if any, required to obtain regular crossings.

Recall that by (3.2.15) the domain of the crossing form $\Gamma(\Psi, s)$ is isomorphic to the kernel of the unbounded self-adjoint operator $A(s) = -J_0 \partial_t - S(s)$ on the Hilbert space $L^2 = L^2(\mathbb{S}^1, \mathbb{R}^{2n})$. Thus in the present context a crossing s corresponds to $A(s)$ having non-trivial kernel. The quadratic form

$$\Gamma(A, s) : \ker A(s) \to \mathbb{R}, \quad \zeta \mapsto \langle \zeta, A'(s)\zeta \rangle_{L^2},$$

is called the **crossing form of the family of selfadjoint operators** $A(s)$. One can show, see [Sal99a, Le. 2.6], that the two crossing forms are isomorphic under the natural isomorphism (3.2.15). Thus the integer

$$\mu^{\mathrm{spec}}(A) := \sum_{s_i} \operatorname{sign} \Gamma(A, s_i) = \mu_{\mathrm{RS}}\left(s \mapsto \Psi_s(1)\right)$$

called the **spectral flow of the operator family** $A = \{A(s)\}$, fits in (3.3.39).

Exercise 3.3.23. a) Check that the signature of the crossing form $\Gamma(A, s_i)$ at a regular crossing s_i measures the number of eigenvalues of $A(s)$, with multiplicities, that change from negative to positive minus those changing from positive to negative. So the total change $\mu^{\mathrm{spec}}(A)$ is a **relative Morse index**.
b) Suppose the asymptotic operators $A^{\mp}$ have finitely many negative eigenvalues – the number of which, including multiplicities, is called the **Morse index** of $A^{\mp}$ and denoted by $\mathrm{ind}(A^{\mp})$. Check that $\mu^{\mathrm{spec}}(A) = \mathrm{ind}(A^-) - \mathrm{ind}(A^+)$.

To conclude the proof of the index formula (3.3.34) we cite the result in [RS95, Thm. 4.21] that the spectral flow of the family $A = \{A(s)\}$ is equal to the Fredholm index of the operator $D = \frac{d}{ds} + A(s)$ given by (3.3.35).[36]

This concludes the (sketch of the) proof of Theorem 3.3.22.

[36] The sign conventions in [RS95] differ from ours in two locations neutralizing each other.

3.3.4 Thom-Smale transversality

Assume $\ell \geq 1$ is an integer. Suppose $\mathcal{U}$ and $\mathcal{V}$ are Banach manifolds of class C^ℓ each of which admits a *countable atlas* and is modeled on a *separable*[37] Banach space. Consider a Banach space bundle $\mathcal{E} \to \mathcal{U} \times \mathcal{V}$ of class C^ℓ and denote by $\mathcal{E}_{(u,V)}$ the fiber over (u, V). Suppose $\mathcal{F}$ is a section of $\mathcal{E}$ of class C^ℓ. It is convenient to introduce the notation

$$\mathcal{F}_V(u) := \mathcal{F}(u, V) =: \mathcal{F}_u(V), \quad u \in \mathcal{U},\, V \in \mathcal{V}.$$

Recall that the tangent bundle of a vector bundle splits naturally along the zero section. In what follows we use the notation D to denote the **linearization of a section** at a point of the zero section: The (Fréchet) derivative as a map, from now on called **differential**, composed with projection onto the fiber. For instance, at $(u, V) \in \mathcal{F}^{-1}(0)$ the symbol

$$D\mathcal{F}(u, V) : T_u\mathcal{U} \times T_V\mathcal{V} \to \mathcal{E}_{(u,V)}$$

denotes the composition of the differential

$$d\mathcal{F}(u, V) : T_u\mathcal{U} \times T_V\mathcal{V} \to T_{\mathcal{F}(u,V)}\mathcal{E} \simeq (T_u\mathcal{U} \times T_V\mathcal{V}) \oplus \mathcal{E}_{(u,V)}$$

followed by projection onto the second component $\mathcal{E}_{(u,V)}$. Note that restriction of $\mathcal{E}$ to $\mathcal{U} \times \{V\}$ yields a bundle $\mathcal{E}^V \to \mathcal{U}$ of which $\mathcal{F}_V$ is a section and with fibers $\mathcal{E}^V_u = \mathcal{E}_{(u,V)}$; similarly $\mathcal{E}^u \to \mathcal{V}$ with section $\mathcal{F}_u$.[38] Note also that

$$\begin{aligned} D\mathcal{F}(u, V) = \overbrace{D\mathcal{F}_V(u)}^{=:D} \oplus \overbrace{D\mathcal{F}_u(V)}^{=:L} : T_u\mathcal{U} \oplus T_V\mathcal{V} &\to \mathcal{E}_{(u,V)} \\ (\xi, v) &\mapsto D\xi + Lv \end{aligned} \tag{3.3.40}$$

at every zero (u, V) of $\mathcal{F}$.

Theorem 3.3.24 (Thom-Smale transversality)**.** *Let $\mathcal{F}$ be a C^ℓ section of $\mathcal{E}$ with*

- **(F)** $D = D\mathcal{F}_V(u) : T_u\mathcal{U} \to \mathcal{E}^V_u$ *is Fredholm and* $\ell \geq \max\{1, 1 + \operatorname{index} D\mathcal{F}_V(u)\}$ *for each $V \in \mathcal{V}$ and every $u \in \mathcal{F}_V^{-1}(0)$.*
- **(S)** $D\mathcal{F}(u, V) = D \oplus L : T_u\mathcal{U} \times T_V\mathcal{V} \to \mathcal{E}_{(u,V)}$ *is surjective,* $\forall (u, V) \in \mathcal{F}^{-1}(0)$.

Then the subset of the parameter manifold $\mathcal{V}$ *given by*

$$\mathcal{V}_{\mathrm{reg}} := \{V \in \mathcal{V} \mid D\mathcal{F}_V(u) \text{ surjective } \forall u \in \mathcal{F}_V^{-1}(0)\}$$

is residual, hence dense, in $\mathcal{V}$.

Remark 3.3.25. In practice, instead of verifying the conditions (F) & (S) for a given section $\mathcal{F}$, it is often more convenient to verify the Fredholm condition (F) and the trivial-annihilator condition:

[37] A Banach space is called **separable** if it admits a dense sequence.

[38] To make this precise, consider the inclusion $\iota_V : \mathcal{U} \to \mathcal{U} \times \mathcal{V}$, $u \mapsto (u, V)$, and denote the pull-back bundle $(\iota_V)^*\mathcal{E} \to \mathcal{U}$ by $\mathcal{E}^V$; analogously for $\mathcal{E}^u$.

(A) At every zero (u, V) of the section $\mathcal{F}$ the **annihilator**

$$\begin{aligned}\mathrm{Ann}_{(u,V)} :&= \{\eta \in \mathcal{E}^*_{(u,V)} \mid \eta(\xi) = 0 \text{ for every } \xi \in \operatorname{im} D\mathcal{F}(u, V)\}\\ &= \{0\}\end{aligned}$$

of the image of the linearization $D\mathcal{F}(u, V)$ is trivial.[39]

Lemma 3.3.26. *If $\mathcal{F}$ satisfies (F) & (A), then it satisfies (F) & (S).*

Proof. Suppose $\mathcal{F}(u, V) = 0$. As a consequence of the Hahn-Banach theorem, see [Bre83, Cor. I.8], triviality of the annihilator $\mathrm{Ann}_{(u,V)}$ is equivalent to density of the image of $D\mathcal{F}(u, V)$. But the image is also closed by Exercise 3.3.13 i). □

The proof of Theorem 3.3.24 rests on two pillars. Firstly, the slightly miraculous equality of the set $\mathcal{V}_{\mathrm{reg}}$ of parameters for which the Fredholm operator $D = D\mathcal{F}_V(u)$ is surjective for all $u \in \mathcal{F}_V^{-1}(0)$ and the set of regular values of the projection $\pi : \mathcal{U} \times \mathcal{V} \supset \mathcal{F}^{-1}(0) \to \mathcal{V}$ onto the second component. Secondly, the generalization of Sard's theorem to infinite dimensions, due to Smale [Sma65]. Applied locally to π, Sard-Smale yields the assertion of Theorem 3.3.24.

Theorem 3.3.27 (Sard-Smale)**.** *Suppose Y and Z are separable Banach spaces and $W \subset Y$ is an open connected*[40] *subset. If $f : W \to Z$ is a Fredholm map of class C^ℓ with $\ell \geq 1 + \mathrm{index}(f)$,*[41] *then the subset $Z_{\mathrm{reg}} \subset Z$ of regular values,*

$$Z_{\mathrm{reg}} := \{z \in Z \mid \operatorname{im} df(x) = Z \text{ for every } x \in W \text{ with } f(x) = z\},$$

is a residual subset of Z, thus dense, hence non-empty.[42]

Remark 3.3.28. The Fredholm condition is necessary; cf. [Sma65]. Also separability is essential in order to localize the proof: *In a separable Banach space every open cover admits a countable subcover*; cf. [Sal99b, Prop. B.14].

Proof of Theorem 3.3.24 (Thom-Smale transversality)

Suppose $\mathcal{F}$ is a C^ℓ section with $\ell \geq 1$ of the Banach space bundle $\mathcal{E} \to \mathcal{U} \times \mathcal{V}$ and it satisfies conditions (F) & (S). Let us call $\mathcal{F}$ a **universal section**, just to emphasize that $\mathcal{F}_V$ and $\mathcal{F}_u$ are its restrictions. The proof takes four steps I–IV.

Step I. Suppose $\mathcal{F}(u, V) = 0$ and recall the notation $D\mathcal{F}(u, V) = D \oplus L$ from (3.3.40) where $D = D\mathcal{F}_V(u)$ is Fredholm from $X = T_u\mathcal{U}$ to $Y = \mathcal{E}_{(u,V)}$ by assumption (F) and $L = D\mathcal{F}_u(V)$ is bounded from $Z = T_V\mathcal{V}$ to Y due to $\ell \geq 1$. As $D\mathcal{F}(u, V)$ is surjective by assumption (S), Exercise 3.3.13 ii) asserts that

a) $\ker D\mathcal{F}(u, V) = \ker(D \oplus L)$ admits a complement, thus a right inverse;

[39] Here $\mathcal{E}^*_{(u,V)} := \mathcal{L}(\mathcal{E}_{(u,V)}, \mathbb{R})$ is the **dual space** of the (real) vector space $\mathcal{E}_{(u,V)}$.
[40] If W is not connected, simply impose one condition $\ell \geq 1 + \mathrm{index}(f)$ for each component.
[41] Recall that being a Fredholm map already requires $\ell \geq 1$.
[42] However, note that Z_{reg} contains any element $z \in Z$ with empty pre-image $f^{-1}(z) = \emptyset$.

b) projection onto the second component

$$\Pi = \Pi_{(u,V)} : T_u\mathcal{U} \oplus T_V\mathcal{V} \supset \ker D\mathcal{F}(u,V) = \ker(D \oplus L) \to T_V\mathcal{V}$$

is a Fredhom operator with $\mathrm{index}(\Pi) = \mathrm{index}(D)$.

Step II. Zero is a regular value of $\mathcal{F}$ by a). Thus by the regular value theorem, see Remark 3.3.14, the **universal moduli space**, namely the zero set

$$\mathcal{M} := \mathcal{F}^{-1}(0)$$

of the universal section, is a C^ℓ Banach manifold modeled on a separable Banach space,[43] say M, and with tangent spaces

$$T_{(u,V)}\mathcal{M} = \ker D\mathcal{F}(u,V) = \{(\xi, v) \mid D\xi + Lv = 0\} \subset T_u\mathcal{U} \oplus T_V\mathcal{V}.$$

Since $\mathcal{M}$ is a C^ℓ manifold, projection to the second component

$$\pi : \mathcal{U} \times \mathcal{V} \supset \mathcal{M} \to \mathcal{V}, \quad (u,V) \mapsto V,$$

is a map of class C^ℓ. But the linearization of this map at (u,V) is precisely the Fredholm operator $\Pi_{(u,V)}$. Thus π is a C^ℓ Fredholm map whose Fredholm index along the **component** $\mathcal{M}^{(u,V)}$ of (u,V) is equal to the index of $D\mathcal{F}_V(u)$.

Step III. For each of the countably many[44] local coordinate charts (φ, C) of $\mathcal{M}$ and (ψ, B) of $\mathcal{V}$ the representative[45]

$$\pi_{\varphi,\psi} := \psi \circ \pi \circ \varphi^{-1} : M \supset W := \varphi(C) \to \psi(B) \subset Z$$

of the C^ℓ Fredholm map π satisfies the assumptions of the Sard-Smale Theorem 3.3.27. Therefore the set $\mathcal{R}(\pi_{\varphi,\psi})$ of regular values of $\pi_{\varphi,\psi}$ is residual in Z. Hence so is, by Exercise 3.2.17, the countable intersection $\cap_{\varphi,\psi}\mathcal{R}(\pi_{\varphi,\psi})$ which is then dense in Z by the Baire Category Theorem 3.2.18 (C). But each point in this intersection corresponds precisely to a regular value of π in $\mathcal{V}$.

Step IV. Theorem 3.3.24 is now reduced to the following Wonder-Lemma:

Lemma 3.3.29. *For $\mathcal{F}$ as in Theorem 3.3.24 the map $\pi : \mathcal{M} \to \mathcal{V}$ is defined and*

$$\begin{aligned}
\mathcal{R}(\pi) &:= \{\textit{regular values of } \pi\} \\
&= \{V \in \mathcal{V} \mid D\mathcal{F}_V(u) \textit{ surjective } \forall u \in \mathcal{F}_V^{-1}(0)\} \\
&=: \mathcal{V}_{\mathrm{reg}}.
\end{aligned}$$

Proof. Suppose $\mathcal{F}(u,V) = 0$. It suffices to show

$$d\pi(u,V) \text{ surjective} \quad \Longleftrightarrow \quad D := D\mathcal{F}_V(u) \text{ surjective.}$$

[43] Any subspace of a separable metric vector space is separable. But $T_{(u,V)}\mathcal{M} \subset T_u\mathcal{U} \times T_V\mathcal{V}$.

[44] The Banach manifold $\mathcal{U} \times \mathcal{V}$ admits a countable atlas since each of $\mathcal{U}$ and $\mathcal{V}$ does. The Banach submanifold $\mathcal{M} \hookrightarrow \mathcal{U} \times \mathcal{V}$ is a closed subspace.

[45] Abusing notation denote again by Z the separable Banach space on which $\mathcal{V}$ is modeled.

Let us use the notation $D\mathcal{F}(u,V) = D \oplus L$ with $L := D\mathcal{F}_u(V)$; see (3.3.40).
'⇒' Pick $\eta \in \mathcal{E}_{(u,V)}$, then $\eta = D\mathcal{F}(u,V)(\xi_0, v) = D\xi_0 + Lv$ for some tangent vector $(\xi_0, v) \in T_{(u,V)}(\mathcal{U} \times \mathcal{V})$ by the surjectivity assumption (S). Given v, then $v = d\pi(u,V)(\xi_1, v)$ by surjectivity of $d\pi(u,V)$ for some element $(\xi_1, v) \in T_{(u,V)}\mathcal{M} = \ker D\mathcal{F}(u,V)$, that is $D\xi_1 + Lv = 0$. Hence $D(\xi_0 - \xi_1) = \eta$.
'⇐' Pick $v \in T_V\mathcal{V}$, then $Lv = D\xi$ for some $\xi \in T_u\mathcal{U}$ by surjectivity of D. Thus the pair $(-\xi, v)$ lies in $T_{(u,V)}\mathcal{M}$. It gets mapped to v under $d\pi(u,V)$. □

Thom-Smale transversality – openness

Theorem 3.3.30. *Consider a C^ℓ section $\mathcal{F}$ of $\mathcal{E} \to \mathcal{U} \times \mathcal{V}$ that satisfies the assumptions of Theorem 3.3.24. Suppose, in addition, that the restriction π_a of the projection $\pi : \mathcal{M} \to \mathcal{V}$ to some open subset $\mathcal{M}^a \subset \mathcal{M}$ is a proper map.*[46] *Then the set of regular parameters*

$$\mathcal{V}^a_{\mathrm{reg}} := \{V \in \mathcal{V} \mid D\mathcal{F}_V(u) \text{ is surjective whenever } (u,V) \in \mathcal{M}^a\}$$

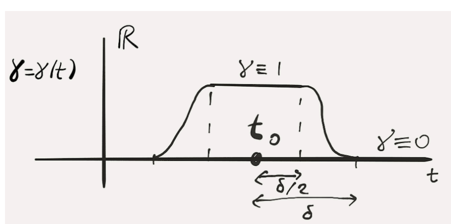
is open and dense in the set $\mathcal{V}$ of all parameters.

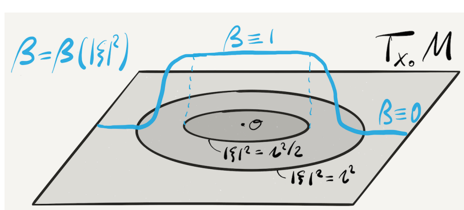
Example: Classical action $\mathcal{S}_V$ is Morse for generic potential V

Details of this have been worked out in [Web02].

Exercise: Symplectic action $\mathcal{A}_H$ is Morse for generic Hamiltonian H

Work out the proof of Theorem 3.2.22. The following figures suggest choices of useful cutoff functions $\gamma \in C^\infty(\mathbb{R}, [0,1])$ and $\beta \in C^\infty(\mathbb{R}, [0,1])$.





[46] **proper map**: pre-images of compact sets are compact

3.4 Floer chain complex and homology

In this section we define the Floer complex associated to a regular pair (H, J) on a *closed* symplectic manifold (M, ω). We only consider the simplest case in which ω and $c_1(M)$ vanish over $\pi_2(M)$; cf. (3.0.1). The corresponding homology, called Floer homology, transforms by isomorphisms when changing regular pairs and in the end of the day represents, again naturally, the singular homology of M. For simplicity we take $\mathbb{Z}_2$ coefficients in all homology theories in Section 3.4.

Definition 3.4.1. A **regular pair** (H, J) consists of a Hamiltonian $H \in C^\infty(\mathbb{S}^1 \times M)$, that is a periodic family of functions $H_{t+1} = H_t := H(t, \cdot)$ of functions on M, and a periodic family $J_{t+1} = J_t \in \mathcal{J}(M, \omega)$ of ω-compatible almost complex structures such that the following is true: The action functional $\mathcal{A}_H : \mathcal{L}M \to \mathbb{R}$ given by (3.2.11) is Morse and the linearized operators D_u given by (3.3.31) are surjective for all connecting trajectories $u \in \mathcal{M}(x, y; H, J)$ and all Hamiltonian loops $x, y \in \mathcal{P}_0(H) = \mathrm{Crit}\mathcal{A}_H$.

Earlier we showed how to obtain a regular pair: To satisfy the Morse condition, pick an element H^0 of the residual subset $\mathcal{H}_{\mathrm{reg}} \subset C^\infty(\mathbb{S}^1 \times M)$ provided by Theorem 3.2.22. Now pick a family $J_{t+1} = J_t$ of ω-compatible almost complex structures on M. Then, either perturb H^0 away from its critical points, see Theorem 3.3.19, or stay with H^0 and perturb the family J_t, see [FHS95, Thm. 5.1], to obtain a regular pair denoted by (H, J).

Definition 3.4.2. Pick a regular pair (H, J). Then the $\mathbb{Z}_2$ vector spaces

$$\mathrm{CF}_k(H) = \mathrm{CF}_k(M, \omega, H) := \bigoplus_{\substack{z \in \mathcal{P}_0(H) \\ {\mu^{\mathrm{CZ}}}_H(z) = k}} \mathbb{Z}_2 z$$

graded by the canonical, that is clockwise normalized, Conley-Zehnder index, see (2.1.5) and (2.3.25), are called the **Floer chain groups** associated to the Hamiltonian H. By convention the empty set generates the trivial vector space. The set $\mathcal{P}_0(H) = \mathrm{Crit} := \mathrm{Crit}\mathcal{A}_H$ of all contractible periodic trajectories is finite by Proposition 3.2.24. The subset Crit_k of those of index k is a basis of $\mathrm{CF}_k(H)$, called the **canonical basis (over $\mathbb{Z}_2$)**.

The **Floer boundary operator** is given on a basis element $x \in \mathrm{Crit}_k$ by

$$\begin{aligned} \partial = \partial^{\mathrm{F}}(H, J) : \mathrm{CF}_*(H) &\to \mathrm{CF}_*(H) \\ x &\mapsto \sum_{y \in \mathrm{Crit}_{k-1}} \#_2(m_{xy})\, y \end{aligned} \tag{3.4.41}$$

where $\#_2(m_{xy})$ is the number (mod 2) of connecting flow lines.[47]

Proposition 3.4.3 (Boundary operator)**.** *It holds that $\partial^2 = 0$.*

[47] A **flow line** is an unparametrized curve (the image of a trajectory trajectory $\mathbb{R} \to \mathcal{L}M$, $s \mapsto u_s$). Flow lines are in bijection with the set m_{xy} of those $u \in \mathcal{M}(x, y; H, J)$ with $\mathcal{A}_H(u_0) = \frac{1}{2}(\mathcal{A}_H(x) + \mathcal{A}_H(y))$. This is a finite set by Exercise 3.4.12. Let $\#_2(m_{xy})$ be the number of elements modulo 2.

Exercise 3.4.4. Given $x \in \mathrm{Crit}_{k+1}$, show that $\partial^2 x$ is equal to the sum over all $z \in \mathrm{Crit}_{k-1}$ where the coefficient of each z is the number (mod 2) of 1-fold broken trajectories (u, v) that start at x, end at z, and pass an intermediate critical point y of index k at which u and v meet.

So to prove $\partial^2 = 0$ it suffices to show that the number of such 1-fold broken orbits (u, v) between x and z is even. To see this one shows that for each (u, v) there exists precisely one partner pair $(\tilde{u}, \tilde{v})$ which is determined by the property that there is a 1-dimensional connected non-compact manifold, i.e. an open interval, of trajectories running straight from x to z and whose two ends correspond to the two partner pairs; see Figure 3.1. The sense in which the two ends correspond to partner pairs is introduced and detailed in Section 3.4.1 on compactness up to broken trajectories. The gluing procedure developed in Section 3.4.2 excludes that two families converge to the same broken orbit (u, v). To summarize, the set of 1-fold broken orbits from x to z is in bijection with the ends of finitely many open intervals, so the number of them is even.

Definition 3.4.5. The chain complex

$$\mathrm{CF}(H) = \mathrm{CF}(M, \omega, H, J) := \big(\mathrm{CF}_*(H), \partial^{\mathrm{F}}(H, J)\big)$$

is called the **Floer complex** associated to a regular pair (H, J). The corresponding homology groups, called **Floer homology groups**, are graded $\mathbb{Z}_2$ vector spaces. They are denoted by

$$\mathrm{HF}_*(H) = \mathrm{HF}_*(M, \omega, H; J).$$

Theorem 3.4.6 (Continuation)**.** *For any two regular pairs (H^α, J^α) and (H^β, J^β) there is a natural*[48] *isomorphism*

$$\Psi^{\beta\alpha} : \mathrm{HF}_*(\alpha) \to \mathrm{HF}_*(\beta).$$

Furthermore, given a third regular pair (H^γ, J^γ), then

$$\Psi^{\gamma\beta}\Psi^{\beta\alpha} = \Psi^{\gamma\alpha}, \qquad \Psi^{\alpha\alpha} = \mathbb{1}.$$

Theorem 3.4.7 (Calculation)**.** *Suppose (M, ω) is a closed symplectic manifold such that ω and $c_1(M)$ vanish over $\pi_2(M)$. Then for any regular pair (H^α, J^α) there is an isomorphism of degree n denoted by*

$$\Psi^\alpha : \mathrm{HF}_{\ell-n}(\alpha) \to \mathrm{H}_\ell(M)$$

and these isomorphisms are natural in the sense that

$$\Psi^\beta \Psi^{\beta\alpha} = \Psi^\alpha.$$

Corollary 3.4.8. *The weak non-degenerate Arnol′d conjecture (1.0.2) is true.*

[48] Here **natural isomorphism** means that there are no (further) choices involved.

3.4.1 Compactness – bubbling off analysis

Throughout let $\mathcal{A}_H$ be Morse. Let $\mathcal{M}_{x,y}$ be the space $\mathcal{M}(x, y; H, J)$, see (3.3.21), of connecting trajectories between contractible 1-periodic trajectories $x, y \in \mathrm{Crit}\mathcal{A}_H$.

Definition 3.4.9 (Convergence to broken trajectory)**.** We say that a sequence $(u^\nu) \subset \mathcal{M}_{xy}$ of connecting trajectories **converges to a $(k-1)$-fold broken trajectory**[49] $(u_k, \dots, u_1)$ if the following holds true. There are

- pairwise different periodic trajectories $z_k = x, z_{k-1}, \dots, z_0 = y$,
- (non-constant) connecting trajectories $u_j \in \mathcal{M}_{z_j z_{j-1}}$ for $j = 1, \dots, k$,
- sequences $(s_j^\nu) \subset \mathbb{R}$ of reals for $j = 1, \dots, k$,

such that the sequence of maps $\mathbb{R} \times \mathbb{S}^1 \to M$, $(s, t) \mapsto u^\nu(s + s_j^\nu, t)$ converges to u_j in C^∞_{loc}, as $\nu \to \infty$, i.e. uniformly with all derivatives on compact sets. By

$$u^\nu \to (u_k, \dots, u_1), \quad \text{as } \nu \to \infty,$$

we henceforth denote **convergence to a broken trajectory**; see Figure 3.8.

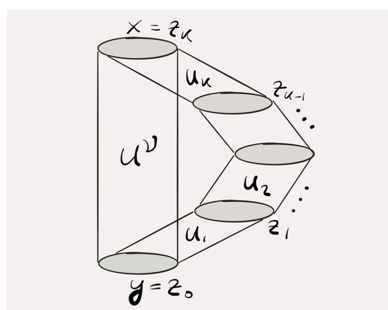


Figure 3.8: Convergence of sequence u^ν to broken trajectory $(u_k, \dots, u_1)$

Exercise 3.4.10 (Strictly decreasing index in case of a regular pair)**.** Show that the index μ^{CZ}, strictly decreases along the members of a broken trajectory $(u_k, \dots, u_1)$ whenever (H, J) is a regular pair.
[Hint: By non-constancy each u_j comes in a family of dimension at least 1. But this dimension is a Fredholm index. Now recall (3.3.34).]

Proposition 3.4.11 (Compactness up to broken trajectories)**.** *Suppose $\mathcal{A}_H$ is Morse. Then any sequence $(u^\nu) \subset \mathcal{M}_{xy}$ of connecting trajectories admits a subsequence which converges to a broken trajectory $(u_k, \dots, u_1)$, as $\nu \to \infty$.*

It is interesting to observe that by the proposition the space $\mathcal{M}$ of finite energy trajectories, see (3.3.27), is compact in case of a regular pair.

[49] or, alternatively, **converges to a broken trajectory with k components**

Exercise 3.4.12 (Finite set in case of a regular pair (H, J))**.** Show that in case of index difference one, that is $\mu^{\mathrm{CZ}}(x) - \mu^{\mathrm{CZ}}(y) = 1$, the set m_{xy} of those connecting trajectores $u \in \mathcal{M}_{xy}$ whose initial loop $u_0 = u(0, \cdot)$ lies on a fixed intermediate action level, say $r = \frac{1}{2}(\mathcal{A}_H(x) + \mathcal{A}_H(y))$, is a finite set.[50]

Proof of Proposition 3.4.11 (Compactness up to broken trajectories)

The proof has four steps the first of which is highly trivial, at least in the present case of a *compact* manifold M. We only sketch the main ideas. Actually the first three steps do not require that $\mathcal{A}_H$ is Morse, not even that x, y are non-degenerate. The Morse requirement enters in Step IV through Theorem 3.3.5.

Step I. (Uniform C^0 bound for u^ν)

Obvious by compactness of M.

Step II. (Uniform C^0 bound for $\partial_s u^\nu$ – bubbling off analysis)

In fact, to carry out Step III below a uniform $W^{1,p}$ bound, for some constant $p > 2$, for the sequence of connecting trajectories u^ν would be sufficient. Note that we get a uniform $W^{1,2}$ bound for free due to the energy identity (3.3.25). But this is not good enough to get uniform C^0 bounds; cf. also Remark 3.3.21.

On the positive side, investing some work even leads to a uniform C^1 bound, that is

$$\sup_\nu \|\partial_s u^\nu\|_{L^\infty} < \infty.$$

To prove this, assume by contradiction there was a sequence of points ζ^ν along which the derivative $|\partial_s u^\nu(\zeta^\nu)| \to \infty$ explodes, as $\nu \to \infty$. It is convenient to view maps defined on $\mathbb{R} \times \mathbb{S}^1$ likewise as maps on $\mathbb{C} \ni s + it$ which are 1-periodic in the imaginary part t. Exploiting the invariance of $\mathcal{M}_{xy}$ under shifts in the s variable in order to replace $u^\nu(s, t)$ by the shifted sequence $u^\nu(s - s^\nu, t)$ and then by compactness of $\mathbb{S}^1$ picking a subsequence, we assume without loss of generality that $\zeta^\nu = 0 + i\tau^\nu \to 0 + it_0 =: z_0$, as $\nu \to \infty$. Now there appears yet another Wonder-Lemma, called the **Hofer-Lemma** [HZ11, §6.4 Le. 5]. When applied for each ν to the continuous non-negative function $g := |\partial_s u^\nu|$ on the complete metric space $X := [-1, 1] \times \mathbb{S}^1$, the point $x_0 := \zeta^\nu \in X$, and the constant $\varepsilon_0 := |\partial_s u^\nu(\zeta^\nu)|^{-1/2} > 0$, the Hofer-Lemma yields a sequence of points $z^\nu \in X$ and constants $\varepsilon^\nu > 0$ such that

- $0 < \varepsilon^\nu \le |\partial_s u^\nu(\zeta^\nu)|^{-1/2} \to 0$
- $R^\nu \varepsilon^\nu := |\partial_s u^\nu(z^\nu)|\, \varepsilon^\nu = g(z^\nu)\varepsilon^\nu \ge \frac{g(\zeta^\nu)}{|\partial_s u^\nu(\zeta^\nu)|^{1/2}} = |\partial_s u^\nu(\zeta^\nu)|^{1/2} \to \infty$
- $\|z^\nu - \zeta^\nu\| \le \frac{2}{|\partial_s u^\nu(\zeta^\nu)|^{1/2}} \to 0$
- $\sup_{B_{\varepsilon^\nu}(z^\nu)} |\partial_s u^\nu| \le 2\, |\partial_s u^\nu(z^\nu)| = 2R^\nu$

[50] The set m_{xy} parametrizes the flow lines from x to y, in other words, the set of connecting trajectores u modulo time shift. As maps u and $u_{(\sigma)} := u(\sigma + \cdot, \cdot)$ are different, but their image curve in the loop space, i.e. their flow line, is the same.

Note that $R^\nu := |\partial_s u^\nu(z^\nu)| \to \infty$, so the derivative explodes as well along the new sequence of points z^ν which also converges to z_0, but for which we have more information than we had for ζ^ν. The key step is then to consider the sequence of rescaled smooth maps

$$v^\nu : \mathbb{C} \to M, \quad z \mapsto u^\nu(z^\nu + (R^\nu)^{-1} z).$$

These maps have the property that they are non-constant and uniformly C^1 bounded on balls $B_{R^\nu \varepsilon^\nu}(0)$ whose radius $R^\nu \varepsilon^\nu$ tends to infinity. More precisely,

- $|\partial_s v^\nu(0)| = 1$;
- $|\partial_s v^\nu| \le 2$ on $B_{R^\nu \varepsilon^\nu}(0)$, see Figure 3.9;
- $\partial_s v^\nu + J_{t^\nu + (t/R^\nu)}(v^\nu) \partial_t v^\nu = \frac{1}{R^\nu} \nabla H_{t^\nu + (t/R^\nu)}(v^\nu)$.

Now one shows that there is a smooth map $v : \mathbb{C} \to M$ and a subsequence, still denoted by v^ν, such that $v^\nu \to v$ in $C^\infty(\mathbb{C}, M)$, that is uniformly with all derivatives on $\mathbb{C}$. In view of the Arzelà-Ascoli Theorem 3.2.10 it suffices to establish uniform in ν bounds on compact sets for v^ν and each of its derivatives. In view of the Sobolev embedding theorem, see e.g. [MS04, Thm. B.1.11], it suffices to establish uniform $W^{k,p}_{\mathrm{loc}}$ bounds for v^ν, that is for each $k \in \mathbb{N}$ and each compact set $K \subset \mathbb{C}$ find a $W^{k,p}$ bound c for the restriction $v^\nu|_K$ to K such that c serves for all ν. The key input is the Calderón-Zygmund inequality (3.4.43) for the Cauchy-Riemann operator $\bar\partial$. The desired bounds are established by induction on k. For details see e.g. [Sal90, Le. 5.2], using elliptic bootstraping techniques, or [HZ11, §6.4 Le. 6], using the **Gromov trick** to first get rid off the Hamiltonian term followed by a proof by contradiction.

The limit map $v : \mathbb{C} \to M$ satisfies the equation

$$\partial_s v + J(v) \partial_t v = 0 \tag{3.4.42}$$

where $J(v) = J_{t_0}(v)$. The solutions $w : \mathbb{C} \to M$ of this elliptic PDE are called **J-holomorphic planes** or **pseudo-holomorphic planes**. They have been introduced in Gromov's 1985 landmark paper [Gro85]. Since

$$|\partial_s v(0)| = 1, \qquad |\partial_s v| \le 2, \qquad \int_{\mathbb{C}} v^* \omega = \|\partial_s v\|_{L^2}^2 > 0,$$

we arrive at a contradiction as soon as we can show that v extends continuously from $\mathbb{C}$ to the Riemann sphere, that is to a continuous map $\tilde v : \mathbb{S}^2 = \mathbb{C} \cup \{\infty\} \to M$. Indeed in this case the proof of Proposition 3.4.11 is complete since

$$0 = \int_{\mathbb{S}^2} \tilde v^* \omega = \int_{\mathbb{C}} v^* \omega > 0.$$

Here the first identity uses that by our standing assumption (C2) the evaluation map $\mathrm{I}_\omega = 0$ of ω over $\pi_2(M)$ vanishes, see (3.0.2), and the second identity holds since a point has measure zero.

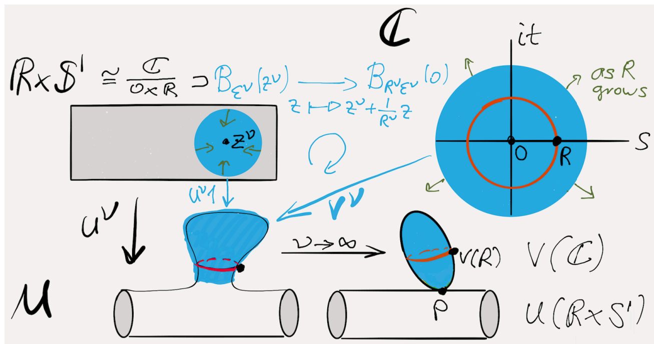


Figure 3.9: Bubbling off of spheres requires that ω does not vanish over $\pi_2(M)$

So it remains to construct the continuous extension $\tilde{v}$. For each radius $R>0$ the map $v:\mathbb{C}\to M$ gives rise to a loop in M by restriction to the radius-R sphere in $\mathbb{C}$ centered at the origin; notation $v_R:\mathbb{C}\supset\{|z|=R\}\to M$; indicated red in Figure 3.9. In [Sal90, Pf. of Prop. 4.2] it is shown, see also [HZ11, paragraph before (6.94)], that the lengths of the image circles γ_R in M of the maps v_R tends to zero, as $R\to\infty$. While along any sequence $R_j\to\infty$ the family of points $\{v(R)=v(R+i0)\}_{R>0}$ admits a convergent subsequence by compactness of M, continuity of v together with the length shrinking property imply uniqueness of the limit p and independence of the choice of the sequence $R_j\to\infty$. Clearly $p:=\lim_{R\to\infty} v(R)$ completes the image $v(\mathbb{C})$ to be a 2-sphere and we are done.[51]

Step III. (Uniform C^∞_{loc} bound for u^ν – elliptic bootstrapping)

First of all, in view of the uniform C^1 bound for our sequence of connecting trajectories u^ν, obtained in Steps I and II, the Arzelà-Ascoli Theorem 3.2.10 provides a continuous map $u:\mathbb{R}\times\mathbb{S}^1\to M$ to which some subsequence, still denoted by u^ν, converges uniformly on compact sets. This allows to analyze the problem in local coordinates on M, hence for maps taking values in $\mathbb{R}^{2n}$.
To derive uniform C^∞ estimates for the sequence u^ν on compact sets is, surely without surprise, an iterative procedure. To illustrate the basic mechanism behind, let us *incorrectly oversimplify* things by assuming that (i) each u^ν is a map $\mathbb{C}\to(\mathbb{R}^{2n},J_0)$, $s+it\mapsto u^\nu(s,t)$, supported in a compact set K – which it clearly isn't, given the non-linear target M and the asymptotic boundary conditions x and y – and that (ii) each u^ν satisfies the much simpler PDE

$$\bar\partial u^\nu=\nabla H(u^\nu),\qquad \bar\partial:=\partial_s+J_0\partial_t.$$

The iteration rests on the **Calderón-Zygmund type inequality**

$$\|v\|_{W^{1,p}}\le c_p\left\|\bar\partial v\right\|_{L^p} \tag{3.4.43}$$

[51] As p lies in the closure of the image of v, it lies in the closure of the union of all images $u^\nu(\mathbb{R}\times\mathbb{S}^1)$. So the bubble is attached at p to whatever is the limit, see Step IV, of the u^ν's.

and its immediate consequence (replace v by derivatives of v)

$$\|v\|_{W^{k,p}} \le c_{p,k} \left\|\bar\partial v\right\|_{W^{k-1,p}} .$$

These hold true for any compactly supported map $v \in C_0^1(\mathbb{C},\mathbb{R}^{2n})$ whenever $k \in \mathbb{N}$ and $1 < p < \infty$ and where the constant $c_p > 0$ only depends on p; see e.g. [Ste70, III §1 Prop. 4] or [Wen15, Thm. 2.6.1]. By continuity the estimates continue to hold for v in the closure $W_0^{k,p}$ of C_0^∞ with respect to the $W^{k,p}$ norm.

Application to our oversimplified maps $u^\nu \in W_0^{1,p}(K,\mathbb{R}^{2n})$ shows that

$$\|u^\nu\|_{W^{k,p}} \le c_p \left\|\nabla H(u^\nu)\right\|_{W^{k-1,p}} \le C(p,k,K,H)$$

where the constant C does not depend on ν. More precisely, starting with $k = 2$ we get a uniform $W^{2,p}$ bound on K which then leads to a uniform $W^{3,p}$ bound and so on for every $k \in \mathbb{N}$. The Sobolev embedding theorem, see e.g. [MS04, Thm. B.1.11], then provides uniform C^k bounds on K for every k. Now apply the Arzelà-Ascoli Theorem 3.2.10 to each derivative of u^ν.

However, the general case is much harder, of course, mainly due to the non-linearity $J(u^\nu)$ in front of the highest order term $\partial_t u^\nu$. The fact that the maps u^ν are not of compact support at all, requires the use of cutoff functions leading to additional terms as well. For details see e.g. [HZ11, §6.4 Le. 6] or [Sal90, Le. 5.2].

STEP IV. (The limit broken trajectory)

Pick $T > 0$ and consider the restrictions of u^ν to $Z_T := [-T,T]\times\mathbb{S}^1$. By Step III there are uniform C^∞ bounds for u^ν on Z_T, thus there is a subsequence, still denoted by u^ν converging uniformly with all derivatives to some smooth map $u : Z_T \to M$ which solves the Floer equation (3.6) as well. Replacing T by $2T$ and choosing subsequences if necessary one concludes that u^ν restricted to Z_{2T} converges to a smooth solution $Z_{2T} \to M$ which coincides with u on Z_T. Iteration leads to a smooth limit solution again denoted by $u : \mathbb{R}\times\mathbb{S}^1 \to M$. If we knew that u was of finite energy we would apply Theorem 3.3.5 to obtain existence of periodic trajectories $z^\mp \in \mathcal{P}_0(H)$ sitting at the ends of u, that is $u \in \mathcal{M}_{z^-z^+}$. With this understood apply the finite[52] iteration detailed in [Sal90, Pf. of Prop. 4.2] to get the desired limit broken orbit $(u_k,\dots,u_1)$.

Let us check that the energy of the limit solution $u : \mathbb{R}\times\mathbb{S}^1 \to M$ is indeed finite. By the energy identity (3.3.25) for connecting trajectories we get

$$\begin{aligned}
\mathcal{A}_H(x) - \mathcal{A}_H(y) &= E(u^\nu)\\
&\ge \int_0^1\int_{-T}^T \left|\partial_s u^\nu(s,t)\right|^2 ds\,dt =: E_{[-T,T]}(u^\nu)\\
&= \mathcal{A}_H(u^\nu_{-T}) - \mathcal{A}_H(u^\nu_T)
\end{aligned}$$

where $u^\nu_T := u^\nu(T,\cdot)$ and the last identity is true since $\partial_s u^\nu$ is the downward L^2 gradient of $\mathcal{A}_H$. By uniform C^∞ convergence we can take the limit over ν

[52] The downward procedure can only end at y and it ends after at most $|\mathrm{Crit}\mathcal{A}_H|$ many steps: The action strictly decreases from z_ℓ to $z_{\ell-1}$ iff $z_\ell \ne z_{\ell-1}$ iff $E(u_\ell) > 0$, but there are only finitely many periodic trajectories by Proposition 3.2.24.

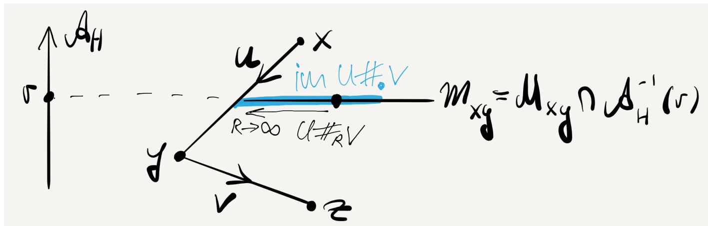


Figure 3.10: (Gluing map) Convergence $u\#_R v \to (u,v)$, as $R \to \infty$

to get the estimate

$$\mathcal{A}_H(x) - \mathcal{A}_H(y) \geq \mathcal{A}_H(u_{-T}) - \mathcal{A}_H(u_T) = E_{[-T,T]}(u) \tag{3.4.44}$$

for every T, in particular, for $T = \infty$. Thus $\infty > \mathcal{A}_H(x) - \mathcal{A}_H(y) \geq E(u)$.[53]

This concludes the proof of Step IV and Proposition 3.4.11.

3.4.2 Gluing

Suppose (H, J) is a regular pair and pick periodic trajectories x, y, z of μ^{CZ}-indices $k+1, k, k-1$, respectively. To conclude the proof of Proposition 3.4.3 ($\partial^2 = 0$) let us construct a continuous map[54]

$$\cdot\#_\cdot \cdot : m_{xy} \times [R_0, \infty) \times m_{yz} \to m_{xz}, \quad (u, R, v) \mapsto u\#_R v,$$

called the **gluing map**, whose image lies in one component of the 1-dimensional manifold m_{xz} of which it covers one 'end' in the sense that $u\#_R v \to (u,v)$, as $R \to \infty$. In other words, the broken trajectory (u,v) represents the boundary point of that end. Furthermore, and most importantly, as it concludes the proof of $\partial^2 = 0$, no sequence in $m_{xz} \setminus u\#_{[R_0,\infty)} v$, that is no sequence away from the image of the glued family $R \mapsto u\#_R v$, converges to (u,v); see Figure 3.10.

The construction is by the **Newton method** to find a zero of a map $\mathcal{F}_H$ near a given approximate zero $\tilde{w}_R$: Roughly speaking, one needs that $\mathcal{F}_H(\tilde{w}_R)$ is 'small', the derivative $D_R := D_{\tilde{w}_R} := D\mathcal{F}_H(\tilde{w}_R)$ at $\tilde{w}_R$ is 'steep',[55] and does not vary 'too much' near the approximate zero; see Figure 3.11.

We only outline the construction and refer to [Sal99a, §3.3] for details. Obviously $\mathcal{F}_H$ is the Floer section (3.3.20) of the Banach bundle $\mathcal{E}^p \to \mathcal{B}^{1,p}(x,z)$; cf. (3.3.32). To start with pick $u \in m_{xy}$ and $v \in m_{yz}$ and consider the approximate zero $\tilde{w}_R = \tilde{w}_R(u,v)$ used in [Sal99a] and illustrated by Figure 3.12. Next,

---

[53] The case $E_{[-T,T]}(u) = 0$ for every $T > 0$ is not excluded: Given $v \in \mathcal{M}_{xy}$, consider the upward shifts $u^\nu(s,t) := v(s-\nu,t)$. So $u^\nu_{-T} = v(-\nu-T,\cdot)$ and $u^\nu_T = v(-\nu+T,\cdot)$. Then $\mathcal{A}_H(u^\nu_{-T})$ and $\mathcal{A}_H(u^\nu_T)$ both converge to $\mathcal{A}_H(x)$, as $\nu \to \infty$. Indeed on *compact* sets u^ν converges uniformly with all derivatives to the constant trajectory $u(s,\cdot) = x(\cdot)$.

[54] *Warning.* Here $u\#_R v$ denotes the true zero, in [Sal99a, §3.4] the approximate zero $\tilde{w}_R$.

[55] More precisely, in our context D_R needs to admit a uniformly bounded right inverse T.

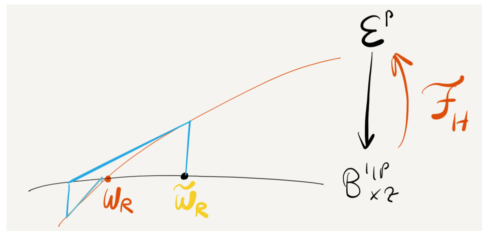


Figure 3.11: (Newton method) Find true zero w_R nearby approximate zero $\tilde{w}_R$

as we wish to use the implicit function theorem, we need to move from Banach bundles to Banach spaces. So, we replace $\mathcal{F}_H$ by the map

$$f_R := f_{\tilde{w}_R} : L^p_{\tilde{w}_R} \supset W^{1,p}_{\tilde{w}_R} \to L^p_{\tilde{w}_R}, \quad \xi \mapsto \mathcal{T}_{\tilde{w}_R}(\xi)^{-1}\mathcal{F}_H(\exp_{\tilde{w}_R}\xi),$$

where $\mathcal{T}$ denotes parallel transport; cf. (3.2.13) and (3.3.32).

Exercise 3.4.13. a) The map $\xi \mapsto \exp_{\tilde{w}_R}\xi$ is a bijection between the zeroes of f_R and those of $\mathcal{F}_H$. b) The linearization $df_R(0)\xi := \frac{d}{d\tau}\big|_{\tau=0} f_R(\tau\xi)$ is D_R. [Hint: Compare [Web99, Pf. of Thm. A.3.1].]

So the tasks at hand are to

(a) show that f_R admits a unique zero η_R whenever R is sufficiently large;

(b) show that $w_R := \exp_{\tilde{w}_R}\eta_R \to (u, v)$, as $R \to \infty$;

(c) define $u\#_R v := w_R$.

Obviously we start with task (c). Next, concerning task (b), let us argue geometrically by looking at Figure 3.12 where $\beta : \mathbb{R} \to [0, 1]$ is a cutoff function that equals zero for $s \le 0$ and one for $s \ge 1$. Observe that the curve $s \mapsto \tilde{w}_R(s\cdot)$ follows more and more, the larger R, all of u and all of v. Thus $\tilde{w}_R$ restricted to a fixed compact subdomain, say of the form $[-T, T] \times \mathbb{S}^1$, runs into the constant in s solution y, unless it is shifted appropriately backward or forward in s in which case it runs towards a piece of u or v, respectively.

Concerning task (a) let us discuss informally the three ingredients needed to carry out Newton's method.

I. Approximate zero. The L^p norm of $f_R(\tilde{w}_R)$, hence of $\mathcal{F}_H(\tilde{w}_R)$ since parallel transport $\mathcal{T}$ is an isometry, is equal to the L^p norm of $\mathcal{F}_H$ applied to the exp parts of $\tilde{w}_R$ along the length 1 domain $[-\frac{R}{2} - 1, -\frac{R}{2}] \times \mathbb{S}^1$ and an analogous length 1 domain in the positive half cylinder. Let us pick the ∂_s-piece of $\mathcal{F}_H$ to illustrate what happens.[56] The derivative $\partial_s(\exp_y \beta\xi)$ is a sum of two terms: Term one approaches zero, as $s \to \infty$, since u_s approaches y uniformly in t and term two approaches zero, as $s \to \infty$, since $\partial_s u_s$ does. By `(exp. decay)`

[56] Concerning the other piece of $\mathcal{F}_H$, check how $J(\cdot)\partial_t \cdot -\nabla H_t(\cdot)$ applied to $\exp_y \beta\xi$ approaches $J(x)\dot{x} - \nabla H_t(x) = 0$, as $s \to \infty$.

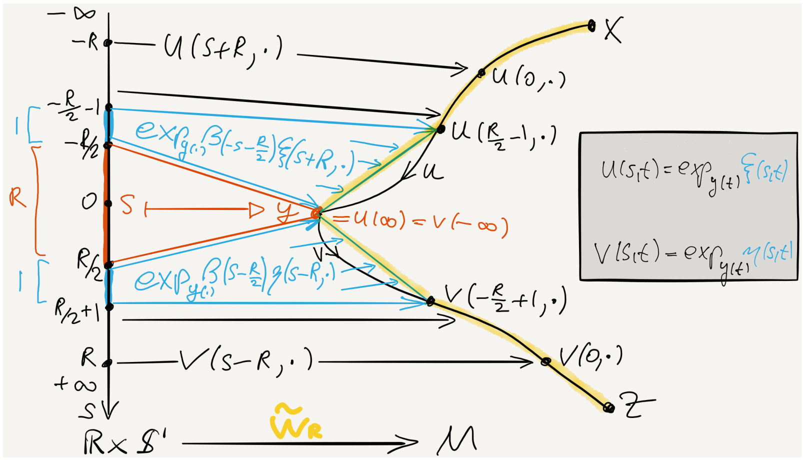


Figure 3.12: The approximate zero $\tilde{w}_R = \tilde{w}_R(u, v)$ of the Floer section $\mathcal{F}_H$

in Theorem 3.3.5 both terms converge to 0 exponentially, as $s \to \infty$. So the integral over the length 1 interval $[-\frac{R}{2} - 1, -\frac{R}{2}]$ – which moves to $+\infty$ with R – becomes as small as desired by picking R large.

II. Right inverse. By assumption both Fredholm operators D_u and D_v, see (3.3.31), are surjective. Salamon proves in [Sal99a, Prop. 3.9] that there are constants $c > 0$ and $R_0 > 2$ such that for any $R > R_0$ the Fredholm operator $D_R := D_{\tilde{w}_R}$ based on the approximate zero is surjective as well and, moreover, there is an injectivity estimate for D_R on the range of ${D_R}^*$, namely

$$\|{D_R}^*\eta\|_{W^{1m,p}} \le c \,\|D_R {D_R}^*\eta\|_{L^p}$$

for every $\eta \in W^{2,p}_{\tilde{w}_R}$. It is, of course, crucial that the estimate is uniform in R.

Exercise 3.4.14. Suppose $R > R_0$. (i) Show that $D_R {D_R}^* : W^{2,p}_{\tilde{w}_R} \to L^p_{\tilde{w}_R}$ is a bijection and admits a continuous inverse. (ii) Show that

$$T := {D_R}^* \left(D_R {D_R}^*\right)^{-1} : L^p_{\tilde{w}_R} \to W^{1,p}_{\tilde{w}_R}$$

is a right inverse of D_R and calculate its operator norm. [Hint: Recall (3.3.38).]

III. Quadratic estimates. To see what is meant by quadratic estimates have a look at [SW06, Prop. 4.5].

With all three preparations I-III in place, apply to f_R the implicit function theorem in the form of [SW06, Prop. A.3.4] with $x_0 = x_1 = 0$ to obtain a unique $\eta_R \in \operatorname{im} T$ such that $f_R(\eta_R) = 0$, equivalently, such that $\mathcal{F}_H(w_R) = 0$ where

$$w_R := \exp_{\tilde{w}_R} \eta_R.$$

This concludes the construction of the gluing map and thereby the proof of Proposition 3.4.3 ($\partial^2 = 0$).

3.4.3 Continuation

Suppose throughout that $J \in \mathcal{J}(M,\omega)$ is fixed and that (H^α, J), (H^β, J), and (H^γ, J) are *regular pairs*.[57] By $H^{\alpha\beta}$ or $\{H^{\alpha\beta}_{s,t}\}$ we denote a **homotopy between Hamiltonians**, that is a smooth map $\mathbb{R} \times \mathbb{S}^1 \times M \to \mathbb{R}$ such that

$$H^{\alpha\beta}_{s,t} = \begin{cases} H^\alpha_t & , s \le -1, \\ H^\beta_t & , s \ge +1. \end{cases}$$

The key idea is to replace the s-independent Hamiltonians in the Floer equation (3.3.20) by s-dependent homotopies, thereby destroying the occasionally disturbing symmetry under s-shifts; see footnote to (3.4.44). Consider the PDE

$$\partial_s u + J_t(u)\partial_t u - \nabla H^{\alpha\beta}_{s,t}(u) = 0 \tag{3.4.45}$$

for smooth cylinders $u : \mathbb{R} \times \mathbb{S}^1 \to M$. It is called the **homotopy Floer equation** and its solutions u **homotopy trajectories**. Impose the usual asymptotic boundary conditions (3.3.22) for two periodic trajectories $z^- = x^\alpha \in \mathcal{P}_0(H^\alpha)$ and $z^+ = x^\beta \in \mathcal{P}_0(H^\beta)$ of *different* Hamiltonians and denote the set of such u by

$$\mathcal{M}_{x^\alpha x^\beta} = \mathcal{M}(x^\alpha, x^\beta; H^{\alpha\beta}).$$

Just as before, for a generic, called **regular**, **homotopy** this moduli space is a smooth manifold for any choice of x^α, x^β and the dimension is the index difference $\mu^{\mathrm{CZ}}(x^\alpha) - \mu^{\mathrm{CZ}}(x^\beta)$. There are also analogous compactness and gluing properties. The difference is that due to the missing invariance under shifts in the s-variable one uses the *index difference zero* moduli spaces $\mathcal{M}_{x^\alpha x^\beta}$ (compact, thus finite, sets) to define maps which are given on $x^\alpha \in \mathrm{Crit}_k \mathcal{A}_{H^\alpha}$ by

$$\begin{aligned} \psi^{\beta\alpha}(H^{\alpha\beta}) : \mathrm{CF}_*(H^\alpha) &\to \mathrm{CF}_*(H^\beta) \\ x^\alpha &\mapsto \sum_{x^\beta \in \mathrm{Crit}_k \mathcal{A}_{H^\beta}} \#_2(\mathcal{M}_{x^\alpha x^\beta})\, x^\beta. \end{aligned} \tag{3.4.46}$$

The *index difference one* moduli spaces $\mathcal{M}_{x^\alpha y^\beta}$ lead to the chain map property

$$\psi^{\beta\alpha}\partial^\alpha = \partial^\beta \psi^{\beta\alpha}.$$

This identity is equivalent to all 1-fold broken trajectories from x^α to y^β appearing as partner pairs as indicated by Figure 3.13. Just as before the partner pair property follows from compactness up to 1-fold broken orbits and a corresponding gluing construction. The **induced morphism on homology**

$$\Psi^{\beta\alpha} := [\psi^{\beta\alpha}(H^{\alpha\beta})] : \mathrm{HF}_*(H^\alpha) \to \mathrm{HF}_*(H^\beta). \tag{3.4.47}$$

are called **Floer continuation maps**. They do not depend on the homotopy.

[57] E.g. pick three elements $H^\alpha, H^\beta, H^\gamma$ of the set $\mathcal{H}_{\mathrm{reg}}(J)$ provided by Theorem 3.3.19.

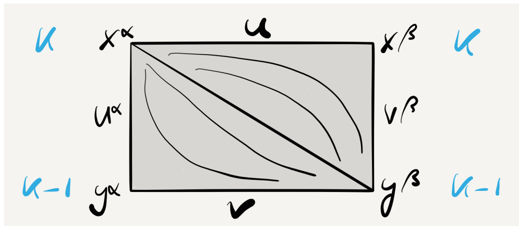


Figure 3.13: Partner pair property $(u^\alpha, v) \sim (u, v^\beta)$ implies $\psi^{\beta\alpha}\partial^\alpha = \partial^\beta\psi^{\beta\alpha}$

Exercise 3.4.15. Denote the **constant homotopy** $H^{\alpha\alpha} \equiv H^\alpha$ again by H^α for simplicity. Show that $\psi^{\alpha\alpha}(H^\alpha) = \mathbb{1}$ is the identity on $\mathrm{CF}_*(H^\alpha)$.

Lemma 3.4.16 (Salamon [Sal99a, Le. 3.11])**.** *Given regular homotopies $H^{\alpha\beta}$ from H^α to H^β and $H^{\beta\gamma}$ from H^β to H^γ, define a map*

$$H_R^{\alpha\gamma} = H_{R,s,t}^{\alpha\gamma} := \begin{cases} H_{s+R,t}^{\alpha\beta} & , \; s \le 0, \\ H_{s-R,t}^{\beta\gamma} & , \; s \ge 0, \end{cases}$$

where $R \ge 2$ is a constant; see Figure 3.14. Then there is a constant $R_0 > 0$ such that for $R > R_0$ the map $H_R^{\alpha\gamma}$ is a regular homotopy from H^α to H^γ and the induced morphism $\psi_R^{\gamma\alpha} : \mathrm{CF}_*(H^\alpha) \to \mathrm{CF}_*(H^\gamma)$ is given by

$$\psi_R^{\gamma\alpha} = \psi^{\gamma\beta} \circ \psi^{\beta\alpha}. \tag{3.4.48}$$

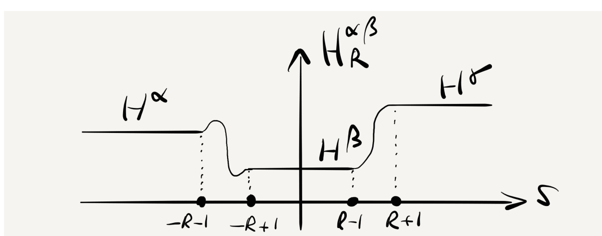


Figure 3.14: The homotopy $H_R^{\alpha\gamma}$ from H^α to H^γ

Exercise 3.4.17 (Induced morphisms $\Psi^{\beta\alpha}$ are independent of homotopy)**.** Given two regular homotopies $H_0^{\alpha\beta}$ and $H_1^{\alpha\beta}$ from H^α to H^β, show that $\psi_0^{\beta\alpha}$ and $\psi_1^{\beta\alpha}$ are chain homotopy equivalent: Define a homomorphism $T : \mathrm{CF}_*(H^\alpha) \to \mathrm{CF}_*(H^\beta)$ such that

$$\psi_1^{\beta\alpha} - \psi_0^{\beta\alpha} = \partial^\beta T + T\partial^\alpha. \tag{3.4.49}$$

Note that such T raises the grading by $+1$.
[Hint: Pick a regular homotopy $\{H_\lambda^{\alpha\beta}\}$ of homotopies $H_\lambda^{\alpha\beta} = H_{\lambda,s,t}^{\alpha\beta}$ from H^α

to H^β which agrees with $H_0^{\alpha\beta}$ for $\lambda = 0$ and with $H_1^{\alpha\beta}$ for $\lambda = 1$. In case of *index difference* -1 the parametrized moduli spaces

$$\mathcal{M}(y^\alpha, x^\beta; \{H_\lambda^{\alpha\beta}\}) := \{(\lambda, u) \mid \lambda \in [0,1], u \in \mathcal{M}(y^\alpha, x^\beta; H_\lambda^{\alpha\beta})\}$$

are 0-dimensional manifolds, in fact finite sets. Count them appropriately to define T. Analyze compactness up to 1-fold broken orbits of the (1-dimensional) *index difference* 0 moduli spaces $\mathcal{M}(x^\alpha, x^\beta; \{H_\lambda^{\alpha\beta}\})$ and set up corresponding gluing maps to prove the desired identity (3.4.49); cf. [Sal99a, Pf. of Le. 3.12].]

Proof of Theorem 3.4.6 (Continuation)

To prove that $\Psi^{\beta\alpha}$ is an isomorphism with inverse $\Psi^{\alpha\beta}$ pick a regular homotopy $H^{\alpha\beta} = H_{s,t}^{\alpha\beta}$ from H^α to H^β and denote the (regular) reverse homotopy by $H^{\beta\alpha} := H_{-s,t}^{\alpha\beta}$. With the associated homotopy $H_R^{\alpha\alpha}$ of Lemma 3.4.16 we get

$$\begin{aligned} \psi^{\alpha\beta}(H_{-s,t}^{\alpha\beta}) \circ \psi^{\beta\alpha}(H_{s,t}^{\beta\alpha}) &= \psi^{\alpha\alpha}(H_R^{\alpha\alpha}) \\ &= \psi^{\alpha\alpha}(H^\alpha) + \partial^\alpha T + T\partial^\alpha \\ &= \mathbb{1} + \partial^\alpha T + T\partial^\alpha \end{aligned}$$

where identity two is by (3.4.49) since the homotopies $H_R^{\alpha\alpha} \sim H^\alpha$ are homotopic and identity three is by Exercise 3.4.15. Hence $\Psi^{\alpha\beta}\Psi^{\beta\alpha} = \mathbb{1}$. Repeat the argument starting with a homotopy from H^β to H^α to get $\Psi^{\beta\alpha}\Psi^{\alpha\beta} = \mathbb{1}$. This shows that $\Psi^{\beta\alpha}$ is an isomorphism with inverse $\Psi^{\alpha\beta}$. That $\Psi^{\alpha\alpha} = \mathbb{1}$ follows from Exercises 3.4.15 and 3.4.17. The identity $\Psi^{\gamma\beta}\Psi^{\beta\alpha} = \Psi^{\gamma\alpha}$ holds by (3.4.48). This proves Theorem 3.4.6.

3.4.4 Isomorphism to singular homology

Recall that we use $\mathbb{Z}_2$ coefficients. After introducing in great length a new homology theory for the data (M, ω, H, J) – which does not depend on (H, J) as we saw in the previous section on continuation – it is natural to ask if Floer homology relates to any known homology theory and if so, how? The answer is that Floer homology relates to *singular homology of the closed manifold M itself* by isomorphisms $\Psi^{\varepsilon H^\alpha} = \Psi^{\mathrm{PSS}}$ compatible with the continuation maps $\Psi^{\beta\alpha}$.

Method 1 (C^2 small Morse functions)

Floer observed in [Flo89, Thm. 2] that, roughly speaking, if one chooses a sufficiently C^2 small Morse function $h : M \to \mathbb{R}$ as Hamiltonian, then not only the 1-periodic trajectories are precisely the critical points of the Morse function h, but also the connecting Floer trajectories $u = u(s, t)$ will not depend on s and turn into connecting Morse trajectories $\gamma = \gamma(s)$.[58]

[58] As it is already not obvious to prove this for constant in s trajectories $u = u(t)$, that is constant periodic trajectories, see Proposition 2.3.16, it should be less obvious for general $u = u(s, t)$.

Remark 3.4.18 (C^2 small: Floer trajectories reduce to Morse – not always!)**.** Actually the situation is surprisingly more complex as we said right above: Hofer and Salamon proved in [HS95, §7] that for sufficiently C^2 small Morse functions all connecting trajectories of index difference *one or less* are independent of t and therefore connecting Morse trajectories. But index difference one (and zero) *is all that is needed* in either chain complex. (See [HS95, Ex. 7.2] for an example how t-independence fails in case of index difference two or larger. See also [HS95, Rmk. 7.5] saying that their proof does not work for symplectic manifolds of minimal Chern number $n-1$.)

Suppose from now on that $h : M \to \mathbb{R}$ is Morse and C^2 small. A closer look shows that the Floer equation (3.3.20) actually turns for t-independent trajectories $u(s,\cancel{t}) =: \gamma(s)$ into the *up*ward gradient equation

$$\gamma' = \partial_s u = -\operatorname{grad} \mathcal{A}_h(u) = \nabla h \tag{3.4.50}$$

for curves $\gamma =: \mathbb{R} \to M$; cf. Figure 3.6. Note that $\operatorname{Crit}\mathcal{A}_h = \operatorname{Crit} h$ by Proposition 2.3.16 and that $\mathcal{A}_h = -h$ along the critical set. Furthermore, by (2.3.26) the Floer grading z as a constant periodic trajectory and the Morse index of z as a critical point of h are related by $\mu^{\mathrm{CZ}}(z) = n - \operatorname{ind}_h(z)$.

Recall from Section 3.1.2, see [Web] for details, that the Morse cochain groups of a Morse function h on a closed manifold are generated by the critical points of h, graded by their Morse index ind_h, and the coboundary operator is given by counting the flow lines of the upward gradient ∇h, that is from critical points x of index, say $2n-k-1 = \operatorname{ind}_h(x)$, equivalently $k+1-n = \mu^{\mathrm{CZ}}(x)$ to those of index $2n-k = \operatorname{ind}_h(y)$, equivalently $k-n = \mu^{\mathrm{CZ}}(y)$.

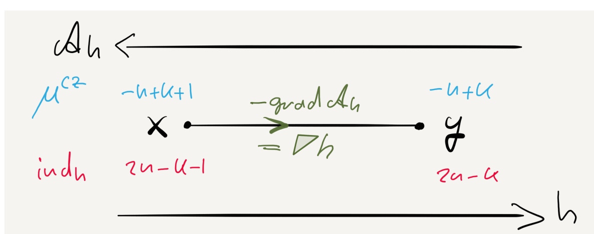


Figure 3.15: Downward Floer gradient degenerates to upward Morse gradient

Note that $\partial^{\mathrm{F}} x = \delta^{\mathrm{M}} x$ by (3.4.50). Consequently for any sufficiently $\boldsymbol{C^2}$ **small Morse function**, denoted for distinction by

$$\varepsilon h : M \to \mathbb{R},$$

the following chain and cochain complexes are naturally equal

$$\left(\mathrm{CF}_{*-n}(\varepsilon h), \partial^{\mathrm{F}}(\varepsilon h)\right) \equiv \left(\mathrm{CM}^{2n-*}(\varepsilon h), \delta^{\mathrm{M}}(\varepsilon h)\right), \tag{3.4.51}$$

up to shifting the grading, that is $\mathrm{CF}_{n-\ell}(\varepsilon h)$ is naturally equal to $\mathrm{CM}^{\ell}(\varepsilon h)$.

To prove Theorem 3.4.7 consider a regular pair (H^α, J^α) and define the desired isomorphism, say denoted by $\Psi^{\varepsilon H^\alpha}$, by composing the following three isomorphisms: Firstly, the Floer continuation map

$$\mathrm{HF}_{k-n}(H^\alpha) \to \mathrm{HF}_{k-n}(\varepsilon h)$$

for some C^2 small Morse function εh. Secondly, the isomorphism

$$\Psi^{\varepsilon h} : \mathrm{HF}_{k-n}(\varepsilon h) \to \mathrm{HM}^{2n-k}(\varepsilon h)$$

induced by the chain level identity (3.4.51), followed by Poincaré duality[59]

$$\mathrm{HM}^{2n-k}(\varepsilon h) \simeq \mathrm{HM}_k(\varepsilon h).$$

Thirdly, the fundamental isomorphism (3.1.5) of Morse homology given by

$$\mathrm{HM}_k(\varepsilon h) \simeq \mathrm{H}_k(M)$$

and compatible with the Morse continuation maps. We use $\mathbb{Z}_2$ coefficients.

As a consequence $\Psi^{\varepsilon H^\alpha}$ is compatible with the Floer continuation maps as stated in Theorem 3.4.7. See also [SZ92] and [MS04, Thm. 12.1.4]. For simplicity and in view of (3.4.52) let us denote $\Psi^{\varepsilon H^\alpha}$ by Ψ^α from now on.

Method 2 (Spiked disks –PSS isomorphism)

Piunikhin, Salamon, and Schwarz came up in [PSS96] with a rather different idea to construct an isomorphism between Floer and Morse homology, denoted by

$$\Phi^{\mathrm{PSS}} : \mathrm{HM}_*(M) \to \mathrm{HF}_{*-n}(M, \omega).$$

Fix a Morse function f and a (generic) Riemannian metric on M such that the Morse complex is defined and pick a generator $x \in \mathrm{Crit}_k f$. Suppose (H, J) is a regular pair, so Floer homology is defined, and pick a generator $z \in \mathrm{Crit}_{k-n}\mathcal{A}_H$. The idea is to relate Morse and Floer flow lines by first following a Morse flow line γ that comes from x and then, say at time $s = 0$, change over to a Floer flow line that goes to z. Of course, the transition from a family of points to a family of circles requires some interpolation, i.e. some thought. The idea is to look at J-holomorphic planes $v : \mathbb{C} \to M$, see (3.4.42), and homotop, as the polar radius s of points in $\mathbb{C}$ traverses the interval $[1, 2]$, the zero Hamiltonian inside the unit circle (polar radius $s = 1$) to the given Hamiltonian H_t on and outside the circle of polar radius $s = 2$. Denote such a homotopy by $H_s = H_{s,t}$. Moreover, one requires $v(e^{2\pi(s+i\cdot)})$ to converge to the given periodic trajectory z, as $s \to \infty$. The key condition that couples the two worlds is then that γ meets v at $s = 0$. Such a configuration, called a spiked disk, is illustrated by Figure 3.16. Now the codimension of the set of all possible points $\gamma(0)$ is $2n - \mathrm{ind}_f(x)$. And the set of all v which satisfy the homotopy Floer equation (3.4.45) for H_s and converge

[59] Any symplectic manifold is naturally *oriented* and our M is *closed* by assumption.

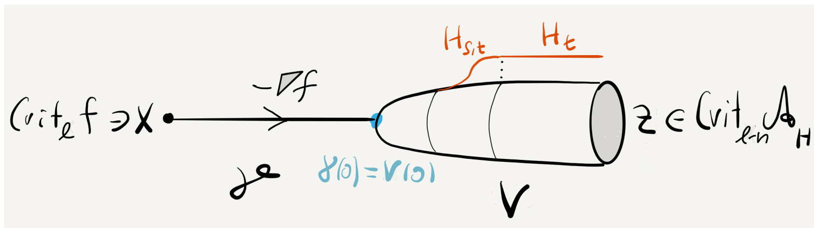


Figure 3.16: Counting spiked disks defines the PSS chain map $\phi^{\mathrm{PSS}}(H, f)$

to z, as $s \to \infty$, has dimension $n - \mu^{\mathrm{CZ}}(z)$.[60] Thus the moduli space of spiked disks is (for generic homotopy H_s) a 0-dimensional manifold in case the Morse index of x is equal to $\mu^{\mathrm{CZ}}(z) + n$. Also there is 'no index left' for broken flow lines, so one has compactness, thus finiteness of index difference zero moduli spaces. It is now clear that on the chain level the homomorphism

$$\phi^{\mathrm{PSS}} = \phi^{\mathrm{PSS}}(H, f) : \mathrm{CM}_\ell(f) \to \mathrm{CF}_{\ell-n}(H)$$

is defined by counting (modulo 2) the finitely many spiked disks from x to z. The index difference one moduli spaces are compact up to 1-fold broken orbits, the breaking can happen on either side, which together with a gluing construction shows that the 1-fold broken orbits come in pairs, namely as partner pairs. This shows that $\phi^{\mathrm{PSS}}(H, f)$ is a chain map. But why is the induced homomorphism Φ^{PSS} on homology an isomorphism?

Exercise 3.4.19. Replace spiked disks by disks with spikes to define chain maps

$$\psi^{\mathrm{PSS}} = \psi^{\mathrm{PSS}}(f, H) : \mathrm{CF}_k(H) \to \mathrm{CM}_{k+n}(f).$$

Show by picture that both compositions $\phi^{\mathrm{PSS}}(H, f) \circ \psi^{\mathrm{PSS}}(f, H) \sim \mathbb{1}_{\mathrm{CF}}$ and $\psi^{\mathrm{PSS}}(f, H) \circ \phi^{\mathrm{PSS}}(H, f) \sim \mathbb{1}_{\mathrm{CM}}$ are chain homotopic to the identity.
[Hint: Draw a configuration for one of the compositions and see how it can degenerate, that is identify the configurations in the boundary of moduli space. Consult [MS04, §12.1] in case you get stuck.]

Exercise 3.4.20. Show that both methods lead to the same isomorphisms

$$\Psi^{\mathrm{PSS}} = \Psi^{\varepsilon H^\alpha} =: \Psi^\alpha. \tag{3.4.52}$$

[Hint: Consult [MS04, §12.1], prior to Rmk. 12.1.7, in case you get stuck.]

3.4.5 Action filtered Floer homology

We summarize the main features in the form of an exercise; use $\mathbb{Z}_2$ coefficients.

[60] The dimension formula $2n - \mu_H(z)$ in [Sal99a, § 3.5] involves $\mu_H(z) := n - \mu_{\mathrm{CZ}}(z) = n + \mu^{\mathrm{CZ}}(z)$.

Exercise 3.4.21. Prove the **energy identity for connecting homotopy trajectories**, that is

$$E(u) = \|\partial_s u\|_{L^2}^2 = \mathcal{A}_{H^\alpha}(x^\alpha) - \mathcal{A}_{H^\beta}(x^\beta) - \int_0^1 \int_{-\infty}^{\infty} \left(\partial_s H_{s,t}\right)(u)\, ds\, dt$$

for every connecting homotopy trajectory $u \in \mathcal{M}(x^\alpha, x^\beta; H^{\alpha\beta})$ where $E(u)$ is defined by (3.3.23) for $H_{s,t}$. A **monotone homotopy** is a homotopy $H_{s,t}$ such that $\partial_s H_{s,t} \geq 0$ pointwise on $\mathbb{S}^1 \times M \times \mathbb{R}$. Show that for monotone homotopies the action of x^α is strictly larger than that of x^β, unless the connecting homotopy flow line u is constant in s in which case $u \equiv x^\alpha = x^\beta$. Suppose that a and b are regular values of $\mathcal{A}_{H^\alpha}$ and define the **action filtered Floer chain groups**

$$\mathrm{CF}_*^{(a,b)}(H^\alpha)$$

as usual except for only employing critical points whose actions lie in the **action window** (a, b). Define the boundary operator as before in (3.4.41) except for only employing critical points of action in (a, b). Show that $\partial^2 = 0$. The homology $\mathrm{HF}_*^{(a,b)}(H^\alpha)$ of this chain complex is called **action filtered Floer homology**.

Given a monotone homotopy $H^{\alpha\beta}$, define the corresponding **monotone continuation map** $\psi^{\beta\alpha}(H^{\alpha\beta})$ as before in (3.4.46) except for only employing critical points of action in (a, b). Check that this defines a chain map. The induced homomorphism $\Psi^{\beta\alpha}$ on homology is the **monotone continuation map**. Now consider Hamiltonians that have the property that $H^\alpha \leq H^\beta$ pointwise on $\mathbb{S}^1 \times M$. Pick a smooth cutoff function $\rho : \mathbb{R} \to [0, 1]$ which is 0 for $s \leq -1$ and 1 for $s \geq 1$. Check that for any Hamiltonians with $H^\alpha \leq H^\beta$ the **convex combination** $H_\rho^{\alpha\beta} := (1 - \rho)H^\alpha + \rho H^\beta$ is a monotone homotopy. Show that the corresponding monotone continuation maps have composition properties

$$\Psi^{\gamma\beta}\Psi^{\beta\alpha} = \Psi^{\gamma\alpha}, \qquad \Psi^{\alpha\alpha} = \mathbb{1},$$

whenever $H^\alpha \leq H^\beta \leq H^\gamma$ analogous to Theorem 3.4.6.

Recall from Proposition 2.3.16 that on a closed symplectic manifold M there are Hamiltonians $H : \mathbb{S}^1 \times M \to \mathbb{R}$ without non-contractible 1-periodic trajectories. So by continuation Floer homology is trivial on components of the free loop space of M other than the component $\mathcal{L}_0 M$ consisting of contractible loops. However, sometimes combining a suitable action window with a geometric constraint leads to nontrivial results; see e.g. [Web06a] for such a situation, although for a class of non-closed symplectic manifolds, namely cotangent bundles.

3.4.6 Cohomology and Poincaré duality

Recall that we work with $\mathbb{Z}_2$ coefficients. Cohomology arises from homology by dualization. In Section 3.1.2 this is explained in detail including the geometric realization, just replace Morse co/homology by Floer co/homology. So here we just summarize the geometric realization of the Floer cochain complex. Given a regular pair (H, J), the **Floer cochain group** is the $\mathbb{Z}_2$ vector space

$$\mathrm{CF}^*(H) := \mathrm{Hom}(\mathrm{CF}_*(H), \mathbb{Z}_2).$$

Obviously the canonical basis $\mathcal{B}_H$ of the $\mathbb{Z}_2$ vector space $\mathrm{CF}_*(H)$ is the set of generators $\mathrm{Crit}\mathcal{A}_H = \mathcal{P}_0(H)$, namely the finite set of contractible 1-periodic Hamiltonian trajectories. Thus the (finite) set $\mathcal{B}_H^{\#} := \{\eta^x \mid x \in \mathcal{P}_0(H)\}$ of Dirac δ-functionals[61] on $\mathrm{CF}_*(H)$ is a basis of $\mathrm{CF}^*(H)$, called the **canonical basis of** $\mathrm{CF}^*(H)$. Thus $\mathrm{CF}^*(H)$ is a $\mathbb{Z}_2$ vector space of finite dimension. Let $\mathrm{CF}^k(H)$ be the subspace generated by those Dirac functionals whose canonical Conley-Zehnder index $\mu^{\mathrm{CZ}}(\eta^x) := \mu^{\mathrm{CZ}}(x)$ is equal to k. As we saw in (3.1.10), the **Floer coboundary operator**

$$\delta^k := (\partial_{k+1})^{\#} : \mathrm{CF}^k(H) \to \mathrm{CF}^{k+1}(H)$$

acts on basis elements by

$$\delta^k \eta^y = \sum_{x \in \mathrm{Crit}_{k+1}\mathcal{A}_H} \#_2(m_{xy})\, \eta^x.$$

Here $\#_2(m_{xy})$ is the number (mod 2) of connecting Floer flow lines, cf. (3.4.41), and $\mathrm{Crit}_k\mathcal{A}_H$ is the set of critical points x with $\mu^{\mathrm{CZ}}(x) = k$. By Proposition 3.4.3 we get that $\delta^2 = (\partial^2)^{\#} = 0$.

Tu put it in a nut shell, the Floer cochain groups are generated by the contractible 1-periodic trajectories and graded by the canonical Conley-Zehnder index, whereas the Floer coboundary operator is given by the (mod 2) upward count of connecting Floer flow lines between critical points of index difference one.

The quotient space

$$\mathrm{HF}^k(H) = \mathrm{HF}^k(M, \omega, H; J) := \frac{\ker \delta^k}{\mathrm{im}\, \delta^{k-1}}$$

is called the $\boldsymbol{k^{\mathrm{th}}}$ **Floer cohomology** with $\mathbb{Z}_2$ coefficients associated to $H \in \mathcal{H}_{\mathrm{reg}}(J)$. Given regular pairs (H^α, J^α) and (H^β, J^β), continuation isomorphisms of degree zero are given by the transposes

$$(\Psi^{\beta\alpha})^{\#} = [\psi^{\beta\alpha}(H^{\alpha\beta})^{\#}] : \mathrm{HF}^*(\beta) \to \mathrm{HF}^*(\alpha)$$

of the continuation maps in Theorem 3.4.6; see (3.4.46) and (3.1.7). The transpose of the natural isomorphism Ψ^α to singular homology in Theorem 3.4.7 provides the isomorphism

$$(\Psi^\alpha)^{\#} : \mathrm{H}^*(M) \to \mathrm{HF}^{*-n}(\alpha)$$

which is of degree $-n$ and compatible with the continuation maps.

[61] Define η^x on basis elements $y \in \mathcal{B}_H$ by $\eta^x(y) = 1$, if $y = x$, and by $\eta^x(y) = 0$, otherwise.

Poincaré duality

Suppose (H, J) is a regular pair and $x \in \mathrm{Crit}\mathcal{A}_H$ is a contractible 1-periodic Hamiltonian trajectory. Consider the maps

$$\hat{x}(t) := x(-t), \qquad \hat{H}_t := -H_{-t}, \qquad \hat{J}_t := J_{-t}.$$

Exercise 3.4.22. Given a regular pair (H, J) and $x, y \in \mathrm{Crit}\mathcal{A}_H$, show that

$$\mu^{\mathrm{CZ}}(\hat{x}; \hat{H}) = -\mu^{\mathrm{CZ}}(x; H), \qquad \mathcal{A}_{\hat{H}}(\hat{\gamma}) = -\mathcal{A}_H(\gamma),$$

for every contractible loop $\gamma : \mathbb{S}^1 \to M$ and that

$$u \in \mathcal{M}(x, y; H, J) \quad \Leftrightarrow \quad \hat{u} \in \mathcal{M}(\hat{y}, \hat{x}; \hat{H}, \hat{J})$$

where $\hat{u}(s, t) := u(-s, -t)$.

Exercise 3.4.23. Given a regular pair (H, J), pick $x \in \mathrm{Crit}_{k+1}\mathcal{A}_H$. Check that the horizontal maps in the diagram

$$\begin{tikzcd}[column sep=large]
 & x \arrow[r, mapsto] & \hat{x} \arrow[r, mapsto] & \eta^{\hat{x}} \\
\mathrm{Pd}_{k+1}: & \mathrm{CF}_{k+1}(H) \arrow[r, "\widehat{\ }"] \arrow[d, "\partial_{k+1}(H,J)"', "-\operatorname{grad}\mathcal{A}_H(u_s)"] & \mathrm{CF}_{-k-1}(\hat{H}) \arrow[r, "\#"] & \mathrm{CM}^{-k-1}(\hat{H}) \arrow[d, "\hat{\delta}^{-k-1}(\hat{H},\hat{J})"] \\
\mathrm{Pd}_k: & \mathrm{CF}_k(H) \arrow[r, "\widehat{\ }"] & \mathrm{CF}_{-k}(\hat{H}) \arrow[r, "\#"] \arrow[u, "\hat{\partial}_{-k}", "\substack{-\operatorname{grad}\mathcal{A}_{\hat{H}}(\hat{u}_s) \\ =\operatorname{grad}\mathcal{A}_H(u_s)}"'] & \mathrm{CF}^{-k}(\hat{H}) \\
 & \sum_y \#_2 m_{xy} \cdot y \arrow[r, mapsto] & \sum_{\hat{y}} \#_2 m_{xy} \cdot \hat{y} \arrow[r, mapsto] & \sum_{\hat{y}} \substack{\#_2 m_{\hat{y}\hat{x}} \\ \#_2 m_{xy}} \cdot \eta^{\hat{y}}
\end{tikzcd}$$

are isomorphisms and that the diagram commutes,[62] that is

$$\hat{\delta}^{-k-1} \circ \mathrm{Pd}_{k+1} = \mathrm{Pd}_k \circ \partial_{k+1}.$$

[Hint: Compare the mod 2 counts $\#_2 m_{xy}(H, J)$ and $\#_2 m_{\hat{y}\hat{x}}(\hat{H}, \hat{J})$; cf. (3.4.41).]

Definition 3.4.24 (Poincaré duality)**.** By Exercise 3.4.23 the chain level isomorphisms $\mathrm{Pd}_k^{\hat{\alpha}\alpha}$, where α abbreviates (H, J), descend to isomorphisms

$$\mathrm{PD}_k^{\hat{\alpha}\alpha} := [\mathrm{Pd}_k^{\hat{\alpha}\alpha}] : \mathrm{HF}_k(\alpha) \xrightarrow{\simeq} \mathrm{HF}^{-k}(\hat{\alpha})$$

which together with continuation provide the Poincaré duality isomorphisms

$$\mathrm{PD}_{k-n}^{\alpha} := \left(\Psi^{\hat{\alpha}\alpha}\right)^{\#} \circ \mathrm{PD}_{k-n}^{\hat{\alpha}\alpha} : \underbrace{\mathrm{HF}_{k-n}(\alpha)}_{\simeq \mathrm{H}_k(M)} \xrightarrow{\simeq} \mathrm{HF}^{n-k}(\hat{\alpha}) \xrightarrow{\simeq} \underbrace{\mathrm{HF}^{n-k}(\alpha)}_{\simeq \mathrm{H}^{2n-k}(M)}$$

for every k and any regular pair (H^α, J^α) and where $2n = \dim M$.

[62] The sums in the last line are over all critical points of canonical Conley-Zehnder index equal to the grading of the corresponding (co)chain group in the previous line.

3.5 Cotangent bundles and loop spaces

Suppose (Q, g) is a closed Riemannian manifold. Pick a smooth function V on $\mathbb{S}^1 \times M$, called **potential energy**, and set $V_t(q) := V(t, q)$. For $v \in T_qQ$ we abbreviate $g_q(v, v)$ by $|v|_q^2 =: |v|^2$.

The Lagrange function or **Lagrangian** $L_{V_t}(q, v) = \frac{1}{2}|v|_q^2 - V_t(q)$, defined on $\mathbb{S}^1 \times TQ$, is the difference of *kinetic* and *potential energy*. The functional $\mathcal{S}_V(x) := \int_0^1 L_{V_t}(x(t), \dot{x}(t))\, dt$ defined on the free loop space $\mathcal{L}Q := C^\infty(\mathbb{S}^1, Q)$ of Q is called the **classical action functional**. Explicitly the functional $\mathcal{S}_V$ is given by

$$\mathcal{S}_V = \mathcal{S}_{V,g} : \mathcal{L}Q \to \mathbb{R}, \quad x \mapsto \int_0^1 \frac{1}{2} |\dot{x}(t)|^2 - V_t(x(t))\, dt. \tag{3.5.53}$$

Its extremals, that is the critical points of $\mathcal{S}_V$, are the perturbed[63] closed geodesics on the Riemannian manifold Q. In fact, there is a bijection

$$\operatorname{Crit} \mathcal{S}_V = \mathcal{P}(V) \simeq \operatorname{Crit} \mathcal{A}_V, \quad x \mapsto z_x := (x, \dot{x}),$$

where $\mathcal{P}(V) = \{-\nabla_t \dot{x} - \nabla V_t(x) = 0\}$, see (1.0.6), and the functional given by

$$\mathcal{A}_V : \mathcal{L}TQ \to \mathbb{R}, \quad z = (q, v) \mapsto \int_0^1 \langle v(t), \dot{q}(t)\rangle - H_{V_t}(q(t), v(t))\, dt,$$

is called the (perturbed) **symplectic action functional**. The critical set $\operatorname{Crit}\mathcal{A}_V$ coincides with the set of 1-periodic trajectories of the Hamiltonian $H_V(q, v) := \frac{1}{2}|v|_q^2 + V_t(q)$ on $TQ \simeq T^*Q$, $(q, v) \mapsto g_q(v, \cdot)$. Moreover, for generic V both functionals are Morse and the Morse index

$$\operatorname{ind}_{\mathcal{S}_V}(x) = \mu^{\mathrm{CZ}}(z_x) \tag{3.5.54}$$

of a critical point x of $\mathcal{S}_V$ coincides with the canonical Conley-Zehnder index, see (1.0.11), of the corresponding critical point $z_x = (x, \dot{x})$ of $\mathcal{A}_V$ whenever the vector bundle $x^*TQ \to Q$ is orientable; otherwise a correction term $\sigma(x) = +1$ adds to $\mu^{\mathrm{CZ}}(z_x)$. For proofs of these facts see [Web02]. It turns out, see [SW06], that the downward L^2 gradient equation of $\mathcal{S}_V$ is the **heat equation**

$$\partial_s u - \nabla_t \partial_t u - \nabla V_t(u) = 0 \tag{3.5.55}$$

for smooth cylinders $u : \mathbb{R} \times \mathbb{S}^1 \to Q$ in the manifold Q. Imposing on u asymptotic boundary conditions $x^\pm \in \mathcal{P}(V)$, similarly to (3.3.22), the operators D_u obtained by linearizing the heat equation (3.5.55) are Fredholm whenever the 1-periodic Hamiltonian trajectories $x^\mp$ are non-degenerate. If all $x^\pm \in \mathcal{P}(V)$ are non-degenerate and all linearizations D_u are surjective, in other words, if the **Morse-Smale condition** holds for (3.5.55), then counting flow lines between

[63] Here the *perturbation* is V. If $V \equiv const$, the critical points are the closed geodesics.

critical points of Morse index difference one, say modulo 2, defines a boundary operator on the Morse chain groups. With $\mathbb{Z}_2$ coefficients these are defined by

$$\mathrm{CM}_*(\mathcal{S}_{V,g}) := \bigoplus_{x\in\mathcal{P}(V)} \mathbb{Z}_2 x$$

and they are graded by the Morse index $\mathrm{ind}_{\mathcal{S}_V}$, whatever coefficient ring. With integer coefficients the Morse complex $\mathrm{CM}(\mathcal{S}) = (\mathrm{CM}_*, \partial_*)$ has been constructed in [Web13b, Web13a]. In [Web] it is shown that there is a natural isomorphism

$$\mathrm{HM}_*(\mathcal{S}) \simeq \mathrm{H}_*(\mathcal{L}Q)$$

to singular homology of the free loop space. In [SW06] a natural isomorphism

$$\mathrm{HF}_*(T^*Q, H_V) \simeq \mathrm{HM}_*(\mathcal{S}_V)$$

was established which, however, over the integers not only requires that Q is orientable ($w_1(Q) = 0$), but also that the second Stiefel-Whitney class $w_2(Q)$ vanishes over (2-)tori; cf. subsection below. The fact that the homology groups

$$\mathrm{HF}_*(T^*Q, \omega_{\mathrm{can}}, \mathcal{A}_V) \simeq \mathrm{H}_*(\mathcal{L}Q) \tag{3.5.56}$$

are isomorphic is called **Viterbo's theorem**; see [Vit98] for the Viterbo proof and [SW06, Web] and [AS06, AM06], otherwise.

Exercise 3.5.1 (Pendulum watched by uniformly rotating observer)**.** Consider the simplest closed manifold $\mathbb{S}^1$ and work out explicitely the Morse and Floer chain complexes leading to (3.5.56). While one quickly sees that a pendulum subject to gravity is described by a Hamiltonian of the form $H_V(q,v) = \frac{1}{2}|v|^2 + V(q)$ on $T^*\mathbb{S}^1 = (\mathbb{R}/\mathbb{Z}) \times \mathbb{R}$, how could one change the system in order to make the potential V not only time-dependent, but even time-1-periodic?
[Hint: Consult [Web96] in case you get stuck.]

Orientations

For general closed manifolds Q the relation between the Morse and the Conley-Zehnder index of $x \in \mathcal{P}(V)$ has been established in [Web02].

That even for orientable closed manifolds Q there is a problem to construct coherent orientations of the spaces of connecting flow lines has been discovered by Kragh [Kra07]; cf. [Sei10]. The problem arises when the second Stiefel-Whitney class does not vanish over (2-)tori. In such cases Abouzaid [Abo11] resolved the problem by using local coefficients to construct the Floer homology goups of the cotangent bundle; see also [AS14, AS15] and [Kra13]. The case of general closed manifolds Q, orientable or not, is treated in [Abo15].

Remark 3.5.2. Suppose Q is a closed manifold. Then Q is orientable iff the first Stiefel-Whitney class (of its tangent bundle) is trivial, that is $w_1(Q) = 0 \in \mathrm{H}^1(Q;\mathbb{Z}_2)$; see e.g. [MS74, p.148]. An orientable manifold is called **spin** if it carries what is called a spin structure and this is equivalent to $w_2(Q) \in \mathrm{H}^1(Q;\mathbb{Z}_2)$ being trivial; see e.g. [LM89, II Thm. 2.1]. In other words, a manifold being spin is equivalent to both $w_1(Q)$ and $w_2(Q)$ being trivial.

In particular, the isomorphism (3.5.56) holds true over the integers

- for orientable closed manifolds Q such that $w_2(Q)$ vanishes on all (2-)tori;
- in particular, for closed manifolds that carry a spin structure.

Examples of spin manifolds Q are

- all closed orientable manifolds of dimension $n \le 3$;
- all spheres (which is non-obvious only for two-spheres);
- all odd complex projective spaces $\mathbb{C}\mathrm{P}^{2n+1}$, e.g. the Riemann sphere $\mathbb{C}\mathrm{P}^1$.

The even complex projective spaces $\mathbb{C}\mathrm{P}^{2n}$ are not spin, in particular, the complex projective plane $\mathbb{C}\mathrm{P}^2$ is not.

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Part II

Reeb dynamics

Chapter 4

Contact geometry

From now on we consider *autonomous* Hamiltonians $F : M \to \mathbb{R}$ on symplectic manifolds (M, ω) and restrict our search for periodic trajectories to closed regular level sets $S = F^{-1}(c)$ equipped with the Hamiltonian vector field X_F. One says that F defines S. If K defines S, too, then $X_F = f X_K$ for some non-vanishing function f. So, given S, the set $\mathcal{C}(S)$ of closed flow lines P, called closed characteristics of S, does not depend on the defining Hamiltonian F. But the natural parametrizations of these embedded circles P depend on F; just multiplicate F by constants $\alpha > 1$ to run faster, $\alpha \in (0, 1)$ to run slower, or $\alpha < 0$ to run in the opposite direction along P. So in this context it doesn't make sense to fix the period. One looks for periodic trajectories, any period. But, even in $(\mathbb{R}^{2n}, \omega_0)$, not every regular level set S admits a periodic trajectory.

To guarantee existence of periodic trajectories one needs to impose geometric conditions on a closed hypersurface in (M, ω). Firstly, co-orientability and, secondly, there must exist a contact form α on the hypersurface which is compatible with the ambient symplectic manifold in the sense that the 2-form $d\alpha$ coincides with the restriction of ω. Such co-orientable closed hypersurfaces are called *hypersurfaces of contact type* and we denote them by Σ or (Σ, α) for distinction from ordinary energy surfaces S. Our main reference in Chapter 4 is [HZ11]. We also recommend the excellent overviews and surveys, [Etn16] and [Gei01], respectively, and the very nicely written introduction in [Wen15, §1.6].

Notation 4.0.3. We use the notation (M^{2n}, ω) for symplectic and (W^{2n-1}, α) for contact manifolds. Exact symplectic manifolds are denoted by (V, λ) with symplectic form $\omega := d\lambda$. For *autonomous* Hamiltonians and their flows we use the letters $F : M \to \mathbb{R}$ and $\phi = \phi^F$, whereas for potentially time-dependent quantities we write H and $\psi = \psi^H$; see also Notation 1.0.5. Energy surfaces are closed hypersurfaces S of the form $F^{-1}(c)$ where c is a regular value of F. Hypersurfaces of contact type in a symplectic manifold (M, ω) are denoted by (Σ, α), where the contact form α has to satisfy a compatibility condition with ω which can be formulated, equivalently, in terms of existence of a Liouville vector field Y near Σ. We assume that the submanifolds S and Σ are closed.

4.1 Energy surfaces in $(\mathbb{R}^{2n}, \omega_0)$

Unless mentioned differently, let $\mathbb{R}^{2n}$ be equipped with the standard symplectic form $\omega_0 = d\lambda_0 =: dx \wedge dy$; cf. (1.0.8). In fundamental difference to Chapter 3 we consider now *autonomous* Hamiltonians $F : \mathbb{R}^{2n} \to \mathbb{R}$. A **hypersurface** is a submanifold of codimension one, sometimes called a **regular hypersurface**.

Definition 4.1.1. A hypersurface Σ in a non-compact manifold V is called **bounding** if it is closed and its complement $V \setminus \Sigma$ consists of two connected components, one of which, called the **inside**, has compact closure, say M. Then M is a compact manifold-with-boundary and $\partial M = \Sigma$ is connected and closed. In this case we say that $\boldsymbol{\Sigma}$ **bounds** $\boldsymbol{M}$.

To understand the dynamics of the flow $\phi = \phi^F$ on $(\mathbb{R}^{2n}, \omega_0)$ generated by the Hamiltonian vector field X_F defined by (2.3.16) recall that, by level preservation (2.3.18), it suffices to understand the dynamics on each level set $F^{-1}(c)$. To have a realistic goal, still ambitious though, we only consider compact level sets. In addition, we require that c is a regular value of F. Hence $F^{-1}(c) \subset \mathbb{R}^{2n}$ is a closed submanifold of codimension one by the regular value theorem. Thus $(F^{-1}(c), \phi^F)$ is a compact smooth dynamical system. This is true for almost every $c \in \mathbb{R}$. (By Sard's theorem the non-regular values form a measure zero subset of $\mathbb{R}$.) The converse is somewhat less obvious.

Lemma 4.1.2. *Every connected closed hypersurface $S \subset \mathbb{R}^m$ is (co-)orientable and of the form $f^{-1}(0)$ for some smooth function $f : \mathbb{R}^m \to \mathbb{R}$ with $\nabla f \pitchfork S$.*
Smooth Jordan Brouwer separation theorem. *S is bounding.*

Note that $\nabla f \pitchfork S$ tells that 0 is a regular value of f and $[\nabla f]$ co-orients S.

Proof. Based on the fact that any closed hypersurface in $\mathbb{R}^m$ is orientable, a construction of a function f on $\mathbb{R}^m$ such that $f^{-1}(0) = S$ and $\nabla f \pitchfork S$ and such that both sets $\{f < 0\}$ and $\{f > 0\}$ are connected is given in [Lim88].[1] By compactness of S there is a radius R such that S lies inside the radius R ball B centered at the origin. Suppose by contradiction that there are elements $x \in \{f < 0\}$ and $y \in \{f > 0\}$ that *both* lie outside the ball B. Connect x and y by a continuous path that lies outside B. Then f must be zero somewhere along the path. Contradiction. $\square$

To summarize, in $\mathbb{R}^{2n}$ connected compact regular level sets $F^{-1}(c)$ and connected closed hypersurfaces S are the same. However, one is related to functions, the other one to geometry. Whereas on $F^{-1}(c)$ dynamics arises by X_H, on the geometry side the dynamical information has its description as well, namely through integral submanifolds of what is called the characteristic line bundle $\mathcal{L}_S \to S$. In the following we discuss both versions.

[1] The construction of f works if S is any codimension one submanifold, compact or not, of any simply-connected manifold; see [Lim88, Rmk.]. Show that 1-connected is *necessary*.

Hamiltonian dynamics on energy surfaces – periodic trajectories

A Hamiltonian $F : \mathbb{R}^{2n} \to \mathbb{R}$ being *autonomous* has useful consequences:

(i) Level sets, called **energy levels**, are preserved by the Hamiltonian flow

$$\phi = \phi^F = \{\phi_t^F\}_{t\in\mathbb{R}}$$

as we saw in (2.3.18). Throughout we assume compactness of level sets, so ϕ is indeed a **complete flow**, i.e. exists for all times in $\mathbb{R}$.

(ii) If c is a **regular value** of F, that is dF nowhere vanishes on $F^{-1}(c)$, and $F^{-1}(c)$ is compact, then we call the closed codimension 1 submanifold

$$S^c := F^{-1}(c) \subset \mathbb{R}^{2n}$$

an **energy surface**. It may have finitely many connected components.

(iii) By Lemma 4.1.2 a connected[2] closed hypersurface $S \subset \mathbb{R}^{2n}$ is a level set

$$S := S^0 = F^{-1}(0) \tag{4.1.1}$$

for some smooth function $F : \mathbb{R}^{2n} \to \mathbb{R}$ with regular value 0.

In $\mathbb{R}^{2n}$ connected closed hypersurfaces are energy surfaces.

Observe that compactness is part of our definition of an energy surface. Note that $X_F \neq 0$ everywhere on S, so ϕ admits no fixed points[3] on S.

(iv) An energy surface S is naturally co-oriented ($\nabla F \perp S$), thus oriented.

(v) A non-constant flow line that closes up at some time $T > 0$, that is $\phi_T x = x$ where $x \in S$, describes an *embedded circle* $P = \mathcal{O}(x) := \phi_{\mathbb{R}} x$ in S. Such P comes with a natural parametrization $\gamma = \gamma_x : \mathbb{R} \to S$, $t \mapsto \phi_t x$, whose prime period[4] $\tau_P := \inf\{\text{periods } T > 0 \text{ of } P\} > 0$ is called ***the* period of the closed flow line** P on S. By continuity of the flow $\phi_{\tau_P} y = y$ for any $y \in P$.

Energy preservation shows that Hamiltonian systems describe physical systems without friction, so for instance oscillations never decrease. This indicates that Hamiltonian flows might be rather *complicated*. Indeed, as opposed to gradient flows, under a Hamiltonian flow any particle *returns close to its origin* again and again (for a proof see e.g. [HZ11, §1.4]):

Theorem 4.1.3 (Poincaré recurrence theorem)**.** *Under the Hamiltonian flow ϕ on a (closed) energy surface S almost every*[5] *point on S is a* ***recurrent point****: For almost every $x \in S$ there is a sequence $t_j \to \infty$ such that* $\lim_{j\to\infty} \phi_{t_j} x = x$.

[2] For a connected hypersurface it is much easier to decide whether it is a level set.

[3] A **fixed point of a flow** $\phi = \{\phi_t\}_{t\in\mathbb{R}}$ is a point x such that $\phi_t x = x$ for every $t \in \mathbb{R}$.

[4] A number $T > 0$ such that $\phi_T y = y$ for some, hence any, point $y \in P = \mathcal{O}(x)$ is called ***a* period** of the closed flow line P.

[5] with respect to the regular measure on S associated to the induced volume form on S

It sounds like some points, if not many, might close up in finite time, returning exactly to their origins. For generic Hamiltonians of class C^2 this is indeed true! This result of Pugh and Robinson [PR83] is called the **Closing-Lemma**.

As opposed to this *generic* phenomenon, the general existence question is:

Does any energy surface $S \subset (\mathbb{R}^{2n}, \omega_0)$ admit a periodic trajectory?

This question has only relatively recently been given the answer

$$\text{"No."} \tag{4.1.2}$$

by Ginzburg and Gürel [GG03] for a C^2 smooth energy surface $S \subset \mathbb{R}^4$ diffeomorphic to $\mathbb{S}^3$; see also references therein and the featured review MR2031857. This answered the Hamiltonian version of Seifert's question from 1950, commonly known as the **Seifert conjecture**: Does any non-vanishing vector field on the unit sphere $\mathbb{S}^3$ admit a periodic trajectory? Restricting the classes of vector fields, starting with the original class of C^1 vector fields, one gets a hierarchy of questions. For the history of counterexamples,[6] including references, we refer to the survey [Gin01] and also the review MR1909955.

Geometric reformulation – closed characteristics on energy surfaces

The Hamiltonian vector field $X_F = X_F^{\omega_0}$ is non-zero on an *energy surface* $S = F^{-1}(0)$ by definition. Thus X_F generates an *oriented* line bundle

$$\begin{tikzcd} \mathcal{L}_F = \mathcal{L}_F^{\omega_0} : \mathbb{R} \arrow[r, hook] & \mathbb{R} \cdot X_F \arrow[d] \\ & S \end{tikzcd} \qquad \begin{tikzcd} TS \arrow[d] \\ S \arrow[u, bend right, "X_F \neq 0"'] \end{tikzcd}$$

in other words, a **distribution**[7] of rank one in TS.

Exercise 4.1.4. Consider the inclusion $\iota : S \hookrightarrow \mathbb{R}^{2n}$ and the restriction $\omega_0|_S := \iota^*\omega_0$ of the symplectic form to $S = F^{-1}(0)$. Show that

$$\mathcal{L}_S = \mathcal{L}_S^{\omega_0} := \ker \omega_0|_S = \mathbb{R} \cdot X_F =: \mathcal{L}_F.$$

[Hint: Show this pointwise. Inclusion $\supset$ is easy and $\dim \ker(\omega_0|_S)_x = 1$: Exclude $= 0$ by odd dimension of S and ≥ 2 by non-degeneracy of the 2-form.]

So $\mathcal{L}_S$ is a line bundle with a non-vanishing section, namely

$$\begin{tikzcd} \mathcal{L}_S : \mathbb{R} \arrow[r, hook] & \ker \omega_0|_S \arrow[d] \\ & S \arrow[u, bend right, "X_F \neq 0 \text{ whenever } F^{-1}(0) = S \text{ regular}"'] \end{tikzcd} \tag{4.1.3}$$

[6] Examples for "No" are referred to as **counterexamples to the Seifert conjecture**.
[7] A **distribution of rank k** in a tangent bundle is a subbundle of rank k.

Definition 4.1.5. One calls $\mathcal{L}_S = \mathcal{L}_F^{\omega_0}$ the **characteristic line bundle** of the energy surface S in (R^{2n}, ω_0). A **closed characteristic** of $\mathcal{L}_S$ is a closed integral curve of the distribution $\mathcal{L}_S$, i.e. an embedded circle $C \subset S$, likewise denoted by $P \subset S$, whose tangent bundle TC is equal to the restriction $\mathcal{L}_S|_C$.

Remark 4.1.6 (Energy surfaces in symplectic manifolds)**.** The same constructions work if $S = F^{-1}(0)$ is a closed regular level set in a general symplectic manifold (M, ω); such S is called an **energy surface in $\boldsymbol{(M, \omega)}$**. However, not every connected closed hypersurface in a manifold is a level set. A sufficient condition is simply-connectedness of the manifold; see Exercise 4.3.1.

Definition 4.1.7. Given a symplectic manifold (M, ω), suppose $S \subset M$ is a closed hypersurface. A function $F : M \to \mathbb{R}$ is called a **defining Hamiltonian for $\boldsymbol{S}$** if $S = F^{-1}(c)$ for some regular value c of F. Let $\mathcal{H}(S)$ be the **set of defining Hamiltonians for $\boldsymbol{S}$**.

Summarizing, in $\mathbb{R}^{2n}$ any connected closed hypersurface S is an energy surface for some Hamiltonian, that is $\mathcal{H}(S) \neq \emptyset$. Furthermore, for any energy surface $F^{-1}(0) =: S \subset (\mathbb{R}^{2n}, \omega_0)$ closed characteristics of the line bundle $\mathcal{L}_S^{\omega_0}$ coincide with closed flow lines of the Hamiltonian vector field $X_F^{\omega_0}$, as X_F is a (non-vanishing) section of $\mathcal{L}_S$.

Exercise 4.1.8 (Line bundle $\mathcal{L}_S^{\omega_0}$ independent of defining Hamiltonian)**.** Suppose S is closed hypersurface of $(\mathbb{R}^{2n}, \omega_0)$ and $F, K \in \mathcal{H}(S)$. Then $X_F = f X_K$ for some non-vanishing function f on S.
[Hint: Note that $\nabla F(x)$ and $\nabla K(x)$ are both non-zero and orthogonal to the codimension-1 subspace $T_x S \subset \mathbb{R}^{2n}$. Alternatively, their co-vectors are colinear, as the kernel of each is precisely $T_x S$, and non-zero, as $\operatorname{codim} T_x S > 0$.]

Since the line bundle $\mathcal{L}_S$ only depends on the closed (smooth) hypersurface $S \subset (\mathbb{R}^{2n}, \omega_0)$, we denote the set of closed characteristics of $\mathcal{L}_S$ by $\mathcal{C}(S) = \mathcal{C}(S; \omega)$. The earlier question can now be reformulated geometrically as follows:

> Is $\mathcal{C}(S)$ non-empty for any connected closed hypersurface $S \subset (\mathbb{R}^{2n}, \omega_0)$?

As we saw above, in this generality the answer is "No".

Energy surfaces of contact type – Weinstein conjecture

The previous question was answered positively for convex and star-shaped hypersurfaces by Weinstein [Wei78] and Rabinowitz [Rab78], respectively. Weinstein isolated key geometric features of a star-shaped hypersurface in $\mathbb{R}^{2n}$ and introduced the notion of *hypersurface of contact type* in a symplectic manifold.

Conjecture 4.1.9 (**Weinstein conjecture** [Wei79])**.** *A closed hypersurface*[8] *of contact type with trivial first real cohomology carries a closed characteristic.*

[8] in general, in any symplectic manifold (M, ω).

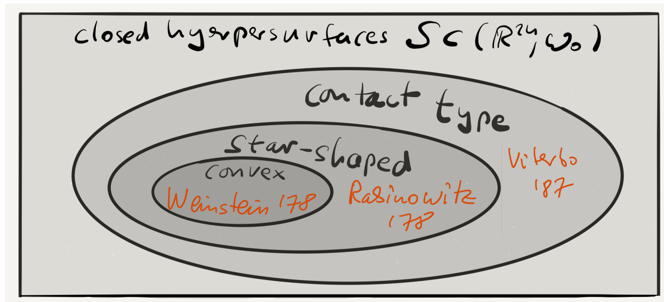

Figure 4.1: Classes of closed hypersurfaces $\Sigma \subset (\mathbb{R}^{2n}, \omega_0)$ with $\mathcal{C}(\Sigma) \neq \emptyset$

The Weinstein conjecture in $(\mathbb{R}^{2n}, \omega_0)$ was confirmed by Viterbo [Vit87], even without any assumption on the first cohomology. More generally, existence of a Reeb loop for any contact form on the 3-sphere, even for any closed orientable contact 3-manifold with trivial π_2, was shown by Hofer [Hof93] and generalized to arbitrary π_2 by Taubes [Tau07]; cf. [Hut10]. Let us now follow Weinstein identifying the key geometric features. Consider the radial vector field

$$Y_0 : \mathbb{R}^{2n} \to \mathbb{R}^{2n}, \quad z = (x, y) \mapsto z = \sum_{j=1}^{n} \left(x_j \partial_{x_j} + y_j \partial_{y_j} \right). \tag{4.1.4}$$

The (local) flow $\theta = \theta^{Y_0} = \{\theta_t^{Y_0}\}$ generated by Y_0 is called **Liouville flow**.

Exercise 4.1.10. Show that Y_0 is a *Liouville vector field*, i.e. $L_{Y_0}\omega_0 = \omega_0$.

Definition 4.1.11. A closed hypersurface Σ in $(\mathbb{R}^{2n} \setminus \{0\}, \omega_0)$ is called **star-shaped** (with respect to the origin) if it is transverse to Y_0, see Figure 4.2, or equivalently if the projection $\mathbb{R}^{2n} \setminus \{0\} \to \mathbb{S}^{2n-1}$, $z \mapsto z/\|z\|$, restricts to a diffeomorphism $\Sigma \to \mathbb{S}^{2n-1}$.

Remark 4.1.12 (Contact type energy surfaces $\Sigma \subset (\mathbb{R}^{2n}, \omega_0)$)**.**

(i) Existence of the Liouville vector field Y_0 has strong geometric and dynamical consequences: Transversality of Y_0 along an energy surface Σ induces

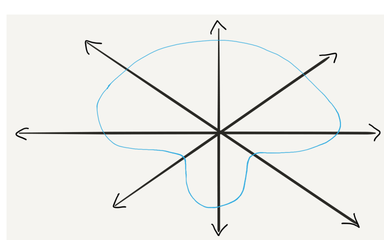

Figure 4.2: A hypersurface $\Sigma \pitchfork Y_0$ not meeting the origin is called star-shaped

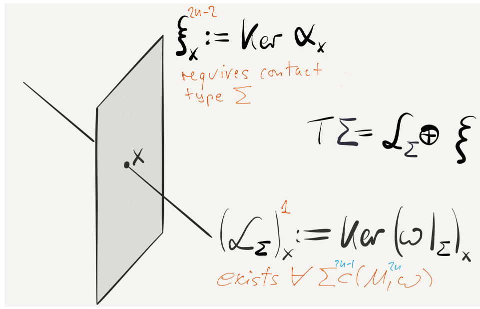

Figure 4.3: Contact structure $\xi := \ker \alpha$ of contact type hypersurface Σ

on some neighborhood the structure of a foliation whose leaves are energy surfaces $\Sigma_\varepsilon := \theta_\varepsilon \Sigma$, i.e. Liouville flow copies of Σ. The apriori slightly obscure condition that Y_0 is a **symplectic dilation**, that is $L_{Y_0}\omega_0 = \omega_0$, causes that the copies Σ_ε are even *dynamical* copies in the sense that the linearized diffeomorphisms $d\theta_\varepsilon : T\Sigma \to T\Sigma_\varepsilon$ identify the characteristic foliations $\mathcal{L}_\Sigma := \ker \omega_0|_\Sigma$ and $\mathcal{L}_{\Sigma_\varepsilon}$ isomorphically; cf. (4.4.8) and Figure 4.4.

(ii) Conclusion: Given an energy surface $\Sigma \subset (\mathbb{R}^{2n}, \omega_0)$, the two key structures are, firstly, existence near Σ of a **Liouville vector field** Y, that is having the dilation property $L_Y\omega_0 = \omega_0$, which is, secondly, transverse to Σ. Such pair (Σ, Y) is called an **energy surface of contact type**.

(iii) Alternative definition: One can show, see Section 4.4, that existence of Y in (ii) is equivalent to existence of a 1-form α *on* Σ *itself* such that, firstly, the restriction $\omega_0|_\Sigma$ is $d\alpha$ (thus $\ker d\alpha$ is the characteristic line bundle $\mathcal{L}_\Sigma$ in (4.1.3)), and such that, secondly, the 1-form α is non-vanishing on $\mathcal{L}_\Sigma$ (evaluation $\alpha_x(\mathcal{L}_\Sigma)_x = \mathbb{R}$ is non-trivial $\forall x \in \Sigma$,[9] see Figure 4.3).

(iv) Given data (Σ, α) as in the previous item (iii), the two conditions

$$R_\alpha \in \mathcal{L}_{d\alpha} := \ker d\alpha, \qquad \alpha(R_\alpha) = 1,$$

uniquely determine a vector field R_α on the contact type energy surface Σ, called the **Reeb vector field** associated to α.

Exercise 4.1.13 (For contact type energy surfaces Hamiltonian and Reeb dynamics coincide up to reparametrization)**.** Given the data (Σ, α) and R_α in the previous item (iv) where $\Sigma = F^{-1}(0)$ for some $F : \mathbb{R}^{2n} \to \mathbb{R}$. Show that $R_\alpha = f\, X_F$ for some non-vanishing function f on Σ. [Hint: $\mathcal{L}_F = \mathcal{L}_\Sigma = \mathcal{L}_{d\alpha}$.]

Example 4.1.14 (Non-contact type)**.** An energy surface $S \subset (\mathbb{R}^{2n}, \omega_0)$ diffeomorphic to the sphere, but not of contact type is shown in [HZ11, §4.3 Fig. 4.8].

[9] Thus the rank of the vector bundle $\xi := \ker \alpha \to \Sigma$ is $2n-2$ and consequently $T\Sigma = \mathcal{L}_\Sigma \oplus \ker \alpha = \mathcal{L}_\Sigma \oplus \xi$, in particular $\mathcal{L}_\Sigma \pitchfork \xi$ in $T\Sigma$.

4.2 Contact manifolds

Exercise 4.1.13 shows that the dynamics on a contact type hypersurface Σ is determined, up to reparametrization, by the 1-form α on Σ itself, independent of the ambient symplectic manifold.

Definition 4.2.1. A **contact form** on a $(2n-1)$-dimensional manifold W is a 1-form α on W such that $d\alpha$ is at any point x a non-degenerate skew-symmetric bilinear form on the subspace $\xi_x := \ker \alpha_x$ of the tangent space. The hyperplane distribution ξ is called a **contact structure** of the **contact manifold** (W, ξ).

The **Gray stability theorem**, see e.g. [Gei08, Thm. 2.2.2], tells that for any given smooth family $\{\xi_t\}_{t\in[0,1]}$ of contact structures there is a smooth family $\{\varphi_t\}_{t\in[0,1]}$ of diffeomorphisms of W such that $(\varphi_t)_*\xi_0 = \xi_t$. But there is no such result for contact forms. This indicates that contact forms are objects less geometrical than contact structures.

Exercise 4.2.2 (α contact $\Leftrightarrow \alpha\wedge(d\alpha)^{n-1}\neq 0$)**.** Given a contact form α, show:

(a) At every $x \in W$ the co-vector α_x is non-zero, thus the vector space ξ_x is necessarily of dimension $2n-2$, in other words a hyperplane.

(b) The defining condition of a contact structure $\xi = \ker\alpha$, namely $d\alpha$ being non-degenerate on ξ, is equivalent to $\alpha\wedge(d\alpha)^{n-1}$ being a **volume form** on W, i.e. this $(2n-1)$-form is at no point x of M the zero-form.

(c) The kernel $\mathcal{L}_{d\alpha} := \ker d\alpha$ is a line bundle over W and $\mathcal{L}_{d\alpha}\oplus\ker\alpha = TS$. There is a unique 'unit' section R_α of the line bundle $\mathcal{L}_{d\alpha}$ determined by $\alpha(R_\alpha) = 1$ and called the **Reeb vector field** associated to α. The (local) flow generated by R_α on W is called **Reeb flow associated to** α and denoted by $\vartheta = \vartheta^{R_\alpha} = \{\vartheta_t^{R_\alpha}\}$.[10]

(d) The distribution $\xi = \ker\alpha$ is nowhere integrable.[11]

(e) The distribution $\xi = \ker\alpha$ is co-oriented.

Exercise 4.2.3 (Contact manifolds are orientable)**.** Suppose $(W, \xi = \ker\alpha)$ is a contact manifold of dimension $2n-1$. Show that α induces an orientation of W. If n is even, then this orientation only depends on the hyperplane distribution ξ, but not on the choice of contact form whose kernel is ξ. [Hint: Pick $-\alpha$.]

Exercise 4.2.4 (Reeb flow preserves contact structure)**.** Show that the Reeb flow preserves the contact form α, thus the contact structure ξ.
[Hint: Check that $L_{R_\alpha}\alpha = 0$.]

Exercise 4.2.5 (Standard contact structure on $\mathbb{R}^3$)**.** Consider $\mathbb{R}^3$ with coordinates (x, y, z) and set $\alpha := dz - y dx$ and $\xi := \ker\alpha$. Check that α is a contact form on $\mathbb{R}^3$ and ξ is spanned by the vector fields ∂_y and $\partial_x + y\partial_z$ whose commutator is $R_\alpha = \partial_z \notin \xi$.

---

[10] The dynamical behavior of Reeb vector fields associated to two contact forms representing the same contact structure $\xi = \ker\alpha = \ker\alpha'$ is in general very different.

[11] Hint: Frobenius; see e.g. [Lan01, Ch. VI] or [Ste83, Sec. II.5] or [War83, Ch. 1].

4.3 Energy surfaces S in (M,ω)

Throughout (M,ω) denotes a symplectic manifold of dimension $2n$. This section parallels Section 4.1 on *energy surfaces* $F^{-1}(0)$ *in* $\mathbb{R}^{2n}$. We don't repeat proofs.

Define the **characteristic line bundle** of a closed hypersurface $S \subset M$ by

$$\mathcal{L}_S := \ker \omega|_S \to S.$$

Whereas any connected closed hypersurface in $\mathbb{R}^{2n}$ is a regular level set of some Hamiltonian $F : \mathbb{R}^{2n} \to \mathbb{R}$, the situation is slightly different in a manifold.

Exercise 4.3.1. Suppose N is a simply-connected manifold. Then any connected closed hypersurface $S \subset N$ is a regular level set $S = S^0 := F^{-1}(0)$ for some function $F : B \to \mathbb{R}$. [Hint: Cf. Lemma. 4.1.2.]

Exercise 4.3.2. For a closed hypersurface $S \subset (M,\omega)$ are equivalent:

(i) S is orientable.

(ii) S is co-orientable.

(iii) $\mathcal{L}_S$ is orientable.

(iv) $S = F^{-1}(0)$ is a regular level set with F defined on a neighborhood $U(S)$.

(v) There exists a **parametrized family of hypersurfaces modeled on S**, that is a diffeomorphism

$$\Phi : (-\delta,\delta) \times S \to U \subset M, \quad (\varepsilon, x) \mapsto \Phi(\varepsilon, x) =: \Phi_\varepsilon(x), \tag{4.3.5}$$

onto some neighborhood U of S that has compact closure and such that $\Phi_0 = \mathrm{id}_S$. We abbreviate $S_\varepsilon := \Phi_\varepsilon S$ and sometimes we denote Φ by (S_ε).

Remark 4.3.3. An **energy surface in a symplectic manifold** is a closed regular level set S of the form $S = S^c := F^{-1}(c)$ where $F : M \to \mathbb{R}$ is a function and c is a regular value. We may assume that S is of the form $F^{-1}(0)$; otherwise, add a constant to F. Note that an energy surface S is a closed co-/orientable hypersurface that may consist of finitely many components. As earlier in Section 4.1, a non-vanishing section of $\mathcal{L}_S$ is provided by the Hamiltonian vector field X_F of any Hamiltonian having S as regular level set. A **closed characteristic on an energy surface S** is an embedded circle $C \subset S$, likewise denoted by the letter P, such that $TC = \mathcal{L}_S|_C$. Recall from (2.3.22) that $\mathcal{C}(S) = \mathcal{C}(S;\omega)$ denotes the set of closed characteristics on the energy surface S.

4.3.1 Stable hypersurfaces

Definition 4.3.4. A closed hypersurface $S \subset (M,\omega)$ is called **stable** if there is a parametrized family (S_ε) modeled on S such that for each ε the linearization

$$d\Phi_\varepsilon : \mathcal{L}_S \to \mathcal{L}_{S_\varepsilon}$$

is a bundle isomorphism.

4.4 Contact type hypersurfaces Σ in (M, ω)

In [Wei79] Weinstein introduced the following notion; cf. Conjecture 4.1.9.

Definition 4.4.1. A closed hypersurface $\iota : \Sigma \hookrightarrow (M, \omega)$ in a symplectic manifold is said to be of **contact type** if Σ is co-orientable[12] and there is a 1-form α on Σ such that

(i) $d\alpha = \iota^*\omega$, that is the restriction $\omega|_\Sigma := \iota^*\omega$ to Σ is exact.

(ii) α is non-vanishing on the characteristic line bundle (except zero section)

$$\mathcal{L}_\Sigma := \ker \omega|_\Sigma = \ker d\alpha.$$

More precisely, condition (ii) means that evaluation $\alpha_x(\mathcal{L}_\Sigma)_x = \mathbb{R}$ is non-trivial at any point $x \in \Sigma$, likewise (cf. Figure 4.3)

$$\mathcal{L}_\Sigma \oplus \ker \alpha = T\Sigma.$$

Exercise 4.4.2 (Contact type hypersurfaces are contact manifolds)**.** To see that a contact type hypersurface (Σ, α) in a symplectic manifold (M, ω) is a contact manifold with contact structure $\xi = \ker \alpha$ show the following.
a) The linear functional $\alpha_x \in T_x^*\Sigma$ is non-zero at any point $x \in \Sigma$. So $\xi := \ker \alpha$ is a $(2n-2)$-plane distribution in the tangent bundle $T\Sigma$.
b) Show that $d\alpha$ restricts to a non-degenerate two-form on ξ, denoted by $d\alpha|\xi$.
c) Show that $\alpha \wedge (d\alpha)^{\wedge(n-1)}$ is a volume form on Σ. (Thus such α orients Σ.)
[Hint: b) $\mathcal{L}_\Sigma = \ker d\alpha$. c) $\ker d\alpha \oplus \ker \alpha = T\Sigma$. Recall Exercise 4.2.2 (b).]

Remark 4.4.3 (Reeb flow on contact type hypersurface)**.** The characteristic line bundle of a contact type hypersurface (Σ, α) admits by co-orientability a natural section, namely the Reeb vector field R_α normalized by $\alpha(R_\alpha) \equiv 1$.

There is a second, of course equivalent, definition of contact type which reveals more of the interaction of contact type with the ambient symplectic manifold. The key element is a vector field Y transverse to Σ – a useful property to generate copies Σ_ε of Σ – and which *dilates the symplectic form*:

Definition 4.4.4. A vector field Y on a symplectic manifold (M, ω) is called a **Liouville vector field**, or an **Euler vector field**, if $L_Y\omega = \omega$.

Exercise 4.4.5. If Y is Liouville on an open subset $U \subset M$, then so is $Y + X_h$ for any function $h : U \to \mathbb{R}$.

Definition 4.4.6. A closed[13] hypersurface $\Sigma \subset (M, \omega)$ is said to be of **contact type** if some neighborhood U of Σ admits a Liouville vector field Y transverse to Σ, in symbols $Y \pitchfork \Sigma$. In case of global existence, namely $U = M$, the hypersurface Σ is said to be of **restricted contact type**. The **Liouville flow** is the flow $\theta = \theta^Y$ of Y on U. It satisfies $\theta_t^*\omega = e^t\omega$ wherever it is defined.[14]

[12] Co-orientability of Σ enables extensions from Σ to neighborhoods; cf. Exercise 4.3.2.
[13] Assuming *closed*, that is compact and without boundary, makes several things so much simpler: Firstly, transversality is much easier to handle and, secondly, flows on Σ are complete.
[14] Thus a closed symplectic manifold cannot admit a global Liouville vector field.

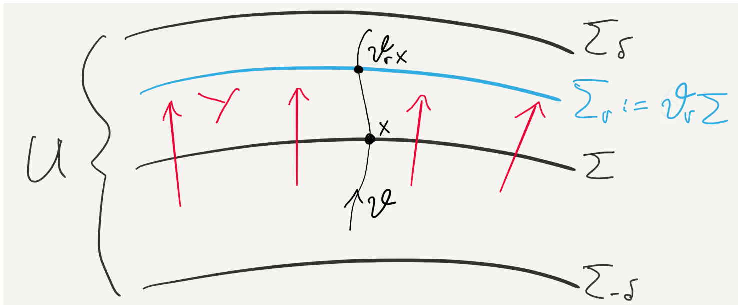

Figure 4.4: Contact type induces foliation (Σ_r) and isomorphisms $\mathcal{L}_\Sigma \cong \mathcal{L}_{\Sigma_r}$

Both definitions of contact type are equivalent

$\boldsymbol{Y \mapsto \alpha_Y}$: Given a Liouville vector field Y on U, define

$$\lambda_Y := i_Y \omega, \qquad \alpha_Y := \iota^* \lambda_Y, \tag{4.4.6}$$

to obtain the desired 1-form α_Y on Σ. Abbreviate $\lambda = \lambda_Y$ and $\alpha = \alpha_Y$. Then indeed $d\lambda = di_Y\omega = L_Y\omega = \omega$ and for any non-zero $v \in (\mathcal{L}_\Sigma)_x := (\ker \omega|_\Sigma)_x \subset T_x\Sigma$, together with $Y(x) \notin T_x\Sigma$ as $Y \pitchfork \Sigma$, we have that

$$0 \neq \omega(Y(x), v) = \lambda_x(v) = \alpha_x(v). \tag{4.4.7}$$

$\boldsymbol{\alpha \mapsto Y_\alpha}$: This way is harder as it involves in the first step to extend the 1-form α from Σ to a 1-form $\lambda = \lambda_\alpha$ on some neighborhood U of Σ such that $d\lambda = \omega|U$; see [HZ11, §4.3 Le. 3]. Once one has the extension, the identity $\lambda_\alpha = \omega(Y, \cdot)$ determines $Y = Y_\alpha$. On U one has $L_Y\omega = di_Y\omega = d\lambda = \omega$. To show $Y \pitchfork \Sigma$ observe that now in (4.4.7) the right hand side is non-zero, thus $Y(x)$ cannot be in $T_x\Sigma$ as v already is.

Geometrical and dynamical consequences of contact type

Suppose $\Sigma \subset (M, \omega)$ is a closed hypersurface of contact type. Let $Y = Y_\lambda$ and $\lambda = \lambda_Y$ be the two associated structures, one corresponding to the other one (see discussion right above), on some neighborhood U of Σ.

The contact type property has powerful geometrical and dynamical consequences illustrated by Figure 4.4: Transversality $Y \pitchfork \Sigma$ leads to a foliation of a neighborhood of Σ by copies of Σ under the Liouville flow $\theta = \theta^Y$, that is by the hypersurfaces defined by $\Sigma_r := \theta_r\Sigma$ for $r > 0$ small. These copies are even *dynamical copies* in the sense that the linearization of the diffeomorphism $\theta_r : \Sigma \to \Sigma_r$ identifies the characteristic foliations $\mathcal{L}_\Sigma$ and $\mathcal{L}_{\Sigma_r}$ isomorphically.

To see this consider the parametrized family of hypersurfaces modeled on Σ:

$$\Phi : (-\delta, \delta) \times \Sigma \to U \subset M, \quad (r, x) \mapsto \theta_r x. \tag{4.4.8}$$

Here $\delta > 0$ is a sufficiently small constant whose existence is guaranteed by compactness of Σ; choose for U the image of Φ. Use $L_Y\omega = \omega$ in (2.3.20) and $\theta_0 = \mathrm{id}$ to obtain the identity

$$(\theta_r)^*\omega = e^r\omega.$$

Given some non-zero vector $v \in (\mathcal{L}_\Sigma)_x = \ker(\omega|_\Sigma)_x$, then

$$0 = \omega\,(v,w) = e^r\omega\,(v,w) = (\theta_r)^*\omega\,(v,w) = \omega\big(d\theta_r(x)v, \underbrace{d\theta_r(x)w}_{\in T_{\theta_r x}\Sigma_r}\big)$$

for every $w \in T_x\Sigma$. So the non-zero vector $d\theta_r(x)v$ lies in $\ker(\omega|_{\Sigma_r})_{\theta_r x}$. Thus

$$d\theta_r : \mathcal{L}_\Sigma \to \mathcal{L}_{\Sigma_r} \tag{4.4.9}$$

is an isomorphism of line bundles proving that *contact type implies stable.* Hence θ_r induces a bijection $\mathcal{C}(\Sigma) \cong \mathcal{C}(\Sigma_r)$, $C \mapsto \theta_r C$. This motivates Definition 4.3.4.

Exercise 4.4.7 (Stable, but not of contact type)**.** Show that the hypersurface $\Sigma := \mathbb{S}^2 \times \mathbb{S}^1$ in the symplectic manifold $(M,\Omega) := (\mathbb{S}^2 \times \mathbb{R}^2, \omega \oplus \omega_0)$ is stable, but not of contact type. Here ω is any symplectic form on $\mathbb{S}^2$; cf. Exercise 2.0.10.

4.4.1 Energy surfaces of contact type

Proposition 4.4.8. *Suppose a closed hypersurface Σ in a symplectic manifold (M,ω) is both, firstly, of contact type with respect to some Liouville vector field Y on some neighborhood U of Σ and, secondly,*[15] *a regular level set $\Sigma = F^{-1}(c)$ of some function $F : U \to \mathbb{R}$. In this case the Reeb vector field and the Hamiltonian vector field are pointwise co-linear along Σ. In symbols, along Σ it holds that*

$$R_{\alpha_Y} = f\,X_F \tag{4.4.10}$$

for some non-vanishing function f on Σ. In other words, on Σ the Reeb flow and the Hamiltonian flow coincide up to reparametrization.

Proof. Set $\alpha := \alpha_Y$, cf. (4.4.6), so $\ker d\alpha = \ker\omega|_\Sigma =: \mathcal{L}_\Sigma = \mathcal{L}_{F^{-1}(c)}$. But R_α is a section of $\ker d\alpha$ by definition and X_F is one of $\mathcal{L}_{F^{-1}(c)}$ by Remark 4.3.3. □

Exercise 4.4.9 (Reeb flows on level sets are Hamiltonian near[16] the level set)**.** Suppose the regular value c of $F : U \to \mathbb{R}$ in Proposition 4.4.8 is *zero*; otherwise replace F by $F - c$. Firstly, extend the non-vanishing function f in (4.4.10) from $\Sigma = F^{-1}(0)$ to a non-vanishing function on some open neighborhood, still denoted by U and f, constant outside a compact neighborhood $D \subset U$ of Σ. [Hint: Co-orientability of Σ, tubular neighborhoods D.] Secondly, show that

- zero is a regular value of the product function $fF : U \to \mathbb{R}$;
- the pre-image $(fF)^{-1}(0)$ is still Σ;
- along Σ there are the identities $R_{\alpha_Y} = f\,X_F = X_{fF}$.

[Hint: As $X_{fF} = fX_F + FX_f$ it helps that $\Sigma = F^{-1}(0)$ is the pre-image of *zero*.]

[15] This assumption is void if Σ is bounding: One easily constructs F even globally on M.
[16] Reeb flows on *bounding* contact type hypersurfaces are Hamiltonian for some $F : M \to \mathbb{R}$.

4.5 Restricted contact type – exact symplectic

Exercise 4.5.1. Let Σ be a closed hypersurface in a symplectic manifold (M, ω).
a) If Σ is of restricted contact type, see Definition 4.4.6, with respect to a Liouville vector field Y on M, then ω is an exact symplectic form with primitive $\lambda_Y := i_Y\omega := \omega(Y, \cdot)$. In particular, the manifold M cannot be closed.
b) If $\omega = d\lambda$ is exact, then every simply-connected closed hypersurface of contact type is of restricted contacttype.

Definition 4.5.2. An **exact symplectic manifold** (V, λ) is a manifold V with a 1-form λ such that $\omega := d\lambda$ is a symplectic form.

Exercise 4.5.3. Show the following. An exact symplectic manifold (V, λ) is necessarily non-compact. The boundary ∂M of a compact exact symplectic manifold-with-boundary (M, λ) is necessarily non-empy. In either case, the **associated vector field** Y_λ determined by the identity $i_{Y_\lambda} d\lambda = \lambda$ is **Liouville**.

Exercise 4.5.4 (Liouville vector fields are outward pointing)**.** Suppose $(M, \omega = d\lambda)$ is a compact exact symplectic manifold-with-boundary and consider the associated Liouville vector field Y_λ defined by $\omega(Y_\lambda, \cdot) = \lambda$ on $U = M$. Suppose that $Y_\lambda \pitchfork \partial M$. (In other words, suppose that the boundary ∂M is of (restricted) contact type with respect to Y_λ.) Let $\iota : \partial M \hookrightarrow M$ be inclusion and set $\alpha := \iota^*\lambda$.

(i) Use the fact that $\lambda(Y_\lambda) = 0$ to prove the relation

$$i_{Y_\lambda}\omega^n = \lambda \wedge (d\lambda)^{n-1}$$

between the natural volume form ω^n on M and its primitive. Recall that the restriction $\iota^*\left(\lambda \wedge (d\lambda)^{n-1}\right) = \alpha \wedge (d\alpha)^{n-1}$ is a volume form on ∂M.

(ii) Let M be equipped with the orientation provided by ω^n and let the boundary ∂M be equipped with the induced orientation according to the 'put outward normal first' rule; cf. [GP74, Ch. 3 §2]. Verify that

$$\begin{aligned} 0 &< \int_M \omega^n \\ &= \int_{\partial M} \iota^*\left(\lambda \wedge (d\lambda)^{n-1}\right) \\ &= \int_{\partial M} \iota^* i_{Y_\lambda}\omega^n \end{aligned}$$

where the first identity is Stoke's theorem; see [GP74, Ch. 4 §7]. Let ν be an outward pointing vector field along ∂M. Then verify that the integral $\int_M \omega^n$ is a positive multiple of $\int_{\partial M} \iota^* i_\nu \omega^n$ due to the 'put outward normal first' rule. So Y_λ points in the same half-space as ν, that is the outer one. Hence Y_λ generates a complete backward flow.

4.5.1 Bounding hypersurfaces of restricted contact type

An exact symplectic manifold (V, λ) already comes equipped with the globally defined **associated Liouville vector field** Y_λ determined by the identity

$$d\lambda(Y_\lambda, \cdot) = \lambda.$$

Proposition 4.5.5 (Defining Hamiltonians)**.** *Suppose Σ is a bounding hypersurface in an exact symplectic manifold (V, λ) transverse to the associated Liouville vector field Y_λ on V. (Thus Σ is of restricted contact type with $\alpha := \alpha_{Y_\lambda} := \lambda|_\Sigma$.) Denote the closure of the inside of Σ by M. Then there is a global Hamiltonian $F : V \to \mathbb{R}$ with regular level set $F^{-1}(0) = \Sigma$ such that F is negative*[17] *inside Σ, equal to a positive constant outside some compact neighborhood of M, and such that the Hamiltonian vector field*

$$X_F = R_\alpha \tag{4.5.11}$$

*coincides along Σ with the Reeb vector field. Such F is called a **defining Hamiltonian** for Σ; see Figure 5.1. The **space of defining Hamiltonians**, denoted by $\mathcal{F}(\Sigma) = \mathcal{F}(\Sigma, V, \lambda)$, is convex.*

Proof. Use a tubular neighborhood of Σ to define F near Σ and then, using that Σ bounds, extend that function, say by $-1/+1$, to the remaining parts of the inside/outside of Σ. Then $R_\alpha = f X_F$ along Σ for some non-vanishing f on Σ by (4.4.10). Use again that Σ bounds to extend f to a non-vanishing function on V. Then the product function $fF : V \to \mathbb{R}$ has the desired properties; cf. Exercise 4.4.9. Convexity essentially follows from the identity $X_{F+G} = X_F + X_G$ and the fact that one has chosen *zero* as regular value. □

Definition 4.5.6. A Hamiltonian F as in Proposition 4.5.5, see Figure 5.1, is called a **defining Hamiltonian** for the bounding hypersurface Σ of restricted contact type in an exact symplectic manifold (V, λ).

4.5.2 Convexity

One gets a natural ambience of Rabinowitz-Floer homology by replacing

bounding hypersurfaces of restricted contact type in exact symplectic manifolds

by

convex exact hypersurfaces Σ in convex exact symplectic manifolds (V, λ).

What is the difference? Whereas the restriction $\lambda|_\Sigma$ to Σ needs to become contact only after adding some exact 1-form on Σ, non-compactness of exact symplectic manifolds is tamed and made 'controlable' outside compact parts by requiring what is called the manifold being *convex*, or *cylindrical*, near infinity; see [EGH00] and [BEH+03]. From now on Σ and V are connected manifolds.

[17] If one does not prescribe the same sign for all F's inside Σ and the opposite sign outside, then one looses convexity of the space of such, since $F + (-F) = 0$.

Definition 4.5.7. A **convex exact symplectic manifold** (V, λ) consists of a connected manifold V of dimension $2n$ equipped with a 1-form λ such that

(i) $\omega := d\lambda$ is a symplectic form on V and

(ii) the exact symplectic manifold (V, λ) is **convex at infinity**, that is there is an exhaustion $V = \cup_k M_k$ of V by compact manifolds-with-boundary $M_k \subset M_{k+1}$ such that $\alpha_k := \lambda|_{\partial M_k}$ is a contact form on ∂M_k for every k.

Remark 4.5.8. Suppose (V, λ) is a convex exact symplectic manifold. Given any *compactly* supported (smooth) function f on V, then $(V, \lambda + df)$ is also a convex exact symplectic manifold: Indeed the 1-forms λ and $\lambda + df$, called **equivalent** 1-forms, generate the same symplectic form ω on V. One obtains a suitable exhaustion by forgetting the first M_k's, use only those on which $df = 0$.

By Exercise 4.5.4 the associated Liouville vector field Y_λ points out of M_k along ∂M_k. So the Liouville flow is automatically *backward* complete. A convex exact symplectic manifold (V, λ) is called **complete** if the vector field Y_λ generates a complete flow on V. If $Y_\lambda \neq 0$ outside some compact set one says that (V, λ) has **bounded topology**.[18] Call a subset $A \subset V$ **displaceable** if

$$A \cap \psi_1^H A = \emptyset$$

for some compactly supported Hamiltonian $H : [0, 1] \times V \to \mathbb{R}$.

Exercise 4.5.9. a) If (V, λ) is a convex exact symplectic manifold, then so is its **stabilization** $(V \times \mathbb{C}, \lambda \oplus \lambda_{\mathbb{C}})$. The 1-form $\lambda_{\mathbb{C}}$ on $\mathbb{C}$ is given by $\frac{1}{2}(x\,dy - y\,dx)$.
b) In $(V \times \mathbb{C}, \lambda \oplus \lambda_{\mathbb{C}})$ every compact subset is displaceable.

Main examples of convex exact symplectic manifolds are

- Euclidean space $\mathbb{R}^{2n}$ equipped with the 1-form λ_0 given by (1.0.8). Indeed the radial Liouville vector field $Y_0(z) = z$ in (4.1.4) is transverse to the boundary of each ball M_k about the origin of radius k.

- cotangent bundles T^*Q equipped with the Liouville form λ_{can} and the canonical fiberwise radial Liouville vector field Y_{can}, see (4.5.12). These are complete and of bounded topology whenever the base manifold is closed. More generally,

- Stein manifolds, see [Eli90, EG91] or [CE12, Thm. 1.5].

[18] To understand this choice of terminology recall that Morse theory describes the change of topology of sublevel sets $\{f \leq c\}$ when c crosses a critical level. But critical points of f are the zeroes of the gradient vector field ∇f, whatever Riemannian metric one picks.

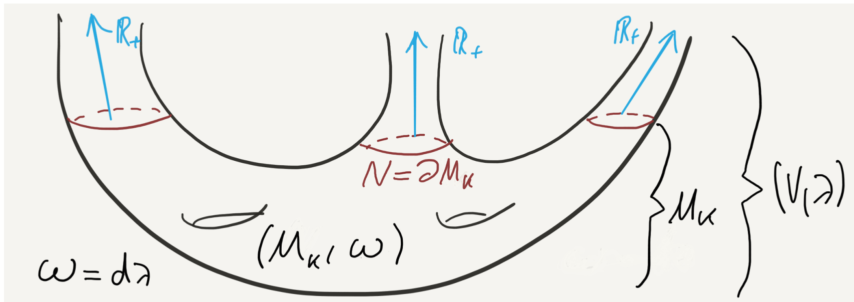

Figure 4.5: Convex exact symplectic manifold (V, λ) with 3 cylindrical ends

Cylindrical ends

Symplectic manifolds with cylindrical ends have been introduced to construct *symplectic field theory (SFT)* in [EGH00].

Exercise 4.5.10. Show that a convex exact symplectic manifold (V, λ) is complete and of bounded topology iff there exists an embedding $\phi : N \times \mathbb{R}_+ \to V$, for some closed, not necessarily connected, manifold N, such that $\phi^*\lambda = e^r \alpha_N$ with contact form $\alpha_N := \phi^*\lambda|_{N\times\{0\}}$ and such that $V \setminus \phi(N \times \mathbb{R}_+)$ is compact. [Hint: Apply the Liouville flow to $N := \partial M_k$ for some large k; cf. Figure 4.5.]

Each connected component $N_j \times \mathbb{R}_+$ of $N \times \mathbb{R}_+$ is called a **cylindrical end** of (V, λ) and comes equipped with the symplectic form $\phi^* d\lambda = d(e^r \alpha_{N_j})$.

Definition 4.5.11. A $d\lambda$-compatible almost complex structure J on (V, λ), i.e. $J \in \mathcal{J}(V, d\lambda)$, is called **cylindrical** if it is cylindrical on the cylindrical ends: Namely, the corresponding almost complex structure ϕ^*J on $N \times \mathbb{R}_+$

- couples Liouville and Reeb vector field, that is $J\partial_r = R_{\alpha_N}$ along N;
- leaves $\ker \alpha_N$ invariant;
- is invariant under the semi-flow $(x, 0) \mapsto (x, r)$ for $(x, r) \in N \times \mathbb{R}_+$.

Concerning existence of cylindrical almost complex structures see [BEH⁺03, §2 §3] or [Abb14, §2.1].

Convex exact hypersurfaces

Definition 4.5.12. A **convex exact hypersurface** in a convex exact symplectic manifold (V, λ) is a <u>connected</u> closed hypersurface $\Sigma \subset V$ such that

(i) there is a contact 1-form α on Σ such that $\alpha - \lambda|_\Sigma$ is exact and

(ii) the hypersurface Σ is bounding,[19] say M. (So $V \setminus \Sigma$ has two connected components, one of compact closure, namely M, the other one not.)

[19] Assuming that Σ bounds, in particular being connected and closed, together with being of codimension 1 in a symplectic manifold, is a sufficient condition that Σ admits a defining Hamiltonian – thereby relating Reeb and Hamiltonian dynamics.

Remark 4.5.13. The next Exercise 4.5.14 shows that a convex exact hypersurface Σ in a convex exact symplectic manifold (V, λ) and with associated contact form α is of restricted contact type with respect to an equivalent 1-form $\mu := \lambda + dh$ which restricts to the same contact form $\alpha = \mu|_\Sigma$. Moreover, the new Liouville vector field, given by $Y_\mu = Y_\lambda - X_h^\omega$ where $\omega = d\lambda = d\mu$, is still transverse to $\Sigma = \partial M$ and still outward pointing; see Exercise 4.5.4.

Exercise 4.5.14. Consider a convex exact hypersurface Σ in (V, λ) with associated contact form α. Show the following.

(a) There is a compactly supported function $h : V \to \mathbb{R}$ such that the 1-form $\mu := \lambda + dh$ on V restricts to $\alpha = \mu|_\Sigma$.

(b) The new Liouville vector field is given by $Y_\mu = Y_\lambda - X_h^\omega$ and it is transverse to Σ whenever Y_λ is.

[Hint: (a) Consult [CF09, p. 253 Rmk. (2)] if you get stuck.]

For a list of further useful consequences see [CF09, p. 253]. For instance, a closed hypersurface is bounding whenever $\mathrm{H}_{2n-1}(V; \mathbb{Z}) = 0$. This holds, for example, if V is Stein of dimension > 2 or a stabilization.

4.5.3 Cotangent bundles

Given a closed manifold Q of dimension n, consider the cotangent bundle $(T^*Q, \omega_{\mathrm{can}} = d\lambda_{\mathrm{can}})$ with its canonical exact symplectic form.

Exercise 4.5.15. Show that Y_{can} determined by $\omega_{\mathrm{can}}(Y_{\mathrm{can}}, \cdot) = \lambda_{\mathrm{can}}$ takes in natural local coordinates the form of a *fiberwise radial* vector field, namely

$$Y_{\mathrm{can}} = 2 \sum_{i=1}^n p_i \partial_{p_i} = 2Y_{\mathrm{rad}}. \tag{4.5.12}$$

It is called the **canonical** or **fiberwise radial Liouville vector field**. The **fiberwise radial vector field** $Y_{\mathrm{rad}} : T^*Q \to TT^*Q$ is given by the derivative

$$Y_{\mathrm{can}}(\eta) = 2Y_{\mathrm{rad}}(\eta) := 2 \left.\frac{d}{d\tau}\right|_{\tau=1} \tau\eta \tag{4.5.13}$$

of the curve $\tau\eta$ in the manifold T^*Q at time 1. Note that Y_{rad} exists on any co/tangent bundle; cf. wiki/Tangent_bundle.

Definition 4.5.16. A hypersurface $\Sigma \subset T^*Q$ is called **fiberwise star-shaped** (with respect to the zero section) if Σ is bounding, disjoint from the zero section, and transverse $\Sigma \pitchfork Y_{\mathrm{can}}$ to the fiberwise radial vector field.[20]

Exercise 4.5.17. The intersection of a fiberwise star-shaped hypersurface $\Sigma \subset T^*Q$ with each fiber T_q^*Q is diffeomorphic to a sphere of dimension $n-1$.

[20] Why is *fiberwise* radial fine, whereas in $\mathbb{R}^{2n}$ one uses the *fully* radial vector field (4.1.4)?

Exercise 4.5.18. Pick a Riemannian metric g and a smooth function V on Q, consider the Hamiltonian $F(q,p) = \frac{1}{2}g_q(p,p) + V(q)$. Show that if $c > \max_Q V$, then $\Sigma^c := F^{-1}(c)$ is a fiberwise star-shaped hypersurface; cf. [HZ11, (4.11)].

Example 4.5.19 (Canonical contact structure on S^*Q)**.** The unit sphere cotangent bundle S^*Q of (Q,g) is fiberwise star-shaped and of restricted contact type in $(T^*Q, \omega_{\mathrm{can}} = d\lambda_{\mathrm{can}})$. In other words, the boundary of the unit disk cotangent bundle D^*Q of the closed Riemannian manifold (Q,g) is fiberwise star-shaped and of restricted contact type.

4.6 Techniques to find periodic trajectories

For convenience of the reader we enlist and summarize, following [HZ11], some key techniques to find closed flow lines of (autonomous) Hamiltonian systems.

4.6.1 Via finite capacity neighborhoods

Based on the Hofer-Zehnder capacity function c_0 established in [HZ11, Ch. 3] one derives the following existence results for closed Hamiltonian flow lines.

- Nearby existence [HZ11, Thm. 4.1, p.106]. Given a closed regular level set $S = S^1 := F^{-1}(1) \subset (M,\omega)$ that admits a (bounded) neighborhood U of finite capacity $c_0(U,\omega) < \infty$, then the set of closed characteristics $\mathcal{C}(S^{r_j};\omega) \neq \emptyset$ is non-empty for some sequence $r_j \to 1$. (There is even a dense subset of $(1-\varepsilon, 1+\varepsilon)$ of such r's.)

  Idea of proof: Use freedom in choosing the Hamiltonian representing S to pick a certain 'radial' one $H = H(r)$.

- Existence on S itself. If the periods T_j of the canonically parametrized[21] closed characteristics P_j on $S^{r_j} = F^{-1}(r_j)$ are bounded, then S itself admits a periodic trajectory, too.

  Idea of proof: Apply the Arzelà-Ascoli Theorem 3.2.10.

- One-parameter families [HZ11, Prop. 4.2, p.110]. Consider a Hamiltonian loop $z^* : \mathbb{R} \to F^{-1}(E^*) \subset (M,\omega)$, said of energy E^*, of period T^* which admits precisely two Floquet multipliers[22] equal to 1. Application of the Poincaré continuation method shows that z^* belongs to a unique smooth family of periodic trajectories z^E parametrized by their energy E and whose periods T^E converge to T^*, as $E \to E^*$.

  Idea of proof: Construct a Poincaré section map for z^*, investigate how the eigenvalues of its linearization along z^* are related to the Floquet multipliers of z^*, apply the implicit function theorem.

[21] For each $p \in P_j$ the solution $z : \mathbb{R} \to F^{-1}(r_j)$ to $\dot z(t) = X_F \circ z(t)$ with $z(0) = p$ traces out P_j and comes back to itself for the first time at some positive time, say T_j.

[22] the eigenvalues of the linear map $d\phi_{T^*}(p) : T_pM \to T_pM$

4.6.2 Via characteristic line bundles

Consider the characteristic line bundle $\mathcal{L}_S$, see (4.1.3), over a closed co-orientable hypersurface S in a symplectic manifold (M, ω). Such $S = F^{-1}(0)$ is an energy surface of some Hamiltonian F defined *near* S; see Exercise 4.3.2. While the set of closed characteristics $\mathcal{C}(S)$ is empty in certain situations, e.g. for the **Zehnder tori** [Zeh87], cf. [HZ11, §4.5], for large classes of closed co-orientable hypersurfaces existence of closed characteristics is guaranteed.

They key concept is that of a parametrized family (S_ε) of hypersurfaces modeled on a closed hypersurface S, introduced earlier in (4.3.5).

Nearby existence listed above fits into this framework whenever U is of finite c_0 capacity (construct the diffeomorphism Φ using the normalized gradient flow of F, say with respect to an ω-compatible Riemannian metric on M).

Two classes of hypersurfaces which do admit periodic trajectories are the following.

- Bounding hypersurfaces. Suppose $S \subset (M, \omega)$ bounds a compact submanifold-with-boundary B and (S_ε) is a parametrized family modeled on S. Then each $S_\varepsilon = \Phi_\varepsilon S$ is the boundary of the symplectic manifold-with-boundary $B_\varepsilon = \Phi_\varepsilon B$. Now the key property is monotonicity[23] of the function $C(\varepsilon) := c_0(B_\varepsilon, \omega)$ which holds by the `(monotonicity)` axiom of the Hofer-Zehnder capacity c_0. For details see [HZ11, Thm. 4.3, p.116].

- Stable hypersurfaces. A closed hypersurface $S \subset (M, \omega)$ is called a **stable hypersurface** if it admits a rather nice parametrized family (S_ε) modeled on S, namely one for which each linearization $d\Phi_\varepsilon : TS \to TS_\varepsilon$ restricts to a line bundle isomorphism $\mathcal{L}_S \to \mathcal{L}_{S_\varepsilon}$.

  Advantage. It suffices to detect a closed characteristic on any member S_ε of the family in order to obtain a closed characteristic of the original dynamical system $(S = F^{-1}(0), X_F)$ itself. For instance, if in the nearby existence result mentioned above S was stable, e.g. of contact type, then $\mathcal{C}(S) \neq \emptyset$. Since a closed hypersurface in $(\mathbb{R}^{2n}, \omega_0)$ admits a bounded neighborhood U of finite[24] c_0 capacity, this confirms the Weinstein conjecture for contact type hypersurfaces in $(\mathbb{R}^{2n}, \omega_0)$; cf. Figure 4.1.

  As we saw earlier in (4.4.9), a hypersurface S of contact type is stable with respect to the parametrized family (S_ε) produced by the Liouville flow $\Phi_\varepsilon := \theta^Y_\varepsilon$. The linearized flow $d\theta^Y_\varepsilon : \mathcal{L}_S \to \mathcal{L}_{S_\varepsilon}$ is a bundle isomorphism between the characteristic line bundles of S and $S_\varepsilon = \theta^Y_\varepsilon S$.

  For an example of a stable hypersurface which is not of contact type see [HZ11, p.122].

[23] By Lebesgue's last theorem monotonicity of a function implies differentiability, thus Lipschitz continuity, almost everywhere in the sense of measure theory; for a proof see e.g. [Pug02].

[24] To see that $c_0(U) < \infty$, pick a ball around $U \subset \mathbb{R}^{2n}$ and apply the axioms `(monotonicity)` and `(non-triviality)` of a symplectic capacity in [HZ11, §1.2].

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Chapter 5

Fixed energy – Rabinowitz-Floer homology

In 2007 Cieliebak and Frauenfelder considered a version of the action functional that involves an additional real variable τ (revealing periodic solutions of whatever period), namely the **Rabinowitz action functional**[1]

$$\mathcal{A}^F : \mathcal{L}V \times \mathbb{R}, \quad (z,\tau) \mapsto \int_{\mathbb{S}^1} z^*\lambda - \tau \int_0^1 F(z(t))\, dt$$

associated to *autonomous* Hamiltonians $F : V \to \mathbb{R}$ on certain exact[2] symplectic connected manifolds (V, λ). The extra parameter τ, together with time independence of F, causes that a critical point (z, τ) of $\mathcal{A}^F$ corresponds to

- either a τ-periodic Hamiltonian loop that lies on the level set $F^{-1}(0)$;
- or a constant (period $\tau = 0$) loop sitting at a point $q = z(0) \in F^{-1}(0)$.

A negative period $\tau < 0$ tells that the Hamiltonian loop follows $-X_F$.

Now one can exploit this *fixed energy property* to study dynamical properties of connected closed hypersurfaces Σ in V which can be represented as the regular level-zero set of an autonomous Hamiltonian F on V with compactly supported differential dF, called a general defining Hamiltonian for Σ. A sufficient condition that Σ is of the form $F^{-1}(0)$ is that $\Sigma = \partial M$ bounds a compact manifold-with-boundary M. However, not all connected bounding energy surfaces admit closed orbits, as we saw in (4.1.2). The tool to prove existence of periodic solutions is a version of Floer homology for the Rabinowitz functional $\mathcal{A}^F$ on the extended domain $\mathcal{L}V \times \mathbb{R}$. Recall from Chapter 3 that the analytic key to set up Floer homology is compactness, up to broken trajectories, of the spaces of connecting trajectories. The extra parameter τ causes non-compactness in

[1] For an exposition of the classical free period action functional see [Abb13].

[2] Exactness of a symplectic manifold implies non-compactness, an inconvenient property which one gets under control by imposing additional conditions, for instance *convexity*.

certain cases – giving way to non-existence (4.1.2). A sufficient condition to fix this is to require the bounding hypersurface Σ to be of **restricted contact type**, i.e. with contact form $\alpha = \lambda|_\Sigma$. Consequently by (4.5.11) there is a convex set $\mathcal{F}(\Sigma) \neq \emptyset$ of **defining Hamiltonians** F whose Hamiltonian vector field is simply equal, along Σ, to one and the same Reeb vector field R_α – but Σ is compact. The identity $X_F = R_\alpha$ is furthermore extremely beneficial in the sense that it allows to utilize the analysis carried out in Chapter 3 in the Hamiltonian setting. In fact, one can allow slightly more general hypersurfaces; cf. Definitions 4.5.7 and 4.5.12 and Remark 4.5.13.

Assumption 5.0.1. In Chapter 5, unless mentioned otherwise, we assume that

- (V, λ) is a convex exact symplectic manifold of bounded topology whose associated Liouville vector field $Y = Y_\lambda$ generates a complete flow on V;[3]
- $\Sigma \overset{\iota}{\hookrightarrow} V$ is a convex exact hypersurface. Let α denote the contact form on Σ and M the compact manifold-with-boundary bounded by $\Sigma = \partial M$.

According to our conventions both V and Σ are *connected*. By Exercise 4.5.4 the Liouville vector field Y is outward pointing along the boundary Σ of M. By Remark 4.5.13 we may assume whenever convenient that $\alpha = \lambda|_\Sigma := \iota^*\lambda$ is the restriction of the primitive λ of the symplectic form $\omega := d\lambda = d\mu$.[4]

Differences to Chapter 3. Now the closed orbits cannot lie anywhere in the symplectic manifold (V, ω), they are constrained to a *fixed* regular energy surface $\Sigma = F^{-1}(0)$ required to be a contact manifold with respect to the restriction $\lambda|_\Sigma =: \alpha$; after changing the primitive λ of ω, if necessary. (Equivalently the Liouville vector field Y determined by $d\lambda(Y, \cdot) = \lambda$ is transverse to Σ.) In exchange, now the periods τ are free – no restriction to period 1 any more. Furthermore, whatever defining Hamiltonian one picks, the non-constant Hamiltonian loops are simultaneously Reeb loops of the contact manifold (Σ, α) and their images are called **closed characteristics**. So what one is really counting are geometric objects associated to the contact manifold (Σ, α) and, as $\alpha = \lambda|_\Sigma$, the way it sits in the exact symplectic manifold (V, λ). Last, not least, as defining Hamiltonians are autonomous, non-constant periodic solutions come at least in $\mathbb{S}^1$-families. So the functional $\mathcal{A}^F$ is at best Morse-Bott, as opposed to Morse.

Under Assumption 5.0.1, appropriately taking account of the, at best, Morse-Bott nature of the Rabinowitz action functional $\mathcal{A}^F$, Cieliebak and Frauenfelder proved the following

Theorem 5.0.2 (Existence and continuation, [CF09])**.** *Under Assumption 5.0.1 Floer homology for the Rabinowitz action functional and with $\mathbb{Z}_2$ coefficients*

$$\mathrm{HF}(\mathcal{A}^F) = \mathrm{HF}(\mathcal{A}^F; \mathbb{Z}_2)$$

is defined. If $\{F_s\}_{s\in[0,1]}$ is a smooth family of defining Hamiltonians of convex exact hypersurfaces Σ_s, then $\mathrm{HF}(\mathcal{A}^{F_0})$ and $\mathrm{HF}(\mathcal{A}^{F_1})$ are canonically isomorphic.

[3] By [CF09, Le. 1.4] *bounded topology* and *complete flow* can be achieved for any convex exact symplectic manifold (V, λ) by modifications outside of Σ.

[4] Otherwise, replace λ by $\mu := \lambda + dh$ where (V, μ) inherits the properties of (V, λ).

In particular, as the space of defining Hamiltonians is convex, see (4.5.11), Floer homology $\mathrm{HF}(\mathcal{A}^F)$ does not depend on the defining Hamiltonian, but on the pair (Σ, V), at most. In fact, in [CFO10, Prop. 3.1] it is shown independence on the unbounded component of $V \setminus \Sigma$, that is only $\Sigma = \partial M$ and its inside, the compact manifold-with-boundary M, are relevant for $\mathrm{HF}(\mathcal{A}^F)$. This justifies the following notation where $\mathrm{RFH}(\Sigma)$ just serves to abbreviate $\mathrm{RFH}(\partial M, M)$.

Definition 5.0.3. The **Rabinowitz-Floer homology** of a convex exact hypersurface $\Sigma \subset V$ bounding M, see Assumption 5.0.1, is the $\mathbb{Z}_2$ vector space[5]

$$\mathrm{RFH}(\Sigma) = \mathrm{RFH}(\partial M, M) := \mathrm{HF}(\mathcal{A}^F) \tag{5.0.1}$$

where $F \in \mathcal{F}(\Sigma)$ is a defining Hamiltonian for $\Sigma = F^{-1}(0)$.

By Theorem 5.0.2 Rabinowitz-Floer homology does not change under homotopies of convex exact hypersurfaces. An integer grading μ of RFH exists, see (5.3.30), if Σ is simply connected and $c_1(V)$ vanishes over $\pi_2(V)$. The following deep result has major consequences, e.g. it reconfirms the Weinstein conjecture for displaceable Σ; see [CF09, Cor. 1.5] and Section 5.4.2.

Theorem 5.0.4 (Vanishing theorem, [CF09])**.** *If Σ is displaceable, then Rabinowitz-Floer homology* $\mathrm{RFH}(\Sigma) = 0$ *vanishes.*

The idea of proof is to decompose life $[0,1]$ into two parts $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$, identifying $\mathbb{S}^1$ and $[0,1]/\{0,1\}$. Then push the Hamiltonian flow of F into the first part of life using a young (support in $[0, \frac{1}{2}]$) cutoff function χ and, in the second part of life, allow for elderly (support in $[\frac{1}{2}, 1]$) but time-experienced Hamiltonian perturbations $H \in \mathcal{H}^\dagger$. This way one gets to the perturbed Rabinowitz action $\mathcal{A}_H^{F\chi}$ in (5.4.33) and proving Theorem 5.0.4 reduces to a smart homotopy argument; cf. (5.4.41).

Albers and Frauenfelder [AF10a] realized that the critical points of $\mathcal{A}_H^{F\chi}$ are Moser's [Mos78] leaf-wise intersection points (LIPs). They obtained existence results for LIPs by associating (Rabinowitz) Floer homology groups $\mathrm{HF}(\mathcal{A}_H^{F\chi})$ to the perturbed action; see Section 5.4.3.

Outline. In Sections 5.1–5.3 we indicate the proof of Theorem 5.0.2 following closely the original, excellently written, paper [CF09]. We also recommend the survey [AF12b]. Section 5.4.3 is on the perturbed action functional and LIPs. In Section 5.5 very briefly we state the relation to loop spaces.

Notation 5.0.5. Cf. Notation 4.0.3 and Notation 1.0.5. The elements of $\mathcal{L}\Sigma \times \mathbb{R}$, in particular, the critical points of $\mathcal{A}^F$, are denoted by (z, τ). Critical points (z, τ) correspond to the points of Σ via constant loops $z_q \equiv q \in \Sigma$ whenever $\tau = 0$, otherwise to τ-periodic Reeb loops $r = (r_z)_\tau : \mathbb{R}/\tau\mathbb{Z} =: \mathbb{S}^1_\tau \to \Sigma$; see (5.1.4).[6] The notation is meant to indicate that for *negative* τ the loop $t \mapsto z(t)$, equivalently r, follows $-X_F = -R_\alpha$. Connecting trajectories are pairs "**upsilon**" $\upsilon = (u, \eta)$ where $u : \mathbb{R} \times \mathbb{S}^1 \to V$ and $\eta : \mathbb{R} \to \mathbb{R}$.

[5] To define RFH with integer coefficients is an *open problem.*

[6] Reeb loops are non-constant, as the Reeb vector field is nowhere zero by $\alpha(R_\alpha) = 1$.

5.1 Rabinowitz action $\mathcal{A}^F$ – free period

Throughout let $\Sigma \overset{\iota}{\hookrightarrow} V$ be a convex exact hypersurface in a convex exact symplectic manifold with symplectic structure $\omega = d\lambda$; see Assumption 5.0.1. Recall from Section 4.5 that $\Sigma = \partial M$ bounds a compact manifold-with-boundary M and comes with the contact form $\alpha = \lambda|_\Sigma := \iota^*\lambda$, whereas V carries the Liouville vector field $Y \pitchfork \Sigma$ that is determined by the identity $i_Y\omega = \lambda$.
Let us repeat from Proposition 4.5.5 the notion of defining Hamiltonian for Σ.

Definition 5.1.1. A **defining Hamiltonian for Σ** is an autonomous Hamiltonian $F : V \to \mathbb{R}$, negative on the inside $\mathring{M}$ of the hypersurface Σ, zero on Σ, positive outside Σ, constant outside some compact set, and with $X_F = R_\alpha$ along Σ, i.e. the Hamiltonian vector field extends the Reeb vector field; cf. (4.5.11) and Figure 5.1. Let $\mathcal{F}(\Sigma)$ be the space of Hamiltonians defining Σ.

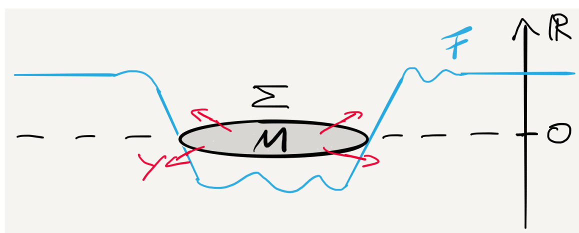

Figure 5.1: Defining Hamiltonian F for hypersurface $\Sigma = F^{-1}(0)$ bounding M

The space of defining Hamiltonians $\mathcal{F}(\Sigma) \neq \emptyset$ is non-empty and convex by Proposition 4.5.5.

Remark 5.1.2. As the Reeb vector field R_α nowhere vanishes, zero is automatically a regular value of $F \in \mathcal{F}(\Sigma)$. Furthermore, the manifold V and the compact manifold-with-boundary M bounded by Σ are canonically oriented by the volume form ω^n. Moreover, the orientation of

$$\Sigma = F^{-1}(0) = \partial M$$

as boundary of M by the put-outward-normal-first convention coincides with the orientation provided by the volume form $\alpha \wedge (d\alpha)^{n-1}$ on Σ; see Exercise 4.5.4 which also shows that the Liouville vector field Y points outward along $\Sigma = \partial M$. The gradient of F with respect to any Riemannian metric points also outward.

Definition 5.1.3. For a defining Hamiltonian $F \in \mathcal{F}(\Sigma)$ the **(unperturbed) Rabinowitz action functional** is defined by

$$\mathcal{A}^F : \mathcal{L}V \times \mathbb{R}, \quad (z,\tau) \mapsto \int_{\mathbb{S}^1} z^*\lambda - \tau \int_0^1 F(z(t))\, dt, \tag{5.1.2}$$

where $\mathcal{L}V := C^\infty(\mathbb{S}^1, V)$ is the free loop space of V.

Bringing in the real numbers τ causes that critical points (z,τ) of $\mathcal{A}^F$ not only correspond to integral loops of X_F, as was the case in (3.2.11), but constrains[7] them to lie on energy level zero and allows for periods τ other than 1.

Exercise 5.1.4 (Critical points correspond to Reeb loops and points of Σ)**.** Show that the critical points (z,τ) of $\mathcal{A}^F$ are the solutions $\tau\in\mathbb{R}$ and $z:\mathbb{R}\to V$ with $z(1)=z(0)$ of the ODE and the constraint given by

$$\begin{cases}\dot z(t)=\tau\, X_F(z(t)) & ,\ t\in\mathbb{S}^1,\\ z(t)\in F^{-1}(0) & ,\ t\in\mathbb{S}^1,\end{cases}$$

or, equivalently, of

$$\begin{cases}\dot z(t)=\tau\, R_\alpha(z(t)) & ,\ t\in\mathbb{S}^1,\\ P:=z(\mathbb{S}^1)\subset\Sigma & .\end{cases}\tag{5.1.3}$$

[Hint: Show $\int_0^1 F(z(t))\,dt=0$. But $z(t)=\phi_t z(0)$ where $\phi=\phi^F$ preserves F.]

Thus a critical point (z,τ) of $\mathcal{A}^F$ corresponds either, in case $\tau>0$, to a τ-periodic[8] Reeb loop $z(\cdot/\tau)$ on the contact manifold (Σ,α), or in case $\tau<0$ to one that runs backwards following $-R_\alpha$, or in case $\tau=0$ to a constant loop $z_p\equiv p=z(0)$ sitting at any point p of Σ.[9] A critical point of the form $(z,0)$ is called a **constant critical point**. By (5.1.3) the critical points of $\mathcal{A}^F$ *do not* depend on the defining Hamiltonian. In the notation (2.3.12) there is a bijection

$$\operatorname{Crit}\mathcal{A}^F\to\mathcal{P}_\pm(\Sigma)\cup\Sigma,\quad (z,\tau)\mapsto(r_z)_\tau:=\begin{cases}z(\cdot/|\tau|) & ,\ \tau>0,\\ z(0) & ,\ \tau=0,\\ z(\cdot/-|\tau|) & ,\ \tau<0,\end{cases}\tag{5.1.4}$$

onto the set $\mathcal{P}_\pm(\Sigma)\cup\Sigma$ given by (and then identified with a set of pairs)

$$\begin{aligned}&\{r_\tau:\mathbb{R}/\tau\mathbb{Z}\to\Sigma\mid \dot r_\tau=R_\alpha(r_\tau),\ \tau\neq0\}\cup\Sigma\\ &\simeq\{(r,\tau)\in\mathbb{C}^\infty(\mathbb{R},\Sigma)\times\mathbb{R}\mid \dot r=\operatorname{sign}(\tau)R_\alpha(r),\ \tau\in\operatorname{Per}(r)\}\,.\end{aligned}\tag{5.1.5}$$

Here $\mathcal{P}_\pm(\Sigma)$ is the **set of signed Reeb loops**, that is the set of (closed) forward or backward[10] Reeb loops on Σ. To see that the map $\simeq$ given by $r_\tau\mapsto(r,\tau)$ is a bijection use the convention $\operatorname{sign}(0):=0$ and recall from (2.3.12) that r_τ stands for a map $r:\mathbb{R}/\tau\mathbb{Z}\to\Sigma$ subject to direction reversal in case $\tau<0$.

[7] To find critical points of a function $f=f(x,y)$, say on $\mathbb{R}^2$ for illustration, whose domain is cut out by a constraint, say $g(x,y)=c$, one introduces a dummy variable $\lambda\in\mathbb{R}$ called **Lagrange multiplier** and determines the critical points of the function $\Lambda(x,y,\lambda)=f+\lambda(g-c)$; cf. Wikipedia. In our case τ plays the role of the Lagrange multiplier and c is zero.

[8] Here τ is just *a* period, not necessarily the prime period τ_r of the Reeb loop $r=z(\cdot/\tau)$.

[9] The appearance of constant loops in addition to Reeb loops – at first glance seemingly an annoying irregularity – is actually the **power plant** of the whole theory; cf. Remark 5.3.7 (ii).

[10] Allowing also for periodic solutions of $\dot r=-R_\alpha(r)$ simplifies things, e.g. the above map (5.1.4) is simply a “bijection” instead of a “$2:1$ map on non-constant critical points”.

Definition 5.1.5 (Simple critical points and their covers)**.** Pick a *non-constant* critical point (z, τ) of $\mathcal{A}^F$ and consider the (embedded) image circle $P := z(\mathbb{S}^1)$. Suppose $\tau > 0$, otherwise take $(\hat{z}, \hat{\tau})$. Observe that $1 \in \mathrm{Per}(z : \mathbb{R} \to \Sigma) = \tau_z \mathbb{Z}$ where $\tau_z > 0$ is the prime period; see (2.3.10). Thus $1 = \ell \tau_z$ for some integer $\ell = \ell(z) \geq 1$. Rescale z and τ by

$$z_P := z^{\tau_z} = z(\tau_z \cdot), \qquad \sigma_P := \tau \tau_z, \quad \tau = \ell \sigma_P. \tag{5.1.6}$$

The prime period of $z_P : \mathbb{R} \to \Sigma$ is 1 and

$$c_P := (z_P, \sigma_P) \in \mathrm{Crit}\, \mathcal{A}^F, \quad z_P : \mathbb{S}^1 \hookrightarrow \Sigma, \quad z_P(\mathbb{S}^1) = P.$$

As $z_P : \mathbb{S}^1 \hookrightarrow \Sigma$ is a simple loop, we call c_P **a simple critical point** of $\mathcal{A}^F$.[11] The other ones with image P are obtained by subjecting z_P to time shifts leading to an $\mathbb{S}^1$-family denoted by $\mathbb{S}^1 * c_P$ or S_{c_P}. As σ_P divides the speed factor τ of z, we call it the **prime speed** of the critical points with image P.

The k-fold covers of a simple critical point c_P defined by

$$c_P^k := \left(z_P^k, k\sigma_P\right), \quad z_P^k := z_P(k \cdot) \qquad k \in \mathbb{Z}, \tag{5.1.7}$$

are critical points of $\mathcal{A}^F$ as well and, up to the $\mathbb{S}^1$-action by time shift, there are no other critical points whose image is P. Observe that $z_P^0 \equiv z_P(0)$ is constant.

Exercise 5.1.6. Check the assertions in Definition 5.1.5. Show that σ_P, modulo time shift also z_P, is independent of $(z, \tau) \in \mathrm{Crit}\, \mathcal{A}^F$ as long as $z(0) \in P$.
[Hints: Let (y, χ) also be a critical point with $\chi > 0$ and $y(\mathbb{S}^1) = P$. Check that both paths $z^{1/\tau}$ and $y^{1/\chi}$ are Reeb solutions and their prime periods are $\tau\tau_z$ and $\chi\tau_y$, respectively. Hence $\tau\tau_z = \chi\tau_y =: \sigma_P$. But now the paths z^{τ_z} and y^{τ_χ} satisfy the same ODE $\dot{x} = \sigma_P R_\alpha(x)$, so they are equal up to time shift.]

Exercise 5.1.7 (Simple Reeb loop associated to (z, τ) via time of first return)**.** Show the assertions of Exercise 5.1.6 as follows. Pick $(z, \tau) \in \mathrm{Crit}\, \mathcal{A}^F$ with $\tau \neq 0$ and set $P := z(\mathbb{S}^1)$. Now set $p := z(0) \in P$ and apply the Reeb flow ϑ_t to get the Reeb path $r(t) := \vartheta_t p$ for $t \in \mathbb{R}$ whose image is P. (Hence the images $P = z(\mathbb{S}^1)$ of non-constant critical points (z, τ) are closed characteristics.) Let $T_P > 0$ be the time of first return. Check that it does not depend on the initial point in P, thereby justifying the notation T_P, as opposed to T_p. Since ϑ is a one-parameter group T_P is a period of r and as it is the smallest positive one $T_P = \tau_r$ is the prime period of r. Thus to (z, τ) belongs the simple Reeb loop

$$r_P : \mathbb{R}/T_P\mathbb{Z} \to \Sigma, \quad t \mapsto \vartheta_t p, \quad p = z(0), \tag{5.1.8}$$

with diffeomorphic image $P = z(\mathbb{S}^1)$.

a) Show that $\tau = kT_P$ for some integer $k \neq 0$.

b) Show that $T_P = \sigma_P$ is the prime speed, hence $(r_P^{T_P}, T_P) = (z_P, \sigma_P) =: c_P$.

[11] Let us call $\hat{c}_P := (z_P(-\cdot), -\sigma_P)$ the **corresponding backward simple critical point**.

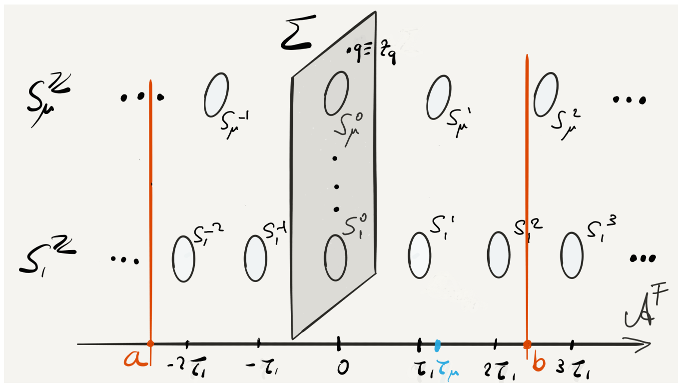

Figure 5.2: Finitely many critical circle towers: $\operatorname{Crit}\mathcal{A}^F \cong \Sigma \cup S_1^{\mathbb{Z}} \cup \ldots S_\mu^{\mathbb{Z}}$

[Hints: a) Show that $r = z^{1/\tau} : \mathbb{R} \to \Sigma$ and observe that $\tau \in \operatorname{Per}(z^{1/\tau})$. b) The pair $(r_P^{T_P}, T_P)$ is a critical point and $r_P^{T_P} : \mathbb{S}^1 \hookrightarrow \Sigma$ is an embedding with image P. Hence $T_P = \ell\sigma_P$ for some $\ell \in \mathbb{N}$ and $z_P := r_P^{T_P}(\cdot/\ell)$ is, in particular, of period 1. But $z_P(1) = z_P(0)$, equivalently $r_P^{T_P}(1/\ell) = r_P^{T_P}(0)$, implies $\ell = 1$.]

Exercise 5.1.8 (Action spectrum)**.** Show that the action value

$$\mathcal{A}^F(c_P^k) = \mathcal{A}^F(z_P^k, k\sigma_P) = k\sigma_P, \qquad \mathcal{A}^F(z, 0) = 0, \tag{5.1.9}$$

of the k-fold cover, $k \neq 0$, of a simple critical point c_P is given by k times the prime speed σ_P. So by Exercise 5.1.7 the **action spectrum** $\mathfrak{S}(\mathcal{A}^F)$, i.e. the set of critical values of $\mathcal{A}^F$, consists of all integer multiples of the prime periods of the Reeb loops, in symbols $\mathfrak{S}(\mathcal{A}^F) = \mathbb{Z}\,\mathfrak{S}(\Sigma)$. The set $\mathfrak{S}(\Sigma)$ of periods of the *simple* Reeb loops is the **prime period spectrum** of the contact manifold Σ.

Remark 5.1.9 (Critical towers)**.** By (5.1.7) any simple critical point of $\mathcal{A}^F$, that is a pair of the form $c_P = (z_P, \sigma_P)$, gives rise to a whole *critical point tower* $c_P^{\mathbb{Z}} := \left(z_P^k, k\sigma_P\right)_{k\in\mathbb{Z}}$. The circle acts on the k-fold cover c_P^k of c_P by time shifts

$$T * c_P^k := \left(T * z_P^k, k\sigma_P\right) := \left(z_P^k(T + \cdot), k\sigma_P\right), \quad T \in \mathbb{S}^1 = \mathbb{R}/\mathbb{Z}.$$

Thus point towers come, at least, as *critical circle towers* $C_P^{\mathbb{Z}} := \left(C_P^k\right)_{k\in\mathbb{Z}}$ where

$$C_P^k := S_P^k \times \{k\sigma_P\}, \quad S_P^k := \mathbb{S}^1 * z_P^k := \left\{T * z_P^k\right\}_{T\in\mathbb{S}^1}, \quad S_P^{\mathbb{Z}} := \left(S_P^k\right)_{k\in\mathbb{Z}}.$$

So the action functional $\mathcal{A}^F$ can be Morse-Bott, at best, but not Morse. The set of simple critical points $c_P = (z_P, \sigma_P)$ corresponds to the set $\mathcal{C}(\Sigma)$ of *closed characteristics* P with distinguished point $p \in P$: a) Associate to z_P the embedded circle $P := z_P(\mathbb{S}^1)$ with distinguished point $p := z_P(0)$. Indeed P is a Reeb orbit, thus a closed characteristic, by Exercise 5.1.7. b) Vice versa, given P and $p \in P$, consider the Reeb path $r(t) := \vartheta_t p$. Its prime period is denoted by τ_r. Then $(z_P, \sigma_P) := (r^{\tau_r}, \tau_r)$ is a simple critical point of $\mathcal{A}^F$ which gets mapped back to P and p by a). That the map in a) followed by the one in b) is the identity as well holds by Exercise 5.1.7 b). Observe that, in general, the simple critical points can appear in families larger than circles.

5.1.1 Transverse non-degeneracy

Recall that a function f is called a **Morse-Bott function** if, firstly, its critical set $C := \mathrm{Crit} f$ is a submanifold (whose components might be of different dimensions) and, secondly, the tangent space T_pC at every point p of C is precisely the kernel of the Hessian $\mathrm{Hess}_p f$ of f at p. In view of invariance of the functional $\mathcal{A}^F$ under the time shift circle action, non-degeneracy is achievable *at most* in directions transverse to the circles in $\mathcal{L}V$ generated by the $\mathbb{S}^1$-action.

Theorem 5.1.10 (Transverse non-degeneracy, [CF09])**.** *There is a residual*[12] *subset $\mathcal{F}_{\mathrm{reg}}$ of the complete metric space $\mathcal{F} := C^\infty_{0'}(V)$ of smooth functions on V with compactly supported* differential *such that the following is true. For every $f \in \mathcal{F}_{\mathrm{reg}}$ the Rabinowitz action functional $\mathcal{A}^f$ is Morse-Bott and its critical set consists of $f^{-1}(0)$ together with a disjoint union of circles.*

Proof. See [CF09, Thm. B.1, p.298]. The proof uses the machinery of Thom-Smale transversality, see Section 3.3.4, with the *additional difficulty* that the universal section cannot be surjective due to the unavoidable critical circles arising by time shifting non-constant critical points. □

Definition 5.1.11. A contact type hypersurface is called **non-degenerate** if all Reeb loops r are **transverse non-degenerate** in the sense that the linearized Reeb flow $d\vartheta_T(p) : \xi_p \to \xi_p$ at $p = r(0)$, where $\xi_p := \ker \alpha(p)$ and $T > 0$ is the prime period of r, does not have 1 amongst its eigenvalues.

Consider from now on the situation of Assumption 5.0.1, in particular, suppose the bounding hypersurface $\Sigma \subset (V, \lambda)$ is of restricted contact type. Fix a defining Hamiltonian $F \in \mathcal{F}(\Sigma)$.

Exercise 5.1.12. Check that $F \in \mathcal{F}_{\mathrm{reg}}$ iff Σ is non-degenerate.
[Hint: Cf. Exercise 3.2.13.]

Exercise 5.1.13. a) Show that there is a convex closed neighborhood $\mathcal{U}^F$ of F in $\mathcal{F} = C^\infty_{0'}(V)$ such that the zero set $f^{-1}(0)$ of any element $f \in \mathcal{U}^F$ is a convex exact hypersurface of (V, λ). (In fact, it is of restricted contact type, if Σ is.)
b) Check that $\mathcal{U}^F_{\mathrm{reg}} := \mathcal{U}^F \cap \mathcal{F}_{\mathrm{reg}}$ is residual in $\mathcal{U}^F$.
[Hint: a) Given the by $d\lambda(Y, \cdot) = \lambda$ globally determined Liouville vector field Y, the contact type property of Σ is equivalent to transversality $Y \pitchfork \Sigma$. But transversality $Y \pitchfork \tilde{\Sigma}$ survives, where $\tilde{\Sigma} := f^{-1}(0)$, whenever f is sufficiently close to F in $\mathcal{F}$. Define $\tilde{\alpha} := \tilde{\iota}^* \lambda$; see (4.4.6).]

[12] cf. Section 3.2.4

Remark 5.1.14. Given regular Hamiltonians $F_0, F_1 \in \mathcal{U}^F_{\mathrm{reg}}$ near the defining Hamiltonian F of Σ, by Exercise 5.1.13 a) convex combination provides a family

$$\Sigma_s := {F_s}^{-1}(0), \quad F_s := (1-s)F_0 + sF_1, \quad s \in [0,1], \tag{5.1.10}$$

of bounding restricted contact type hypersurfaces of (V, λ). While the endpoints of the family are non-degenerate, there is no reason that all members Σ_s be.

Remark 5.1.15 (Crit $\mathcal{A}^F$ consists of $\mu < \infty$ circle towers and Σ)**.** If Σ is *non-degenerate*, i.e. $F \in \mathcal{F}_{\mathrm{reg}}$, then $\mathcal{C}(\Sigma)$ consists of *finitely many* closed characteristics $P_1, \dots, P_\mu$ since Σ is compact. The embedded circles $P_i \hookrightarrow \Sigma$ correspond, up to fixing a point $p_i \in P_i$, likewise modulo time shifts, to simple Reeb loops

$$r_i := r_{P_i} = \vartheta . p_i : \mathbb{R}/T_i\mathbb{Z} \to \Sigma, \quad T_i := T_{P_i},$$

see (5.1.8), so by Exercise 5.1.7 b) to simple critical points

$$c_i := (z_i, \tau_i) := (r_i^{T_i}, T_i) = (z_{P_i}, \sigma_{P_i}) \in \operatorname{Crit} \mathcal{A}^F, \qquad i = 1, \dots, \mu. \tag{5.1.11}$$

Observe that $\tau_i := T_i = \sigma_{P_i}$ is the prime period (time of first return) of the Reeb loop r_i and simultaneously the prime speed of the simple critical point $z_i = r_i^{T_i}$. Time shifting r_i and z_i moves the initial point $p_i = r_i(0) = z_i(0)$ around the circle P_i. Let c_i^k be the k-fold cover (5.1.7) of c_i. Let $S_i^k := S_{P_i}^k = \mathbb{S}^1 * z_i^k$, so

$$C_i^k := C_{P_i}^k = S_i^k \times \{k\tau_i\}, \quad k \in \mathbb{Z}^* := \mathbb{Z} \setminus \{0\}, \tag{5.1.12}$$

are the corresponding circles in $\mathcal{L}\Sigma \times \mathbb{R}$ arising by time shift; cf. Remark 5.1.9. The components of the critical set $C := \operatorname{Crit} \mathcal{A}^F$ are the circles C_i^k along which the action is constantly $k\tau_i$ and the component $C_0 = \Sigma \times \{0\}$ of constant loops in Σ where the action is zero; cf. (5.3.18). Note that each critical component is compact and that by non-degeneracy only finitely many of them lie on action levels inside any given compact interval $[a, b]$. This is illustrated by Figure 5.2.

5.2 Upward gradient flow – downward count

Pick a defining Hamiltonian $F \in \mathcal{F}(\Sigma)$. So F is constant, i.e. $X_F = 0$, outside a compact subset K of V. In order to turn the differential $d\mathcal{A}^F$ of the Rabinowitz action functional into a gradient, we need to choose a metric on $\mathcal{L}V \times \mathbb{R}$. The fact that V is non-compact, as opposed to the symplectic manifold in Section 3.4.1 on fixed-period Floer homology, causes serious difficulties when it comes to prove compactness of moduli spaces of connecting trajectories $\upsilon = (u, \eta)$, due to the lack of an apriori C^0 bound; cf. Theorem 5.3.6. The key idea to obtain nevertheless uniform C^0 bounds for connecting trajectories is to choose a family $J = (J_t)_{t \in \mathbb{S}^1}$ of ω-compatible almost complex structures J_t which are *cylindrical*, see Definition 4.5.11, along the cylindrical ends $N \times \mathbb{R}_+$ of (V, λ).

Remark 5.2.1 ($X_F = 0$ on cylindrical ends – causes uniform C^0 bound for u's)**.** Recall from Exercise 4.5.10 that $N = \partial M_k$ is chosen as boundary of one of the members of the exhaustion of $V = \cup_k M_k$. Choose k larger, if necessary, such that M_k contains K. This guarantees that X_F vanishes on the cylindrical ends.

For $(\xi_i, \tau_i) \in T_{(z,\tau)}(\mathcal{L}V \times \mathbb{R})$ an L^2 metric on $\mathcal{L}V \times \mathbb{R}$ is defined by

$$\langle(\xi_1, \tau_1), (\xi_2, \tau_2)\rangle_J := \int_0^1 \omega\left(\xi_1(t), J_t(z(t))\xi_2(t)\right) dt + \tau_1\tau_2. \tag{5.2.13}$$

Exercise 5.2.2. The gradient of $\mathcal{A}^F$ with respect to the L^2 metric is given by

$$\operatorname{grad} \mathcal{A}^F(z, \tau) = \begin{pmatrix} -J_t(z)\left(\partial_t z - \tau X_F(z)\right) \\ -\int_0^1 F(z)\, dt \end{pmatrix}. \tag{5.2.14}$$

So for the norm-square of the gradient one gets

$$\begin{aligned} \left\|\operatorname{grad} \mathcal{A}^F(z, \tau)\right\|^2 &= \left\langle(\operatorname{grad} \mathcal{A}^F(z, \tau), \operatorname{grad} \mathcal{A}^F(z, \tau)\right\rangle_J \\ &= \|\partial_t z - \tau X_F(z)\|_2^2 + \operatorname{mean}^2(F \circ z) \\ &\le \Big(\|\partial_t z - \tau X_F(z)\|_2 + |\operatorname{mean}(F \circ z)|\Big)^2 \end{aligned} \tag{5.2.15}$$

where[13] we used that J_t is compatible with ω and where

$$\operatorname{mean}(F \circ z) := \int_0^1 F(z(t))\, dt.$$

The *upward* gradient trajectories of $\operatorname{grad} \mathcal{A}^F$ are the solutions denoted by the letter "**upsilon**"

$$\upsilon = (u, \eta) \in C^\infty(\mathbb{R} \times \mathbb{S}^1, V) \times C^\infty(\mathbb{R}, \mathbb{R})$$

of the elliptic PDE given by

$$\partial_s \upsilon = \operatorname{grad} \mathcal{A}^F(\upsilon), \qquad \upsilon = (u, \eta), \tag{5.2.16}$$

or, equivalently, the zeroes of a section, cf. (3.3.33), namely

$$\mathcal{F}_F(u, \eta) := \begin{pmatrix} \partial_s u + J_t(u)\left(\partial_t u - \eta X_F(u)\right) \\ \partial_s \eta + \int_0^1 F(u)\, dt \end{pmatrix} = 0. \tag{5.2.17}$$

Remark 5.2.3 (Homology – upward vs. downward gradient flow)**.** The advantage of using the *upward* gradient equation is that on cylindrical ends, where $X_F = 0$, component one of (5.2.17) becomes the well known J-holomorphic curve equation[14] in which case to obtain an apriori C^0 bound for u one can simply refer to the literature; see [CF09, p.268, pf. of Thm. 3.1]. In order to define nevertheless *homology*, as opposed to cohomology, one must ensure that the action *decreases* along the *boundary operator* ∂. This leads to the non-standard order in the coefficients $n(c_-, c_+)c_-$ of ∂c_+ in (5.3.23) below. In other words, to define an action decreasing boundary operator ∂c_+, one can either

(standard) count downward flow lines emanating from c_+ at time $-\infty$ or

(present) count upward flow lines that end at c_+ at time $+\infty$.

Figure 5.3 *illustrates*[15] the downward count of upward flows.

[13] Here $\|\xi\|_2^2 := \int_0^1 \omega\left(\xi(t), J_t(z(t))\xi(t)\right) dt$ for smooth vector fields ξ along the loop z.
[14] The downward gradient leads to the anti J-holomorphic curve equation $\partial_s u - J(u)\partial_t u = 0$.
[15] Be aware that ∂c_+ will not count connecting flow lines $u/\mathbb{R} := \tilde{u}$ of $\operatorname{grad} \mathcal{A}^F$, but con-

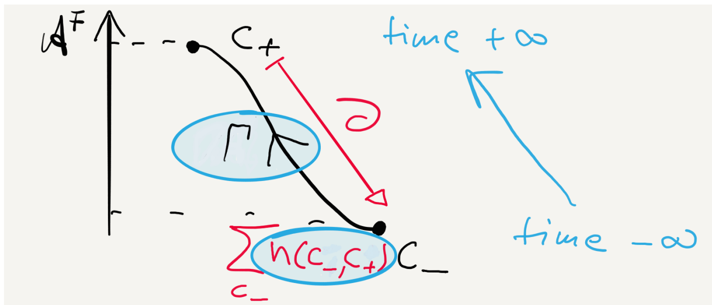

Figure 5.3: *Downward* count of upward flows Γ meeting at c_+

5.3 Rabinowitz-Floer chain complex

Consider a convex exact hypersurface $\Sigma \subset (V, \lambda)$, say of restricted contact type, in a convex exact symplectic manifold, as in Assumption 5.0.1. Fix a family $J = (J_t)_{t \in \mathbb{S}^1}$ of almost complex structures J_t compatible with the symplectic structure $\omega := d\lambda$ on V and cylindrical along the cylindrical ends; see Definition 4.5.11. Choose the associated L^2 inner product on $\mathcal{L}V \times \mathbb{R}$ given by (5.2.13).

There are two goals in Section 5.3. Firstly, to associate to a defining Hamiltonian $F \in \mathcal{F}(\Sigma)$ of Σ Floer homology groups $\mathrm{HF}(\mathcal{A}^F)$ with $\mathbb{Z}_2$ coefficients. Secondly, to show that these are independent, up to natural isomorphism, not only on the choice of defining Hamiltonian F, but also under convex exact homotopies of the hypersurface Σ itself. Cieliebak and Frauenfelder [CF09] defined

$$\mathrm{RFH}(\Sigma, V) := \mathrm{HF}(\mathcal{A}^F)$$

called **Rabinowitz-Floer homology** of the convex exact hypersurface $\Sigma = \partial M$ which, by assumption, bounds a compact manifold-with-boundary, say M.

To construct Floer homology one usually slightly perturbs relevant quantities in a first step, see Section 3.3.4, in order to get to a Morse situation, so one can use the then discrete critical points themselves as generators of the Floer chain groups. Here this is impossible: Since F, being defining for Σ, is necessarily *time-independent* the Rabinowitz action functional $(z, \tau) \mapsto \mathcal{A}^F(z, \tau)$ will always be invariant under the $\mathbb{S}^1$-action on $\mathcal{L}V \times \mathbb{R}$ given by time shifting the loop component z. While $\mathcal{A}^F$ is therefore never Morse, it is of the simplest Morse-Bott type for generic, called regular, defining Hamiltonian F by Theorem 5.1.10. For $F \in \mathcal{F}_{\mathrm{reg}}$ the critical set is the union of $C_0 \cong \Sigma$ and μ critical point towers

$$\begin{aligned} C := \operatorname{Crit} \mathcal{A}^F = C_0 \,\dot\cup\, \dot{\bigcup_{k \in \mathbb{Z}^*}} C_1^k \dots \dot\cup\, \dot{\bigcup_{k \in \mathbb{Z}^*}} C_\mu^k \\ \subset \mathcal{L}\Sigma \times \mathbb{R} \end{aligned} \tag{5.3.18}$$

necting *cascade* flow lines $\Gamma = (\gamma_-^0, \tilde{v}^1, \gamma^1, \dots, \tilde{v}^\ell, \gamma_+^\ell)$; cf. Figure 5.4. The cascades are unparametrized curves $v^j/\mathbb{R} := \tilde{v}^j = (\tilde{u}^j, \tilde{\tau}^j)$, the $\gamma_{(\pm)}^j$ are (semi-)finite time, thus parametrized, Morse gradient trajectories along components of the Morse-Bott manifold $C := \operatorname{Crit} \mathcal{A}^F$.

as illustrated by Figure 5.2. The floors are compact connected manifolds, namely the ground floor $C_0 \cong \Sigma$ together with the upper ($k > 0$) and lower ($k < 0$) circle floors $C_i^k \cong \mathbb{S}^1$ given by (5.1.12). Throughout it is convenient to identify constant loops in Σ (period and action zero) with the points of the compact hypersurface Σ itself. Remark 5.1.15 describes how each closed characteristic together with a chosen point $p_i \in P_i \subset \Sigma$ corresponds to a simple critical point z_i of $\mathcal{A}^F$ with $z_i(0) = p_i$ whose prime speed σ_i is the prime period $\tau_i := \tau_{r_i}$ of the Reeb loop r_i that parametrizes P_i and is determined by the initial condition $r_i(0) = p_i$; see also Remark 5.1.15.

Remark 5.3.1 (Critical set)**.** The restrictions $\mathcal{A}^F|_{C_i^k} \equiv k\tau_i$ and $\mathcal{A}^F|_{C_0} \equiv 0$ of the action functional to components are constant by Exercise 5.1.8. So it makes sense to speak of the *action value of a critical component.* As $i = 1, \dots, \mu$ only runs through a *finite* set, caused by compactness and transverse non-degeneracy of Σ, there can only be finitely many critical components with actions in a given bounded interval $[a, b]$; see Figure 5.2. In other words, the set of critical points

$$C^{[a,b]} := \mathrm{Crit}^{[a,b]}\mathcal{A}^F := \{a \le \mathcal{A}^F \le b\} \cap \mathrm{Crit}\,\mathcal{A}^F$$

whose actions lie in an interval $[a, b]$ form a *closed* submanifold $C^{[a,b]} \subset \mathcal{L}\Sigma \times \mathbb{R}$ diffeomorphic to a finite union of embedded circles whenever $0 \notin [a, b]$; otherwise, there is in addition one connected component diffeomorphic to Σ.

Perturbing F amounts to perturbing $\Sigma = F^{-1}(0)$, of course. However, by Exercise 5.1.13 a), small perturbations will not leave the class of convex exact hypersurfaces and even the restricted contact type property of Σ will be preserved.[16] So from now on

- we assume that Σ in Assumption 5.0.1 is, in addition, non-degenerate[17]
- with defining Hamiltonian $F \in \mathcal{F}(\Sigma) \cap \mathcal{F}_{\rm reg}$; cf. Exercise 5.1.13 b).

Given such $\Sigma = F^{-1}(0)$, the goal is to find suitable auxiliary data (h, g) that allows to define Floer homology[18] and has the property that different choices lead to naturally isomorphic homology groups. A suitable candidate is

Frauenfelder's implementation of Morse-Bott theory by cascades [Fra04]

which, in addition, requires to fix a Morse-Smale pair (h, g) consisting of

- a Morse function $h : C \to \mathbb{R}$ and
- a Riemannian metric g on the critical manifold $C = \mathrm{Crit}\,\mathcal{A}^F$.

While the critical set $\mathrm{Crit}\,h \subset \mathrm{Crit}\,\mathcal{A}^F$ of h is not necessarily finite, its subset

$$\mathrm{Crit}^{[a,b]}h := C^{[a,b]} \cap \mathrm{Crit}\,h = \mathrm{Crit}\,h|_{C^{[a,b]}} \tag{5.3.19}$$

is finite, because the manifold $C^{[a,b]}$ is closed, see Remark 5.3.1, and h is Morse.

[16] The contact condition is open; cf. Exercise 4.2.2 (b).

[17] The non-degeneracy assumption on Σ is justified by invariance of RFH under smooth variations of Σ up to natural isomorphism; see Section 5.3.2 and Remark 5.1.14.

[18] meaning that the chain groups should be generated by some finite, or at least discrete, critical set and the boundary should count isolated flow lines connecting critical points

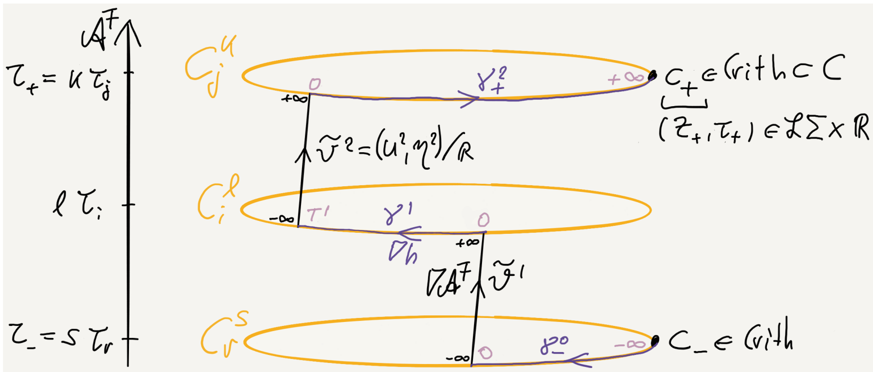

Figure 5.4: Cascade flow $\Gamma = \left(\gamma_-^0, \tilde{v}^1, \gamma^1, \tilde{v}^2, \gamma_+^2\right) \in \mathcal{M}_{c_- c_+}(\mathcal{A}^F, h, J, g)$

Chain groups

For $F \in \mathcal{F}(\Sigma) \cap \mathcal{F}_{\mathrm{reg}}$ the Floer chain group $\mathrm{CF}(\mathcal{A}^F, h)$ is defined as the vector space over $\mathbb{Z}_2$ that consists of all formal sums

$$x = \sum_{c \in \mathrm{Crit}\, h} x_c c$$

of critical points of h such that the $\mathbb{Z}_2$-coefficients $x_c = x_c(x)$ in such a formal sum x satisfy the **upward finiteness condition**

$$\left|\left\{c \in \mathrm{Crit}\, h \mid x_c(x) \neq 0 \text{ and } \mathcal{A}^F(c) \geq \kappa\right\}\right| < \infty, \quad \forall \kappa \in \mathbb{R}. \tag{5.3.20}$$

In words, given a formal sum x, one requires finiteness of the number of non-zero coefficients $x_c(x)$ above any given action value κ.

Definition 5.3.2. The Floer chain group $\mathrm{CF}(\mathcal{A}^F, h)$ is the $\mathbb{Z}_2$-vector space generated by **upward finite formal sums** x of critical points of the Morse function h on the critical set C of the Rabinowitz action functional $\mathcal{A}^F$.

Connecting cascade trajectories – upward flows

Given $F \in \mathcal{F}(\Sigma) \cap \mathcal{F}_{\mathrm{reg}}$, on the critical manifold $C := \mathrm{Crit}\, \mathcal{A}^F$ (whose components are circles and one component is given by the closed hypersurface Σ according to Theorem 5.1.10) consider the *Morse gradient flow* generated by the Morse function $h : C \to \mathbb{R}$ and the Riemannian metric g on C through the ODE $\dot{\gamma} = \nabla^g h(\gamma)$ for smooth maps $\gamma : \mathbb{R} \to \mathbb{C}$. Given two critical points $c_\pm = (z_\pm, \tau_\pm) \in \mathrm{Crit}\, h \subset C$, a **connecting upward trajectory with cascades**, also called an **upward connecting cascade trajectory**, is a tuple of the form

$$\Gamma = \left(\gamma_-^0, \tilde{v}^1, \gamma^1, \ldots, \tilde{v}^{\ell-1}, \gamma^{\ell-1}, \tilde{v}^\ell, \gamma_+^\ell\right)$$

where where $\ell \in \mathbb{N}_0$ is the number of cascades $\tilde{v}^j$ and

$$\dot{\gamma}^j = \nabla h(\gamma^j), \quad \partial_s v^j = \operatorname{grad} \mathcal{A}^F(v^j), \quad \tilde{v}^j = [v^j : \mathbb{R} \to \mathcal{L}V \times \mathbb{R}].$$

The notation $\tilde{v} = [v] =$ "$\{v\}/\mathbb{R}$" indicates unparametrized flow lines or, equivalently, flow trajectories that only differ by shifting the s variable are considered equivalent. See Figure 5.4 for a connecting trajectory with $\ell = 2$ cascades. Actually each intermediate Morse trajectory comes with a finite time $T^j \geq 0$ and is defined on a *finite time* interval, namely

$$\gamma^j : [0, T^j] \to C.$$

In contrast, the two ends $\gamma_-^0 : (-\infty, 0] \to C$ and $\gamma_+^\ell : [0, \infty) \to C$ are *semi-infinite* Morse trajectories. Neither of the Morse trajectories $\gamma_{(\pm)}^j$ is invariant under the time-s shift $\mathbb{R}$ action.

More precisely, the tuple Γ starts with a semi-infinite Morse trajectory γ_-^0 backward asymptotic to the given critical point $c_- \in \operatorname{Crit} h$ and whose position at time zero, namely $\gamma_-^0(0)$, is the backward asymptote v_-^1 of some $\operatorname{grad} \mathcal{A}^F$ flow trajectory $v^1 \in \tilde{v}^1 = [v^1]$ defined on the whole real line, see (5.2.17), where

$$v_\mp^j = \lim_{s \to \mp\infty} v^j(s) \in C, \quad v^j \in [v^j], \quad v^j = (u^j, \eta^j) : \mathbb{R} \to \mathcal{L}V \times \mathbb{R}. \tag{5.3.21}$$

The other, positive, asymptote $v_+^1 \in C$ provides the initial point $\gamma^1(0)$ of a finite time Morse trajectory $\gamma^1 : [0, T^1] \to C$ whose endpoint $\gamma^1(T^1)$ is a backward asymptote v_-^2. Continuing this way one reaches the final piece, namely, the semi-infinite Morse trajectory $\gamma_+^\ell : [0, \infty) \to C$ that starts at the previous backward asymptote point v_+^ℓ at time 0 and is itself forward asymptotic to the second given critical point c_+. The unparametrized Morse-Bott flows $\tilde{v}^j$ are called **cascades**.

The **moduli space of connecting cascade trajectories**

$$\mathcal{M}_{c_- c_+} = \mathcal{M}_{c_- c_+}(\mathcal{A}^F, h, J, g) \tag{5.3.22}$$

consists of all cascade trajectories Γ connecting c_- and c_+. The following are non-trivial – even in finite dimension – although well known: See [Fra04, App. A] and [CF09, App. A]. Firstly, for generic J and g these moduli spaces are smooth manifolds. Secondly, it is a consequence of the Compactness Theorem 5.3.6 below that the 0-dimensional part $\mathcal{M}_{c_- c_+}^0$ of $\mathcal{M}_{c_- c_+}$ is compact, so a finite set.

Boundary operator and Floer homology

Pick $F \in \mathcal{F}(\Sigma) \cap \mathcal{F}_{\text{reg}}$. Since for generic J and g the 0-dimensional part $\mathcal{M}_{c_- c_+}^0$ of the space of connecting trajectories $\mathcal{M}_{c_- c_+}(\mathcal{A}^F, h, J, g)$ is compact, hence a finite set, the number of elements modulo two

$$n(c_-, c_+) := \#_2 \mathcal{M}_{c_- c_+}^0$$

is well defined. Then the Floer boundary operator

$$\partial : \mathrm{CF}(\mathcal{A}^F, h) \to \mathrm{CF}(\mathcal{A}^F, h)$$

is defined as the linear extension of

$$\partial c_+ := \sum_{c_- \in \mathrm{Crit}\, h} n(c_-, c_+) c_- \tag{5.3.23}$$

for $c_+ \in \mathrm{Crit}\, h$. This is illustrated by Figure 5.3; cf. also Remark 5.2.3.

Exercise 5.3.3. The formal sum in the definition of ∂ satisfies the finiteness condition (5.3.20). [Hint: The action non-decreases along cascade trajectories, (5.3.19), Exercise 5.1.8, Arzelà-Ascoli Theorem 3.2.10.]

The boundary operator property $\partial^2 = 0$ follows by standard gluing and compactness arguments, compare Proposition 3.4.3, once one has the Compactness Theorem 5.3.6 for the 1-dimensional part of moduli space, together with finiteness of the number $|\mathrm{Crit}^{[a,b]} h| = |\mathrm{Crit}\, h|_{C^{[a,b]}}|$ of critical points of the Morse function h in any finite action interval $[a, b]$ or, equivalently, on the closed manifold $C^{[a,b]}$. By definition **Floer homology of the Rabinowitz action functional** $\mathcal{A}^F$ is the homology of this chain complex, namely

$$\mathrm{HF}(\mathcal{A}^F) := \frac{\ker \partial}{\mathrm{im}\, \partial}, \qquad F \in \mathcal{F}(\Sigma) \cap \mathcal{F}_{\mathrm{reg}}. \tag{5.3.24}$$

We already dropped (h, J, g) from the notation since $\mathrm{HF}(\mathcal{A}^F)$ does not depend on that choice, up to canonical isomorphism, as follows by the standard continuation techniques detailled in Section 3.4.3. Via continuation one also shows independence of the regular defining Hamiltonian $F \in \mathcal{F}(\Sigma) \cap \mathcal{F}_{\mathrm{reg}}$.

Rabinowitz-Floer homology of a convex exact hypersurface

Remark 5.3.4. Given Σ as in Assumption 5.0.1, observe the following: There is no regular defining Hamiltonian $F \in \mathcal{F}(\Sigma)$ iff Σ itself is already degenerate; see Exercise 5.1.12. Hence fixing non-regularity of F is equivalent to perturbing Σ. In practice pick F_0 in the dense subset $\mathcal{U}^F_{\mathrm{reg}} := \mathcal{U}^F \cap \mathcal{F}_{\mathrm{reg}}$ of the small open neighborhood $\mathcal{U}^F$ of F in $\mathcal{F}$ provided by Exercise 5.1.13 b). In particular, the zero set $\Sigma_0 := F_0^{-1}(0)$ is a non-degenerate convex exact hypersurface nearby Σ. The Floer homology of $\mathcal{A}^{F_0}$ is then defined by (5.3.24). For any two such choices F_0 and F_1 there is the standard continuation isomorphism on homology.

Rabinowitz-Floer homology of a convex exact hypersurface $\Sigma \subset (V, \lambda)$ with defining Hamiltonian $F \in \mathcal{F}(\Sigma)$ is defined by

$$\mathrm{RFH}(\Sigma) := \mathrm{HF}(\mathcal{A}^{F_0}), \qquad F_0 \in \mathcal{U}^F_{\mathrm{reg}}. \tag{5.3.25}$$

By continuation $\mathrm{RFH}(\Sigma)$ does not depend on the choice of F_0; cf. Remark 5.1.14. Observe that F_0 lies in $\mathcal{F}_{\mathrm{reg}}$ and in $\mathcal{F}(\Sigma_0)$: It is defining for the non-degenerate convex exact hypersurface $\Sigma_0 := F_1^{-1}(0)$ nearby Σ. For $\mathbb{S}^1$-equivariant Rabinowitz-Floer homology see [FS16].

5.3.1 Compactness of moduli spaces

Definition and property $\partial^2 = 0$ of the boundary operator ∂ both hinge on compactness properties of the moduli spaces of connecting trajectories with cascades. Recall that a connecting cascade trajectory consists of (semi-)finite Morse gradient trajectories[19] γ^j along the critical set C of $\mathcal{A}^F$ and the cascades $\tilde{v}^j$ themselves. Cascades are, modulo s-shift, connecting trajectories $v = (u, \eta)$ of the gradient $\operatorname{grad} \mathcal{A}^F$ in $\mathcal{L}V \times \mathbb{R}$ between two critical sets.[20] While compactness up to broken trajectories for Morse trajectories can even be handled within finite dimensional dynamical systems, see e.g. [Web], in the case of cascades compactness of the loop space component u is also standard, but this is not so for the Lagrange multiplier component η. Let us detail this: To prove compactness up to broken trajectories of the set of $\operatorname{grad} \mathcal{A}^F$ trajectories $v = (u, \eta)$ that are subject to a uniform action bound

$$a \le \mathcal{A}^F(v(s)) \le b \tag{5.3.26}$$

the following apriori bounds are sufficient: Namely,

(i) a uniform C^0 bound on u;

(ii) a uniform C^0 bound on η;

(iii) a uniform C^0 bound on $\partial_s u$ (thus on $\partial_t u$).

(i) is fine due to our choice of *cylindrical* almost complex structures; see Remark 5.2.3. (ii) was new and required new techniques when it was established in [CF09, §3.1]. Thus we shall outline their argument below. (iii) follows from the exactness assumption of the symplectic form $\omega = d\lambda$ via standard bubbling-off analysis as indicated in Section 3.4.1.

Exercise 5.3.5. Doesn't one need a uniform C^0 bound on the derivative η' as well? And why is the C^0 bound requirement (iii) for the derivative $\partial_s u$ not listed directly after condition (i) for u itself?

Let us state the compactness theorem in a rather general form that will serve simultaneously

- the continuation problem, where s-dependent Hamiltonians appear;
- the proof of the Vanishing Theorem 5.0.4, where the rescalings χ help.

Of course, the following compactness theorem for families requires three C^0 bounds as in (i–iii) which are also uniform in the family parameters.

[19] in general not connecting ones: Morse trajectories have finite life time, except the initial and ending one which are semi-infinite, also called semi-connecting.

[20] *Connecting* trajectories necessarily live on the infinite domain $(\mathbb{R} \times \mathbb{S}^1) \times \mathbb{R}$ and their moduli spaces are subject to division by the free $\mathbb{R}$-action given by shifting the s-variable.

Theorem 5.3.6 (Cascade compactness)**.** *Suppose that $F:[0,1]\times V\to\mathbb{R}$ is a smooth function such that each Hamiltonian $F_\sigma:=F(\sigma,\cdot)$ defines a convex exact hypersurface Σ_σ and $\chi:[0,1]\times\mathbb{S}^1\to[0,\infty)$ is a smooth function such that each $\chi_\sigma:=\chi(\sigma,\cdot)$ integrates to one. Let $\varepsilon>0$ and $c<\infty$ be constants as provided by [CF09, Prop. 3.4],[21] and suppose that the following inequality holds*

$$\left(c+\frac{\|F\|_\infty}{\varepsilon^2}\right)\cdot\left(\|\partial_\sigma F\|_\infty+\|\partial_\sigma\chi\|_\infty\cdot\|F\|_\infty\right)\le\frac{1}{8}. \tag{5.3.27}$$

Then the following is true. Assume that $\upsilon_\nu=(u_\nu,\eta_\nu)\in C^\infty(\mathbb{R}\times\mathbb{S}^1,V)\times C^\infty(\mathbb{R},\mathbb{R})$ is a sequence of trajectories of the s-dependent gradient $\operatorname{grad}\mathcal{A}^{\chi_sF_s}$ and there are bounds $a,b\in\mathbb{R}$ such that

$$a\le\lim_{s\to-\infty}\mathcal{A}^{\chi_sF_s}(\upsilon_\nu(s))$$

and

$$\lim_{s\to\infty}\mathcal{A}^{\chi_sF_s}(\upsilon_\nu(s))\le b.$$

Then there is a subsequence, still denoted υ_ν, and a trajectory υ of $\operatorname{grad}\mathcal{A}^{\chi_sF_s}$ such that the subsequence $\upsilon_\nu=(u_\nu,\eta_\nu)$ converges to $\upsilon=(u,\eta)$ in the C^∞_{loc}-topology.

Proof. [CF09, Thm. 3.6]. □

Uniform bounds for multiplier paths η – contact type enters

For simplicity we only consider the case of trajectories of the s-independent gradient $\operatorname{grad}\mathcal{A}^F$. Moreover, we only sketch proofs; for details see [CF09, §3.1].

Remark 5.3.7 (Contact type of Σ enters)**.** (i) So far the geometric condition on Σ to be not just any bounding hypersurface in (V,λ), but a convex exact one,[22] has not been used. That will change now in order to obtain a uniform C^0 bound for the Lagrange multiplier components η of all trajectories $\upsilon=(u,\eta)$ subject to the same action bounds a and b as in (5.3.26).

(ii) Some condition on the energy surface Σ is indeed necessary, given that there are energy surface counterexamples Σ' in $\mathbb{R}^{2n}$ to the Hamiltonian Seifert conjecture; cf. (4.1.2). But even in the absence of closed characteristics the constant loops corresponding to the points of Σ are still critical points of $\mathcal{A}^F$ thereby giving rise to nontriviality $\mathrm{RFH}(\Sigma')\simeq\mathrm{H}(\Sigma;\mathbb{Z}_2)\neq 0$; cf. Exercise 5.3.17 and [CF09, Pf. of Cor. 1.5]. But this contradicts the Vanishing Theorem 5.0.4 since in $\mathbb{R}^{2n}$ any compact subset is displaceable.

Proposition 5.3.8. *Let $F\in\mathcal{F}(\Sigma)$ be a defining Hamiltonian. Then there are constants $\varepsilon>0$ and $c<\infty$ such that for pairs $(z,\tau)\in\mathcal{L}V\times\mathbb{R}$ it holds*

$$\left\|\operatorname{grad}\mathcal{A}^F(z,\tau)\right\|\le\varepsilon\quad\Rightarrow\quad|\tau|\le c\left(|\mathcal{A}^F(z,\tau)|+1\right).$$

[21] [CF09, Prop. 3.4] is a generalization of Proposition 5.3.8 to families $(F_\sigma,\Sigma_\sigma,\chi_\sigma)$.
[22] We assume contact type for Σ, otherwise, add an exact form to λ; cf. Exercise 4.5.14.

The proposition tells that near critical points the multiplier part of a pair $(z,\tau)\in\mathcal{L}V\times\mathbb{R}$ is bounded in terms of the pair action. So if (z,τ) is one element of a whole trajectory $(u,\eta):\mathbb{R}\to\mathcal{L}V\times\mathbb{R}$ subject to the bound (5.3.26), thus

$$|\mathcal{A}^F(u(s),\eta(s))|\le\kappa:=\max\{|a|,|b|\},\quad s\in\mathbb{R},\tag{5.3.28}$$

then the proposition provides a uniform bound on $|\eta(s)|$ for s near $\pm\infty$. So it remains to deal with $|\eta|$ along compact intervals in $\mathbb{R}$, one compact interval for each trajectory of the, generally non-compact, family under consideration.

Corollary 5.3.9 (Uniform η bound)**.** *Let $v=(u,\eta)\in C^\infty(\mathbb{R}\times\mathbb{S}^1,V)\times C^\infty(\mathbb{R},\mathbb{R})$ be a trajectory of* $\operatorname{grad}\mathcal{A}^F$ *along which the action remains in a compact interval, say in $[a,b]$; cf. (5.3.26). Then the L^∞-norm of η is bounded uniformly in terms of a constant c that depends on a,b, but not on v.*

Sketch of proof of Proposition 5.3.8. (For details see [CF09, Prop. 3.2].) The key input, due to Σ being both an energy surface $F^{-1}(0)$ *and* of contact type with respect to α, is the coupling $X_F=R_\alpha$ of Hamiltonian and Reeb dynamics along Σ; see (4.5.11). To illustrate the effect of the coupling note that for a critical point (z,τ) of $\mathcal{A}^F$

$$\mathcal{A}^F(z,\tau)=\int_0^1\Bigl(\underbrace{\lambda|_z(\dot z)}_{\alpha(\tau R_\alpha)\equiv\tau}-\tau\underbrace{F\circ z}_{0}\Bigr)dt=\tau.\tag{5.3.29}$$

This actually solves Exercise 5.1.8. Let's see how much of this identity survives for a general pair (u,η) whose only restriction is that the loop part u must stay in a small neighborhood U_δ of Σ.

I. *There are constants $\delta>0$ and $c_\delta<\infty$ with the following significance. For every pair $(z,\tau)\in\mathcal{L}V\times\mathbb{R}$ whose loop part z remains δ-near to Σ in the sense that $z(\mathbb{S}^1)\subset U_\delta:=F^{-1}(-\delta,\delta)$ the Lagrange multiplier satisfies the estimate*

$$|\tau|\le 2\left|\mathcal{A}^F(z,\tau)\right|+c_\delta\left\|\operatorname{grad}\mathcal{A}^F(z,\tau)\right\|.$$

The key step is to obtain the two constants. By compactness of $\Sigma=F^{-1}(0)$ and zero being a regular value of F the closure of U_δ is compact for sufficiently small $\delta>0$. Now choose $\delta>0$ smaller, if necessary, such that

$$\lambda_p\left(X_F(p)\right)\ge\tfrac12+\delta,\quad p\in U_\delta.$$

Such δ exists since $\lambda(X_F)=\alpha(R_\alpha)\equiv1$ along $\Sigma=F^{-1}(0)$: To see this use contact type, see (4.5.11), and the definition of R_α, see Exercise 4.2.2 (c). The constant $c_\delta:=2\|\lambda|_{U_\delta}\|_\infty$ is finite since U_δ is of compact closure. The desired estimate is a rather mild generalization of (5.3.29), see [CF09, p.264].

II. *For each $\delta>0$ there is a constant $\varepsilon=\varepsilon(\delta)>0$ such that whenever a pair (z,τ) satisfies $\|\operatorname{grad}\mathcal{A}^F(z,\tau)\|\le\varepsilon$, then the loop part z remains in U_δ.*

To show II one first analyses z in two cases, in each case forgetting one of the two components of $\|\operatorname{grad}\mathcal{A}^F(z,\tau)\|$; see (5.2.15). Excluding both cases yields II.

Case 1: There are times $t_0, t_1 \in \mathbb{S}^1$ with $|F(z(t_0))| \ge \delta$ and $|F(z(t_1))| \le \delta/2$. In this case by periodicity of z and continuity of F there are two points, again denoted by $t_0, t_1 \in \mathbb{S}^1$, with $t_0 < t_1$ and $|F \circ z(t)| \in [\frac{\delta}{2}, \delta]$ on $[t_0, t_1]$. Set $\mu := \max_{x \in \overline{U}_\delta, t \in \mathbb{S}^1} |\nabla^{J_t} F(x)|_{g_{J_t}}$ and forget component *two* in (5.2.15) to get[23]

$$\begin{aligned} \left\|\operatorname{grad} \mathcal{A}^F(z,\tau)\right\| &\ge \|\dot z - \tau X_F(z)\|_2 \\ &\ge \int_{t_0}^{t_1} |\dot z - \tau X_F(z)|\, dt \\ &\ge \dots \\ &\ge \frac{\delta}{2\mu}. \end{aligned}$$

The omitted steps, using e.g. $dF(X_F) = 0$, are detailled in [CF09, p.265].
Case 2: The loop z lives outside $U_{\delta/2}$. Forgetting component *one* in (5.2.15)[24]

$$\left\|\operatorname{grad} \mathcal{A}^F(z,\tau)\right\| \ge \Bigl|\int_0^1 F(z(t))\, dt\Bigr| = \int_0^1 \underbrace{|F(z(t))|}_{\ge \delta/2}\, dt \ge \frac{\delta}{2}.$$

To prove Step II pick any $\delta > 0$, set

$$\varepsilon := \frac{\delta}{4\max\{1,\mu\}} < \min\{\frac{\delta}{2}, \frac{\delta}{2\mu}\},$$

and assume $(z,\tau) \in \mathcal{L}V \times \mathbb{R}$ satisfies $\|\operatorname{grad} \mathcal{A}^F(z,\tau)\| \le \varepsilon$. Neither case 1 nor 2 applies to z. So z hits $U_{\delta/2}$ ($\neg$ case 2), but then it cannot leave U_δ ($\neg$ case 1).

III. *We prove the proposition.*

Choose the constants δ and $\varepsilon = \varepsilon(\delta)$ of Steps I and II, respectively, and set $c := \max\{2, c_\delta \varepsilon\}$. Suppose $\|\operatorname{grad} \mathcal{A}^F(z,\tau)\| \le \varepsilon$. Then by Step II the loop part z remains in U_δ, hence Step I applies and yields

$$\begin{aligned} |\tau| &\le 2\left|\mathcal{A}^F(z,\tau)\right| + c_\delta \left\|\operatorname{grad} \mathcal{A}^F(z,\tau)\right\| \\ &\le c\left|\mathcal{A}^F(z,\tau)\right| + c_\delta \varepsilon. \end{aligned}$$

This concludes the outline of the proof of Proposition 5.3.8. □

Proof of Corollary 5.3.9 (Trajectories). Pick $\varepsilon > 0$ as in Proposition 5.3.8.[25] Set $u_s := u(s,\cdot)$ and $\eta_s := \eta(s)$. The key tool is the quantity given for $\sigma \in \mathbb{R}$ by

$$\mathcal{T}_\sigma := \inf\{s \ge 0 \mid \|\operatorname{grad} \mathcal{A}^F((u,\eta)_{\sigma+s})\| < \varepsilon\}, \quad (u,\eta)_s := (u_s, \eta_s).$$

This quantity helps twice. Firstly, given a trajectory (u,η) and a time σ element $(u,\eta)_\sigma$ of gradient norm $\ge \varepsilon$ (otherwise $\mathcal{T}_\sigma = 0$), the function $\mathcal{T}_\sigma$ measures for

[23] Use that by Cauchy-Schwarz $\|f\|_2 = \|f\|_2\|1\|_1 \ge \langle f, 1\rangle_{L^2} = \|f\|_1$ for $f \in C^\infty(\mathbb{S}^1, \mathbb{R})$.
[24] To see the identity note that $F(z(t))$ is non-zero, hence will not change sign; cf. (5.4.43).
[25] By Step II above $\varepsilon = \varepsilon(\delta)$ given any sufficiently small constant $\delta > 0$.

how long the gradient norm will remain $\geq \varepsilon$, so for how long the trajectory will not get too close to a critical point. By assumption (5.3.26) we get the estimate

$$\begin{aligned}
b-a &\geq \lim_{s\to\infty} \mathcal{A}^F((u,\eta)_s) - \lim_{s\to-\infty} \mathcal{A}^F((u,\eta)_s) \\
&= \int_{-\infty}^{\infty} \|\operatorname{grad} \mathcal{A}^F(u,\eta)\|^2 \, ds \\
&\geq \int_{\sigma}^{\sigma+\mathcal{T}_\sigma} \underbrace{\|\operatorname{grad} \mathcal{A}^F(u,\eta)\|^2}_{\geq \varepsilon^2 \text{ on } (\sigma,\sigma+\mathcal{T}_\sigma)} \, ds \\
&\geq \mathcal{T}_\sigma \varepsilon^2
\end{aligned}$$

which holds true for every $\sigma \in \mathbb{R}$ and, by the way, also shows finiteness $\mathcal{T}_\sigma < \infty$. Secondly, for any $\sigma \in \mathbb{R}$ the gradient norm of the trajectory element $(u,\eta)_{\sigma+\mathcal{T}_\sigma}$ at time $\sigma + \mathcal{T}_\sigma$ is $\leq \varepsilon$. Hence the pair $(u,\eta)_{\sigma+\mathcal{T}_\sigma}$ satisfies the assumption of Proposition 5.3.8, so together with the action bound κ in (5.3.28) we get that

$$\begin{aligned}
|\eta(\sigma+\mathcal{T}_\sigma)| &\leq c\left(|\mathcal{A}^F((u,\eta))_{\sigma+\mathcal{T}_\sigma}| + 1\right) \\
&\leq c(\kappa+1) =: c_\kappa
\end{aligned}$$

for every $\sigma \in \mathbb{R}$. Putting things together one obtains the desired estimate

$$\begin{aligned}
|\eta(\sigma)| &= \left|\eta(\sigma+\mathcal{T}_\sigma) - \int_\sigma^{\sigma+\mathcal{T}_\sigma} \partial_s \eta(s)\, ds\right| \\
&\leq |\eta(\sigma+\mathcal{T}_\sigma)| + \int_\sigma^{\sigma+\mathcal{T}_\sigma} \underbrace{|\partial_s \eta(s)|}_{\leq \|F\|_\infty \int_0^1 dt} \, ds \\
&\leq c_\kappa + \mathcal{T}_\sigma \|F\|_\infty \\
&\leq c_\kappa + \frac{\|F\|_\infty (b-a)}{\varepsilon^2}
\end{aligned}$$

for every $\sigma \in \mathbb{R}$ using that $\partial_s \eta$ satisfies the gradient flow equation (5.2.17). □

5.3.2 Continuation

To prove invariance of Floer homology $\mathrm{HF}(\mathcal{A}^F)$ defined by (5.3.24) for a convex exact hypersurface Σ with regular defining Hamiltonian F, not only under change of the regular defining Hamiltonian, but even under convex exact deformations of Σ itself, see Theorem 5.0.2, suppose $\{F_s\}_{s\in[0,1]}$ is a smooth family of defining Hamiltonians of convex exact hypersurfaces Σ_s in (V,λ). The construction of natural continuation maps follows precisely the same steps as in Section 3.4.3, see [CF09, §3.2] for details,[26] once appropriate compactness

[26] The r-homotopy of s-homotopies $\overline{H}_s^{\bar{\chi},r}$ in [CF09, p. 276] should be $H^{\chi_{1-s}}_{r(1-s)}$, not $H^{\chi_{1-s}}_{1-rs}$, in order that for each r the initial point (at $s=0$) coincides with the endpoint (at $s=1$) of the s-homotopy $H_s^{\chi,r} := H_{rs}^{\chi_s}$ and so the two homotopies can be concatenated.

properties of the spaces of connecting cascade trajectories for the s-dependent gradient $\operatorname{grad} \mathcal{A}^{F_s}$ have been established.
The only problem is that condition (5.3.27) on smallness of the product $\|F\|_\infty \|\partial_s F\|_\infty$ might not be satisfied in general. A common technique is to carry out the homotopy $\{F_s\}_{s\in[0,1]}$ in N steps, namely sucessively via the homotopies

$$F^j_s := F_{\frac{j-1+s}{N}}, \quad s \in [0,1], \quad j = 1, \dots, N,$$

and show that the continuation map provided by each of them is an isomorphism. Since

$$\left\|\partial_s F^j_s\right\|_\infty = \frac{1}{N} \left\|\partial_s F_{\frac{j-1+\sigma}{N}}\right\|_\infty \le \frac{1}{N} \left\|\partial_s F\right\|_\infty$$

condition (5.3.27) is satisfied indeed for each homotopy F_j whenever N is chosen sufficiently large. But the composition of the continuation isomorphisms provided by each individual F^j is the continuation map, hence isomorphism, provided by F; cf. (3.4.48).

To show that $\mathrm{RFH}(\Sigma)$ is well defined by (5.3.25) requires to pick $F_0, F_1 \in \mathcal{U}^F_{\mathrm{reg}}$ as in Remark 5.3.4 and show that $\mathrm{HF}(\mathcal{A}^{F_0}) \simeq \mathrm{HF}(\mathcal{A}^{F_1})$ by continuation. For this it is sufficient, as mentioned above, to have a smooth family $\{F_s\}_{s\in[0,1]}$ of *defining Hamiltonians* of *convex exact* hypersurfaces Σ_s in (V, λ) interpolating F_0 and F_1. Note that such family is obtained simply by convex combination of F_0 and F_1; see Remark 5.1.14.

In [CFO10, Prop. 3.1] it is even shown independence on the unbounded component of $V \setminus \Sigma$, that is only $\Sigma = \partial M$ and its inside, the compact manifold-with-boundary M, are relevant for $\mathrm{HF}(\mathcal{A}^F)$. This leads to the notation $\mathrm{RFH}(\partial M, M)$ for $\mathrm{HF}(\mathcal{A}^F)$, often abbreviated by $\mathrm{RFH}(\Sigma)$; see Definition 5.0.3.

5.3.3 Grading

Suppose Σ is a convex exact hypersurface, say of restricted contact type, and $F \in \mathcal{F}(\Sigma)$ is defining. Throughout Section 5.3.3 suppose that

(i) the contact manifold (Σ, α) is simply-connected, that is $\pi_1(\Sigma) = 0$;

(ii) $\mathcal{A}^F : \mathcal{L}V \times \mathbb{R} \to \mathbb{R}$ is Morse-Bott;

(iii) $(V, d\lambda)$ has trivial first Chern class over $\pi_2(V)$, that is $\mathrm{I}_{c_1} = 0$;

(iv) $h : C \to \mathbb{R}$ is Morse on the critical manifold $C := \operatorname{Crit} \mathcal{A}^F \subset \mathcal{L}\Sigma \times \mathbb{R}$.

Under these conditions there exists an integer grading $\mu = \mu^{\mathrm{RS}}_\xi + \mathrm{ind}^\sigma_h$ taking values in $\frac{1}{2}\mathbb{Z}$, see (5.3.30), of the Rabinowitz-Floer complex in terms of the sum of the transverse Robbin-Salamon index of (rescaled) Reeb loops and the signature index of a critical point of the Morse function h. For non-degenerate Σ the transverse Robbin-Salamon index reduces to the transverse Conley-Zehnder index μ^{CZ}_ξ and μ will be half-integer valued.

Transverse Robbin-Salamon index

Pick a critical point (z,τ) of $\mathcal{A}^F$, i.e. $z:\mathbb{S}^1\to\Sigma$ satisfies $\dot z=\tau R_\alpha(z)$ by (5.1.3). In words, the loop $z:\mathbb{R}/\mathbb{Z}\to\Sigma$ integrates the rescaled Reeb vector field τR_α.[27] Recall that $\xi:=\ker\alpha\to\Sigma$ defines a – with respect to $d\alpha$ symplectic – vector bundle of rank $2n-2$ and that $T\Sigma=\xi\oplus\mathbb{R}R_\alpha$. By (i) pick a smooth extension $\bar z:\mathbb{D}\to\Sigma$ of the loop z and choose a unitary trivialization of the symplectic vector bundle $(\bar z^*\xi,\bar z^*d\alpha)$. Then the linearization of the flow generated by τR_α, informally called the **linearized rescaled Reeb flow**, provides along the trajectory $z:[0,1]\to\Sigma$ by (ii) a path in the symplectic linear group $\mathrm{Sp}(2n-2)$ with initial point $\mathbb{1}$ and Robbin-Salamon index denoted by

$$\mu_\xi^{\mathrm{RS}}(z,\tau)\in\tfrac{1}{2}\mathbb{Z}.$$

It is called the **transverse Robbin-Salamon index** of the critical point (z,τ) or, alternatively, the 1-periodic solution z of the rescaled Reeb vector field τR_α.

Exercise 5.3.10. Show that the definition of μ_ξ^{RS} is independent of the choice of, firstly, the extending disk by (iii) and, secondly, of the unitary trivialization.

Exercise 5.3.11. Calculate $\mu_\xi^{\mathrm{RS}}(z,0)$ for constant critical points $(z,0)$ of $\mathcal{A}^F$.

Transverse Conley-Zehnder index

Suppose a critical point (z,τ) of $\mathcal{A}^F$ is transverse non-degenerate;[28] cf. Definition 5.1.11 and Exercise 5.1.12. So the image $z(\mathbb{S}^1)$ is an isolated closed characteristic $P\cong\mathbb{S}^1$ of Σ. Moreover, the corresponding path in $\mathrm{Sp}(2n-2)$ ends away from the Maslov cycle, i.e. the path is an element of $\mathrm{SP}^*(2n-2)$, so the Robbin-Salamon index is nothing but the Conley-Zehnder index of this path. In order to implicitly signalling the assumption of transverse non-degeneracy, as opposed to just general Morse-Bott, we use the notation

$$\mu_\xi^{\mathrm{CZ}}(z,\tau)\in\tfrac{1}{2}+\mathbb{Z}$$

for transverse non-degenerate critical points and call this index the **transverse Conley-Zehnder index**.

Exercise 5.3.12 (Half-integers)**.** Show that μ_ξ^{CZ} is half-integer valued.

Signature index and Morse index

Let $f:N\to\mathbb{R}$ be a Morse-Bott function on a finite dimensional manifold N. Given a critical point x of f, recall that the **Morse index** $\mathrm{ind}_f(x)$ is the number of negative eigenvalues, with multiplicities, of the Hessian of f at x. The **signature index** of $x\in\mathrm{Crit}f$ is defined by

$$\mathrm{ind}_f^\sigma(x):=-\frac{1}{2}\mathrm{sign}\,\mathrm{Hess}_x f.$$

[27] Alternatively consider the corresponding τ-periodic Reeb path $r(t)=\vartheta_t p$ with $p=z(0)$ on the τ-dependent interval $[0,\tau]$; see Exercise 5.1.7.

[28] Call (z,τ) **transverse non-degenerate** iff $\tau\neq0$ and the corresponding Reeb loop is.

Exercise 5.3.13 (Morse-Bott)**.** Let $f : N \to \mathbb{R}$ be Morse-Bott. Show that[29]

$$\mathrm{ind}^\sigma_f(C_i) = \mathrm{ind}_f(C_i) - \tfrac{1}{2}\left(\dim N - \dim C_i\right)$$

for every connected component C_i of the critical manifold $C = \mathrm{Crit} f$.

Exercise 5.3.14 (Morse)**.** If $x \in \mathrm{Crit} f$ is a non-degenerate critical point, then

$$\mathrm{ind}^\sigma_f(x) = \mathrm{ind}_f(x) - \tfrac{1}{2}\dim N \in \begin{cases} \mathbb{Z} & , \dim N \text{ even}, \\ \tfrac{1}{2} + \mathbb{Z} & , \dim N \text{ odd}. \end{cases}$$

Morse-Bott grading of the Rabinowitz-Floer complex

Definition 5.3.15 (Grading)**.** For $c \in \mathrm{Crit}\, h \subset C = \mathrm{Crit}\, \mathcal{A}^F$, define

$$\mu(c) := \mu^{\mathrm{RS}}_\xi(c) + \mathrm{ind}^\sigma_h(c) \in \tfrac{1}{2}\mathbb{Z}. \tag{5.3.30}$$

Exercise 5.3.16 (Σ non-degenerate $\Rightarrow$ half-integers)**.** For non-degenerate Σ, show that $\mu(c) \in \frac{1}{2} + \mathbb{Z}$ is half-integer valued for every $c \in \mathrm{Crit}\, h$. Is this also true if $c = (z, 0)$ is a constant critical point?

Given $c_-, c_+ \in \mathrm{Crit}\, h$, consider a connecting cascade trajectory Γ, an element of the moduli space $\mathcal{M}_{c_- c_+}(\mathcal{A}^F, h, J, g)$ defined by (5.3.22). For generic J and g this space is a smooth manifold whose **local dimension**[30] at Γ is given by

$$\begin{aligned} \dim_\Gamma \mathcal{M}_{c_- c_+} &= \mu^{\mathrm{RS}}_\xi(c_+) + \mathrm{ind}^\sigma_h(c_+) - \left(\mu^{\mathrm{RS}}_\xi(c_-) + \mathrm{ind}^\sigma_h(c_-)\right) - 1 \\ &= \mu(c_+) - \mu(c_-) - 1. \end{aligned}$$

The first step is non-obvious even for the Morse-Bott cascade complex in finite dimensions; see [CF09, (65)]. One crucial ingredient to obtain the first step, cf. [CF09, (64)], is the following even less trivial formula. Let $\mathcal{M}$ be the moduli space of finite energy trajectories v of $\mathrm{grad}\, \mathcal{A}^F$. By Morse-Bott, firstly, the asymptotic limits $v^\mp$ in (5.3.21) exist, let $C_\mp$ be their components of C, and secondly the linearization $D\mathcal{F}_F(v)$ of the gradient equation (5.2.17) at a flow trajectory v, cf. (3.3.33), is Fredholm between suitable spaces. By [CF09, Prop. 4.1]

$$\begin{aligned} \dim_v \mathcal{M} &= \mathrm{index}\, D\mathcal{F}_F(v) + \dim C_+ + \dim C_- \\ &= \mu^{\mathrm{RS}}_\xi(v_+) - \mu^{\mathrm{RS}}_\xi(v_-) + \tfrac{\dim C_+ + \dim C_-}{2}. \end{aligned}$$

For further details concerning index computations see [CF09] and [MP11].

5.3.4 Relation to homology of Σ

Exercise 5.3.17. If Σ carries no Reeb loops that are contractible in V, then

$$\mathrm{RFH}_*(\Sigma) \simeq \mathrm{H}_*(\Sigma; \mathbb{Z}_2)$$

is a grading preserving isomorphism where $\mathrm{RFH}_*(\Sigma) := \mathrm{HF}_*(\mathcal{A}^F)$ for any defining Hamiltonian $F \in \mathcal{F}(\Sigma)$; see (5.0.1). Why should there be a grading without assuming triviality of $\pi_1(\Sigma)$ and I_{c_1}?

[29] Set $\mathrm{ind}_f(C_i) := \mathrm{ind}_f(x)$ and $\mathrm{ind}^\sigma_f(C_i) := \mathrm{ind}^\sigma_f(x)$ for some, hence any, $x \in C_i$.

[30] dimension of the component that contains Γ

5.3.5 Example: Unit cotangent bundle of spheres

Consider the unit cotangent bundle $\Sigma := S^*\mathbb{S}^n$ of the unit sphere $Q := \mathbb{S}^n \subset \mathbb{R}^{n+1}$ in euclidean space equipped with the induced Riemannian metric. Observe that Σ is a hypersurface of restricted contact type of the convex exact symplectic manifold $(V, \lambda) = (T^*\mathbb{S}^n, \lambda_{\mathrm{can}})$; see Example 4.5.19. In particular, the restriction $\alpha = \lambda_{\mathrm{can}}|\Sigma$ is a contact form on the hypersurface $\Sigma = S^*\mathbb{S}^n$ that bounds the (compact) unit disk cotangent bundle $M = D^*\mathbb{S}^n$.

Theorem 5.3.18 (Unit cotangent bundle of unit sphere, [CF09])**.** *For $n \geq 4$*

$$\mathrm{RFH}_k(S^*\mathbb{S}^n) = \begin{cases} \mathbb{Z}_2 & , k \in \left\{-n+\tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{1}{2}, n-\tfrac{1}{2}\right\} + \mathbb{Z}\cdot(2n-2), \\ 0 & , \textit{else.} \end{cases}$$

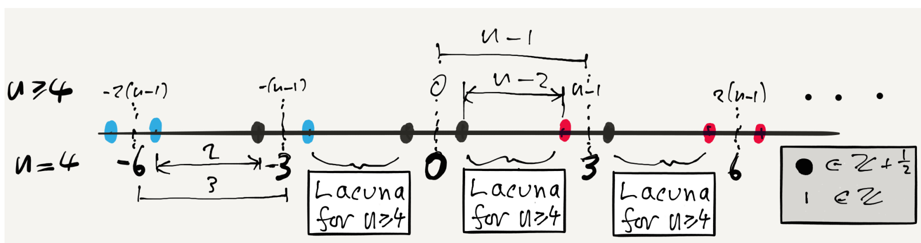

Figure 5.5: Indices $\color{cyan}\bullet$ $\bullet$ $\color{red}\bullet$ $\in \mathbb{Z}+\frac{1}{2}$ for which $\mathrm{RFH}_{\bullet\bullet\bullet}(S^*S^n) = \mathbb{Z}_2$, $n \geq 4$

Idea of proof (Lacunary principle). One exploits the facts that for the round metric on $\mathbb{S}^n$, suitably normalized, all geodesics are periodic with prime period 1, that the functional $\mathcal{A}^F$ is Morse-Bott,[31] and that the critical manifold $C = \operatorname{Crit}\mathcal{A}^F$ corresponds to $\mathbb{Z}$ copies of $S^*\mathbb{S}^n$ where $\mathbb{Z}$ labels the periods of the geodesics. Now one picks a Morse function h_0 on $S^*\mathbb{S}^n$ with precisely four critical points of Morse indices $0, n-1, n, 2n-1$ and defines h to be the Morse function on C which coincides with h_0 on each component. Then

$$\operatorname{Crit} h \cong \operatorname{Crit} h_0 \times \mathbb{Z}.$$

Now one pits action knowledge (5.1.9) in terms of the prime period and positivity of the action difference of two critical points c_- and c_+ sitting on the two ends of a non-constant connecting trajectory against the facts that the boundary operator decreases the grading μ_ξ^{RS} precisely by 1 and that μ_ξ^{RS} can be related via the Morse index theorem for geodesics to the particular Morse indices $0, n-1, n, 2n-1$ encountered among the critical points of h as mentioned above. In the end, for $n \geq 4$, one gets to the conclusion that there cannot be a connecting trajectory between critical points of μ_ξ^{RS}-index difference one. For details see [CF09, p. 293].

[31] Do not confuse with the stronger notion "transverse non-degenerate" in Theorem 5.1.10.

5.4 Perturbed Rabinowitz action $\mathcal{A}_H^F$

To prove the Vanishing Theorem 5.0.4 motivates to allow for more general Hamiltonians in the action functional; cf. [CF09, §3.3]. Pick a defining, thus autonomous, Hamiltonian $F \in \mathcal{F}(\Sigma)$ and a **young** cutoff function[32]

$$\chi : \mathbb{S}^1 \to [0, \infty), \qquad \operatorname{supp} \chi \subset (0, \tfrac{1}{2}), \qquad \int_0^1 \chi(t)\, dt = 1. \tag{5.4.31}$$

The assumption $\chi \geq 0$ is used in (5.4.43). Let us call the Hamiltonian

$$F^\chi := \chi F : \mathbb{S}^1 \times V \to \mathbb{R} \tag{5.4.32}$$

a **young Hamiltonian**, as its flow is active only in the first half $[0, \frac{1}{2}]$ of life. Suppose $H \in C^\infty(\mathbb{S}^1 \times V)$ is a possibly non-autonomous 1-periodic Hamiltonian. The **perturbed Rabinowitz action functional** on $\mathcal{L}V \times \mathbb{R}$ is defined by

$$\mathcal{A}_H^{F^\chi}(z, \tau) := \mathcal{A}^{F^\chi}(z, \tau) - \int_0^1 H_t(z(t))\, dt. \tag{5.4.33}$$

In the notation of (5.2.17) and taking the same choices, such as a cylindrical almost complex structure J with induced L^2 metric g_J on $\mathcal{L}V \times \mathbb{R}$, the upward gradient trajectories of $\operatorname{grad} \mathcal{A}_H^{F^\chi}$ are the solutions $\upsilon = (u, \eta)$ of the PDE

$$\partial_s \upsilon - \operatorname{grad} \mathcal{A}_H^{F^\chi}(\upsilon) = \begin{pmatrix} \partial_s u \\ \partial_s \eta \end{pmatrix} + \begin{pmatrix} J_t(u)\left(\partial_t u - \eta \chi X_F(u) - X_H(u)\right) \\ \int_0^1 \chi F(u)\, dt \end{pmatrix} = 0. \tag{5.4.34}$$

Compactness in Section 5.3.1, thus the definition of Floer homology in Section 5.3, goes through for $\mathcal{A}^{F^\chi}$ with only minor modifications; see [CF09, §3.1].

5.4.1 Proof of Vanishing Theorem

The Vanishing Theorem 5.0.4 asserts triviality $\mathrm{RFH}(\Sigma) = 0$ of Rabinowitz-Floer homology whenever Σ is displaceable.

Proof of Vanishing Theorem 5.0.4 – v1

The first version of the proof is short and rather illustrative, hence well suited to communicate the main ideas. But – it has the disadvantage that it involves a class of Hamiltonians for which we haven't introduced Floer homology $\mathrm{HF}(\mathcal{A}_{H^\rho}^{F^\chi})$; however, this is done in [AF10a]. In other words, in proof v1 the required technical work is moved elsewhere, so we are just left with the nice bits.

The idea is to find Hamiltonians such that there are no critical points

$$\operatorname{Crit} \mathcal{A}_{H^\rho}^{F^\chi} = \emptyset. \tag{5.4.35}$$

[32] The **support of χ**, notation $\operatorname{supp} \chi$, is the closure of its non-vanishing locus $\{\chi \neq 0\}$.

In this case there are no generators of the corresponding Floer complex. Hence

$$0 = \mathrm{HF}(\mathcal{A}^{F^\chi}_{H^\rho}) \simeq \mathrm{HF}(\mathcal{A}^{F^\chi}) \simeq \mathrm{HF}(\mathcal{A}^F) =: \mathrm{RFH}(\Sigma). \tag{5.4.36}$$

The isomorphisms are by continuation. Section 5.3 establishes $\mathrm{HF}(\mathcal{A}^F)$ only, but the same construction goes through for F^χ replacing F, including the construction of the second continuation isomorphism. The first continuation isomorphism and $\mathrm{HF}(\mathcal{A}^{F^\chi}_{H^\rho})$ itself are constructed in [AF10a, §2.3]; see Section 5.4.3.

In preparation of the proof of (5.4.35) we take the following choices, given a compactly supported Hamiltonian $H : [0,1] \times V \to \mathbb{R}$ that displaces Σ. By compactness of Σ the map ψ_1^H not only displaces Σ from itself, but also a small neighborhood U of Σ. Observe that Σ is contained in $\operatorname{supp} X_F$ for any of its defining Hamiltonians F, since 0 is a regular value of F. Change or modify F, if necessary, such that $\operatorname{supp} X_F \subset U$. Hence this support also gets displaced:

$$\operatorname{supp} X_F \cap \psi_1^H(\operatorname{supp} X_F) = \emptyset. \tag{5.4.37}$$

Now pick an **elderly** cutoff function

$$\rho : [0,1] \to [0,1], \qquad \rho|_{[0,1/2]} \equiv 0, \qquad \rho \equiv 1 \text{ near } t = 1, \tag{5.4.38}$$

as illustrated by Figure 5.6 and consider the **elderly Hamiltonian**, defined by

$$H^\rho = H^\rho_t := \dot\rho(t) H_{\rho(t)} \in C^\infty(V), \quad t \in \mathbb{S}^1. \tag{5.4.39}$$

Elderly means that the flow of H^ρ is active only in the second half $[\frac{1}{2}, 1]$ of life.

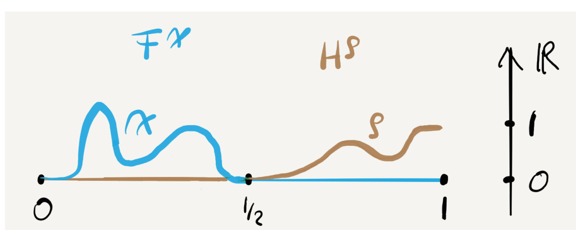

Figure 5.6: Young and elderly cutoff functions and Hamiltonians

Exercise 5.4.1. Show a) $\psi_t^{H^\rho} = \psi^H_{\rho(t)}$ for $t \in \mathbb{R}$ and b) $\psi_1^{\tau F^\chi} = \psi^F_\tau$ for $\tau \in \mathbb{R}$. [Hint: a) Footnote.[33] b) Note that $\tau F^\chi = F^{\tau\chi} = \dot\rho F$ with $\rho(t) := \tau \int_0^t \chi(\sigma)\, d\sigma$.]

The critical points of $\mathcal{A}^{F^\chi}_{H^\rho}$ are the solutions $(z, \tau) \in \mathcal{L}V \times \mathbb{R}$ of the equations

$$\begin{cases} \dot z(t) = \tau\chi(t) X_F(z(t)) + \dot\rho(t) X_{H_{\rho(t)}}(z(t)) & , t \in \mathbb{S}^1, \\ \int_0^1 \chi(t) F(z(t))\, dt = 0 & . \end{cases} \tag{5.4.40}$$

[33] $\frac{d}{dt}\psi^H_{\rho(t)} = \frac{d\,\psi^H_{\rho(t)}}{d\,\rho(t)}\dot\rho(t) = X_{H^\rho} \circ \psi^H_{\rho(t)}$

It remains to prove emptyness (5.4.35). By contradiction suppose that (z,τ) is a solution of (5.4.40). There are two cases.

I. $z(0) \notin \operatorname{supp} X_F$. On $[0,\frac{1}{2}]$ the first equation in (5.4.40) describes a reparametrization of the Hamiltonian flow of F, so F is preserved. Thus

$$F \circ z|_{[0,\frac{1}{2}]} \equiv F(z(0)) =: c \neq 0$$

is non-zero since by assumption $z(0) \notin \operatorname{supp} X_F \supset \Sigma = F^{-1}(0)$. Thus

$$\int_0^1 \chi F(z)\, dt = \int_0^{\frac{1}{2}} \chi F(z)\, dt = c \int_1^{\frac{1}{2}} \chi\, dt = c \neq 0$$

as $\operatorname{supp} \chi$ lies in $[0,\frac{1}{2}]$ where $F \circ z \equiv c$. This contradicts equation two in (5.4.40).

II. $z(0) \in \operatorname{supp} X_F$. As X_{F^χ} is young and X_{H^ρ} is elderly, these two vector fields are supported in disjoint time intervals. Thus the flow of their sum is the composition of their individual flows. Together with Exercise 5.4.1 we get that

$$\begin{aligned}
\overbrace{\operatorname{supp} X_F \ni z(0)}^{\text{hypothesis}} &= z(1) \\
&= \psi_1^{H^\rho} \circ \psi_1^{\tau F^\chi} z(0) \\
&= \psi_1^H \circ \underbrace{\psi_\tau^F z(0)}_{\text{hyp.} \Rightarrow \in \operatorname{supp} X_F} .
\end{aligned}$$

But this is impossible since ψ_1^H displaces $\operatorname{supp} X_F$ by (5.4.37). Contradiction.

Proof of Vanishing Theorem 5.0.4 – v2

This version of proof gets away with Floer homology as introduced in Section 5.3 and is based on continuation and the following stronger version of the absence (5.4.35) of critical points of the perturbed action $\mathcal{A}_{H^\rho}^{F^\chi}$ associated to the young Hamiltonians F^χ in (5.4.32) and the elderly ones H^ρ in (5.4.39).

Lemma 5.4.2 (No critical points)**.** *There is a constant $\gamma = \gamma(J) > 0$ such that*

$$\left\|\operatorname{grad} \mathcal{A}_{H^\rho}^{F^\chi}(z,\tau)\right\| \geq \gamma$$

for every $(z,\tau) \in \mathcal{L}V \times \mathbb{R}$.[34]

Fix a smooth monotone cutoff function $\beta : \mathbb{R} \to [0,1]$ with $\beta(s) = 1$ for $s \geq 1$ and $\beta(s) = 0$ for $s \leq -1$ in order to define a homotopy $H.$, and its reverse $\overline{H}.$, between the zero Hamiltonian 0 and the Hamiltonian $H^\rho = H_t^\rho$, namely

$$H_s := \beta(s)\, H^\rho, \qquad \overline{H}_s := (1-\beta(s))\, H^\rho, \qquad s \in \mathbb{R}.$$

[34] Both grad and $\|\cdot\|$ depend on J; see Exercise 5.2.2.

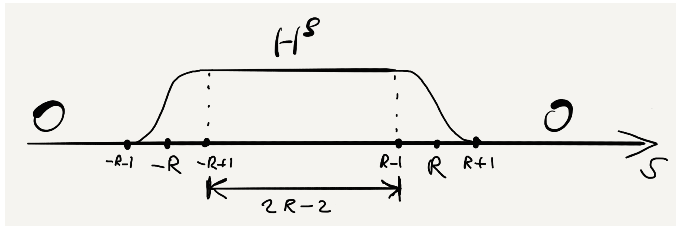

Figure 5.7: Concatenation homotopy $s \mapsto \bar{H}_s \#_R H_s$ from 0 over H^ρ back to 0

For each real parameter value $R \geq 1$ consider the homotopy $\overline{H}.\#_R H.$ from the zero Hamiltonian 0 back to itself which is defined by the concatenation

$$\mathbb{R} \ni s \mapsto \overline{H}_s \#_R H_s := \begin{cases} H_{s+R} & , s \leq 0, \\ \overline{H}_{s-R} & , s \geq 0, \end{cases}$$

of $H.$ followed by $\overline{H}.$ as illustrated by Figure 5.7.

Now consider the homotopy, in $r \in [0,1]$, of homotopies of Hamiltonians $s \mapsto F^\chi + r\overline{H}_s \#_R r H_s$ and their corresponding action functionals

$$\mathcal{A}_{r,s} := \mathcal{A}^{F^\chi}_{r\overline{H}_s \#_R r H_s}.$$

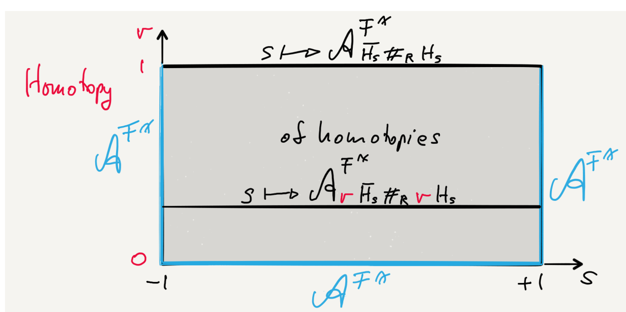

Figure 5.8: Homotopy, in r, of homotopies $s \mapsto \mathcal{A}_{s,r}$, each from $\mathcal{A}^{F^\chi}$ to $\mathcal{A}^{F^\chi}$

The homotopies at $r = 0$ and $r = 1$ have special properties. The one at $r = 0$ is constant, given by $s \mapsto \mathcal{A}^{F^\chi}$, and the one at $r = 1$ has – as a consequence of the no-critical-points Lemma 5.4.2 – no connecting flow lines:

Lemma 5.4.3 (No finite energy trajectories)**.** *There is a constant R_0 depending only on F^χ, $\|H^\rho\|_\infty$, and the action values $\mathcal{A}^{F^\chi}(v^\pm)$ of two fixed critical points such that the following is true. For each real $R \geq R_0$ there are no trajectories of the s-dependent gradient* $\operatorname{grad} \mathcal{A}_{1,s}$ *converging asymptotically to $v^\pm$.*

Lemma 5.4.4 (Uniform C^0 bound)**.** *Pick $R \geq 1$ and assume that $v(s) = (u(s), \eta(s))$ is a trajectory of* $\operatorname{grad} \mathcal{A}_{r,s}$*, for some $r \in [0,1]$, converging asymptotically to $v^\pm = (u^\pm, \eta^\pm) \in \operatorname{Crit} \mathcal{A}^{F^\chi}$. Then $\eta(s)$ is uniformly bounded by a constant that depends only on F^χ, $\|H^\rho\|_\infty$, R, and the action values of $v^\pm$.*

Lemma 5.4.4 implies compactness of the relevant components of the moduli spaces appearing in the definition (3.4.47) of the continuation homomorphisms

$$\Psi^r = [\psi(\overline{H}.\#_R H.)] : \mathrm{HF}(\mathcal{A}^{F^\chi}) \to \mathrm{HF}(\mathcal{A}^{F^\chi}), \quad r \in [0,1].$$

But the Ψ^r are all equal, as the homotopies of any two are homotopic, cf. Exercise 3.4.17, and $\Psi^0 = \mathbb{1}$, because it is induced by the constant homotopy, cf. Exercise 3.4.15. But $\Psi^1 = 0$ since by Lemma 5.4.3 there are just no connecting trajectories of $\operatorname{grad} \mathcal{A}^{F^\chi}_{\overline{H}_s \#_R H_s}$ whenever $R \geq R_0$. Thus we arrive at the conclusion

$$\mathbb{1} = \Psi^0 = \Psi^1 = 0 : \mathrm{HF}(\mathcal{A}^{F^\chi}) \to \mathrm{HF}(\mathcal{A}^{F^\chi}). \tag{5.4.41}$$

But this is only possible if the domain $\mathrm{HF}(\mathcal{A}^{F^\chi}) = 0$ is trivial which is what is claimed by the Vanishing Theorem 5.0.4. Full details are given in [CF09, §3.3].

Let us look at the ideas, at least, how to prove the three lemmas. For details see [CF09, p. 281-284].

No critical points. The proof of Lemma 5.4.2 takes three steps and uses compactness of Σ, that H displaces Σ and that F^χ and H^ρ are young and elderly, respectively. Furthermore, it is crucial that $\operatorname{grad} \mathcal{A}$ has two components, see (5.2.14), whose norms can be played out, one against the other one.

Step 1: There is a constant $\varepsilon_1(J)$ such that if $(z, \tau) \in \mathcal{L}V \times \mathbb{R}$ satisfies

$$\left\| \partial_t z - \tau \chi X_F(z) - \dot{\rho} X_{H_\rho}(z) \right\|_2 \leq \varepsilon_1 \tag{5.4.42}$$

then

$$\left(z(0), z(\tfrac{1}{2})\right) \notin \operatorname{supp} X_F \times \operatorname{supp} X_F.$$

In words, for such loops z not both, birth and midlife crisis, can happen under F-action, at least one of them requires a rest from the defining Hamiltonian F. The proof of this uses that Σ is compact and is displaced by H^ρ only at the end of life $t = 1$ where the young F^χ is not supported.

Step 2: There are $\varepsilon_2, \delta > 0$ such that if (z, τ) satisfies (5.4.42) for ε_2, then

$$|F \circ z| \geq \frac{\delta}{2}, \qquad \text{on } [0, \tfrac{1}{2}].$$

That is F stays away from zero during the first part of life of any such loop z. The proof exploits that H^ρ is elderly and uses Step 1.

Step 3: Pick $(z, \tau) \in \mathcal{L}V \times \mathbb{R}$. To prove that $\left\|\operatorname{grad} \mathcal{A}^{F^\chi}_{H^\rho}(z, \tau)\right\| \geq \gamma =: \min\{\varepsilon_2, \delta/2\}$ involves two cases in each of which one simply forgets one of the two gradient components, cf. (5.2.15).

If $I := \|\partial_t z - \tau\chi X_F(z) - \dot\rho X_{H_\rho}(z)\|_2 \ge \gamma$, we are done: Just forget component two. Otherwise, if $I < \gamma$, Step 2 applies. Forget component one to obtain that

$$\begin{aligned}\left\|\operatorname{grad}\mathcal{A}^{F^\chi}_{H^\rho}(z,\tau)\right\| &\ge |\operatorname{mean}(F^\chi\circ z)|\\ &= \left|\int_0^{\frac{1}{2}} \underbrace{\chi}_{\ge 0}\cdot\underbrace{F(z)}_{\ne 0}\,dt\right|\\ &= \int_0^{\frac{1}{2}} \chi\cdot\underbrace{|F(z)|}_{\ge\delta/2}\,dt \ge \frac{\delta}{2}\ge\gamma.\end{aligned} \tag{5.4.43}$$

Here we used that χ is young and integrates to one. Concerning the second identity use the non-negativity assumption $\chi \ge 0$ in (5.4.31) together with the fact that by Step 2 the function $F\circ z\ne 0$ is non-zero along the interval $[0,\frac{1}{2}]$, in fact, it is either $\ge\frac{\delta}{2}$ or $\le -\frac{\delta}{2}$.

Exercise 5.4.5. It seems to be an *open question*, at least we couldn't localize a proof in the literature, if the conclusion $\|\operatorname{grad}\mathcal{A}^{F^\chi}_{H^\rho}(z,\tau)\|\ge\frac{\delta}{2}$ in (5.4.43) remains valid for real-valued cut-off functions $\chi:\mathbb{S}^1\to\mathbb{R}$ in (5.4.31).

No finite energy trajectories. This is essentially an integrated version of Lemma 5.4.2. In the s-independent case the proof is trivial: Any connecting trajectory (u,η) has finite energy $E(u,\eta) := \int_{-\infty}^{\infty}\|\operatorname{grad}\mathcal{A}(u,\eta)\|^2\,ds$, namely, it is given by the action difference of the asymptotic limits. But a positive lower gradient bound as in Lemma 5.4.2 contradicts finiteness, so such trajectories cannot exist. The s-dependent case is rather similar; see [CF09, p.283, Step 1].

Uniform bound. The proof is similar to the one of the uniform bound on $|\eta|$ in Section 5.3.1, again built on the quantity $\mathcal{T}_\sigma$. That H displaces Σ is not used.

5.4.2 Weinstein conjecture

Theorem 5.4.6. *A displaceable convex exact hypersurface Σ in a convex exact symplectic manifold (V,λ) carries a closed Reeb loop that is contractible in V.*

Proof. Vanishing Theorem 5.0.4 and Exercise 5.3.17. □

The theorem was established in [Sch06] for stably displaceable hypersurfaces of contact type. Recall that if a closed contact type hypersurface $\Sigma\subset(\mathbb{R}^{2n},\omega_0)$ is simply-connected, then it is of restricted contact type by Exercise 4.5.1.

Exercise 5.4.7. Consider a connected closed hypersurface $\Sigma\subset(\mathbb{R}^{2n},\omega_0)$.
a) Show that Σ is bounding and displaceable.
b) Show that if Σ is, in addition, transverse to some global[35] Liouville vector field Y, then it is convex exact.

[35] A locally near Σ defined Y is sufficient whenever $\pi_1(\Sigma)=0$; cf. Exercise 4.5.1 b).

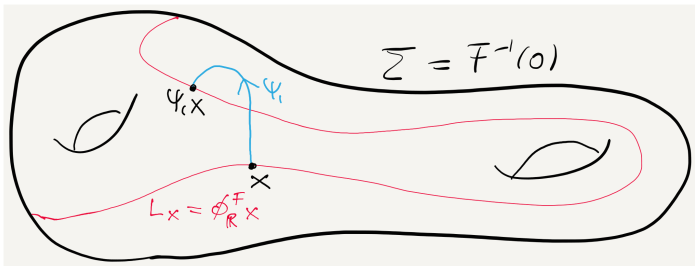

Figure 5.9: Leaf-wise intersection $\psi_1 x \in L_x$:
Satellite at x deviates under sudden comete influence $\psi_t = \psi_t^{H_t}$,
but happens to end up on its unperturbed trajectory $\phi_{\mathbb{R}}^F x$ afterwards

5.4.3 Leaf-wise intersections

Before giving the formal definition of "leaf-wise intersection" let us first switch on light by looking at the **circular restricted three body problem** in celestial mechanics; cf. [AM78, Ch. 10].

Motivation: Satellite perturbed by comet

Following [AF12a] consider an almost massless particle, the satellite s, moving in the gravitational field of two huge massive bodies called primaries, say earth E and moon M. By assumption the whole system is restricted to a given fixed plane and each of the primaries moves along a circle about their common center of mass. So the configuration space of the system is rather restricted. Now suppose the satellite, so far moving peacefully on its energy surface $\Sigma = F^{-1}(0)$ in phase space, gets temporarily influenced by a comet passing by during a time interval, say of length one. Suppose the previously autonomously via $\phi = \phi^F$ on Σ moving satellite receives extra kinetic energy by gravitational attraction as the comet appears in front at time zero and looses energy when the comet disappears behind at time one. The comet's presence is described by a time-dependent Hamiltonian H with Hamiltonian flow $\psi = \psi^H$. In other words, at time zero, say at the phase space location $x \in \Sigma$, the satellite gets lifted off of Σ following ψ for one unit of time after which it gets dropped back onto Σ at $\psi_1 x$. Let $L_x := \phi_{\mathbb{R}} x \subset \Sigma$, called the **leaf of $\boldsymbol{x}$**, denote the whole phase space trajectory of the satellite as it would happen without the comet's appearance. One would probably not expect that the satellite will get dropped back at $\psi_1 x$ to its original trajectory L_x. However if it happens, the unexpected penomenon

$$\psi_1 x \in L_x, \qquad L_x := \phi_{\mathbb{R}} x \subset \Sigma,$$

is called a **leaf-wise intersection**, see Figure 5.9, and x is called a **leaf-wise intersection point** (LIP).

Surprisingly, the phenomenon indeed happens: In certain hypersurfaces of exact symplectic manifolds, for instance such of non-zero Rabinowitz-Floer homology, *for any* comet there is by [AF10a, Thm. C] a satellite position x which ends up on its own unperturbed trajectory afterwards. In certain cotangent bundle situations there are, for generic comets, even infinitely many undestroyable satellite trajectories by [AF10b]; see also [AF12a, Thm. 1] and §2.2 in the survey [AF12b].

Rabinowitz-Floer homology for perturbed action $\mathcal{A}_H^{F\chi}$

Consider the Rabinowitz action functional $\mathcal{A}_H^{F\chi}$ in (5.4.33) for possibly non-autonomous **elderly Hamiltonians**

$$H \in \mathcal{H}^\dagger := \{H \in C^\infty(\mathbb{S}^1 \times V) \mid H_t = 0 \text{ for } t \in [0, \tfrac{1}{2}]\}$$

also called **elderly perturbations** of $\mathcal{A}^{F\chi}$. The perturbed functional $\mathcal{A}_H^{F\chi}$ has a number of useful properties: As time-dependence is allowed for H, the functional $\mathcal{A}_H^{F\chi}$ is Morse for generic $H \in \mathcal{H}^\dagger$, as shown in [AF10a, Thm. 2.13]. Thus no Morse-Bott complex will be needed at all. The critical points of $\mathcal{A}_H^{F\chi}$ correspond to leaf-wise intersections.[36]

Proposition 5.4.8 (Critical points and LIPs, [AF10a])**.** *If $(z, \tau) \in \operatorname{Crit} \mathcal{A}_H^{F\chi}$, then $x := z(\frac{1}{2})$ lies in $\Sigma = F^{-1}(0)$ and $\psi_1^H x$ lies on the Reeb leaf $L_x = \phi_{\mathbb{R}}^F x$.*

The definition of Floer homology $\mathrm{HF}(\mathcal{A}_H^{F\chi})$ proceeds pretty much as in Chapter 3 with the little extra twist of an upward finiteness condition, completely analogous to (5.3.20), that takes care of infinitely many critical points. More precisely, given a defining Hamiltonian $F \in \mathcal{F}(\Sigma)$, pick a young cutoff function χ as in (5.4.31) and a generic elderly perturbation $H \in \mathcal{H}^\dagger$ such that $\mathcal{A}_H^{F\chi}$ is Morse. Let $\mathrm{CF}(\mathcal{A}_H^{F\chi})$ be the $\mathbb{Z}_2$ vector space that consists of all formal sums

$$\xi = \sum_{c \in \operatorname{Crit} \mathcal{A}_H^{F\chi}} \xi_c c$$

such that, given ξ, for each $\kappa \in \mathbb{R}$ there is only a finite number of non-zero $\mathbb{Z}_2$-coefficients ξ_c that belong to critical points c of action $\geq \kappa$; cf. (5.3.20). Let $\mathcal{M}(c_+, c_-)$ be the space of connecting (upward) gradient trajectories, that is solutions $\upsilon = (u, \tau)$ of the PDE (5.4.34), with asymptotic boundary conditions $c_\pm \in \operatorname{Crit} \mathcal{A}_H^{F\chi}$ sitting at $s = \pm\infty$; cf. Figure 5.3. For generic $\mathbb{S}^1$-families of cylindrical almost complex structures J_t the space $\mathcal{M}(c_+, c_-)$ is a smooth finite dimensional manifold that carries a free $\mathbb{R}$-action by s-shift. Let

$$n(c_-, c_+) := \#_2(m_{c_- c_+}), \qquad m_{c_- c_+} := \mathcal{M}(c_+, c_-)/\mathbb{R},$$

be the number (mod 2) of zero-dimensional components of the moduli space of connecting flow lines; cf. (3.4.41). Define the Floer boundary operator on the chain groups $\mathrm{CF}(\mathcal{A}_H^{F\chi})$ analogous to (5.3.23) by linear extension of

$$\partial c_+ := \sum_{c_- \in \operatorname{Crit} \mathcal{A}_H^{F\chi}} n(c_-, c_+) c_-$$

[36] The map $\operatorname{Crit} \to \{\mathrm{LIP}s\}$, $(z, \tau) \mapsto z(\frac{1}{2})$, is injective, unless $L_x \cong \mathbb{S}^1$ for some LIP x.

for $c_+ \in \operatorname{Crit} \mathcal{A}_H^{F^\times}$; cf. Figure 5.3 and Remark 5.2.3.

By definition **Floer homology of the perturbed Rabinowitz action functional** $\mathcal{A}_H^{F^\times}$ is the homology of this chain complex, namely

$$\operatorname{HF}(\mathcal{A}_H^{F^\times}) := \frac{\ker \partial}{\operatorname{im} \partial}.$$

Theorem 5.4.9 (Invariance of RFH under elderly perturbations, [AF10a]). *If $\mathcal{A}_H^{F^\times}$ is Morse for an elderly perturbation $H \in \mathcal{H}^\dagger$, then*

$$\operatorname{HF}(\mathcal{A}_H^{F^\times}) \simeq \operatorname{HF}(\mathcal{A}_0^{F^\times}) = \operatorname{RFH}(\Sigma).$$

The theorem actually concludes the proof of version one of the Vanishing Theorem 5.0.4; cf. (5.4.36).

Remark 5.4.10 (Rabinowitz Floer for Reeb chords and Voyager missions). Coming back to the previous space travel motivation, there is practical interest in so-called consecutive collision orbits – of course, in small perturbations of them. In [FZ17] consecutive collision orbits, interpreted as Reeb chords, are encoded as critical points of an adequate version of the Rabinowitz action functional. Calculation of the corresponding Rabinowitz Floer homology then leads to infinitely many collision orbits.

The general picture: Coisotropic intersections

The previous situation of a closed codimension one submanifold $\Sigma = F^{-1}(0)$ being foliated by flow lines, that is 1-dimensional leaves L_x, is the rather special case $r = 1$ of the general leaf-wise intersection problem described in the masterpiece [Mos78]. It is amazing to see how important results fall off as special cases for particular values of the codimension r of a closed coisotropic submanifold Σ of a simply-connected exact symplectic manifold (V, λ) of dimension $2n$; see the presentation in [Mos78] of consequences 1.–4. of the main theorem that asserts existence of leaf-wise intersections.

A codimension r submanifold Σ of a symplectic manifold of dimension $2n$ is called a **coisotropic submanifold** if every tangent space of Σ is a coisotropic subspace of the corresponding tangent space of V. This implies $r \leq n$. The collection of symplectic complements $(T_p\Sigma)^\omega$ turns out to provide an integrable distribution of rank $r = \dim(T_p\Sigma)^\omega$ in the tangent bundle $T\Sigma$. Thus by Frobenius the $2n - r$ dimensional manifold Σ is foliated by leaves of dimension r.

For $r = 0$ the main theorem in [Mos78] proves existence of at least two fixed points of a symplectic diffeomorphism of a closed simply-connected symplectic manifold, see Remark 1.0.4; the assumption of being simply-connected was removed in [Ban80].

For $r = 1$ one gets to the previously described situation of integral curves of the characteristic line bundle of an energy surface, hence to Reeb dynamics if Σ is of contact type.

For $r = n$ one recovers the Lagrangian intersection problem.

5.5 Symplectic homology and loop spaces

Running out of time and pages, let us just briefly mention that, given a closed Riemannian manifold Q, Rabinowitz-Floer homology of the unit sphere cotangent bundle $\Sigma = S^*Q$, bounding the unit disk cotangent bundle $M = D^*Q$, in the cotangent bundle $(V, \lambda) = (T^*Q, \lambda_{\mathrm{can}})$ encodes both homology and cohomology of the free loop space.[37] It was shown in [CFO10, Thm. 1.10] that

$$\mathrm{RFH}_{*'}(S^*Q) \simeq \begin{cases} \mathrm{H}_{*'}(\mathcal{L}Q) & , *' > 1, \\ \mathrm{H}^{-*'+1}(\mathcal{L}Q) & , *' < 0, \end{cases} \tag{5.5.44}$$

and that for $*' = 0, 1$ there are isomorphisms involving the Euler class of the vector bundle $T^*Q \to Q$. Now the grading $*'$ of RFH is different from the half-integer grading $*$ defined earlier in (5.3.30), namely

$$*' := \begin{cases} * + \frac{1}{2} & , \text{on generators } c \text{ of positive action } \mathcal{A}^F(c) > 0, \\ * - \frac{1}{2} & , \text{on generators } c \text{ of negative action } \mathcal{A}^F(c) < 0. \end{cases}$$

The isomorphism is obtained by relating via a long exact sequence Rabinowitz-Floer homology to symplectic homology and cohomology of D^*Q, aka Floer co/homology of the cotangent bundle, and then use the isomorphism (3.5.56). A proof of (5.5.44) by a direct construction is given in [AS09] and a generalization to twisted cotangent bundles in [Mer11]; see [BF11] for an alternative method.

[37] Recall that in the present text we work with $\mathbb{Z}_2$ coefficients; for field coefficients see the convention prior to Thm. 1.10 in [CFO10].

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