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THE REGULARIZED FREE FALL
II. HOMOLOGY COMPUTATION VIA HEAT FLOW

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Abstract. In [1] Barutello, Ortega, and Verzini introduced a non-local functional which regularizes the free fall. This functional has a critical point at infinity and therefore does not satisfy the Palais–Smale condition. In this article we study the L^2 gradient flow which gives rise to a non-local heat flow. We construct a rich cascade Morse chain complex which has one generator in each degree $k \geq 1$. Calculation reveals a rather poor Morse homology having just one generator. In particular, there must be a wealth of solutions of the heat flow equation. These can be interpreted as solutions of the Schrödinger equation after a Wick rotation.

1. Introduction

The free fall describes the motion of a particle on a line in the gravitational field of a heavy body. The particle will after some time collide with the heavy body. However, collisions can be regularized so that after collision the particle bounces back. An interesting new approach for regularizing collisions was discovered in the recent paper [1] by Barutello, Ortega, and Verzini. Change of time gives rise to a *delayed*, that is non-local, regularized functional $\mathcal{B}$ with an intriguing mathematical structure.

In fact, there are two non-local functionals describing the free fall, namely, a Lagrangian version $\mathcal{B}$, defined in (2.1) below, and a Hamiltonian version $\mathcal{A}_{\mathcal{H}}$. The two functionals are Morse–Bott and related to each other by a non-local

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Legendre transform as studied in [5]. In the present article we compute the Morse homology of the Lagrangian version $\mathcal{B}$ with respect to an L^2 metric. This is a non-local analogue of the heat flow Morse homology of the second author [8], [9], [11]. We overcome the additional present difficulty of Morse–Bott, as opposed to Morse, by using the first authors cascade Morse complex [4], see also [2].

The significance of the free fall lies in the fact that it is the starting point of the exploration of more complicated systems like the Helium problem which is an active topic of research of the first named author with Cieliebak and Volkov [3].

In this paper we introduce the heat flow homology for the Lagrangian functional $\mathcal{B}$ of the free fall and compute it. It turns out that there is a rich interplay between critical points and gradient flow lines. Although the chain groups are infinite dimensional it turns out that in sharp contrast the heat flow Morse homology for the free fall is extremely meager, it is actually concentrated in degree 1. In particular, by the contrast principle "large chain complex – low homology" many solutions of the heat flow must exist.

To be more precise since the functional $\mathcal{B}$ is not Morse, but only Morse–Bott, one has to modify standard Morse homology. We shall choose cascade Morse homology, established by the first author in his PhD thesis, see [4]. because it continues to work with the original gradient flow of $\mathcal{B}$; this is not the case if one perturbs $\mathcal{B}$ and does Morse homology with a nearby Morse functional $\widetilde{\mathcal{B}}$.

THEOREM 1.1. *The cascade Morse homology of the Morse–Bott functional $\mathcal{B}$ is*

$$\mathrm{HM}_*(\mathcal{B};\mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & \text{if } * = 1, \\ 0 & \text{else.} \end{cases}$$

For the proof see Proposition 7.1.

An interesting aspect of the heat flow equation is that after applying a Wick rotation, that is considering imaginary time, one obtains a solution of the Schrödinger equation.

In two planned future articles III and IV we intend to study the Hamiltonian analogue of the heat flow homology in order to obtain a non-local Floer homology and relate the two by an adiabatic limit in the spirit of [7]. In the first step of this project, article I, we proved [5] that the Fredholm indices in both theories agree. The gradient flow equations in the Hamiltonian theory are non-local perturbed holomorphic curve equations which after a Wick rotation become solutions of a transport equation and hence solve a wave equation.

Theorem 1.1 might also be interpreted that the Morse homology of the functional $\mathcal{B}$ computes the homology of a Conley pair (N, L) where N is the domain of $\mathcal{B}$ and $L := \{\mathcal{B} < C_1\}$ is the sub-level set corresponding to the lowest critical value. It is therefore conceivable that Theorem 1.1 can be proved by an infinite dimensional Conley index argument as in [10]; for a short overview see [10].

In the present paper we follow a different approach by arguing directly with the Morse complex without reference to the topology of the underlying space. However, we do not provide a direct existence proof of the heat flow gradient flow lines. We deduce their existence with some tricks. The crucial observation is that if one fixes the asymptotics then the heat flow gradient flow lines lie in some finite dimensional subspaces. This allows us to deduce the existence of gradient flow lines by considering the finite dimensional Morse homology of the restriction of the action functional to the finite dimensional subspaces. Now the crucial step is to chose the auxiliary Morse function on the critical manifold in such a way that the Morse–Smale condition holds simultaneously on the full space as well as on the finite dimensional subspaces.

Although triviality of the negative bundles $\mathcal{V}^-C_k$ over the critical manifolds C_k is not used in the present article (since we use $\mathbb{Z}_2$ coefficients only), triviality is relevant for $\mathbb{Z}$ coefficients and this is why we include the proof as an appendix. In the appendix we also include the short argument, implicit in [5], that the functional $\mathcal{B}$ is indeed Morse–Bott of nullity 1.

Idea of proof. The functional $\mathcal{B}$ is Morse–Bott and its critical manifold C consists of countably many circles $C_k \cong \mathbb{S}^1$ of odd Morse indices $2k-1$ for $k \in \mathbb{N}$, as illustrated on the left hand side of Figure 1.

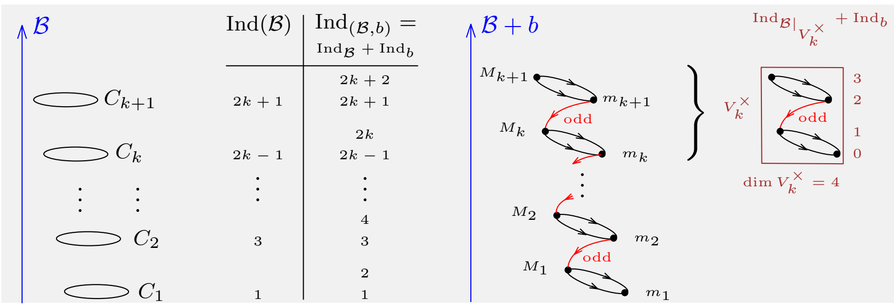

Figure 1. Cascade complex - the cycles are m_1 and all $M_k = \partial m_{k+1}$.

We choose on each of the circles C_k an auxiliary Morse function b_k having exactly one maximum M_k and exactly one minimum m_k, as illustrated on the right hand side of Figure 1. The cascade indices $\mathrm{Ind}_{(\mathcal{B},b)}$ are defined and given by

$$\mathrm{Ind}_{(\mathcal{B},b)}(M_k) := \mathrm{Ind}_{\mathcal{B}}(M_k) + \mathrm{Ind}_b(M_k) = 2k,$$
$$\mathrm{Ind}_{(\mathcal{B},b)}(m_k) := \mathrm{Ind}_{\mathcal{B}}(m_k) + \mathrm{Ind}_b(m_k) = 2k-1.$$

Therefore the cascade chain groups have exactly one generator in each degree

$$\mathrm{CM}_\ell(\mathcal{B}, b; \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2\langle M_k\rangle & \text{for } \ell = 2k, \\ Z_2\langle m_k\rangle & \text{for } \ell = 2k-1, \end{cases}$$

whenever $\ell \in \mathbb{N}$ and they are zero else. On each circle C_k there are two gradient flow lines from M_k to m_k. Since we count gradient flow lines modulo two we have $\partial M_k = 0$. More subtle is to count the cascades from m_{k+1} to M_k. These are solutions of the non-local heat flow equation. We do not construct them directly, but deduce their existence indirectly via the following *crucial observation*: the heat flow gradient flow lines lie in finite dimensional subspaces $V_k^\times$, in fact $\dim V_k^\times = 4$. The restriction $\mathcal{B}_k$ of the functional $\mathcal{B}$ to $V_k^\times$ has as critical point set precisely the circles C_{k+1} and C_k. We prove that by careful choice of the auxiliary Morse function b_k on C_k we can achieve that our gradient flow equation satisfies the Morse-Smale condition simultaneously as well on $V_k^\times$ as on the full space. This allows us to consider the cascade complex of $\mathcal{B}_k$ as a sub-complex of the cascade complex of $\mathcal{B}$ whose degree is however shifted by $2k-1$. On the finite dimensional subspaces $V_k^\times$ we can use topology to compute the cascade Morse homology which turns out to be the homology of the 3-dimensional sphere $\mathbb{S}^3$ which has one generator in degree 0 and one generator in degree 3. Therefore we can conclude that on the finite dimensional subspaces $V_k^\times$ there is an *odd number* of gradient flow lines from m_{k+1} to M_k.

By our crucial observation the gradient flow lines of $\mathcal{B}$ from m_{k+1} to M_k are precisely the gradient flow lines of $\mathcal{B}_k$ from m_{k+1} to M_k. Thus $\partial m_{k+1} = M_k$. Here and throughout we count modulo two. In particular, the minima m_{k+1} are no cycles, while the maxima M_k are cycles but boundaries as well. Hence the only cycle which is not a boundary is the overall minimum m_1. Therefore the homology has a single generator and this generator sits in degree 1.

2. The Morse–Bott functional $\mathcal{B}$

A quite new approach to the regularization of collisions was discovered in the recent paper [1] by Barutello, Ortega, and Verzini where the change of time leads to a *delayed* functional. In the case of the 1-dimensional Kepler problem this functional attains the following form

$$\mathcal{B}\colon W^{1,2}_\times := W^{1,2}(\mathbb{S}^1, \mathbb{R}) \setminus \{0\} \to \mathbb{R}, \qquad q \mapsto 4\|q\|^2\,\frac{1}{2}\|\dot q\|^2 + \frac{1}{\|q\|^2}, \tag{2.1}$$

where $\|\cdot\|$ is the L^2 norm associated to the L^2 inner product $\langle\,\cdot\,,\,\cdot\,\rangle$. One might interpret this functional as a non-local mechanical system consisting of kinetic minus potential energy. As shown in [5] the differential

$$d\mathcal{B}\colon W^{1,2}_\times \times W^{1,2} := W^{1,2}(\mathbb{S}^1, \mathbb{R}) \setminus \{0\} \times W^{1,2}(\mathbb{S}^1, \mathbb{R}) \to \mathbb{R}$$

is given by

$$\begin{aligned} d\mathcal{B}(q,\xi) &= 4\langle q,\xi\rangle \|\dot q\|^2 + 4\|q\|^2 \langle \dot q, \dot\xi\rangle - 2\,\frac{\langle q,\xi\rangle}{\|q\|^4} \\ &= 4\|q\|^2 \Big\langle -\ddot q + \underbrace{\left(\frac{\|\dot q\|^2}{\|q\|^2} - \frac{1}{2\|q\|^6}\right)}_{=:\alpha} q, \xi \Big\rangle \end{aligned}$$

where identity two is valid for sufficiently regular q, say $q \in W^{2,2}(\mathbb{S}^1,\mathbb{R}) \setminus \{0\}$.

Lemma 2.1 (Critical points, [5]). *The functional $\mathcal{B}\colon W^{1,2}_\times \to \mathbb{R}$. The set* $\mathrm{Crit}\mathcal{B}$ *of critical points of $\mathcal{B}$ consists of the functions*

$$q_k(t) = c_k \cos 2\pi k t, \quad c_k = \frac{1}{2^{1/6}(\pi k)^{1/3}} \in (0,1), \quad k \in \mathbb{N} \tag{2.2}$$

and their time shifts

$$(\sigma_* q_k)(t) := q_k(t+\sigma) = c_k\big(\cos 2\pi k\sigma \underbrace{\cos 2\pi k t}_{=:\phi_k(t)} - \sin 2\pi k \sigma \underbrace{\sin 2\pi k t}_{=:\psi_k(t)}\big) \tag{2.3}$$

where $\sigma, t \in \mathbb{S}^1$. The corresponding critical values are given by

$$\mathcal{B}(q_k) = \mathcal{B}(\sigma_* q_k) = 2^{1/3}\, 3 (\pi k)^{2/3} \tag{2.4}$$

and the Morse indices are

$$\mathrm{Ind}(q_k) = \mathrm{Ind}(\sigma_* q_k) = 2k-1 \tag{2.5}$$

3. L^2_q gradient equation and flow lines

We consider the following metric on $W^{1,2}_\times$. Given a point $q \in W^{1,2}_\times$ and two tangent vectors

$$\xi_1, \xi_2 \in T_q W^{1,2}_\times = W^{1,2}(\mathbb{S}^1,\mathbb{R}) =: W^{1,2}$$

we define what we call the L^2_q *inner product* by

$$\langle \xi_1,\xi_2\rangle_q := 4\|q\|^2 \langle \xi_1, \xi_2\rangle, \quad \text{where } \langle \xi_1,\xi_2\rangle := \int_0^1 \xi_1(t)\xi_2(t)\,dt. \tag{3.1}$$

Note that $\langle\cdot,\cdot\rangle$ is the standard L^2 inner product on $L^2(\mathbb{S}^1,\mathbb{R})$. In this notation

$$\mathcal{B}(q) = \frac{1}{2}\langle \dot q, \dot q\rangle_q + \frac{1}{\|q\|^2}, \qquad d\mathcal{B}(q,\xi) = \langle -\ddot q + \alpha q, \xi\rangle_q$$

where identity two is valid for $q \in W^{2,2}_\times := W^{2,2}(\mathbb{S}^1,\mathbb{R}) \setminus \{0\}$. The L^2_q gradient of $\mathcal{B}$ at $q \in W^{2,2}_\times$ is denoted and given by

$$\mathrm{Grad}\,\mathcal{B}(q) = -\ddot q + \alpha q, \qquad \alpha = \alpha_q := \left(\frac{\|\dot q\|^2}{\|q\|^2} - \frac{1}{2\|q\|^6}\right) \in \mathbb{R} \tag{3.2}$$

Flow lines. A smooth cylinder $u\colon \mathbb{R}\times\mathbb{S}^1\to\mathbb{R}$ whose associated path of loops $s\mapsto u_s := u(s,\,\cdot\,)$ avoids the zero loop is called a *heat flow line* ([1]) if it satisfies the scale ode given by

$$\mathcal{F}(u) := \partial_s u - \partial_t\partial_t u + \alpha_s u = 0, \qquad \alpha_s := \frac{\|\partial_t u_s\|^2}{\|u_s\|^2} - \frac{1}{2\|u_s\|^6}. \tag{3.3}$$

Remark 3.1 (Wick rotation). If one considers the above heat flow equation (3.3) in imaginary time is, corresponding to a Wick rotation, one obtains the following non-local Schrödinger equation

$$i\partial_s u - \partial_t\partial_t u + \alpha_s u = 0$$

Remark 3.2 Asymptotic boundary values of heat flow lines). If a heat flow line u, that is any smooth cylinder $u\colon \mathbb{R}\times\mathbb{S}^1\to\mathbb{R}$ such that $\mathcal{F}(u)=0$, admits a non-empty ω-limit set, then this set $\omega_\pm(u)=\{q_\pm\}$ consists of a single critical point (2.2) of the functional $\mathcal{B}$ (this holds since $\mathcal{B}$ is Morse–Bott by Lemma A.1). In this case it is well known ~~(see [?])~~ that the gradient flow line $s\mapsto u_s := u(s,\,\cdot\,)$ converges exponentially to $q_\pm$, as flow time $s\to\pm\infty$. The exponential rate of decay is determined by the spectral gap, namely, the smallest absolute value of a non-zero eigenvalue.

In the case of the functional $\mathcal{B}$, non-emptiness of the ω-limit set $\omega_\pm(u)$ is not guaranteed, neither in the forward direction by trying to exploit the facts that there is a forward semi-flow and $\mathcal{B}$ is bounded below (unfortunately a minimum is not achieved due to escape to infinity), nor in both directions by imposing a finite energy condition on u.

Linearization. We shall linearize the map $\mathcal{F}$ defined by (3.3) at any smooth cylinder $u\colon \mathbb{R}\times\mathbb{S}^1\to\mathbb{R}$ which has as asymptotic boundary conditions two critical points, see (2.3), of the Morse–Bott functional $\mathcal{B}$, in symbols

$$q_\pm := \lim_{s\to\pm\infty} u(s,\,\cdot\,) \in \operatorname{Crit}\mathcal{B} \tag{3.4}$$

where the limit is uniformly in $t\in\mathbb{S}^1$.

Definition 3.3. Suppose H is a separable Hilbert space. Fix a monotone cutoff function $\beta\in C^\infty(\mathbb{R},[-1,1])$ with $\beta(s)=-1$ for $s\le -1$ and $\beta(s)=1$ for $s\ge 1$. Fix a constant $\delta\in(0,4\pi^2)$ ([2]) and (see Figure 2) define a function $\gamma_\delta\colon\mathbb{R}\to\mathbb{R}$ by

$$\gamma_\delta(s) := e^{\delta\beta(s)s}.$$

([1]) A downward gradient flow line in the L^2_q metric of the non-local action functional $\mathcal{B}$.

([2]) The interval $(0,4\pi^2)$ is contained in the spectral gap of any Hessian operator $A_{\sigma_* q_k}$ associated to a critical point; see (A.5).

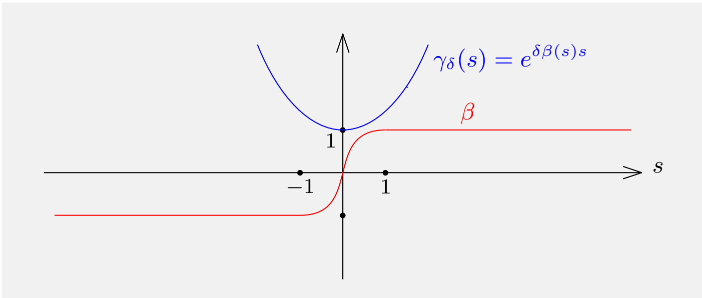

FIGURE 2. Monotone cutoff function β and exponential weight γ_δ.

Pick a constant $p \in (1, \infty)$. Consider the Hilbert space valued Sobolev spaces defined for $k \in \mathbb{N}_0$ by

$$W^{k,p}_\delta(\mathbb{R}, H) := \left\{v \in W^{k,p}(\mathbb{R}, H) \mid \gamma_\delta v \in W^{k,p}(\mathbb{R}, H)\right\} \tag{3.5}$$

with norm $\|v\|_{W^{k,p}_\delta} := \|\gamma_\delta v\|_{W^{k,p}}$. These are Banach spaces, see e.g. [6, Appendix A.2].

Given a smooth cylinder $u\colon \mathbb{R} \times \mathbb{S}^1 \to \mathbb{R}$ subject to asymptotic boundary conditions (3.4) and a smooth compactly supported function $\xi\colon \mathbb{R} \times \mathbb{S}^1 \to \mathbb{R}$, pick a family u^τ such that $u^0 = u$ and $\frac{d}{d\tau}\big|_{\tau=0} u^\tau = \xi$, say $u + \tau\xi$. Abbreviating $W^{k,2} = W^{k,2}(\mathbb{S}^1)$, then the linearization

$$D_u := D\mathcal{F}(u)\colon W^{0,2}_\delta\left(\mathbb{R}, W^{2,2}\right) \cap W^{1,2}_\delta\left(\mathbb{R}, W^{1,2}\right) \to W^{0,2}_\delta\left(\mathbb{R}, L^2\right) \tag{3.6}$$

is of the form

$$\begin{aligned} D_u\xi := D\mathcal{F}(u)\xi := \frac{d}{d\tau}\bigg|_{\tau=0} \mathcal{F}(u^\tau) &= \frac{d}{d\tau}\bigg|_{\tau=0} \left(\partial_s u^\tau - \partial_t^2 u^\tau + \alpha_{s,\tau} u^\tau\right) \\ &= \partial_s\xi - \partial_t\partial_t\xi + \alpha_s\xi + \left(\frac{d}{d\tau}\bigg|_{\tau=0} \alpha_{s,\tau}\right) u \end{aligned}$$

Further calculation shows that at any smooth cylinder u we obtain

$$\begin{aligned} D_u\xi &= \partial_s\xi - \partial_t\partial_t\xi + \alpha_s\xi \\ &\quad - 2\,\frac{\langle \partial_t\partial_t u_s, \xi_s\rangle}{\|u_s\|^2}\, u - 2\left(\frac{\|\partial_t u_s\|^2}{\|u_s\|^4} - \frac{3/2}{\|u_s\|^8}\right) \langle u_s, \xi_s\rangle\, u \\ &= \partial_s\xi - \partial_t\partial_t\xi + \alpha_s\xi \\ &\quad - \frac{2}{\|u_s\|^2}\left(\langle \underbrace{\partial_t\partial_t u_s}_{\partial_s u_s + \alpha_s u_s}, \xi_s\rangle + \underbrace{\left(\frac{\|\partial_t u_s\|^2}{\|u_s\|^2} - \frac{3/2}{\|u_s\|^6}\right)}_{\alpha_s - 1/\|u_s\|^6} \langle u_s, \xi_s\rangle\right) u \\ &= \partial_s\xi - \partial_t\partial_t\xi + \alpha_s\xi \\ &\quad - \frac{2}{\|u_s\|^2}\left(\langle \partial_s u_s, \xi_s\rangle + \left(2\alpha_s - \frac{1}{\|u_s\|^6}\right)\langle u_s, \xi_s\rangle\right) u \end{aligned} \tag{3.7}$$

where the last identity holds whenever u solves the heat equation.

The adjoint of the linearization. Given a smooth cylinder $u\colon \mathbb{R}\times\mathbb{S}^1 \to \mathbb{R}$, consider the L^2_u *inner product* defined by

$$\langle \xi, \eta\rangle_u := \int_{-\infty}^{\infty} \langle \xi_s, \eta_s\rangle_{u_s} ds := \int_{-\infty}^{\infty} 4\|u_s\|^2 \langle \xi_s, \eta_s\rangle \, ds$$

for compactly supported smooth functions $\xi, \eta\colon \mathbb{R}\times\mathbb{S}^1 \to \mathbb{R}$. The L^2_u adjoint operator of D_u, notation D_u^*, is determined by the identity

$$\langle D_u \xi, \eta\rangle_u = \langle \xi, D_u^* \eta\rangle_u$$

for compactly supported smooth vector fields ξ and η along the cylinder u. To get a formula for D_u^* we rewrite the inner product as follows. In the first step we use for $D_u\xi$ the equality (3.7) and in the second step we apply partial integration with respect to s to obtain

$$\begin{aligned}
\langle D_u\xi, \eta\rangle_u &= \int_{-\infty}^{\infty} 4\|u_s\|^2 \cdot \big(\langle \partial_s \xi_s, \eta_s\rangle - \langle \partial_t\partial_t \xi_s, \eta_s\rangle + \langle \alpha_s \xi_s, \eta_s\rangle\big)\, ds \\
&\quad + \int_{-\infty}^{\infty} 4\|u_s\|^2 \left(-2\,\frac{\langle \partial_t\partial_t u_s, \xi_s\rangle}{\|u_s\|^2} - 2\,\frac{\|\partial_t u_s\|^2}{\|u_s\|^4}\langle u_s, \xi_s\rangle + \frac{3}{\|u_s\|^8}\langle u_s, \xi_s\rangle \right) \langle u_s, \eta_s\rangle\, ds \\
&= \int_{-\infty}^{\infty} \big(-8\langle u_s, \partial_s u_s\rangle \langle \xi_s, \eta_s\rangle + 4\|u_s\|^2 \langle \xi_s, -\partial_s \eta_s - \partial_t\partial_t \eta_s + \alpha_s \eta_s\rangle\big)\, ds \\
&\quad - 2\int_{-\infty}^{\infty} \left(\frac{\langle \partial_t\partial_t u_s, \xi_s\rangle_{u_s}}{\|u_s\|^2} + \frac{\|\partial_t u_s\|^2}{\|u_s\|^4}\langle u_s, \xi_s\rangle_{u_s} - \frac{3/2}{\|u_s\|^8}\langle u_s, \xi_s\rangle_{u_s} \right) \langle u_s, \eta_s\rangle ds \\
&= \langle \xi, -\partial_s \eta - \partial_t\partial_t \eta - \alpha\eta\rangle_u - 2\int_{-\infty}^{\infty} \frac{\langle u_s, \partial_s u_s\rangle}{\|u_s\|^2} \langle \xi_s, \eta_s\rangle_{u_s}\, ds \\
&\quad - 2\int_{-\infty}^{\infty} \left(\langle \xi_s, \partial_t\partial_t u_s\rangle_{u_s} \frac{\langle u_s, \eta_s\rangle}{\|u_s\|^2} + \langle \xi_s, u_s\rangle_{u_s} \left(\frac{\|\partial_t u_s\|^2}{\|u_s\|^4} - \frac{3/2}{\|u_s\|^8} \right) \langle u_s, \eta_s\rangle \right) ds \\
&= \langle \xi, D_u^* \eta\rangle_u
\end{aligned}$$

Hence the L^2_u adjoint of the linearization D_u is of the form

$$\begin{aligned}
D_u^*\eta = &-\partial_s \eta - \partial_t\partial_t \eta + \alpha_s \eta - 2\,\frac{\langle u_s, \partial_s u_s\rangle}{\|u_s\|^2}\,\eta \\
&- 2\,\frac{\langle u_s, \eta_s\rangle}{\|u_s\|^2}\,\partial_t\partial_t u - 2\left(\frac{\|\partial_t u_s\|^2}{\|u_s\|^4} - \frac{3/2}{\|u_s\|^8} \right) \langle u_s, \eta_s\rangle u
\end{aligned} \tag{3.8}$$

where the yellow extra term arose when we integrated by parts the s variable.

4. Fourier mode intervals and isolating neighbourhoods

Flow lines. We write $u(s,t)$ for any fixed time $s\in\mathbb{R}$ as a Fourier series in the form

$$u_s(t) := u(s,t) = a_0(s) + \sum_{k=1}^{\infty} \big(a_k(s)\cos 2\pi k t + b_k(s)\sin 2\pi k t\big) \tag{4.1}$$

PROPOSITION 4.1 (Isolating neighbourhood – Fourier mode interval). *Assume that u is a solution of the delayed heat equation* (3.3) *with asymptotic boundary conditions* (3.4), *that is*

$$q_\pm(t) := \lim_{s\to\pm\infty} u(s,t) = a_{k_\pm}\cos 2\pi k_\pm t + b_{k_\pm}\sin 2\pi k_\pm t \tag{4.2}$$

for every $t \in \mathbb{S}^1$, uniformly in t, and for some positive integers $k_\pm \in \mathbb{N}$ and constants $a_{k_\pm}$ and $b_{k_\pm}$ (cf. (2.3)*). Then the Fourier coefficients $a_k(s) \equiv 0$ and $b_k(s) \equiv 0$ vanish identically for all k outside the interval $[k_+, k_-]$.*

PROOF. Pick a Fourier mode $k \in \mathbb{N}_0$. With the constants defined by

$$a_k^\pm := \begin{cases} 0 & \text{if } k \neq k_\pm, \\ a_{k_\pm} & \text{if } k = k_\pm, \end{cases} \qquad b_k^\pm := \begin{cases} 0 & \text{if } k \neq k_\pm, \\ b_{k_\pm} & \text{if } k = k_\pm, \end{cases} \tag{4.3}$$

we obtain the identity $\lim_{s\to\pm\infty} a_k(s) = a_k^\pm$. Taking one s derivative and two t derivatives of the Fourier series (4.1) the heat equation (3.3) implies that

$$a_k'(s) + \left((2\pi k)^2 + \alpha_s\right) a_k(s) = 0 \tag{4.4}$$

for every $s \in \mathbb{R}$. Since (4.4) is a first order ODE we conclude that

$$a_k(0) \neq 0 \;\Rightarrow\; a_k(s) \neq 0 \quad \text{for all } s \in \mathbb{R}.$$

So we assume that $a_k(0) \neq 0$. It is useful to calculate the derivative

$$\frac{d}{ds}\ln\left(a_k(s)^2\right) = \frac{2a_k(s)\,a_k'(s)}{a_k(s)^2} = 2\,\frac{a_k'(s)}{a_k(s)} = -2(2\pi k)^2 - 2\alpha_s$$

where in the last equality we used the ode (4.4).

Step 1. $k < k_+ \Rightarrow a_k \equiv 0$. The proof of Step 1 works by showing that the assumption $a_k(0) \neq 0$ produces a contradiction. Since $k < k_+$ we get that

$$\frac{d}{ds}\left(\ln\left(a_k(s)^2\right) - \ln\left(a_{k_+}(s)^2\right)\right) = 2(2\pi k)^2({k_+}^2 - k^2) > 0$$

This shows that, for $s > 0$, there is the inequality

$$\ln\left(a_k(s)^2\right) - \ln\left(a_{k_+}(s)^2\right) > \ln\left(a_k(0)^2\right) - \ln\left(a_{k_+}(0)^2\right)$$

or equivalently

$$\ln\left(a_k(s)^2\right) - \ln\left(a_k(0)^2\right) > \ln\left(a_{k_+}(s)^2\right) - \ln\left(a_{k_+}(0)^2\right)$$

for every $s > 0$. Exponentiating we get that

$$\frac{a_k(s)^2}{a_k(0)^2} > \frac{a_{k_+}(s)^2}{a_{k_+}(0)^2}$$

Taking the limit, as $s \to \infty$, of the right hand side we obtain

$$\lim_{s\to\infty}\frac{a_{k_+}(s)^2}{a_{k_+}(0)^2} = \frac{a_{k_+}^2}{a_{k_+}(0)^2} > 0$$

since $\lim_{s\to\infty} u_s = q_+$. On the other hand, taking the limit, as $s \to \infty$, of the left hand side we obtain

$$\lim_{s\to\infty} \frac{a_k(s)^2}{a_k(0)^2} = \frac{0}{a_k(0)^2} = 0.$$

Here we used that the Fourier coefficient for k in q_+ vanishes. The last three displayed formulas contradict each other. Therefore the assumption that $a_k(0) \neq 0$ had to be wrong. We conclude that $a_k \equiv 0$ vanishes identically if $k < k_+$.

Step 2. $k > k_- \Rightarrow a_k \equiv 0$. To prove this note that since $k > k_-$ we get that

$$\frac{d}{ds}\left(\ln\left(a_k(s)^2\right) - \ln\left(a_{k_-}(s)^2\right)\right) = 2(2\pi k)^2({k_-}^2 - k^2) < 0.$$

This shows that for $s < 0$ there is the inequality

$$\ln\left(a_k(s)^2\right) - \ln\left(a_{k_-}(s)^2\right) > \ln\left(a_k(0)^2\right) - \ln\left(a_{k_-}(0)^2\right)$$

or equivalently

$$\ln\left(a_k(s)^2\right) - \ln\left(a_k(0)^2\right) > \ln\left(a_{k_-}(s)^2\right) - \ln\left(a_{k_-}(0)^2\right)$$

for every $s > 0$. Exponentiating we get that

$$\frac{a_k(s)^2}{a_k(0)^2} > \frac{a_{k_-}(s)^2}{a_{k_-}(0)^2}.$$

Taking the limit as $s \to -\infty$ of the right hand side we obtain

$$\lim_{s\to-\infty} \frac{a_{k_-}(s)^2}{a_{k_-}(0)^2} = \frac{a_{k_-}^2}{a_{k_-}(0)^2} > 0$$

since $\lim_{s\to-\infty} u_s = q_-$. On the other hand, taking the limit as $s \to -\infty$ of the left hand side we obtain

$$\lim_{s\to-\infty} \frac{a_k(s)^2}{a_k(0)^2} = \frac{0}{a_k(0)^2} = 0$$

Here we used that the Fourier coefficient for k in q_- vanishes. The last three displayed formulas contradict each other. Therefore the assumption that $a_k(0) \neq 0$ had to be wrong. We conclude that $a_k(s) \equiv 0$ vanishes identically if $k < k_-$. The proof for $b_k(s)$ is analogous. □

Linearization. We write $u(s,t)$ for any fixed time $s \in \mathbb{R}$ as a Fourier series in the form of equation (4.1). Similarly we write $\xi\colon \mathbb{R} \times \mathbb{S}^1 \to \mathbb{R}$ for any fixed time $s \in \mathbb{R}$ as a Fourier series in the form

$$\xi_s(t) := \xi(s,t) = A_0(s) + \sum_{k=1}^{\infty} \left(A_k(s)\cos 2\pi kt + B_k(s)\sin 2\pi kt\right) \tag{4.5}$$

Proposition 4.2 (The kernel of D_u has the same Fourier mode interval as u). *Let u be a solution of the delayed heat equation* (3.3) *with asymptotic boundary conditions* (4.2), *namely, two critical points*

$$q_\pm := \lim_{s\to\pm\infty} u(s,\,\cdot\,) \in \operatorname{Crit}\mathcal{B}$$

where $q_\pm$ is determined by a positive integer $k_\pm \in \mathbb{N}$ and two constants $a_{k_\pm}, b_{k_\pm}$. Suppose that ξ is an element of the kernel of D_u, that is $D_u\xi = 0$. For $s \in \mathbb{R}$ write $\xi_s := \xi(s, \cdot)\colon \mathbb{S}^1 \to \mathbb{R}$ in the form of the Fourier series (4.5)*. Then $A_k(s) \equiv 0$ and $B_k(s) \equiv 0$ vanish identically for all k outside the interval $[k_+, k_-]$.*

Proof. Pick a common Fourier mode $k \in \mathbb{N}_0$ of u and ξ. Consider the constants $a_k^\pm$ and $b_k^\pm$ defined by (4.3) and let $A_k^\pm$ and $B_k^\pm$ be defined analogously. Taking one s derivative and two t derivatives of the Fourier series (4.5) the equation $D_u\xi = 0$, see (3.7), and the heat equation (3.3) for u provide the ode

$$A_k'(s) + \big((2\pi k)^2 + \alpha_s\big)A_k(s) - (\Phi^*\xi_s)\cdot a_k(s) = 0 \tag{4.6}$$

for the function $A_k(s)$. Here the function $a_k(s)$ satisfies the ode (4.4) and

$$\Phi^*\xi_s := -\frac{2}{\|u_s\|^2}\left(\langle \partial_t\partial_t u_s, \xi_s\rangle + \left(\alpha_s - \frac{1}{\|u_s\|^6}\right)\langle u_s, \xi_s\rangle\right).$$

Once we recall that for k outside the interval $[k_+, k_-]$ the functions $a_k \equiv 0$ and $b_k \equiv 0$ vanish identically, the proof of the present proposition reduces to the one of Proposition 4.1. Indeed for $k \notin [k_+, k_-]$ the ode (4.6) reduces to the ode

$$A_k'(s) + \big((2\pi k)^2 + \alpha_s\big)A_k(s) = 0$$

for $A_k(s)$. But this is exactly the ode (4.4) for which we already showed the assertion. The proof for $B_k(s)$ is analogous. □

5. Restriction to 4-dimensional subspaces V_k

Fix $k \in \mathbb{N}$ and define functions

$$\phi_k(t) := \cos 2\pi k t, \qquad \psi_k(t) := \sin 2\pi k t.$$

Consider the 4-dimensional vector subspace of the free loop space $W^{1,2}\big(\mathbb{S}^1, \mathbb{R}\big)$ spanned by the following four functions (cf. Lemma 2.1)

$$V_k = \operatorname{span}\{\phi_k, \psi_k, \phi_{k+1}, \psi_{k+1}\}, \qquad V_k^\times := V_k \setminus \{0\}.$$

The following corollary tells that flow lines from C_{k+1} to C_k critical points lie in one and the same V_k.

Corollary 5.1. *Suppose $u\colon \mathbb{R}\times\mathbb{S}^1 \to \mathbb{R}$ is a gradient flow line of* $\operatorname{Grad}\mathcal{B}$*, see* (3.2)*, which asymptotically converges to critical points lying in V_k. Then the whole gradient flow line $u_s := u(s, \cdot)$ lies in $V_k^\times$ for all $s \in \mathbb{R}$.*

See Proposition 4.1 for the proof.

In view of the above corollary we want to study in detail the restriction of the functional

$$\mathcal{B}(q) = 4\|q\|^2\,\frac{1}{2}\|\dot q\|^2 + \frac{1}{\|q\|^2}$$

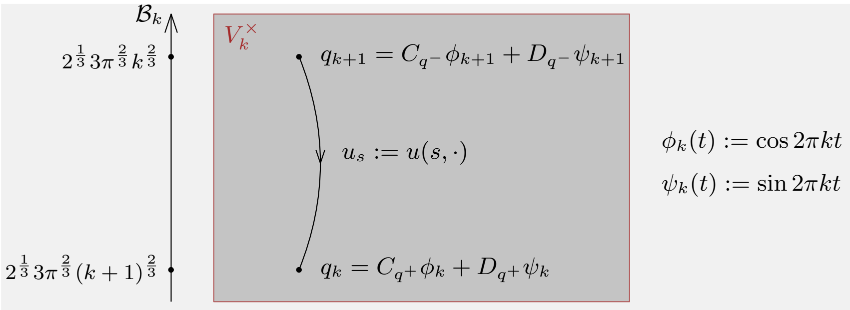

Figure 3. Flow lines connecting consecutive critical manifold components C_{k+1} and C_k lie in a 4-dimensional space V_k.

to the pointed 4-dimensional subspace $V_k^\times$, notation

$$\mathcal{B}_k := \mathcal{B}|_{V_k^\times} : V_k^\times \to (0,\infty).$$

Lemma 5.2 (Morse indices of the restricted functional). *For $k \in \mathbb{N}$ it holds that* $\operatorname{Crit}\mathcal{B}|_{V_k} \subset \operatorname{Crit}\mathcal{B}$ *and*

$$\operatorname{Ind}_{\mathcal{B}|_{V_k}}(q_{k+1}) = 2, \qquad \operatorname{Ind}_{\mathcal{B}|_{V_k}}(q_k) = 0.$$

Proof. This follows from the computation of the eigenvalues and eigenvectors in (A.4). □

We write q as a linear combination of the four basis elements of V_k to obtain the estimate

$$\|\dot q\|^2 \geq (2\pi k)^2 \|q\|^2.$$

Since $k \geq 1$ the restriction of $\mathcal{B}$ to $V_k^\times$ goes to infinity when $\|q\|$ moves to infinity or to zero. In particular, the restriction of $\mathcal{B}$ to $V_k^\times$ is a coercive function (preimages of compacta are compact). Therefore Morse homology of the coercive functional $\mathcal{B}$ represents singular homology of the domain $V_k^\times$ of $\mathcal{B}$. But $V_k^\times$ is homotopy equivalent to the 3-sphere $\mathbb{S}^3$. We summarize these findings in

Lemma 5.3 (Morse complex of the restriction $\mathcal{B}_k\colon V_k \to \mathbb{R}$). *For $k \in \mathbb{N}$ it holds*:

$$\mathrm{HM}_*(\mathcal{B}_k;\mathbb{Z}_2) \simeq \mathrm{H}_*\left(V_k^\times;\mathbb{Z}_2\right) \simeq \mathrm{H}_*\left(\mathbb{S}^3;\mathbb{Z}_2\right) = \begin{cases} \mathbb{Z}_2 & \text{if } * = 0,3, \\ 0 & \text{else.} \end{cases}$$

6. Construction of a cascade Morse complex for $\mathcal{B}$

We choose on each critical manifold C_k a point m_k, that is for each $k \in \mathbb{N}$. We consider the unstable manifold of m_{k+1} with respect to the restriction $\mathcal{B}_k$ of $\mathcal{B}$ to $V_k^\times$, notation $W^u_{-\nabla\mathcal{B}_k}(m_{k+1})$.

Since the Morse index of $\mathcal{B}_k$ along C_{k+1} is 2, this unstable manifold is a 2-dimensional sub-manifold of V_k. Since $\mathcal{B}_k$ is coercive each point $p \neq m_{k+1}$ of the unstable manifold $W^u_{-\nabla \mathcal{B}_k}(m_{k+1})$ converges under the negative gradient flow of $\mathcal{B}_k$ in positive time to a point y on C_k. Hence we obtain a well defined *evaluation map* given by

$$\mathrm{ev}\colon W^u_{-\nabla \mathcal{B}_k}(m_{k+1}) \setminus \{m_{k+1}\} \to C_k, \qquad p \mapsto \lim_{t\to+\infty} \varphi^t_{-\nabla \mathcal{B}_k}(p).$$

We choose a regular value of ev different from m_k, notation M_k.

On each C_k (it is diffeomorphic to $\mathbb{S}^1$) we choose a Morse function b_k with exactly two critical points, namely a maximum at M_k and a minimum at m_k. Let b denote the resulting Morse function on the set $C = \bigcup_k C_k$ of critical points of $\mathcal{B}$. Note that the Morse index of the critical points is zero or one, namely $\mathrm{Ind}_b(m_k) = 0$ and $\mathrm{Ind}_b(M_k) = 1$.

Hence in view of (2.5) for the *cascade index* $\mathrm{Ind}_{(\mathcal{B},b)}$ we obtain

$$\begin{aligned} \mathrm{Ind}_{(\mathcal{B},b)}(M_k) &:= \mathrm{Ind}_{\mathcal{B}}(M_k) + \mathrm{Ind}_b(M_k) = 2k, \\ \mathrm{Ind}_{(\mathcal{B},b)}(m_k) &:= \mathrm{Ind}_{\mathcal{B}}(m_k) + \mathrm{Ind}_b(m_k) = 2k-1. \end{aligned} \tag{6.1}$$

From M_k to m_k there are 2 gradient flow lines of b and since we count modulo 2 we have for the Morse boundary operator

$$\partial M_k = 0. \tag{6.2}$$

It remains to compute ∂m_{k+1}. Before we can do that we have to make sure that we have a well defined count of cascades from m_{k+1} to M_k.

Hence we consider a gradient flow line u of $\mathcal{B}$ from m_{k+1} to M_k and we need to show that D_u is surjective. In view of these specific asymptotic boundary conditions, we know by Proposition 4.1 that $s \mapsto u_s := u(s,\cdot\,)$ takes values in V_k. We consider the restriction of D_u to V_k as an operator

$$D_u|_{V_k}\colon W^{1,2}_\delta(\mathbb{R}, V_k) \to W^{0,2}_\delta(\mathbb{R}, V_k).$$

It follows by Proposition 4.2

$$\ker D_u = \ker D_u|_{V_k} \tag{6.3}$$

Since M_k was chosen as a regular value of the evaluation map ev we have

$$\dim \ker D_u|_{V_k} = \mathrm{index}(D_u|_{V_k}). \tag{6.4}$$

Furthermore, it is well known that the Fredholm index of the linearization is given by the cascade index difference of the asymptotic boundary conditions and this shows the first and the final identity in the following

$$\begin{aligned}\text{(6.5)}\quad \operatorname{index}(D_u|_{V_k}) &= \operatorname{Ind}_{(\mathcal{B}_k,b)}(m_{k+1}) - \operatorname{Ind}_{(\mathcal{B}_k,b)}(M_k)\\ &= \operatorname{Ind}_{\mathcal{B}_k}(m_{k+1}) + \operatorname{Ind}_b(m_{k+1}) - \big(\operatorname{Ind}_{\mathcal{B}_k}(M_k) + \operatorname{Ind}_b(M_k)\big)\\ &= 2+0-(0+1) = 1 = (2(k+1)-1)-2k\\ &= \operatorname{Ind}_{(\mathcal{B},b)}(m_{k+1}) - \operatorname{Ind}_{(\mathcal{B},b)}(M_k) = \operatorname{index}(D_u).\end{aligned}$$

Here equality two is by definition of the cascade index, equality three is by Lemma 5.2, and the penultimate equality is by (6.1).

Summarizing, apply successively the results (6.3), (6.4), and (6.5) to obtain

$$\begin{aligned}\dim\ker D_u &= \dim\ker D_u|_{V_k} = \operatorname{index}(D_u|_{V_k})\\ &= \operatorname{index}(D_u) := \dim\ker D_u - \operatorname{dimcoker} D_u.\end{aligned}$$

Hence $\operatorname{coker} D_u$ is trivial and therefore the linearized operator D_u is surjective.

7. Proof of the main theorem

PROPOSITION 7.1 (Cascade chain complex). *The cascade chain groups of the Morse–Bott functional $\mathcal{B}$ and with respect to the auxiliary Morse function b on $C := \operatorname{Crit}\mathcal{B}$ carefully chosen in Section 6 are given by*

$$\operatorname{CM}_\ell(\mathcal{B},b;\mathbb{Z}_2) = \begin{cases}\mathbb{Z}_2\langle M_k\rangle & \text{if } \ell = 2k,\\ Z_2\langle m_k\rangle & \ell = 2k-1,\end{cases} \tag{7.1}$$

whenever $\ell \in \mathbb{N}$ and they are zero else. All maxima are cycles

$$\partial M_k = 0, \quad M_k = \partial m_{k+1}, \quad k \in \mathbb{N}, \tag{7.2}$$

but also boundaries. There is exactly one more cycle, the lowest minimum

$$\partial m_1 = 0 \tag{7.3}$$

and m_1 is not a boundary. Thus m_1 generates the Morse homology.

PROOF OF PROPOSITION 7.1. Assertion (7.1) follows from (6.1). The first equation in (7.2) follows from (6.2).

It remains to prove the second equation $\partial m_{k+1} = M_k$ in (7.2). In order to do that we consider the cascade complex of the restriction $\mathcal{B}_k$ of the functional $\mathcal{B}$ to the 4-dimensional space $V_k^\times = V_k \setminus \{0\}$. The cascade complex of the pair $(\mathcal{B}_k, b)$ has four generators, namely $M_{k+1}, m_{k+1}, M_k, m_k$. By Lemma 5.2 it holds that

$$\begin{aligned}\operatorname{ind}_{(\mathcal{B}_k,b)}(M_{k+1}) &:= \operatorname{ind}_{\mathcal{B}_k}(M_{k+1}) + \operatorname{ind}_b(M_{k+1}) = 2+1 = 3,\\ \operatorname{ind}_{(\mathcal{B}_k,b)}(m_{k+1}) &:= \operatorname{ind}_{\mathcal{B}_k}(m_{k+1}) + \operatorname{ind}_b(m_{k+1}) = 2+0 = 2,\\ \operatorname{ind}_{(\mathcal{B}_k,b)}(M_k) &:= \operatorname{ind}_{\mathcal{B}_k}(M_k) + \operatorname{ind}_b(M_k) = 0+1 = 1,\\ \operatorname{ind}_{(\mathcal{B}_k,b)}(m_k) &:= \operatorname{ind}_{\mathcal{B}_k}(m_k) + \operatorname{ind}_b(m_k) = 0+0 = 0.\end{aligned}$$

According to Lemma 5.3 the cascade homology of $(\mathcal{B}_k, b)$ on the 4-dimensional space $V_k^\times$ vanishes in degrees 1 and 2. Therefore there has to exist an odd number of gradient flow lines of the restricted functional $\mathcal{B}_k$ from m_{k+1} to M_k. According to Corollary 5.1 these are precisely the gradient flow lines of the unrestricted functional $\mathcal{B}$ from m_{k+1} to M_k. Therefore $\partial m_{k+1} = M_k$.

Because there are no generators of degree lower than the degree one of m_1, it holds that $\partial m_1 = 0$. □

Appendix A. Morse–Bott and trivial negative bundles

The connected components of the critical manifold $C := \operatorname{Crit}\mathcal{B}$ consist of circles C_k labelled by $k \in \mathbb{N}$. In [5] we already showed that the kernel of the Hessian of $\mathcal{B}$ at each point of C_k is 1-dimensional. Since the kernel always contains the tangent space to the critical manifold which in our case is of dimension one, the two are equal. But this is the definition of *Morse–Bott*. Thus from [5] we know that $C_k \simeq \mathbb{S}^1$ is Morse–Bott of index $2k-1$. Since C_k is Morse–Bott there is the splitting

$$T_{C_k} W^{1,2}_\times = TC_k \oplus \mathcal{V}^- C_k \oplus \mathcal{V}^+ C_k$$

which at each point corresponds to the splitting in zero/negative/positive eigenspaces of the Hessian. Note that the rank of $\mathcal{V}^- C_k$ corresponds to the Morse index of C_k. The above argument proves the following lemma.

LEMMA A.1 (Morse–Bott functional). *The functional $\mathcal{B}$ defined by* (2.1) *is Morse–Bott and every critical point is of nullity* 1.

In this section we show additionally that the negative bundle $\mathcal{V}^- C_k$ is trivial for each C_k. This plays an important role in order to compute Conley indices of the critical components C_k.

LEMMA A.2 (Trivial negative normal bundles). *For each $k \in \mathbb{N}$ the negative normal bundle $\mathcal{V}^- C_k$ over the Morse–Bott manifold C_k is*

(a) *trivial, and*
(b) *of rank* $2k-1$.

The proof of this lemma covers the following three pages and ends after equation (A.6). The proof follows the computation of the Morse index in our previous paper [5]. The new aspect that the line bundle is trivial is to choose a global trivialization of the restriction of the tangent bundle of $W^{1,2}_\times$ to C_k which has the property that the eigenvalues and eigenvectors with respect to this global trivialization are independent of the base point. This then proves that the negative and the positive normal bundles are both trivial.

The Hessian operator – with respect to L^2_q. The Hessian operator A_q of the Lagrange functional $\mathcal{B}$ is the derivative of the L^2_q gradient at a critical point q, that is by (3.2) the derivative of the equation

$$0 = \operatorname{Grad}\mathcal{B}(q) = -\ddot{q} + \alpha q, \qquad \alpha = \alpha_q := \left(\frac{\|\dot{q}\|^2}{\|q\|^2} - \frac{1}{2\|q\|^6}\right) \in \mathbb{R}.$$

Here $q \in W^{1,2}(\mathbb{S}^1, \mathbb{R}) \setminus \{0\}$ is automatically smooth since it is a critical point.

LEMMA A.3 ([5]). *The Hessian operator of $\mathcal{B}$ at a critical point q is given by*

$$\begin{gathered} A_q \colon W^{2,2}(\mathbb{S}^1, \mathbb{R}) \to L^2(\mathbb{S}^1, \mathbb{R}), \\ \xi \mapsto -\ddot{\xi} + \alpha\xi - \frac{2}{\|q\|^2}\left(2\alpha - \frac{1}{\|q\|^6}\right)\langle q, \xi\rangle q. \end{gathered} \tag{A.1}$$

By (2.3) the critical points of the functional $\mathcal{B}$ are of the form

$$(\sigma_* q_k)(t) := q_k(t+\sigma) = c_k\big(\cos 2\pi k\sigma \underbrace{\cos 2\pi kt}_{=:\phi_k(t)} - \sin 2\pi k\sigma \underbrace{\sin 2\pi kt}_{=:\psi_k(t)}\big) \tag{A.2}$$

for $k \in \mathbb{N}$ and $\sigma, t \in \mathbb{S}^1$ and where $c_k = 2^{-1/6}(\pi k)^{-1/3} \in (0,1)$. From now on we fix a critical point $\sigma_* q_k$, that is $k \in \mathbb{N}$ and $\sigma \in \mathbb{S}^1$ are fixed from now on. Taking two t derivatives we conclude that

$$\frac{d^2}{dt^2}(\sigma_* q_k) = -(2\pi k)^2 \sigma_* q_k.$$

Since

$$\frac{d^2}{dt^2}(\sigma_* q_k) = \alpha \cdot (\sigma_* q_k)$$

we obtain

$$\alpha = \alpha(\sigma_* q_k) = -(2\pi k)^2 = -\frac{2}{c_k^6},$$

where the last equality is (2.2). The formula of the Hessian operator $A_{\sigma_* q_k}$ involves the L^2 norm of $\sigma_* q_k$ and, in addition, the formula of the non-local Lagrange functional $\mathcal{B}$ involves $\|\sigma_* \dot{q}_k\|^2$. Straightforward calculation shows that

$$\begin{aligned} \|\sigma_* q_k\|^2 = \|q_k\|^2 \overset{(2.2)}{=} \frac{c_k^2}{2} &= \frac{1}{2^{4/3}(\pi k)^{2/3}}, \\ \left\|\frac{d}{dt}(\sigma_* q_k)\right\|^2 = (2\pi k)^2 \frac{c_k^2}{2} &= 2^{2/3}(\pi k)^{4/3}. \end{aligned}$$

Thus

$$\mathcal{B}(\sigma_* q_k) = 2\|\sigma_* q_k\|^2 \|\frac{d}{dt}(\sigma_* q_k)\|^2 + \frac{1}{\|\sigma_* q_k\|^2} = 2^{1/3} 3 (\pi k)^{2/3}.$$

To calculate the formula of $A_{\sigma_* q_k}$ we write ξ as a Fourier series

$$\xi = \xi_0 + \sum_{\substack{n=1 \\ n \neq k}}^{\infty} (\xi_n \cos 2\pi nt + \xi^n \sin 2\pi nt) + \xi_k \cos 2\pi k(t+\sigma) + \xi^k \sin 2\pi k(t+\sigma)$$

where we shifted the k^{th} modes for the reasons explained next. That the coefficients $\xi_\cdot$ and $\xi^\cdot$ do in fact not depend on σ we will see right after (A.4).

Remark A.4 (Global trivialization). The above (partially shifted) Fourier basis depends on the point $\sigma_* q_k \in C_k$ of the k^{th} component of the critical manifold C. But note that this $\mathbb{S}^1$-family of Fourier bases gives a new global trivialization of the restriction of the tangent bundle of $W^{1,2}_\times$ to C_k, namely

$$T_{C_k} W^{1,2}_\times \simeq C_k \times W^{1,2}\bigl(\mathbb{S}^1, \mathbb{R}\bigr).$$

We use the orthogonality relation

$$\langle \sigma_*(\cos 2\pi k \,\cdot\,), \xi\rangle = \langle \cos 2\pi k(\,\cdot\, + \sigma), \xi\rangle = \frac{1}{2}\,\xi_k$$

to calculate the product

$$\langle \sigma_* q_k, \xi\rangle = \langle c_k \cos 2\pi k(\,\cdot\, + \sigma), \xi\rangle = \frac{1}{2}\, c_k \xi_k.$$

Putting everything together we recover for the slightly more general case $\sigma_* q_k$ the result we derived in [5] for q_k, namely

Lemma A.5 (Critical values and Hessian). *The critical points of $\mathcal{B}$ are of the form* (A.2) *and at any such $\sigma_* q_k$ the value of $\mathcal{B}$ is*

$$\mathcal{B}(\sigma_* q_k) = 2^{1/3}\, 3(\pi k)^{2/3}$$

and the Hessian operator (A.1) *of $\mathcal{B}$ is*

$$A_{\sigma_* q_k}\xi = -\ddot{\xi} - (2\pi k)^2 \xi + 12(2\pi k)^2 \xi_k \cos 2\pi k(\,\cdot\, + \sigma) \tag{A.3}$$

for every $\xi \in W^{2,2}\bigl(\mathbb{S}^1, \mathbb{R}\bigr)$.

Eigenvalues and Morse index. Recall that $k \in \mathbb{N}$ and $\sigma \in \mathbb{S}^1$ are fixed, that is we consider the given critical point $\sigma_* q_k$. For the Hessian $A_{\sigma_* q_k}$ given by (A.3) we are looking for solutions of the eigenvalue problem

$$A_{\sigma_* q_k}\xi = \mu\xi$$

for $\mu = \mu(\xi; k, \sigma) \in \mathbb{R}$ and $\xi \in W^{2,2}\bigl(\mathbb{S}^1, \mathbb{R}\bigr) \setminus \{0\}$. Observe that

$$\begin{aligned}
-\ddot{\xi}(t) &= \sum_{\substack{n=1\\ n\neq k}}^{\infty} (2\pi n)^2 \bigl(\xi_n \cos 2\pi n t + \xi^n \sin 2\pi n t\bigr) \\
&\quad + (2\pi k)^2 \xi_k \cos 2\pi k(t+\sigma) + (2\pi k)^2 \xi^k \sin 2\pi k(t+\sigma), \\
-(2\pi k)^2 \xi(t) &= -(2\pi k)^2 \xi_0 - (2\pi k)^2 \sum_{\substack{n=1\\ n\neq k}}^{\infty} \bigl(\xi_n \cos 2\pi n t + \xi^n \sin 2\pi n t\bigr) \\
&\quad - (2\pi k)^2 \xi_k \cos 2\pi k(t+\sigma) - (2\pi k)^2 \xi^k \sin 2\pi k(t+\sigma).
\end{aligned}$$

Comparing coefficients in the eigenvalue equation $A_{\sigma_* q_k}\xi = \mu\xi$ we obtain eigenvectors (left hand side) and eigenvalues (right hand side) as follows

$$\begin{aligned} &\cos 2\pi k(t+\sigma) && \left\{\mu\xi_k = 12(2\pi k)^2\xi_k,\right. \\ &\sin 2\pi k(t+\sigma) && \left\{\mu\xi^k = 0,\right. \\ &\cos 2\pi nt && \left\{\mu\xi_n = 4\pi^2\big(n^2-k^2\big)\xi_n \quad \text{for all } n\in\mathbb{N}_0\setminus\{k\},\right. \\ &\sin 2\pi nt && \left\{\mu\xi^n = 4\pi^2\big(n^2-k^2\big)\xi^n \quad \text{for all } n\in\mathbb{N}\setminus\{k\}.\right. \end{aligned} \tag{A.4}$$

We observe that the right hand sides do not depend on σ. Therefore the eigenvalues μ, as well as the coefficients $\xi_\cdot$ and $\xi^\cdot$ of the σ-dependent Fourier basis, are all σ-independent. Since the σ-dependent Fourier basis gives rise to the global trivialization we observe that the negative and the positive part of the normal bundle are both trivial. This proves part (a) of Lemma A.2.

For the Hessian $A_{\sigma_* q_k}$ one obtains the same eigenvalues and multiplicities as we obtained in [5] in the unshifted case q_k. Indeed the eigenvalues of the Hessian $A_{\sigma_* q_k}$ are given by

$$\mu_n := 4\pi^2(n^2-k^2), \quad n\in\mathbb{N}\setminus\{k\} \tag{A.5}$$

and by

$$\mu_0 := -4\pi^2k^2, \qquad \mu_k := 0, \qquad \widehat{\mu}_k := 12(2\pi k)^2.$$

Moreover, their multiplicity (the dimension of the eigenspace) is given by

$$m(\mu_n) = 2, \quad n\in\mathbb{N}\setminus\{k\}, \qquad m(\mu_0) = m(\mu_k) = m(\widehat{\mu}_k) = 1.$$

Observe that the eigenvalue $\widehat{\mu}_k \neq \mu_n$ is different from μ_n for every $n\in\mathbb{N}_0$. Indeed, suppose by contradiction that $\widehat{\mu}_k = \mu_n$ for some $n\in\mathbb{N}_0$, that is

$$48\pi^2k^2 = 4\pi^2(n^2-k^2) \;\Leftrightarrow\; 13k^2 = n^2 \tag{A.6}$$

which contradicts that n is an integer. This proves part (b) of Lemma A.2.

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