Real Quadratic Rational Maps

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Two Points of View

Any real rational map f can be studied in two different ways:

As a piecewise monotone map from the circle $\mathbb{P}^1(\mathbb{R})$ to itself, f can be studied by very elementary methods.

But f always extends to a rational map $f_{\mathbb{C}}$ from the Riemann sphere $\mathbb{P}^1(\mathbb{C})$ to itself, so that all of the tools of holomorphic dynamics are also available.

This talk will study the very special case of

Real Quadratic Maps, with Real Critical Points.

The Moduli Space $M = V/G$

Let V be the smooth manifold consisting of all real quadratic maps

$$f(x) \;=\; \frac{ax^2 + bx + c}{a'x^2 + b'x + c'}$$

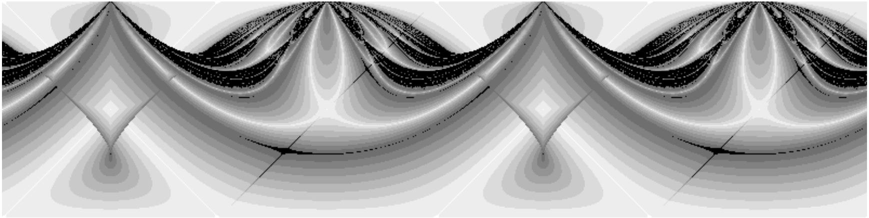
with real critical points. Let G be the group of orientation preserving fractional linear transformations

$$L(x) \;=\; \frac{\alpha x + \beta}{\gamma x + \delta} \qquad \text{with} \quad \alpha\delta - \beta\gamma > 0,$$

acting on V by conjugation $f \mapsto L \circ f \circ L^{-1}$.

Then we can form the quotient space $M = V/G$.

Assertion: M is a Topological Cylinder



This is a picture of part of the universal covering space of M .

The white regions correspond to maps such that both critical orbits converge to strongly attracting orbits of low period;

while the black points correspond to maps for which at least one critical orbit does not converge to an attracting periodic orbit.

Canonical Normal Form

Main Lemma. Every map in V is conjugate under G to a unique map which satisfies three conditions:

(1): f has the form

$$f(x) = \frac{A\,x^2 + B}{C\,x^2 + D}\,,$$

(Proof: Put the critical points at zero and infinity.)

(2): $AD - BC > 0$,

(Conjugate by $L(x) = -1/x$ if necessary.)

(3): $A^2 + C^2 = B^2 + D^2$.

(Conjugate by $L(x) = k\,x$. The required equation is satisfied for one and only one $k > 0$.)

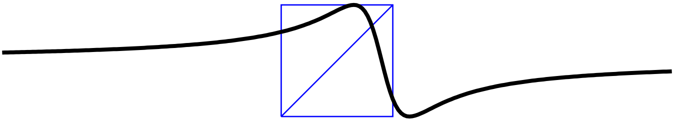
The resulting map f is then uniquely determined! $\square$

(But A, B, C, D are only unique up to multiplication by a common non-zero constant.)

The Invariant Interval $f(\widehat{\mathbb{R}})$.

If we think of f as a map from the circle $\widehat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ to itself, then:

The image $f(\widehat{\mathbb{R}})$ is a closed interval bounded by the two critical values.



Evidently all of the interesting dynamics is concentrated in this interval.

Theorem 1. *For any closed interval $\mathcal{I}$ contained in the circle $\widehat{\mathbb{R}}$ there is one and only one map f in canonical form for which $f(\widehat{\mathbb{R}}) = \mathcal{I}$.*

Thus the moduli space M is homeomorphic to the set of all intervals $\mathcal{I} \subset \widehat{\mathbb{R}}$.

Proof: (Identifying $\widehat{\mathbb{R}}$ with $\mathbb{R}/\mathbb{Z}$)

For any map $f(x) = \dfrac{A\,x^2 + B}{C\,x^2 + D}$, the two critical values are

$$v_0 = f(0) = B/D \qquad \text{and} \qquad v_\infty = f(\infty) = A/C \ .$$

If we normalize the four coefficients A, B, C, D so that $A^2 + C^2 = B^2 + D^2 = 1$, then we can set

$$A = \sin(\pi t_\infty),\ C = \cos(\pi t_\infty); \quad B = \sin(\pi t_0),\ D = \cos(\pi t_0) \ .$$

Thus $v_\infty = \tan(\pi t_\infty)$ and $v_0 = \tan(\pi t_0)$.

By definition, t_∞ and t_0 are the **critical value angles.**

More generally, any angle $t \in \mathbb{R}/\mathbb{Z}$ *corresponds to a unique point* $\tan(\pi t) \in \widehat{\mathbb{R}}$.

Theorem 2. *M* is diffeomophic to $\mathbb{R}/2\mathbb{Z} \times (0,1)$.

Proof. We have two distinct critical value angles

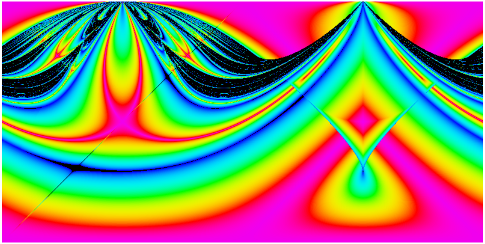
$$t_\infty,\ t_0\ \in\ \mathbb{R}/\mathbb{Z}\ ,$$

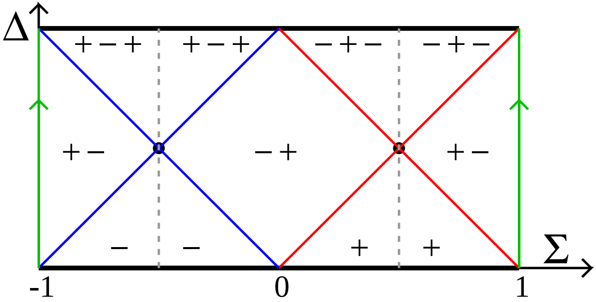
Lift to points $\widehat{t}_\infty,\ \widehat{t}_0\ \in\ \mathbb{R}$ so that $\widehat{t}_0\ <\ \widehat{t}_\infty\ <\ \widehat{t}_0+1\,.$
Then the corresponding point of M is uniquely determined by the two numbers

$$\Sigma = \widehat{t}_\infty + \widehat{t}_0 \pmod{2\mathbb{Z}}\,,\quad \text{and}\quad \Delta = \widehat{t}_\infty - \widehat{t}_0 \in (0,1)\ . \qquad \square$$

Here Δ is precisely the length of the interval $f(\widehat{\mathbb{R}})$, lifted to $\mathbb{R}/\mathbb{Z}$.

Two Pictures of M.

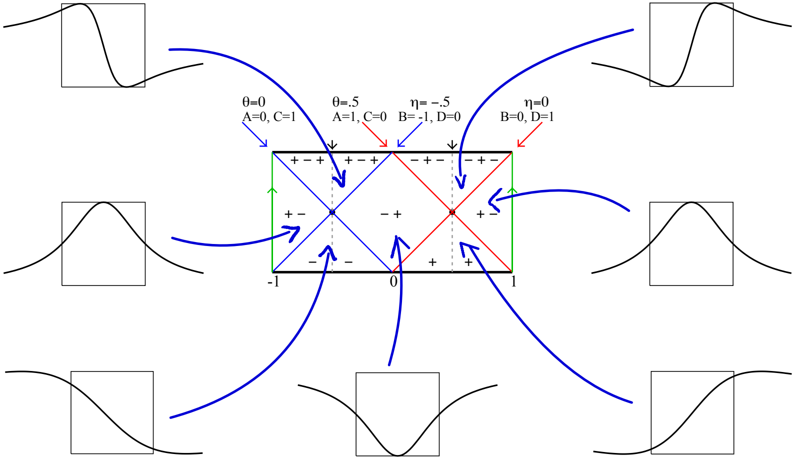




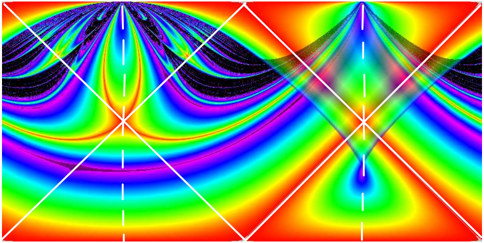
Red Lines: Polynomials (with critical fixed point).

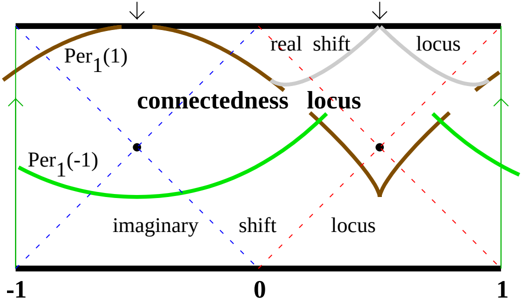
Blue lines: “co-polynomials”. (One critical point maps to the other.)

The Six Regions in Moduli Space



More about M .





Chebyshev curve: with $f(\text{critical value}) =$ fixed point with $\mu > 1$.

The Shift Locus

Every complex quadratic rational map either:

(1) belongs to the **connectedness locus**,

$\Leftrightarrow$ connected Julia set, or

(2) belongs to the **shift locus**,

$\Leftrightarrow$ totally disconnected Julia set.

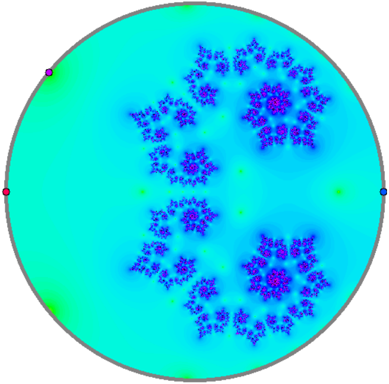
For our real maps f, there is a further distinction:

real shift locus

$\Leftrightarrow \quad J(f_{\mathbb{C}}) \subset \widehat{\mathbb{R}}$,

imaginary shift locus

$\Leftrightarrow \quad J(f_{\mathbb{C}}) \cap \widehat{\mathbb{R}} = \emptyset$.



Critically finite maps fall into three types:

Hyperbolic.

(A quadratic map is **hyperbolic** *if both critical orbits converge to attracting periodic orbits.)*

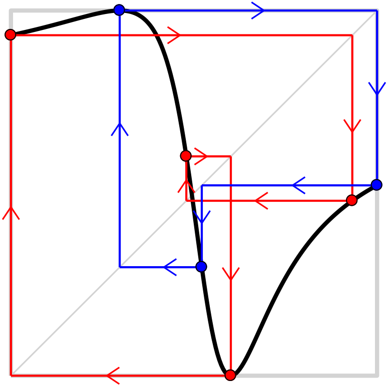
Half-Hyperbolic *if only one critical orbit converges to an attracting periodic orbit.*

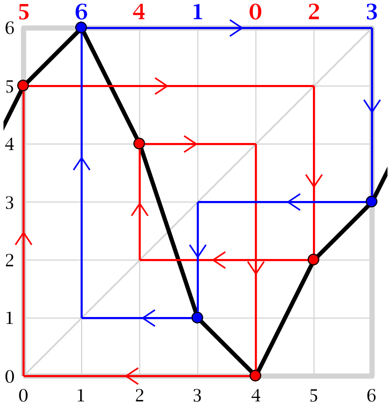
Totally Non-Hyperbolic *if no critical orbit converges to an attracting periodic orbit.*

Every hyperbolic and critically finite f is the center point of a **hyperbolic component** in the connectedness locus.

But if f *is totally non-hyberbolic then* $J(f_{\mathbb{C}})$ *is the entire Riemann sphere.*

Combinatorics by Example: The Wittner Map





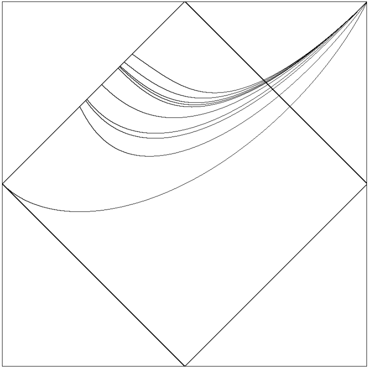
With combinatorics: (5, 6, 4, 1, 0 2, 3).

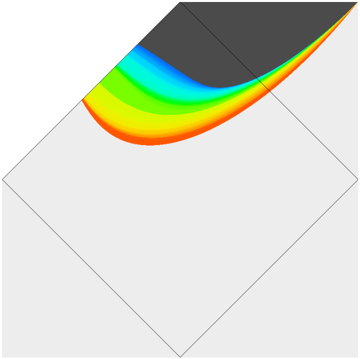
Theorem: Any critically finite point in *M*
is uniquely determined by its combinatorics.

[movie]

By definition, a **bone** in M is a maximal smooth curve on which one of the two critical points is periodic.

In the unimodal region:



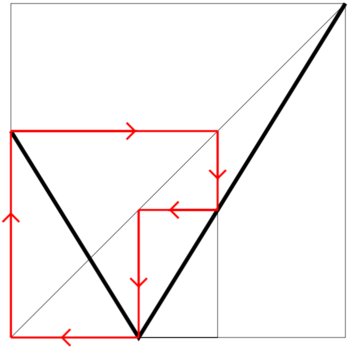


Theorem of Filom + Yan Gao: Each bone in the unimodal region is a smooth curve from polynomial to co-polynomial;

and each locus of constant topological entropy is connected.

Maps with Constant |Slope|.

Misiurewicz showed that a map with constant $|\text{slope}| = s > 1$ has topological entropy $h = \log(s)$.



Here $s = (\sqrt{5}+1)/2$.

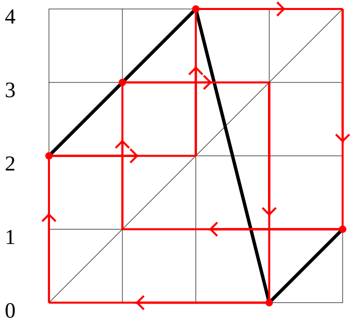
Theorem. To every critically finite co-polynomial there corresponds a critically finite polynomial.

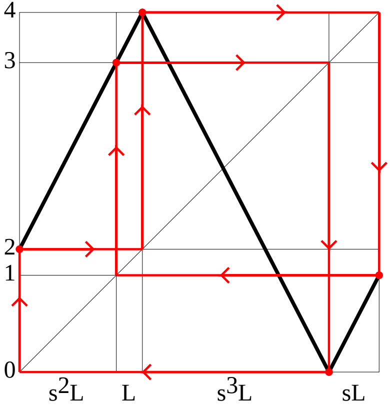
(Conversely to almost every critically finite polynomial there corresponds a critically finite co-polynomial.)

The Filom-Pilgrim family of maps.

Given $0 < p/q < 1$ consider the combinatorics
$(m_0, m_1, \cdots, m_{q-1})$ with $m_k \equiv k + p \pmod{q}$.

Here is the PL model
for $p/q = 2/5$.





Since the longest edge maps onto the entire interval, we get

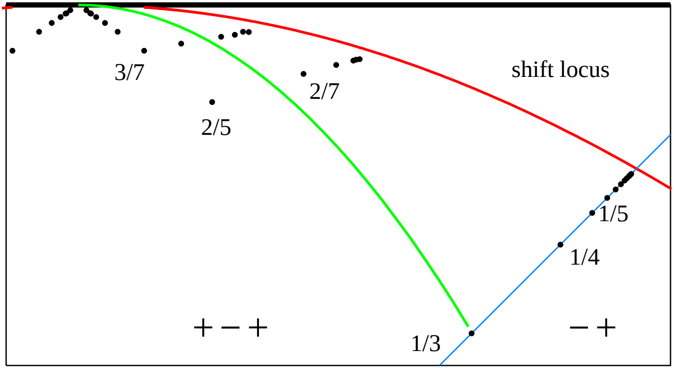
$$s^4 - s^3 - s^2 - s - 1 = 0 .$$

Thus the topological entropy $\log(s_q)$ depends only on q.

Theorem (Filom and Pilgrim).

In the $+-+$ region, loci of constant topological entropy can have arbitrarily many connected components.

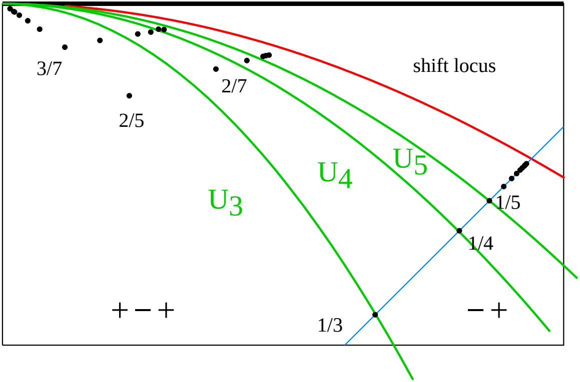
Step 1: Each hyperbolic component $H(p/q)$ contains a curve leading from the center point to the ideal point $(\Sigma,\ \Delta) = (-.5,\ 1)$.



Furthermore these curves depend monitonically on p/q.

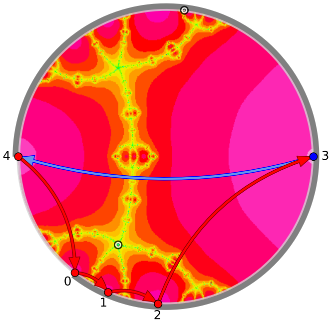
From Ideal Point to Ideal Point:

For each $q \geq 3$ there is a curve $\mathcal{C}_q$ of constant entropy $\log(s_q)$ which extends from the left ideal point $(-.5, 1)$ through $f_{1/q}$ to the right and ideal point $(+.5, 1)$. (Compare Slide 14.)

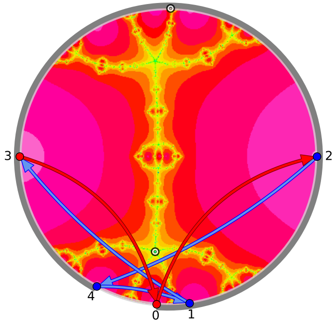


The curves $\mathcal{C}_q$ divide the square region $-.5 \leq \Sigma \leq .5$ into disjoint connected open sets $U_3,\ U_4,\ U_5,\ \cdots$.

If $q > n(n-1)$ is prime, there is a p so that $f_{p/q} \in U_n$.



$H(1/5)$



$H(2/5)$

Key Lemma. As f tends to the ideal point $(-.5,\ 1)$ within the hyperbolic component $H(p/q)$, the multiplier of the fixed point of $f_{\mathcal{C}}$ in the upper half plane tends to $e^{2\pi i\, p/q}$.

Proof that $|\mu_1| \to 1$.

If we are given two of the fixed point multipliers for a quadratic rational map, then the third is given by

$$\mu_3 = \frac{2 - \mu_1 - \mu_2}{1 - \mu_1 \mu_2} .$$

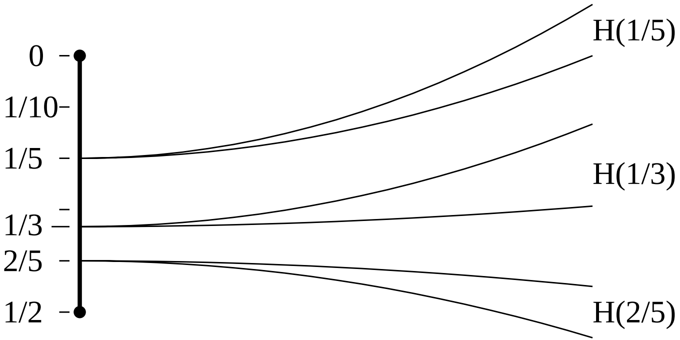
Now suppose that $\mu_1 = r\, e^{i\theta}$ and $\mu_2 = r\, e^{-i\theta}$. Then

$$\mu_3 = \frac{2 - 2\, r \cos(\theta)}{1 - r^2} .$$

If the map has real critical points then we must have $r \geq 1$.

If $\mu_3 \to -\infty$, it follows easily that $r \to 1$. $\square$

Blowing Up the Ideal Point



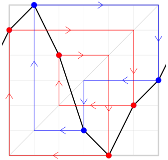
This is a hypothetical picture of what we would get if we replace the ideal point $(-.5,\ 1)$ by an entire vertical interval of angles $0 \le \theta \le 1/2$.

This completes the outlined proof of non-monotonicity.

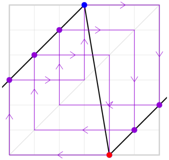
For further details see Filom and Pilgrim's paper.

Examples of the Thurston Pullback Map.

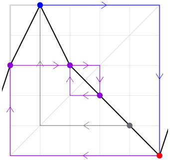
Wittner $((5,6,1,0,2,3))$



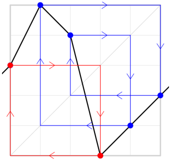
Filom-Pilgrim $3/7$: $((3,4,5,6,0,1,2))$



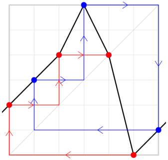
Weakly Obstructed $((3,4,3,2,1,0))$



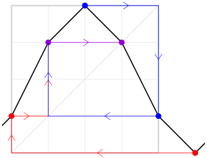
Str. Obstructed $((3,5,4,0,1,2))$



Str. Obstructed $((2,3,4,6,4,0,1))$



Exceptional $((1,3,4,3,1,0))$



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In preparation:

A more detailed manuscript, plus an interactive web site.