Snapshots of Topology in the 50's

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Zum 90. Geburtstag von Beno Eckmann

Beno Eckmann and Heinz Hopf in 1953

(from the Oberwolfach collection)

Paul Alexandroff and Heinz Hopf in 1931

in Zürich
(from the Oberwolfach collection)

Hopf had many students.

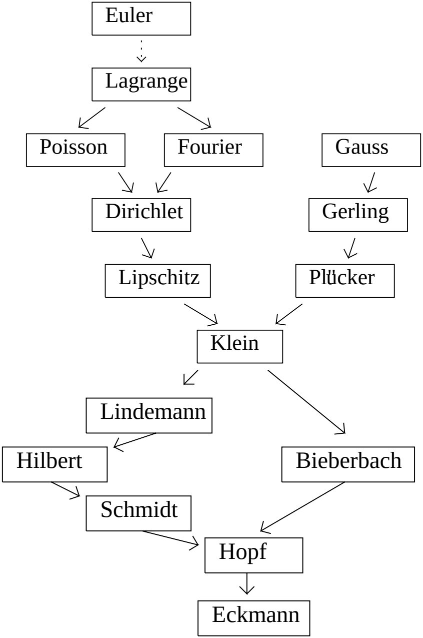
According to the Mathematics Genealogy Project, he has

2212 mathematical descendants

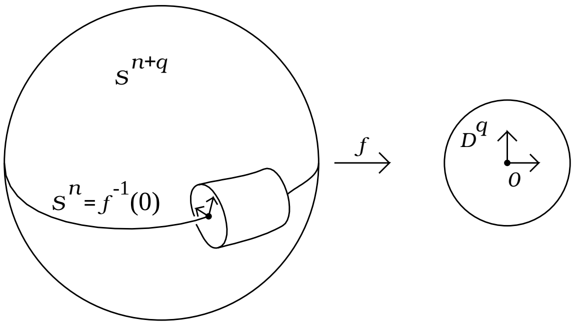
More than one third of these are descended from Hopf

via Beno Eckmann.

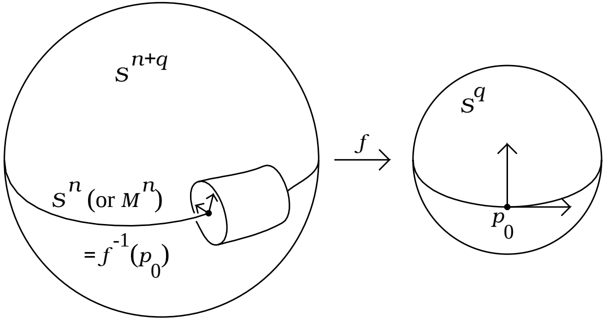
Genealogy


Euler
Lagrange
Poisson
Fourier
Gauss
Dirichlet
Gerling
Lipschitz
Plücker
Klein
Lindemann
Hilbert
Bieberbach
Schmidt
Hopf
Eckmann

$$J : \pi_n(\mathrm{SO}(q)) \to \pi_{n+q}(S^q)$$



Consider a tubular neighborhood N of a great n-sphere in S^{n+q}. A framing of its normal bundle, described by an element of $\pi_n(\mathrm{SO}(q))$, gives rise to a map from N to the unit disk D^q.



Now collapse the boundary of D^q to a point p_1, yielding a sphere S^q, so that

$$(N, \partial N) \xrightarrow{f} (D^q, \partial D^q) \to (S^q, p_1),$$

and map all of $S^{n+q} \setminus N$ to p_1. The resulting map $g : S^{n+q} \to S^q$ represents the required element of $\pi_{n+q}(S^q)$.

George Whitehead, 1950: A generalization of the Hopf invariant, Ann. of Math. **51**.

Based on:

Heinz Hopf, 1931: Über die Abbildungen der dreidimensionalen Sphäre auf die Kugelfläche, Math. Ann. **104**.

Heinz Hopf, 1935: Über die Abbildungen von Sphären auf Sphären niedriger Dimension, Fundamenta Math. **25**.

and also:

Hans Freudenthal, 1937: Über die Klassen der Sphärenabbildungen, Compositio Math **5**.

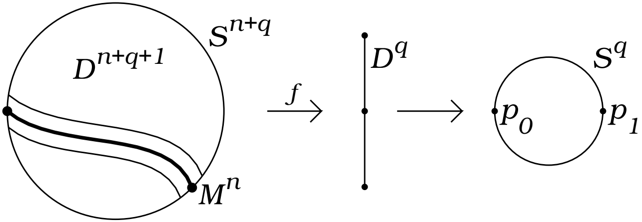
Beno Eckmann, 1942: Über die Homotopiegruppen von Gruppenräumen, Comment. Math. Helv. **14**).

Beno Eckmann, 1943: Stetige Lösungen linearer Gleichungssysteme, Comment. Math. Helv. **15**.

Beno Eckmann, 1943: Systeme von Richtungsfeldern in Sphären und stetige Lösungen komplexer linearer Gleichungen, Comment. Math. Helv. **15**.

More General Construction:

Lev Pontrjagin, 1955: Smooth Manifolds and their Applications in Homotopy Theory (Russian); AMS Translation 1959.



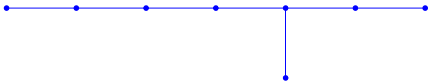
In place of the sphere $S^n \subset S^{n+q}$, consider any closed submanifold $M^n \subset S^{n+q}$ which has trivial normal bundle. Choosing some normal framing, the analogous construction yields a map $g : S^{n+q} \to S^q$, with $g^{-1}(p_0) = M^n$.

Theorem (Pontrjagin)

This g extends to a map $\widehat{g} : D^{n+q+1} \to S^q$ if and only if the framed manifold $M^n \subset S^{n+q}$ is the boundary of a framed manifold $W^{n+1} \subset D^{n+q+1}$.

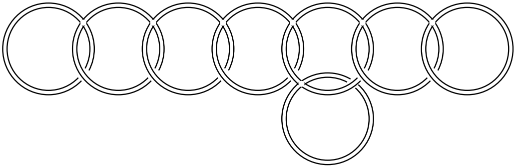
Work with Michel Kervaire.

We construct a manifold W^{4k} having the homotopy type of a bouquet $S^{2k} \vee \cdots \vee S^{2k}$ of eight copies of the $2k$-sphere.



The homology group $H_{2k}(W^{4k})$ is free abelian with one basis element for each dot in this "E_8 diagram," with intersection pairing:

$$e_i \cdot e_j = \begin{cases} +2 & \text{if } i = j \\ +1 & \text{if the dots are joined by a line} \\ 0 & \text{otherwise} \end{cases}$$



This intersection pairing $H_{2k} \otimes H_{2k} \to \mathbb{Z}$ is positive definite, with determinant $+1$.

It follows that $M^{4k-1} = \partial W^{4k}$ has the homology of S^{4k-1}.

In fact, if $k > 1$, then M^{4k-1} has the homotopy type of a sphere, and hence, by Smale, is a topological sphere.

But M^{4k-1} is not diffeomorphic to S^{4k-1}.

Spectral sequences of fibrations provide a powerful tool for studying homotopy groups.

Theorem

The homotopy groups $\pi_{n+q}(S^q)$ *with* $n > 0$ *are finite, except in the cases studied by Hopf:*

$$\pi_{4k-1}(S^{2k}) \cong \mathbb{Z} \oplus (\mathrm{finite}).$$

The set Ω_n of cobordism classes of closed oriented n-manifolds forms a finitely generated abelian group.

Theorem

The class of a $4k$-manifold, modulo torsion, is determined by its Pontrjagin numbers $p_{i_1} \cdots p_{i_r}[M^{4k}] \in \mathbb{Z}$.

Theorem

The signature $\mathrm{sgn}(M^{4k})$ of the intersection number pairing

$$\alpha, \beta \in H_{2k}(M^{4k}) \mapsto \alpha \cdot \beta \in \mathbb{Z}$$

is a cobordism invariant; and hence is determined by Pontrjagin numbers.

worked out the precise formula for signature as a function of Pontrjagin numbers.

In particular, for a manifold with $p_1 = p_2 = \cdots = p_{k-1} = 0$ we have

$$\mathrm{sgn}(M^{4k}) = s_k\, p_k[M^{4k}]\,, \qquad s_k = \frac{2^{2k}(2^{2k-1}-1)B_k}{(2k)!}\,,$$

where

$$B_1 = 1/3\,, \ \ B_2 = 1/30\,, \ \ B_3 = 1/42\,, \ \ \ldots$$

are Bernoulli numbers.

Proof (for small k). Recall that M^{4k-1} was constructed as the boundary of a parallelizable manifold W^{4k} whose intersection pairing is positive definite, of signature +8.

If M^{4k-1} were diffeomorphic to S^{4k-1}, then we could paste on a $4k$-disk to obtain a smooth manifold

$$M^{4k} \;=\; W^{4k} \cup_{S^{4k-1}} D^{4k}$$

with $p_1 = \cdots p_{k-1} = 0$, and with signature $+8$.

Then, according to Hirzebruch

$$p_k[M^{4k}] \;=\; \mathrm{sgn}(M^{4k})/s_k \;=\; 8/s_k\,.$$

But (at least for small $k > 1$), this is not an integer:

$$8/s_2 = 2^3 \cdot 3^2 \cdot 5/7\,, \qquad 8/s_3 = 2^2 \cdot 3^3 \cdot 5 \cdot 7/31\,, \ldots \quad \square$$

Definition

A smooth closed manifold M is **almost parallelizable** if $M^n \smallsetminus \text{(point)}$ is parallelizable.

Then we can write $M^n = M^n_0 \cup_{S^{n-1}} D^n,$ where M^n_0 is parallelizable.

Embedding (M^n_0, S^{n-1}) in (D^{n+q}, S^{n+q-1}), we can frame M^n_0.

Using Pontrjagin's Theorem, the induced framing of S^{n-1} represents an element of the kernel of the J-homomorphism

$$J : \pi_{n-1}(\mathrm{SO}_q) \; \to \; \pi_{n+q-1}(S^q)\,.$$

Conversely, every element in the kernel of J arises in this way, from some almost parallelizable manifold.

Morse Theory can be used to compute the stable homotopy groups of rotation groups. In particular: $\pi_{4k-1}(\mathrm{SO}) \cong \mathbb{Z}\,.$

Theorem

A generator of $\pi_{4k-1}(\mathrm{SO})$ *corresponds to an* SO*-bundle* ξ *over* S^{4k} *with Pontrjagin class*

$$p_k(\xi) \in H^{4k}(S^{4k}) \cong \mathbb{Z}\,,$$

equal to $(2k-1)!\epsilon_k$ *where* $\epsilon_k = \mathrm{GCD}(2, k+1)$.

According to Serre, the stable homotopy groups of spheres are finite. Thus the stable J-homorphism

$$J : \pi_{4k-1}(\mathrm{SO}) \to \Pi_{4k-1}$$

maps a free cyclic group to a finite group.

Let M_0^{4k} be an almost parallelizable manifold with the smallest possible positive signature $\mathrm{sgn}(M^{4k}) > 0$.
It follows that:

$$\left|J\big(\pi_{4k-1}(\mathrm{SO})\big)\right| \;=\; \frac{p_k[M_0^{4k}]}{(2k-1)!\epsilon_k}\,,$$

where $\epsilon_k = \mathrm{GCD}(2, k+1)$.

The following sharp estimate was obtained around this time:

Theorem (Hirzebruch)

For any M^{4k} with $w_2 = 0$, the $\widehat{A}$-genus

$$\widehat{A}(M^{4k}) \;=\; -B_k p_k[M^{4k}]/2(2k)! + \cdots$$

is an integer, divisible by ϵ_k.

Combining these two results, Kervaire and I obtained:

Theorem (1958)

$$|J\big(\pi_{4k-1}(\mathrm{SO})\big)| \equiv 0 \pmod{\mathrm{denominator}(B_k/4k)}.$$

Here:

$k =$	1	2	3	4	5	6	$\cdots$
$\frac{B_k}{4k} =$	$\frac{1}{2^3\cdot 3}$	$\frac{1}{2^4\cdot 3\cdot 5}$	$\frac{1}{2^3\cdot 3^2\cdot 7}$	$\frac{1}{2^5\cdot 3\cdot 5}$	$\frac{1}{2^3\cdot 3\cdot 11}$	$\frac{691}{2^4\cdot 3^2\cdot 5\cdot 7\cdot 13}$	$\cdots$

Later, Frank Adams obtained the precise result:

Theorem (1966)

In fact the order is precisely equal to the denominator of $B_k/4k$.

HAPPY BIRTHDAY BENO!!