Understanding Cubic Maps

John Milnor

Stony Brook University (*www.math.sunysb.edu*)

TOULOUSE, DECEMBER 2009

Parameter Space

PROBLEM: To study cubic polynomial maps F with a critical point which is periodic under F.

—work with Araceli Bonifant and Jan Kiwi—

Normal form:

Any cubic polynomial map is affinely conjugate to a monic centered map

$$F(z) = F_{a,v}(z) = (z-a)^2(z+2a)+v\,.$$

Here a *is the* **marked critical point***,*
and $F(a)=v$ *is the* **marked critical value***.*

The **parameter space** for the family of all such maps is the set of all pairs $(a,v) \in \mathbb{C}^2$.

The Period p Curve

Definition. The **period p curve** $\mathcal{S}_p$ consists of those parameter pairs $(a, v) \in \mathbb{C}^2$ such that that marked critical point a for $F = F_{a,v}$ has period *exactly* p.

(Conjecture: $\mathcal{S}_p$ is irreducible for all $p \geq 1$.)

Degree computation: The set of parameter pairs (a, v) which satisfy the polynomial equation

$$F^{\circ p}(a) = a \qquad (1)$$

forms a smooth affine variety

$$\mathcal{S}_p^{\oplus} = \bigsqcup_{n|p} \mathcal{S}_n \subset \mathbb{C}^2 .$$

Equation (1) has degree 3^{p-1}. Hence the degree d_p of $\mathcal{S}_p$ can be computed inductively from the equation

$$\sum_{n|p} d_n = 3^{p-1} .$$

$$d_1 = 1 , \quad d_2 = 2 , \quad d_3 = 8 , \quad d_4 = 24 , \quad d_5 = 80 , \quad \ldots .$$

Canonical Coordinates for $\mathcal{S}_p$

Define $H_p : \mathbb{C}^2 \to \mathbb{C}$ by

$$H_p(a, v) = F^{\circ p}(a) - a \,, \qquad \text{with} \quad F = F_{a,v} \,.$$

This vanishes everywhere on $\mathcal{S}_p$, with $dH_p \neq 0$ on $\mathcal{S}_p$.

Think of H_p as a complex Hamiltonian function, and consider the Hamiltonian differential equation

$$\frac{da}{dt} = \frac{\partial H_p}{\partial v} \,, \qquad \frac{dv}{dt} = -\frac{\partial H_p}{\partial v} \,.$$

The local solutions $t \mapsto (a, v) = \big(a(t),\ v(t)\big)$ are holomorphic, and lie in curves $H_p = \text{constant}$.

Those solutions which lie in $\mathcal{S}_p$ provide a local holomorphic parametrization, unique up to a translation, $t \mapsto t + \text{constant}$.

Equivalently, the holomorphic 1-form

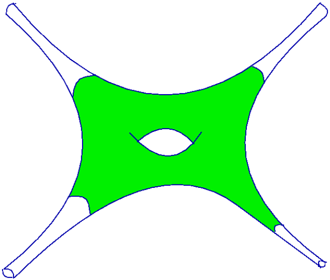
$$dt = \frac{da}{\partial H_p / \partial v} \quad \text{and/or} \quad \frac{-dv}{\partial H_p / \partial a}$$

is well defined and non-zero everywhere on $\mathcal{S}_p$.

Smooth compactification

More generally, any smooth affine curve $\mathcal{S} \subset \mathbb{C}^2$ has such a canonical 1-form dt.

Such a curve can be decomposed (non-uniquely) into a compact subset, together with finitely many end regions $\mathcal{E}_h$, each conformally isomorphic to $\mathbb{C} \smallsetminus \overline{\mathbb{D}}$.



We can compactify, to obtain a smooth compact complex 1-manifold $\overline{\mathcal{S}}$, by adding a single ideal point ∞_h to each end region $\mathcal{E}_h$.

Computation of the Euler characteristic

The holomorphic 1-form *dt* on $\mathcal{S}$ becomes a meromorphic 1-form on $\overline{\mathcal{S}}$, with zeros or poles only at the ideal points.

The **Euler characteristic** of $\overline{\mathcal{S}}$ can be computed as follows:

$$\chi(\overline{\mathcal{S}}) \;=\; \#(\text{poles}) \,-\, \#(\text{zeros})\,,$$

counted with multiplicity.

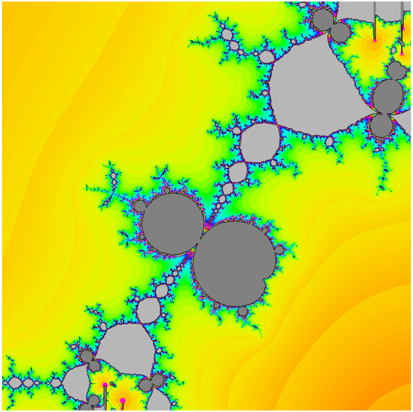
If $\mathcal{S}$ is connected, then

$$\text{genus}(\mathcal{S}) \;=\; \text{genus}(\overline{\mathcal{S}}) \;=\; 1 - \chi(\overline{\mathcal{S}})/2\,.$$

Special properties of the period p curve

There is a dynamically defined compact subset of $\mathcal{S}_p$, namely the **connectedness locus** $\mathcal{C}(\mathcal{S}_p)$ consisting of all maps $F \in \mathcal{S}_p$ such that the Julia set $J(F)$ is connected.

Each connected component $\mathcal{E}_h \subset \mathcal{S}_p \smallsetminus \mathcal{C}(\mathcal{S}_p)$, called an **escape region** in $\mathcal{S}_p$, is conformally isomorphic to $\mathbb{C} \smallsetminus \overline{\mathbb{D}}$.



The winding number

Theorem. *The residue of dt at each ideal point* $\infty_h \in \overline{\mathcal{S}}_p$ is zero:

$$\frac{1}{2\pi i}\oint_{\infty_h} dt \;=\; 0\,.$$

Thus t can be defined as a meromorphic function throughout any simply connected subset of $\overline{\mathcal{S}}_p$.

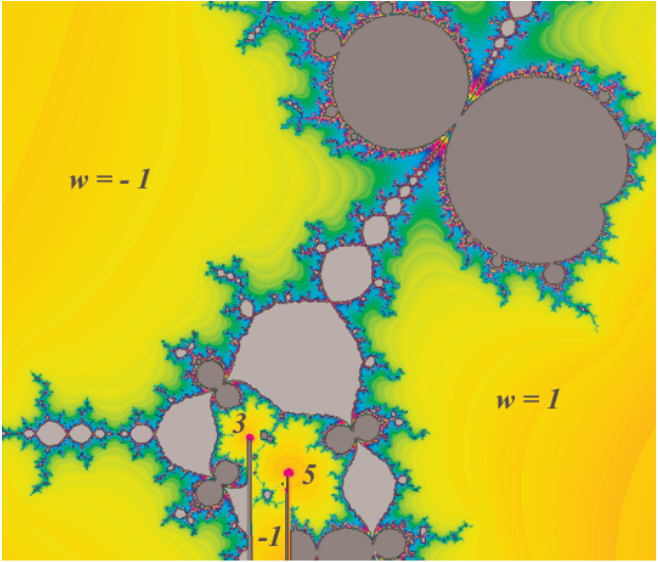
Normal form near an ideal point ∞_h: We can choose a local parameter ζ for $\overline{\mathcal{S}}_p$, and a canonical parameter t, so that

$$t \;=\; \zeta^{w_h}\,, \qquad \text{with} \qquad w_h \in \mathbb{Z}\,,\;\; w_h \neq 0\,.$$

Here w_h is the **winding number** of the t-plane around ∞_h.

$$\text{As} \quad \zeta \to 0\,, \qquad \text{note that} \;\; t \to \begin{cases} 0 & \text{if } w > 0\,, \\ \infty & \text{if } w < 0\,. \end{cases}$$

Winding number: examples in $\mathcal{S}_4$



Euler characteristic formulas

Since $t = \zeta^{w_h}$,

$$dt = d(\zeta^{w_h}) = w_h \zeta^{w_h - 1} d\zeta ,$$

with a zero of order $w_h - 1$ at the ideal point.

Thus the formula $\chi = \#(\text{poles}) - \#(\text{zeros})$ takes the form

$$\chi(\overline{\mathcal{S}}_p) = \sum_h (1 - w_h) ,$$

summed over all ideal points. With a lot of work, this yields

$$\chi(\overline{\mathcal{S}}_p) = (2 - p) d_p + (\text{number of ideal points}) .$$

(Key tool for the proof:
Kiwi's theory of dynamics, including Branner-Hubbard puzzles, over the completion of the field of formal Puiseux series.)

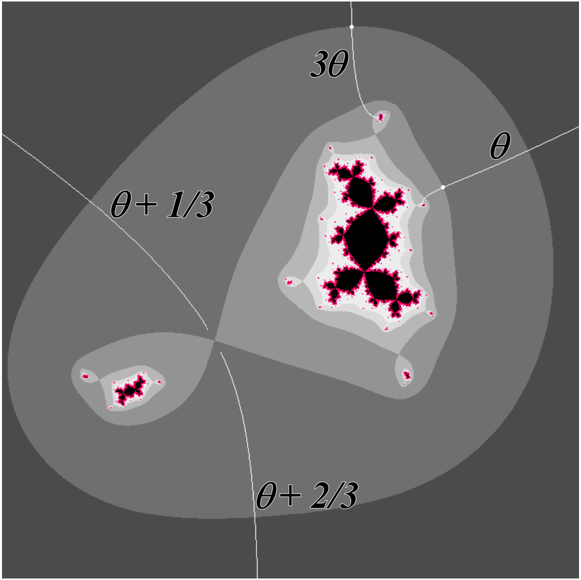
Examples:

$$\chi(\mathcal{S}_1) = \chi(\mathcal{S}_2) = 2 , \ \chi(\mathcal{S}_3) = 0 , \ \chi(\mathcal{S}_4) = -28 .$$

Further computations of $\chi(\overline{\mathcal{S}}_p)$ by Laura DeMarco

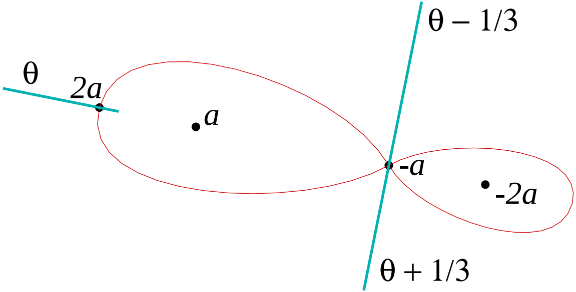
- Period 5: -184
- Period 6: -784
- Period 7: -3236
- Period 8: -11848
- Period 9: -42744
- Period 10: -147948
- Period 11: -505876
- Period 12: -1694848
- Period 13: -5630092
- Period 14: -18491088
- Period 15: -60318292
- Period 16: -195372312
- Period 17: -629500300
- Period 18: -2018178780
- Period 19: -6443997852
- Period 20: -20498523320

Dynamics: A sample Julia set picture



Filled Julia set for a map in the “rabbit” escape region of $\mathcal{S}_3$.

External rays in the dynamic plane



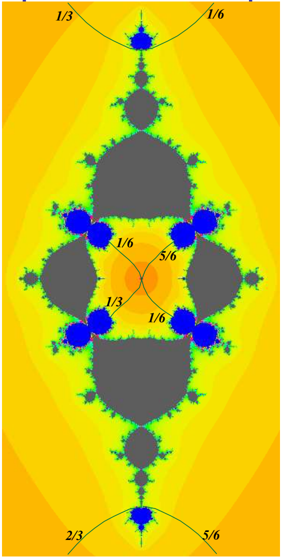
Sketch of the dynamic plane for a map belonging to any escape region $\mathcal{E}_h \subset \mathcal{S}_p$.

Critical points: $a\,,\ -a$.

Cocritical points: $2a\,,\ -2a\,,$ with $F(\pm 2a) = F(\mp a)\,.$

Definition: $\theta = \theta(F) \in \mathbb{R}/\mathbb{Z}$ is the **cocritical angle.**

Rays in parameter space: Examples in $\mathcal{S}_2$



The indicated rays all land at parabolic maps,
and have angles of the form $m/3n$.

Parameter rays

Each external ray in an escape region $\mathcal{E}_h \subset \mathcal{S}_p$ is labeled by its cocritical angle $\theta(F) \in \mathbb{R}/\mathbb{Z}$.

Theorem. *Every parameter ray with rational cocritical angle θ lands at a well defined map F_0 in the topological boundary $\partial\mathcal{E}_h \subset \mathcal{S}_p$.*

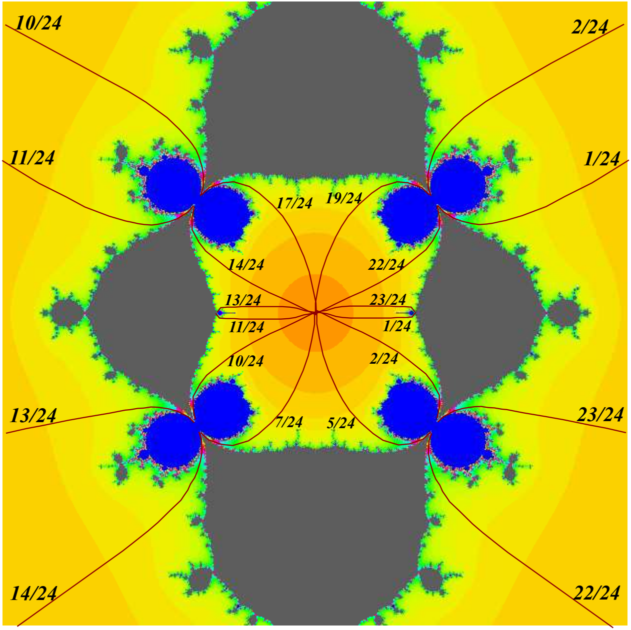
This landing map F_0 has a parabolic orbit

$\Longleftrightarrow$ *one of the two angles $\theta \pm 1/3$ is periodic.*

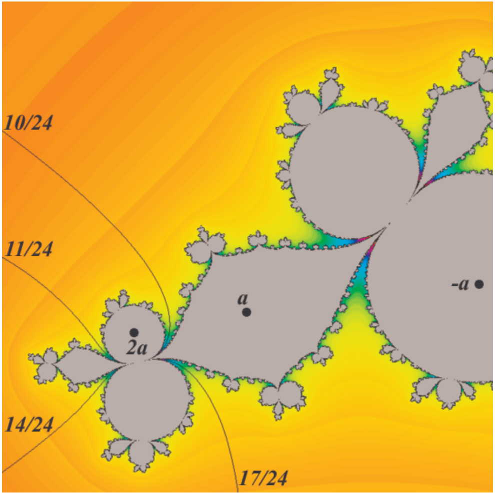
$\Longleftrightarrow$ *θ has the form $\frac{m}{3n}$ with $3 \not| m$ and $3 \not| n$.*

Complication: For each θ, there are μ_h distinct parameter rays in $\mathcal{E}_h$ with label θ, where $\mu_h \geq 1$ is an invariant called the **multiplicity** of $\mathcal{E}_h$.

More examples in $\mathcal{S}_2$



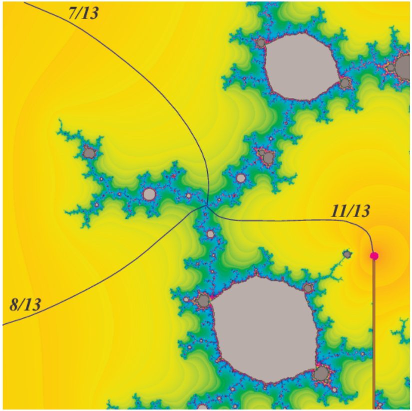
Picture of a corresponding of a Julia set $J(F_0)$

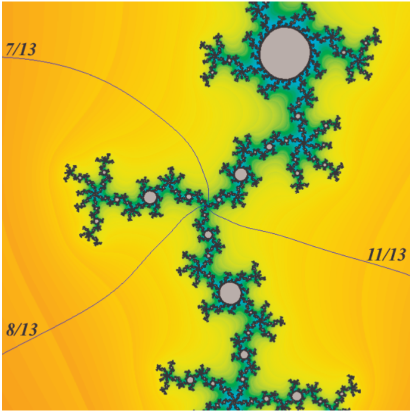


Here F_0 is the landing map for the 10/24, 11/24, 14/24, and 17/24 rays at the upper left of the previous figure.

Critically finite maps

Theorem: If the landing map for a rational parameter ray is not parabolic, then it is critically finite. An example in $\mathcal{S}_4$:





The same rays land at $F \in \mathcal{S}_p$ as at $2a_F \in J(F)$.

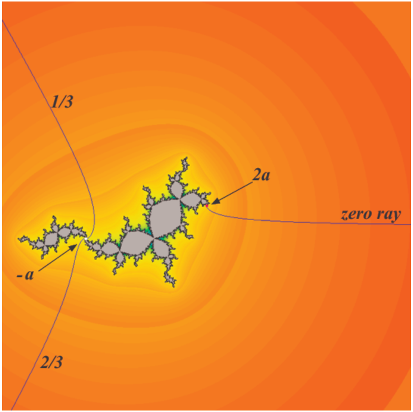
Asymptotic similarity (as in Tan Lei)

$F \in \mathcal{S}_p$ critically finite map, $\eta =$ multiplier of postcritical cycle.

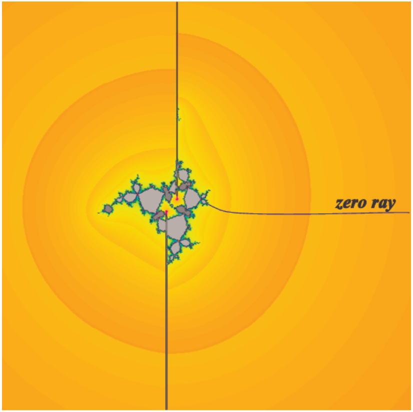
Kœnigs: There is a Hausdoff limit $\lim_{n\to\infty} \eta^n\big(K(F) - 2a\big)$.

Linear equivalence: $\cong \lim_{n\to\infty} \eta^n\big(\mathcal{C}(\mathcal{S}_p) - F\big)$

(interpreting last expression using a local parameter).



Julia set



parameter space (in $\mathcal{S}_3$)

References

J. Kiwi, *Puiseux series polynomial dynamics and iteration of complex cubic polynomials*, Ann. Inst. Fourier (Grenoble) **56** (2006) 1337–1404.

Cubic Polynomial Maps with Periodic Critical Orbit:

Part I, in "Complex Dynamics Families and Friends", ed. D. Schleicher, A. K. Peters 2009, pp. 333-411.

Part II: Escape Regions (with Bonifant and Kiwi), arXiv:0910.1866. To appear in Journal of Conformal Geometry and Dynamics.

Part III: External rays (with Bonifant), in preparation.