Critically Periodic Cubic Polynomials

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IN MEMORY OF ADRIEN DOUADY

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Parameter Space

THE PROBLEM: To study cubic polynomial maps F with a marked critical point which is periodic under F.

—work in progress with Araceli Bonifant—

Any cubic polynomial map with marked critical point is affinely conjugate to one of the form

$$F(z) \;=\; F_{a,v}(z) \;=\; z^3 - 3a^2 z + (2a^3 + v)\,.$$

Here a is the **marked critical point**,
$F(a) = v$ is the **marked critical value**,
$-a$ is the **free critical point.**

The set of all such maps $F = F_{a,v}$ *will be identified with the* **parameter space**, *consisting of all pairs* $(a, v) \in \mathbb{C}^2$.

The Period p Curve

Definition: the **period p curve** $\mathcal{S}_p \subset \mathbb{C}^2$, consists of all maps $F = F_{a,\,v}$ such that the marked critical point a has period exactly p.

Assertion. $\mathcal{S}_p$ is a smooth affine curve in $\mathbb{C}^2$.

Complication: The genus of $\mathcal{S}_p$ increases rapidly with p.

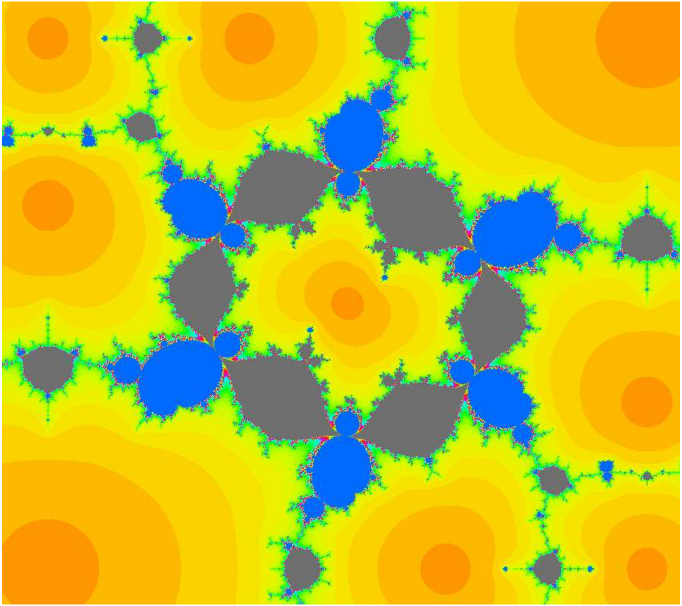
$\mathcal{S}_1$ has genus zero with one puncture ($\cong \mathbb{C}$),
$\mathcal{S}_2$ has genus zero with two punctures,
$\mathcal{S}_3$ has genus one with 8 punctures,
$\mathcal{S}_4$ has genus 15 with 20 punctures, $\cdots$

We can simplify a little by passing to to the **moduli space** $\mathcal{S}_p/\mathcal{I}$ of holomorphic conjugacy classes. Here $\mathcal{I}$ is the involution

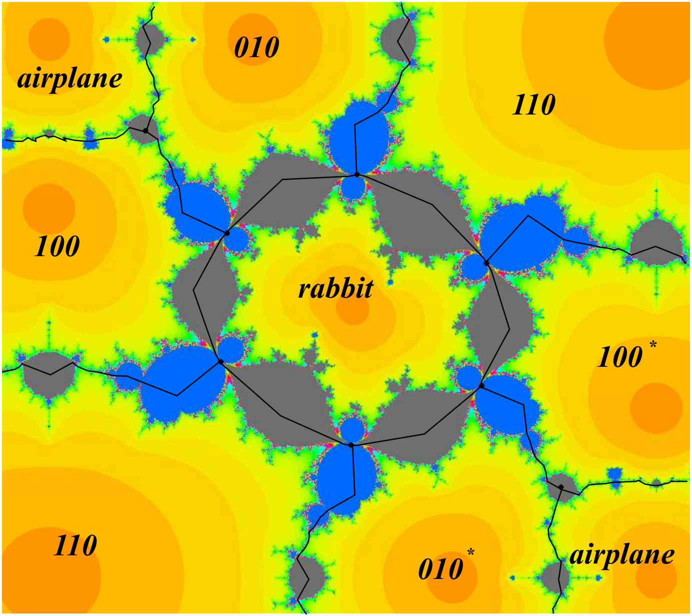
$$F(z) \;\leftrightarrow\; -F(-z)\,, \qquad \text{so that} \qquad F_{a,v} \;\leftrightarrow\; F_{-a,-v}\,.$$

The genus of $\mathcal{S}_p/\mathcal{I}$ is smaller, but still increases with p.

Picture of Part of $\mathcal{S}_3$



Part of $\mathcal{S}_3$, labeled

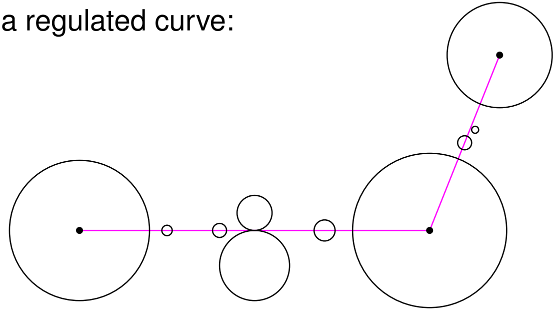


A Cell Structure in $\overline{\mathcal{S}}_p$.

Let $\overline{\mathcal{S}}_p$ be the smooth compact surface obtained from $\mathcal{S}_p$ by filling in each puncture point.

> **Conjecture.** *There is a canonical cell subdivision of each $\overline{\mathcal{S}}_p$. For $p \geq 2$, the 1-skeleton can be identified with the union of all simple closed regulated curves.*

Sketch of a regulated curve:



Escape Regions

Let $\mathcal{C}(\mathcal{S}_p)$ be the **connectedness locus** in $\mathcal{S}_p$.

Each connected component $\mathcal{E}$ of the complement $\mathcal{S}_p \smallsetminus \mathcal{C}(\mathcal{S}_p)$ will be called an **escape region** in $\mathcal{S}_p$.

Theorem. *For each $\mathcal{E}$, there is a canonical covering map*

$$\mathcal{E} \;\to\; \mathbb{C} \smallsetminus \overline{\mathbb{D}}\,.$$

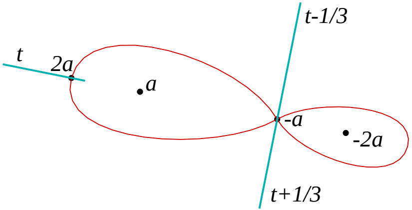
The degree of this covering map will be called the **multiplicity** $\mu \geq 1$ of the escape region.

We can talk about **equipotentials** and **parameter rays** in each escape region.

Notation: *A parameter ray in the escape region $\mathcal{E}$ will be denoted by $\mathcal{R}_{\mathcal{E}}(t)$. Here $t \in \mathbb{R}/\mu\mathbb{Z}$.*

If $\mu > 1$, then t will be called a **generalized angle.**

The Dynamic Plane for a map $F \in \mathcal{E}$.

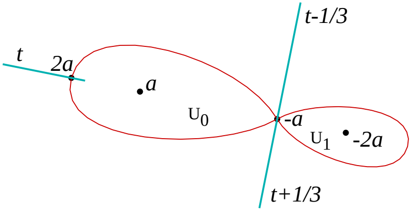


For F in the escape region $\mathcal{E}$, the equipotential through $2a$ and $-a$ is a figure eight curve. Here $2a$ is the free **cocritical point**, with $F(2a) = F(-a)$.

The Böttcher coordinate $\beta(2a) \in \mathbb{C} \setminus \overline{\mathbb{D}}$ of the escaping cocritical point is well defined, and the correspondence $F \mapsto \beta(2a)$ is the required covering map

$$\mathcal{E} \to \mathbb{C} \setminus \overline{\mathbb{D}}.$$

The Kneading Sequence.



Let U_0 and U_1 be the two bounded regions cut out by the figure eight curve, with $a \in U_0$. Any bounded orbit $z_1 \mapsto z_2 \mapsto \cdots$ determines a sequence $\sigma_1, \sigma_2, \ldots$ of zeros and ones with

$$z_j \in U_{\sigma_j} .$$

Now take z_1 equal to the marked critical value $v = F(a)$.

The associated sequence $\{\sigma_j\}$ will be called the **kneading sequence** of the escape region $\mathcal{E}$. Thus

$$F^{\circ j}(a) \in U_{\sigma_j} \qquad \text{for } j \geq 1 .$$

The Associated Quadratic Map.

The kneading sequence of any escape region $\mathcal{E} \subset \mathcal{S}_p$ is clearly periodic: its period p_1 divides p.

Theorem (Branner and Hubbard). *Suppose that F belongs to the escape region $\mathcal{E} \subset \mathcal{S}_p$. Then the Julia set $J(F)$ consists of countably many copies of a quadratic Julia set $J(Q)$, together with uncountably many single point components. Here the quadratic polynomial $Q = Q_{\mathcal{E}}$ is critically periodic of period p_2 where*

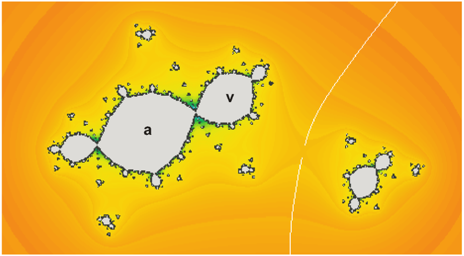
$$p = p_1 p_2 \,.$$

In other words:

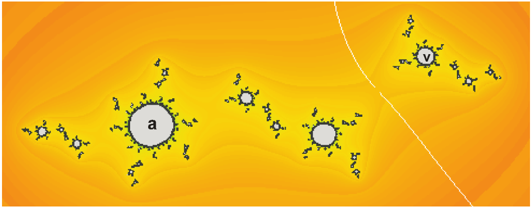
Period of marked critical point

= (kneading period)×(associated quadratic period).

Period 2 Examples



Here the kneading sequence is $\overline{00}$, and the associated quadratic map is $z^2 - 1$ (the “basilica”).



Kneading sequence $\overline{10}$, with associated quadratic z^2.

Canonical Coordinates for $\mathcal{S}_p$.

Consider the function

$$H_p : \mathbb{C}^2 \to \mathbb{C}\,, \qquad\qquad H_p(a,v) \;=\; F_{a,v}^{\circ p}(a) - a$$

which vanishes everywhere on $\mathcal{S}_p$. Think of H_p as a "complex Hamiltonian function", and consider the Hamiltonian differential equation

$$\frac{da}{dt} \;=\; \frac{\partial H_p}{\partial v}\,, \qquad \frac{dv}{dt} \;=\; -\frac{\partial H_p}{\partial v}\,.$$

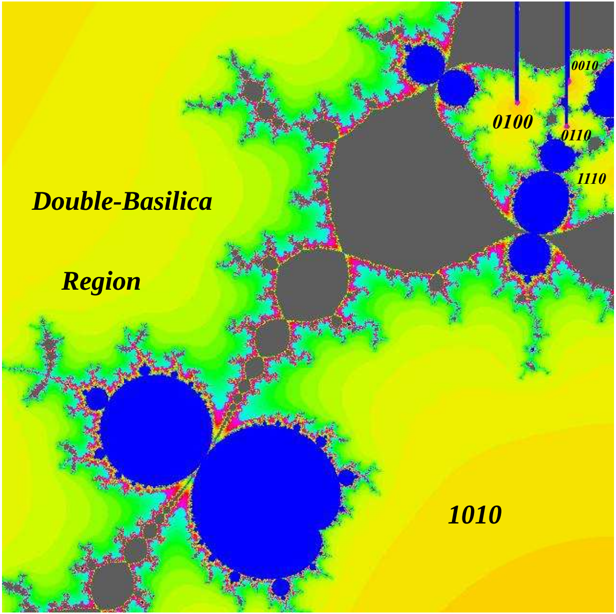
There are holomorphic local solutions

$$t \;\mapsto\; (a,v) \;=\; \Phi(t)\,.$$

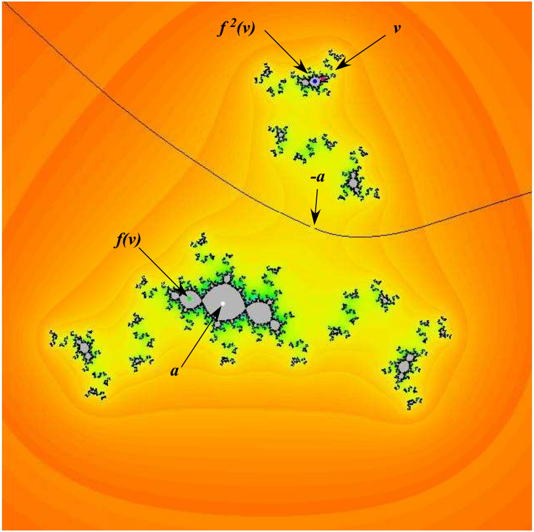
These lie in curves $H_p = \text{constant}$, parallel to $\mathcal{S}_p$. Those solutions which lie in $\mathcal{S}_p$ provide a local holomorphic parametrization, unique up to translation of the t-coordinate.

Equivalent description: *There is a canonical 1-form dt which is well defined and non-zero throughout $\mathcal{S}_p$.*

Part of $\mathcal{S}_4$ in canonical coordinates

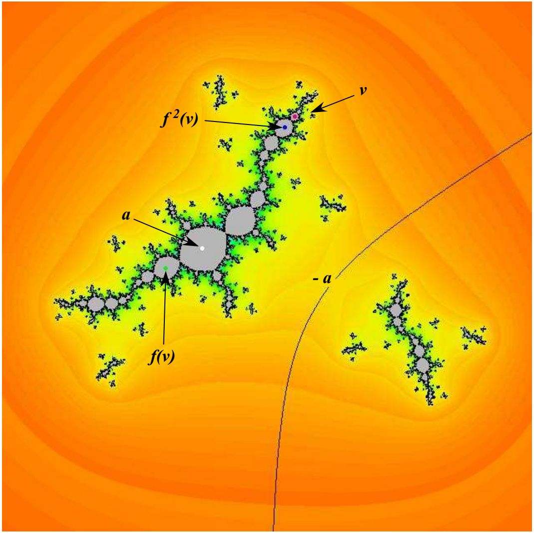


The 1010 Region: sample Julia set.



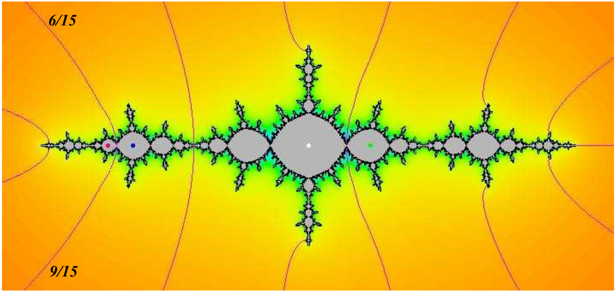
Kneading sequence $1010\cdots$, with period $p_1 = 2$.
$Q(z) = z^2 - 1$ with critical period $p_2 = 2$.

Example in the Double-Basilica Region.

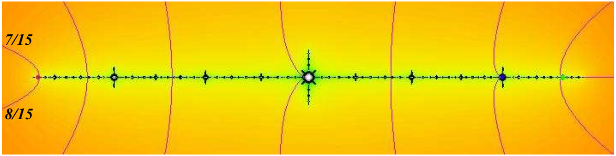


Kneading sequence $0000\cdots$, with period $p_1 = 1$.
$Q(z) = z^2 - 1.3107\ldots$ with critical period $p_2 = 4$.

Quadratic Julia sets:

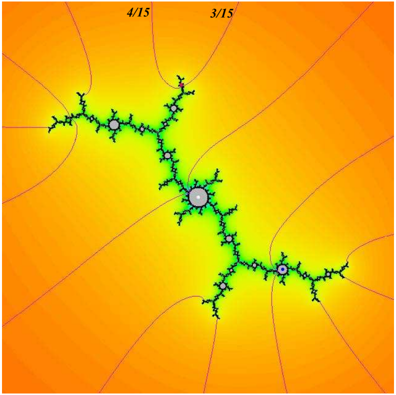


Double-Basilica

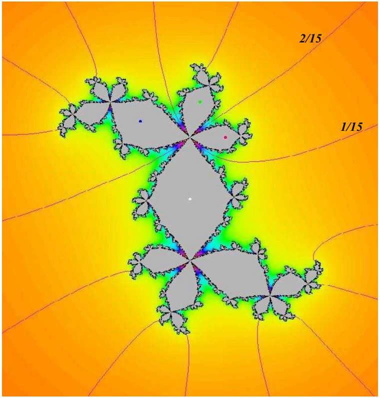


Worm

Two More Quadratic Julia Sets

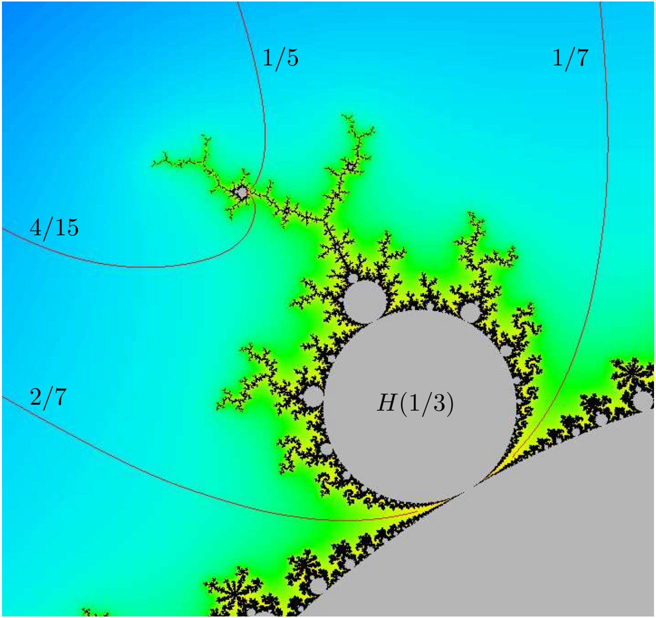


Kokopelli



(1/4)-Rabbit

Comparing Rays in the Mandelbrot Set



Parameter Rays

Let $\mathcal{E} \subset \mathcal{S}_p$ be any escape region.

Theorem. *If the generalized angle t_0 is rational, then the ray $\mathcal{R}_{\mathcal{E}}(t_0)$ lands at a well defined point F_0 in the boundary $\partial\mathcal{E}$. Furthermore, F_0 is either critically finite, or parabolic.*

Define $t \in \mathbb{Q}/\mathbb{Z}$ to be **co-periodic** if:

$t \pm 1/3$ is periodic under angle tripling,

$\Leftrightarrow$ $3t$ is periodic but t is not periodic,

$\Leftrightarrow$ t has the form $\frac{m}{3n}$ where m and n are not divisible by 3.

Theorem. *If $t_0 \pmod{\mathbb{Z}}$ is co-periodic, then the landing point of $\mathcal{R}_{\mathcal{E}}(t_0)$ is parabolic.*

We believe that this should be an if and only if statement:

t_0 co-periodic $\Leftrightarrow$ the landing point is parabolic.

The Period q Decomposition of $\mathcal{S}_p$.

If $t \pm 1/3$ has period q, we say that t has **co-period** q.

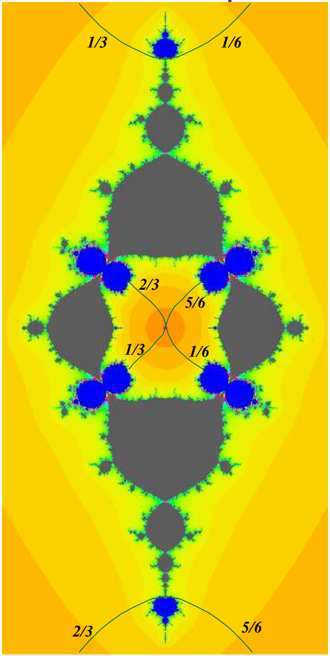
Note that any angle of co-period q can be written as a fraction

$$t = \frac{m}{3(3^q - 1)}.$$

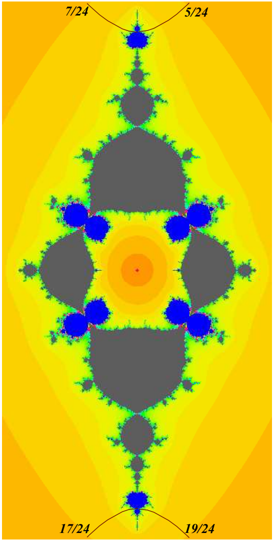
For example, $q = 1 \Rightarrow t = m/6$,
$q = 2 \Rightarrow t = m/24$.

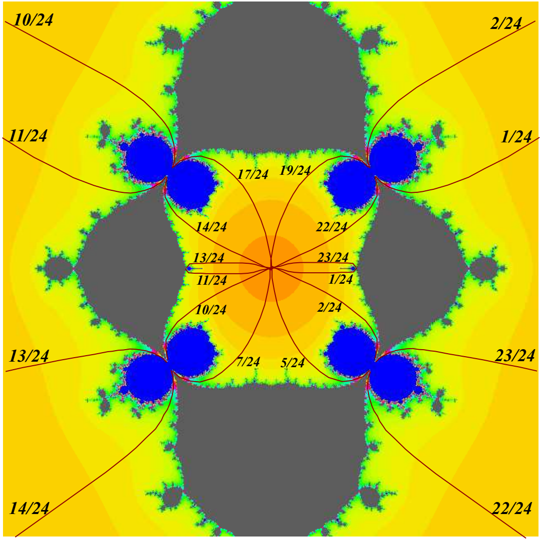
Period q decomposition: *The collection of all rays of co-period q, together with their landing points, decomposes the parameter curve $\mathcal{S}_p$ into a finite number of connected open sets U_j.*

Example: The Period 1 Decomposition of $\mathcal{S}_2$.



Period 2 Decomposition of $\mathcal{S}_2$.





Let U_j be any connected component of

$$\mathcal{S}_p \smallsetminus \bigcup \overline{\text{rays of coperiod } q},$$

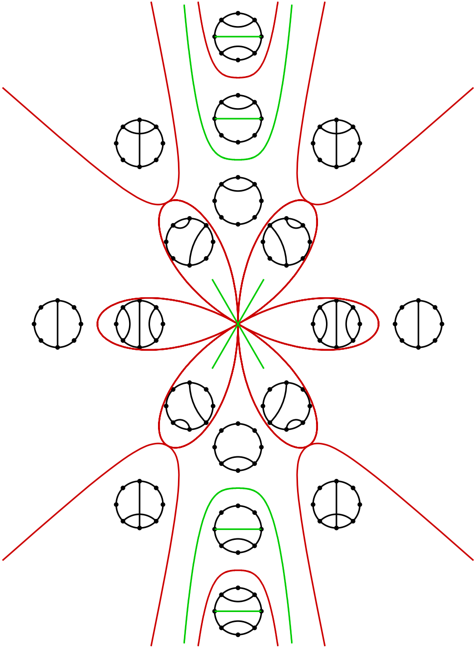
and let $t_0 \in \mathbb{Q}/\mathbb{Z}$ have period q.

As F varies over U_j, the dynamic ray $\mathcal{R}_F(t_0)$ varies smoothly:

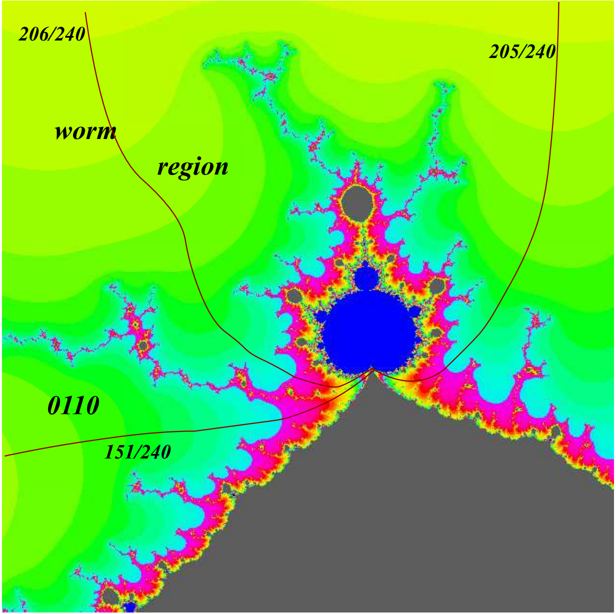
> **Theorem.** *For each $F \in U_j$, and each angle $t_0 \in \mathbb{Q}/\mathbb{Z}$ of period q, the ray $\mathcal{R}_F(t_0)$ lands at a repelling periodic point $z_F \in J(F) \subset \mathbb{C}$.*
> *Furthermore, the correspondence $F \mapsto z_F$ defines a holomorphic function $U_j \to \mathbb{C}$.*
> *The pattern of which dynamic rays of period q have a common landing point is the same for all $F \in U_j$.*

Corollary. Every parabolic map $F_0 \in \mathcal{S}_p$ is the landing point of at least one co-periodic ray.

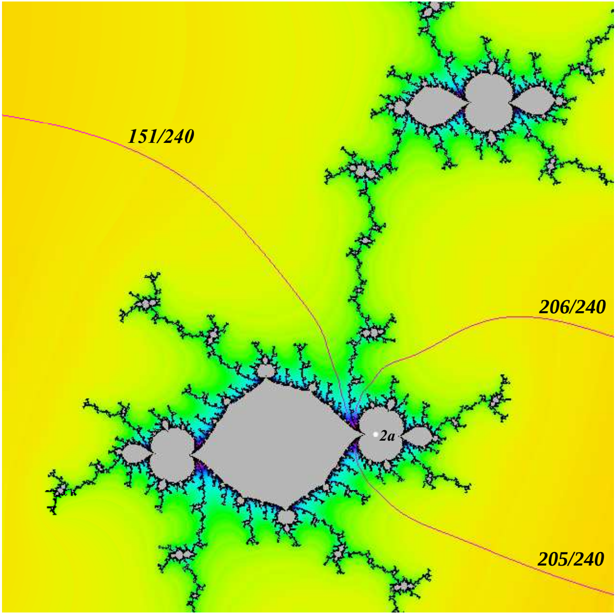
Orbit Portraits for $F \in \mathcal{S}_2$ (Periods 1 and 2).



A Small Mandelbrot Set in $\mathcal{S}_4$



Detail of $J(F_0)$ near $2a$



(Empirical Claims)

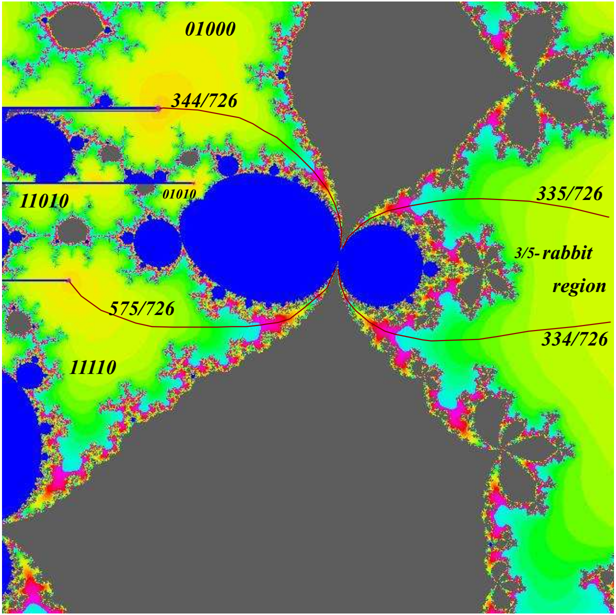
Every Mandelbrot component $\mathcal{M} \subset \mathcal{S}_p$ has a well defined root point F_0, and every parabolic point $F_0 \in \mathcal{S}_p$ is the root point of a unique Mandelbrot component $\mathcal{M} \subset \mathcal{S}_p$.

For $F \in \mathcal{M}$, let r_0 be the root point of the Fatou component $U(2a)$ containing the cocritical point $2a$. Then a neighborhood of F_0 in $\mathcal{S}_p$ is closely related to a neighborhood of r_0 in the dynamic plane for F. More precisely:

- The two closest parameter rays at F_0 which enclose $\mathcal{M}$ have the same angles (modulo $\mathbb{Z}$) as the two closest dynamic rays at r_0 which enclose $U(2a)$.

- Furthermore, *any* parameter ray landing at F_0 has the same angle (modulo $\mathbb{Z}$) as some dynamic ray landing at r_0.

A Small Mandelbrot Set in $\mathcal{S}_5$



Detail of corresponding Julia Set