Topology
through Four Centuries:

Low Dimensional Manifolds

John Milnor

Institute for Mathematical Sciences

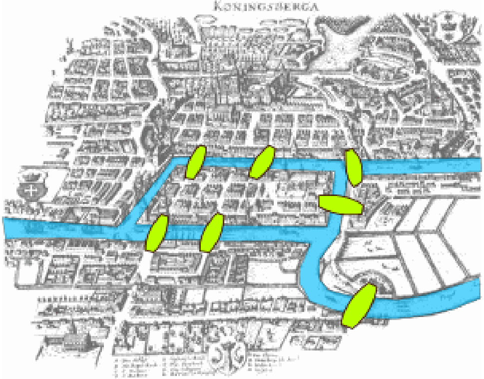
Stony Brook University

International Congress of Mathematicians

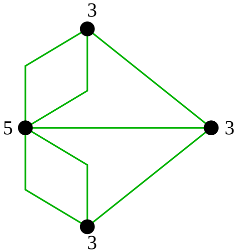
Seoul, August 15, 2014

PART 1. PRELUDE TO TOPOLOGY

Leonhard Euler
St. Petersburg, 1736



Königsberg



Euler's Theorem:

$\exists$ path traversing each edge once

$\Longleftrightarrow$ at most two "odd" vertices.

Euler, Berlin, 1752

For any convex polyhedron,

$$V - E + F = 2\ .$$

$$60 - 90 + 32 = 2$$

19th century:
Define the **Euler characteristic** of any finite cell complex K as

$$\chi(K) \ = \ \#(\text{even dimensional cells}) \ - \ \#(\text{odd dimensional cells})$$

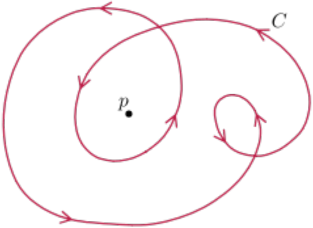
Theorem (20th century): This is a topological invariant.

Fundamental property: If $K = K_1 \cup K_2$, then

$$\chi(K) = \chi(K_1) + \chi(K_2) - \chi(K_1 \cap K_2)\,.$$

Augustin Cauchy, École Polytechnique, Paris, 1825

First precise definition of **continuity**.



$$W_C(p) = \frac{1}{2\pi i} \oint_C \frac{dz}{z-p}$$

Thus he described a topological invariant, the **winding number** of a loop C around p, and computed it as the integral of a holomorphic differential form along C.

Niels Henrik Abel, 1820's

$$\int \frac{dx}{\sqrt{(x-a_1)\cdots(x-a_n)}}$$

Consider the smooth affine variety $V \subset \mathbb{C}^2$ defined by

$$y^2 = f(x) = (x-a_1)\cdots(x-a_n)\,.$$

Then $$\frac{dx}{y} = \frac{2\,dy}{f'(x)}$$

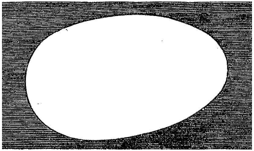
is a holomorphic 1-form or **Abelian differential** on V.

For any closed loop L on V we can integrate, yielding a homomorphism

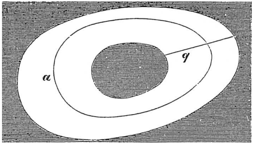
$$L \mapsto \int_L dx/y \quad \text{from} \quad \pi_1(V) \quad \text{to} \quad \mathbb{C}\,.$$

2. TWO DIMENSIONAL MANIFOLDS

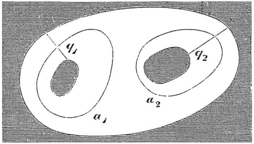
Bernhard Riemann,
Göttingen, 1857



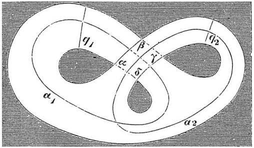
simply connected
“Riemann surface”



doubly connected



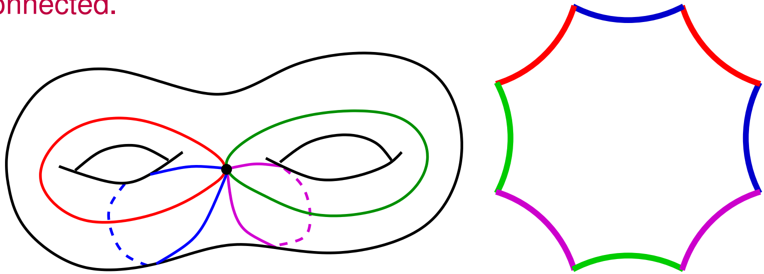
triply connected



triply connected

Closed Surfaces.

Riemann also considered the case of a closed surface $\mathcal{F}$.
He described a procedure for cutting $\mathcal{F}$ open along a number of simple closed curves, which intersect at just one point, so that the cut open surface is connected and simply connected.



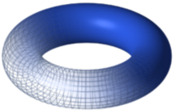
The number of such curves is always an even number $2p$.

This Riemann integer $p \geq 0$ is an invariant now known as the **genus** of $\mathcal{F}$, while $2p$ is known as the **first Betti number**.

August Ferdinand Möbius, Leipzig, 1863

Define the **“class”** of a **closed surface** $\mathcal{F} \subset \mathbb{R}^3$ as the smallest number n such that any n disjoint loops in $\mathcal{F}$ necessarily disconnect it.

[In other words, we can choose $p = n - 1$ closed loops which **do not** disconnect the surface; but one further disjoint loop will necessarily disconnect it.]



Example: $p = 1$.

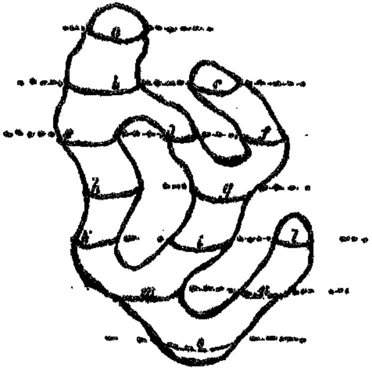
Möbius Theorem. Any two closed surfaces of the same class are “elementarily related”

$= C^1$-diffeomorphic ?

(Möbius continued)

Definition. *Two geometric figures are* ***"elementarily related"*** *if to any infinitely small element of any dimension in one figure there corresponds an infinitely small element in the other figure, such that two neighboring elements in one figure correspond to two elements in the other which also come together, . . .*

The proof:

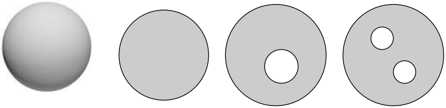


Put the surface in general position in $\mathbb{R}^3$, and cut it open along horizontal planes. Then each of the resulting connected pieces has either one, two, or three boundary curves.

It is either a **2-cell**, an **annulus**, or a **"pair of pants".**

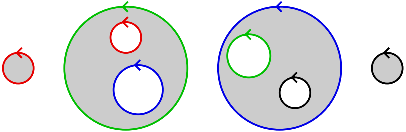
The Möbius proof (continued).

More generally, let E_k denote a region in the 2-sphere with k boundary curves.



E_0 E_1 E_2 E_3 $\cdots$

Thus the Möbius construction splits the surface $\mathcal{F}$ into a disjoint union



$$E = E_{k_1} \sqcup E_{k_2} \sqcup \cdots \sqcup E_{k_N} \quad \text{with} \quad k_j \in \{1, 2, 3\}\,.$$

To get the original surface $\mathcal{F}$ we must identify each boundary curve of E with some other boundary curve, under a prescribed diffeomorphism.

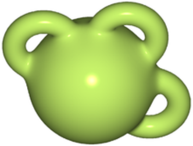
The Möbius proof (conclusion).

Lemma. *If we simplify* $E_k \sqcup E_\ell$ *by identifying one boundary curve of* E_k *with one boundary curve of* E_ℓ *, then the result is diffeomorphic to* $E_{k+\ell-2}$ *.*

Now simplify the set $E = E_{k_1} \sqcup \cdots \sqcup E_{k_N}$ inductively by identifying one pair of boundary curves at a time.

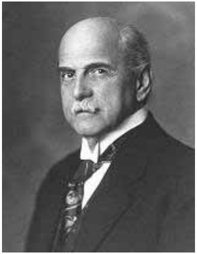
After $N-1$ such identifications,
we obtain a connected set of the form E_{2p}.
(This p is Riemann's invariant: the genus.)

Now each further identification of two curves amounts to adding a "handle" to the 2-sphere $S^2 = E_0$. □



Example: The case $p = 3$.

Walther von Dyck, Munich 1888



Definition: Topology is the study of properties which are invariant under continuous functions with continuous inverse.

The **"Gauss-Bonnet Formula"** as a global theorem about smooth closed 2-dimensional manifolds:

$$\chi(M) = \frac{1}{2\pi} \int\int K\, dA\,.$$

Corollary. If $K > 0$, or $K = 0$, or $K < 0$ everywhere, then $\chi > 0$, or $\chi = 0$, or $\chi < 0$ respectively.

Henri Poincaré, Paris (from 1892 to 1907)

- Homology
- Betti numbers
- Poincaré duality
- Homotopy
- Fundamental group
- Covering spaces
- Uniformization of Riemann surfaces

Paul Koebe, Berlin 1907

Uniformization Theorem:

The universal covering space $\widetilde{S}$ of any Riemann surface S is conformally isomorphic to either:

1. the Riemann sphere $\mathbb{C} \cup \infty$,
2. the complex plane $\mathbb{C}$, or to
3. the open unit disk $\mathbb{D} \subset \mathbb{C}$.

Corollary: Any Riemann surface has a metric of constant curvature:

- $K \equiv +1$ in Case (1), 
- $K \equiv 0$ in Case (2), 
- $K \equiv -1$ in Case (3). 

Hermann Weyl, 1913: “The Concept of a Riemann Surface”.

MATHEMATISCHE VORLESUNGEN AN DER UNIVERSITÄT GÖTTINGEN: V

DIE IDEE
DER RIEMANNSCHEN FLÄCHE

VON

DR. HERMANN WEYL
PRIVATDOZENT AN DER UNIVERSITÄT GÖTTINGEN

MIT 27 IN DEN TEXT GEDRUCKTEN FIGUREN

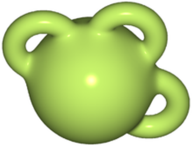
LEIPZIG UND BERLIN
DRUCK UND VERLAG VON B. G. TEUBNER
1913

- First precise definition of a general Riemann surface.
- First use of overlapping coordinate charts.

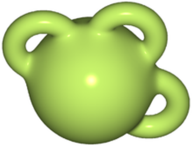
3. THREE DIMENSIONAL MANIFOLDS

Poul Heegaard, Copenhagen 1898

Any closed oriented 3-manifold can be decomposed as a union of two handlebodies of the same genus, which intersect only along their boundaries.



$$f : \partial H \xrightarrow{\cong} \partial H$$



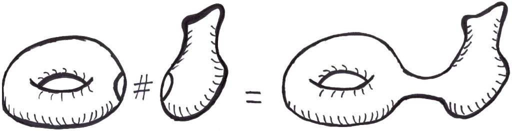
Poincaré, 1904

Question: If a closed three dimensional manifold has trivial fundamental group, is it necessarily homeomorphic to the standard sphere S^3 ?

Hellmuth Kneser, Greifswald 1929

Consider connected oriented piecewise-linear 3-manifolds.

Any two have a well defined connected sum



with the sphere as identity element, $M\#S^3 \cong M$.

Definition: The manifold $M \not\cong S^3$ is **prime** if this is the only way of expressing M as a connected sum.

Theorem: Every compact M is isomorphic to a connected sum $M \cong P_1\#\cdots\#P_k$ of **prime** 3-manifolds.

In fact, the P_j are unique up to order and up to isomorphism. [M. 1962]

Christos Papakyriakopoulos, Princeton 1957

First proof of “Dehn’s Lemma”:

Theorem. Given a PL-map f from the closed 2-disk into $\mathbb{R}^3$ such that $f^{-1} \circ f$ is single valued near the boundary, there exists a PL-embedding which coincides with f near the boundary.

[This result had been claimed by Max Dehn in 1910; but Dehn’s proof had an essential error.]

Corollary: Let $K \subset \mathbb{R}^3$ be a PL simple closed curve. Then K is unknotted if and only if $\pi_1(\mathbb{R}^3 \smallsetminus K) \cong \mathbb{Z}$.

George Mostow, Yale 1968

Rigidity Theorem:
A closed manifold of dimension ≥ 3 with curvature $K \equiv -1$ is uniquely determined **up to isometry** by its fundamental group.

(Also proved by Margulis.
Extended to complete manifolds
of finite volume by Prasad.)

Corollary. The **volume** of such a manifold is a topological invariant.

Open Problem in Number Theory:

What can one say about the numbers which occur as volumes?

Example: The Figure Eight Complement

Robert Riley and Troels Jørgensen in the mid 1970s showed that the complement of the figure eight knot in S^3 has a complete hyperbolic structure of finite volume.

Bill Thurston found many more hyperbolic knot complements.

He computed the volume V of the figure eight complement by "triangulating" it into two regular ideal 3-simplexes. Using methods going back to Lobachevsky, this can be computed as

$$V = -6 \int_0^{\pi/3} \log|2\sin(u)|\, du = 2.02988\cdots .$$

Thurston, Princeton 1978

Theorem:

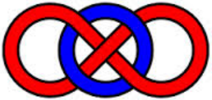
- The volumes of hyperbolic 3-manifolds form a well ordered set.

(That is, any non-empty subset has a smallest element).

- There are at most finitely many non-homeomorphic manifolds for each fixed volume.

The volume of any manifold M with k ends, $k \geq 1$, is an increasing limit of volumes of manifolds with $k-1$ ends.

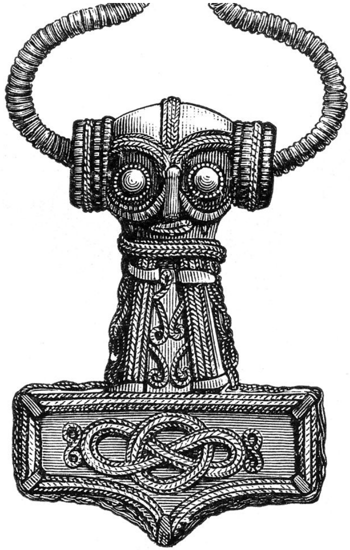
The idea: Each "end" can be identified with a copy of $S^1 \times S^1 \times [0, \infty)$, which can be cut off and replaced by a solid torus $S^1 \times \mathbb{D}$ in many ways.



The Whitehead link

$V = 3.66386 \cdots$

Sweden: One Thousand Years Ago



Mjölnir: the hammer of Thor, with a Whitehead Link.

The JSJ decomposition, late 1970s.



William Jaco, Peter Shalen, and Klaus Johannson

(in Sillwater Oklahoma, Chicago, Frankfurt)

They showed that any PL 3-manifold can be decomposed into simpler pieces by cutting along embeded spheres **and tori.**

The Geometrization Conjecture (Thurston, 1982).

Thurston conjectured that every smooth closed 3-manifold could be decomposed, by embedded spheres and tori, into manifolds M which can be given a locally homogeneous structure, so that $\widetilde{M}$ is a homogeneous space.

Furthermore, there are exactly eight possibilities for $\widetilde{M}$.

Three of these are the three classical geometries:

(1) The Sphere S^3, with curvature $K \equiv +1$.

[*Riemannian manifolds with S^3 as universal covering had been classified by Heinz Hopf in 1925.*]

(2) The Euclidean space $\mathbb{R}^3$, with curvature $K \equiv 0$.

[*The corresponding compact flat manifolds had been classified by Bieberbach in 1911.*]

(3) The Hyperbolic space H^3, with $K \equiv -1$.

[*This is the most interesting and difficult case.*]

The Eight Geometries (continued).

Next, two easy cases:

(4) $\widetilde{M} \cong \mathbb{R} \times S^2$. **Example:** $M = S^1 \times S^2$.

(5) $\widetilde{M} \cong \mathbb{R} \times H^2$. **Example:** $M = S^1 \times (\text{hyperbolic surface})$.

For the remaining geometries, $\widetilde{M}$ will be a three dimensional Lie group G with left invariant metric.

(6) **Nilgeometry**, with nilpotent group $\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}$.

Example: A non-trivial circle bundle over the torus.

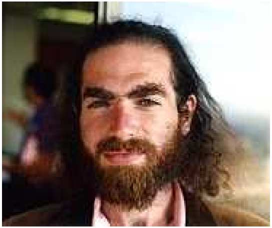
(7) **Solvgeometry**, with solvable group $\begin{pmatrix} 1 & 0 & 0 \\ x & e^z & 0 \\ y & 0 & e^{-z} \end{pmatrix}$.

Example: Most torus bundles over the circle.

(8) $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ **geometry.**

Example: The unit tangent bundle of a hyperbolic surface.

Grigori Perelman, St. Petersburg 2003



Thurston proved many cases of the Geometrization Conjecture. But the hardest cases remained open for twenty years.

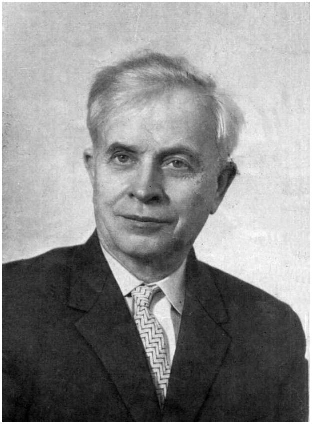
Then Perelman proved the full Conjecture, including the Poincaré Conjecture !

His proof made ingenious use of a simple looking differential equation first studied by Richard Hamilton:

$$\frac{\partial g_{jk}}{\partial t} = -2\, R_{jk} \quad .$$

4. FOUR DIMENSIONAL MANIFOLDS

A. A, Markov Jr., Leningrad 1953



Theorem. The problem of classifying closed 4-manifolds up to homeomorphism is algorithmically unsolvable.

His proof was based on the unsolvability of the **isomorphism problem** for groups presented by finitely many generators and relations.

Thus, for any chance of a reasonable theory, we must consider only manifolds with **known** fundamental group.

J. H. C. Whitehead 1949

Whitehead classified simply connected four dimensional **complexes** up to homotopy type.

Applied to manifolds:

Corollary. A closed oriented simply connected 4-manifold is determined, up to oriented homotopy type, by its **intersection form**

$$H_2(M) \otimes H_2(M) \to \mathbb{Z}\ .$$

This form is symmetric, bilinear, and unimodular (determinant $= \pm 1$).

The classification of such symmetric bilinear forms is an important and non-trivial problem in number theory.

Example (C. L. Siegel): The number of distinct positive definite unimodular forms of rank equal to 30 is $>$ **904,000,000.** (Reference: [M. and Husemoller].)

Michael Freedman, 1982

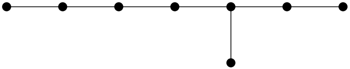
Theorem. A closed oriented simply connected 4-manifold is uniquely determined, up to homeomorphism, by

- its intersection form,
- and its "Kirby-Siebenmann invariant" in $\mathbb{Z}/2$, which is always zero in the smooth case.

Any symmetric bilinear unimodular form can be realized by a **topological** manifold;

but not always by a smooth or PL manifold !

Example:



Simon Donaldson, 1983

Theorem. If a smooth, closed simply connected 4-manifold M has positive definite intersection form, then this form is diagonalizable, hence

$$M \cong \mathbb{CP}^2 \# \cdots \# \mathbb{CP}^2 .$$

Example: Among the more than 904,000,000 topological manifolds with positive definite intersection form of rank 30,

only one is represented by a smooth manifold.

Cliff Taubes, 1984

The combination of Freedman's hard core topology and Donaldson's methods inspired by Mathematical Physics had amazing consequences.

Theorem. The Euclidean space $\mathbb{R}^4$ can be given uncountably many distinct differentiable structures.

[For every $n \neq 4$, the space $\mathbb{R}^n$ has only one differentiable structure up to diffeomorphism.]

Thus dimension four really is different !

To Conclude: An Open Problem.

The smooth analog of the Poincaré Conjecture remains completely open in dimension four:

If a smooth 4-manifold has the homotopy type of S^4
and hence by Freedman is homeomorphic to S^4,
does it follow that it is diffeomorphic to S^4 ?

For any $n \geq 1$ the set of oriented diffeomorphism classes of manifolds homeomorphic to S^n forms a commutative semigroup $\mathscr{S}_n$ under the connected sum operation.

For $n \neq 4$, $\mathscr{S}_n$ is a finite Abelian group
(Kervaire & M. + Perelman),
For $n = 1, 2, 3, \quad 5, 6, \quad 12, \quad 61$ this group is trivial;
but for all other $n \leq 63$, $n \neq 4$ it is **non-trivial.**
However the semigroup $\mathscr{S}_4$ is completely unknown!
If non-trivial, is it at least a group? How big?
If not a group, what kind of semigroup?

- M. Freedman and F. Quinn, "The Topology of 4-manifolds," Princeton University Press 1990.
- M. Freedman, R. Gompf, S. Morrison and K. Walker, *Man and machine thinking about the smooth 4-dimensional Poincaré conjecture,* Quantum Topol. **1** (2010) 171–208.
- R. Gompf, *An exotic menagerie,* J. Differential Geom. **37** (1993) 199–223.
- J. Milnor, *Differential topology forty-six years later,* Notices A.M.S. **58** (2011) 804–809.
- J. Morgan and G. Tian, "Ricci Flow and the Poincaré Conjecture," Clay Mathematics Monographs **3**, A.M.S. 2007.
- J. Morgan and G. Tian, "The Geometrization Conjecture," Clay Mathematics Monographs **5**, A.M.S. 2014.
- W. Thurston, *Three-dimensional manifolds, Kleinian groups and hyperbolic geometry*, Bulletin A.M.S. **6** (1982) 357-381.
- W. Thurston, "Three-Dimensional Geometry and Topology," Princeton University Press 1997.