Hyperbolic Component Boundaries*

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Gyeongju, August 23, 2014

*Revised version. The conjectures on page 16 were problematic, and have been corrected.

The Problem

Hyperbolic components, in a reasonable space of polynomial or rational maps, are well understood.

But their topological boundaries can be very complicated.

This talk will first describe a special case where the boundaries are very well behaved.

It will then speculate about the other cases.

Definitions. Let $\mathrm{Rat}_n \subset \mathbb{P}^{2n+1}(\mathbb{C})$ be the space of all rational maps of degree $n \geq 2$:

$$\left(f(z) = \frac{\sum_0^n a_j z^j}{\sum_0^n b_j z^j}\right) \quad \longleftrightarrow \quad [a_0 : \cdots : a_n : b_0 : \cdots : b_n] \in \mathbb{P}^{2n+1}(\mathbb{C})\,.$$

For any algebraic variety $V \subset \mathbb{P}^{2n+1}(\mathbb{C})$, the intersection $\mathcal{F} = V \cap \mathrm{Rat}_n$ will be described as an

algebraic family of rational maps.

Hyperbolic Components.

Definitions: A rational map is ***hyperbolic*** if the orbit of every critical point converges towards an attracting cycle.

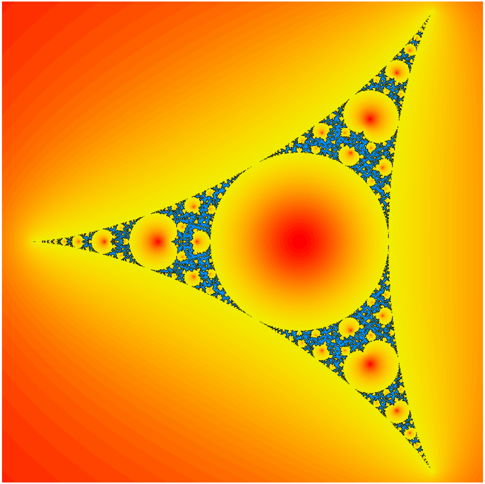
In any algebraic family $\mathcal{F}$, the hyperbolic maps form an open subset.

*Any connected component of this open subset is called a **hyperbolic component*** $\mathcal{H} \subset \mathcal{F}$ ***.***

Two critical points will be called ***Grand Orbit equivalent*** if their forward orbits intersect.

Theorem 1. *Suppose that the maps in* $\mathcal{H}$ *have the property that the basin of every attracting cycle contains* **exactly one** *GO-equivalence class of critical points. Then the closure* $\overline{\mathcal{H}}$ *, the topological boundary* $\partial\mathcal{H}$ *, and* $\mathcal{H}$ *itself are all* ***semi-algebraic sets.***

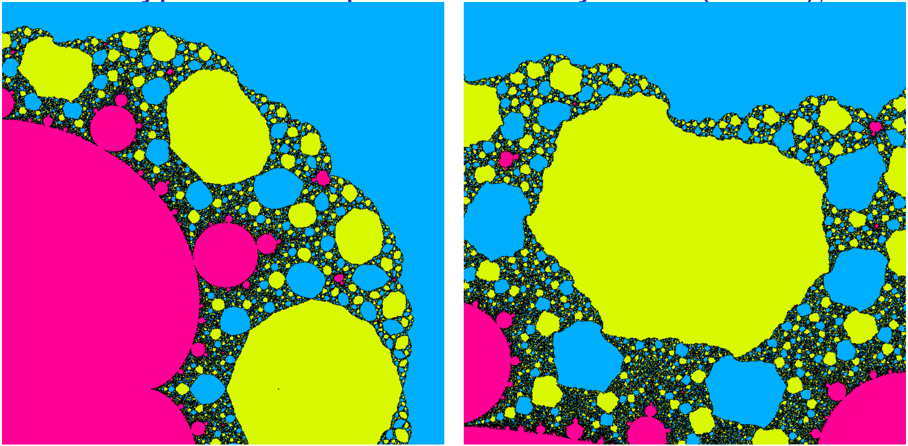
Example: The family $f_a(z) = a + 1/z^2$



Each f_a has critical points 0 and ∞, with $f_a : 0 \mapsto \infty$.
Thus each f_a has **only one critical grand orbit.**

$\Longrightarrow$ every hyperbolic component is semi-algebraic.

A More Typical Example: The family $z \mapsto (z+a)/z^2$.



Critical points: $z = 0$ and $z = -2a$.

The periodic orbit $0 \leftrightarrow \infty$ is always superattractive.

In the Blue and Yellow Regions:

the other critical point is eventually attracted to this orbit.

In the Red Regions: there is a disjoint periodic orbit.

Only the red (Mandelbrot-like) regions are semi-algebraic.

Semi-Algebraic Sets: the Definition.

Consider subsets of $\mathbb{R}^n$ of the form either

$$\{x \in \mathbb{R}^n \,;\ p(x) \geq 0\} \qquad \text{or} \qquad \{x \in \mathbb{R}^n \,;\ p(x) \neq 0\}\ .$$

Here $p : \mathbb{R}^n \to \mathbb{R}$ can be an arbitrary real polynomial.

Definition: Any finite intersection of such sets is called a **basic semi-algebraic set.**

Any finite union of basic semi-algebraic sets is called a **semi-algebraic set.**

[Note that we can obtain equalities by combining two inequalities:
If $p(x) \geq 0$ and $-p(x) \geq 0$, then $p(x) = 0$.]

This definition is applied to subsets of $\mathbb{C}^n$ by simply ignoring the complex structure, identifying $\mathbb{C}^n$ with $\mathbb{R}^{2n}$.

Semi-algebraic Sets: Basic Properties

(Reference:

Bochnak, Coste, and Roy, “Real Algebraic Geometry”.)

- Any finite union or intersection of semi-algebraic sets is itself a semi-algebraic set.
- The complement $\mathbb{R}^n \setminus S$ of a semi-algebraic set is itself a semi-algebraic set.
- A semi-algebraic set has finitely many connected components, and each of them is semi-algebraic.
- The topological closure of a semi-algebraic set is semi-algebraic.
- (Tarski-Seidenberg Theorem.) The image of a semi-algebraic set under projection from $\mathbb{R}^n$ to $\mathbb{R}^{n-k}$ is semi-algebraic.
- Every semi-algebraic set can be triangulated, and hence is locally connected.

Proof of Theorem 1.

Recall the statement:

If the maps $f \in \mathcal{H} \subset \mathcal{F}$ have only one grand-orbit-equivalence class of critical points in the basin of each attracting cycle, then $\mathcal{H}$, $\partial\mathcal{H}$, and $\overline{\mathcal{H}}$ are all semi-algebraic.

First Step:

Let $p_1, p_2, \ldots, p_m$ be the periods of the m attracting cycles.

Let $\mathcal{F}(p_1, p_2, \ldots, p_m)$ *be the set of all*

$$(f, z_1, z_2, \ldots, z_m) \in \mathcal{F} \times \mathbb{C}^m$$

satisfying two conditions:

- Each z_j should have period exactly p_j under the map f ;
- and the orbits of the z_j must be disjoint.

Lemma. *This set* $\mathcal{F}(p_1, p_2, \ldots, p_m) \subset \mathcal{F} \times \mathbb{C}^m$ *is semi-algebraic.*

The proof is an easy exercise. $\square$

Proof (Continued)

Let U be the open set consisting of all

$$(f,\, z_1, \ldots, z_m) \;\in\; \mathcal{F}(p_1\,,\; p_2\,,\; \ldots\,,\; p_m)$$

such that the multiplier of the orbit for each z_j satisfies

$$|\mu_j|^2 < 1\ .$$

This set U is semi-algebraic.

Hence each component $\widetilde{\mathcal{H}} \subset U$ is semi-algebraic.

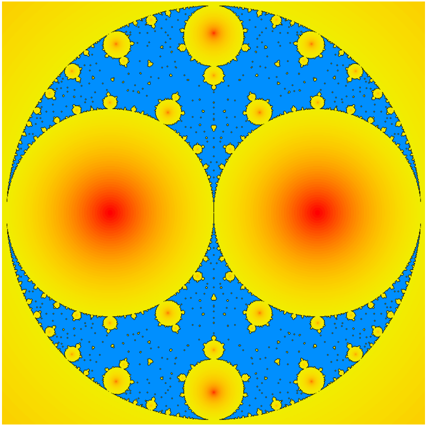
Hence the image of $\widetilde{\mathcal{H}}$ under the projection $\mathcal{F}(p_1\,,\; p_2\,,\; \ldots\,,\; p_m) \to \mathcal{F}$ is a semi-algebraic set $\mathcal{H}$, which is clearly a hyperbolic component in $\mathcal{F}$.

In fact any hyperbolic component $\mathcal{H} \subset \mathcal{F}$ having attracting cycles with periods $p_1\,,\; p_2\,,\; \ldots\,,\; p_m$ can be obtained in this way.

This proves that $\mathcal{H}$, its closure $\overline{\mathcal{H}}$, and its boundary $\partial\mathcal{H} = \overline{\mathcal{H}} \cap (\overline{\mathcal{F} \smallsetminus \mathcal{H}})$ are all semi-algebraic sets. $\square$

Theorem 1 is **not** a best possible result.

k-plane for the family of maps $f_k(z) = k(z + z^{-1})$.



This is a different kind of example with
all hyperbolic components semi-algebraic.

Here the two critical points ± 1 **are not GO-equivalent,**
but are bound together by the symmetry $f_k(-z) = -f_k(z)$.

More General Hyperbolic Components.

Suppose that the maps $f \in \mathcal{H}$ have an attracting cycle with two distinct **free** critical points in its attracting basin

Here “free" is to mean **completely independent**, so that there are at least two complex degrees of freedom.

Conjecture 1:

This implies that the boundary $\partial\mathcal{H}$ is **not locally connected.**

This is a question in two complex dimensions
$\Longrightarrow$ four real dimensions.

It cannot be answered by a 2-dimensional picture.

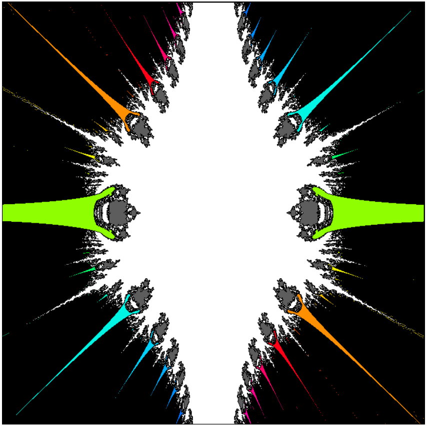
Added Remark (Generalized MLC Problem):
I do not know any example of an $\mathcal{H}$ in a complex one parameter family for which $\partial\mathcal{H}$ is not locally connected. Does such an example exist ?

Example: The family $f(z) = z^2(z-a)/(1+bz)$.

Let $\mathcal{H} \subset \mathbb{C}^2$ be the hyperbolic component centered at

$$a = b = 0 \iff f(z) = z^3 .$$

Consider the **real plane** $\mathcal{P} \cong \mathbb{R}^2 \subset \mathbb{C}^2$ defined by $b = \overline{a}$.



The central white region is $\mathcal{H}_{\mathcal{P}} = \mathcal{H} \cap \mathcal{P}$.

Its complement is $\mathcal{P} \smallsetminus \mathcal{H}_{\mathcal{P}}$.

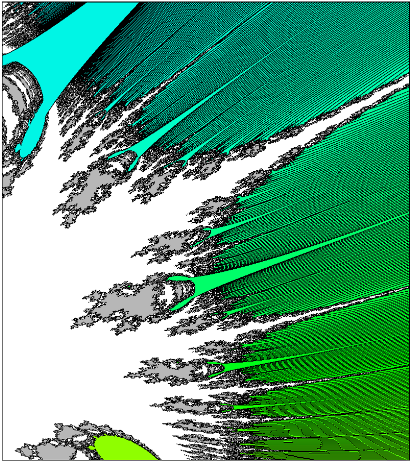
Theorem (with Bonifant & Buff): $\mathcal{H}_{\mathcal{P}}$ is simply connected, and contains infinitely many "fjords" leading out to infinity.

These are separated by infinitely many connected components of the complement $X = \mathcal{P} \smallsetminus \mathcal{H}_{\mathcal{P}}$.

A large disk $|a| \leq r$ intersects infinitely many of these components.

Corollary 1: $\partial\mathcal{H}_{\mathcal{P}}$ is not locally connected.

Proof of Corollary 1: non local-connectivity



Recall that any large disk $\mathbb{D}_r$ intersects infinitely many connected components of the closed set $X = \mathcal{P} \setminus \mathcal{H}_{\mathcal{P}}$.

Let $x_0 \in X \cap \mathbb{D}_r$ be any accumulation point for this collection of components.

Then X is not locally connected at x_0 .

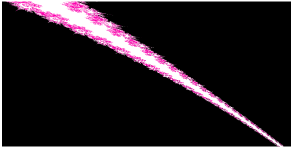
It follows that $\partial X = \partial \mathcal{H}_{\mathcal{P}}$ is not locally connected. $\square$

The full hyperbolic component $\mathcal{H} \subset \mathbb{C}^2$

Corollary 2: The boundary $\partial\mathcal{H} \subset \mathbb{C}^2$ is not **locally contractible.**

Note that the real plane $\mathcal{P} \subset \mathbb{C}^2$ is the fixed point set of an involution $\mathscr{I} : (a, b) \leftrightarrow (\overline{b}, \overline{a})$ of $\mathbb{C}^2$.

The region $\mathcal{H} \cap \mathcal{P}$ contains arbitrarily thin “fjords”:



Choose two points x and y which are arbitrarily close to each other, but lie on opposite banks of such a fjord.

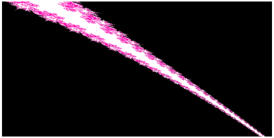
Suppose that $\partial\mathcal{H}$ is locally contractible.

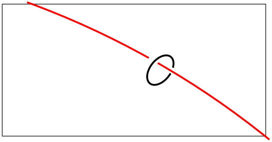
A short path from x to y **within** $\partial\mathcal{H}$ together with its image under $\mathscr{I}$, would form a small $\mathscr{I}$-invariant loop L.

Then L bounds a small disk D in $\partial\mathcal{H}$, and $D \cup \mathscr{I}(D)$ is a small $\mathscr{I}$-invariant singular 2-sphere.

Non Local-**Contractibility**: outline proof continued.

We must show that this singular 2-sphere $S^2 \mapsto \partial\mathcal{H}$ links a central curve within the fjord, and hence is not contractible within $\mathbb{C}^2 \setminus \mathcal{H}$.





This can be proved using the following.

Topological Lemma. If a map f from S^2 to itself fixes both poles, and commutes with the 180° rotation about the poles, then it has odd degree.

Proved by approximating f by a smooth map $\widehat{f}$ satisfying the same conditions, which has the pole p_0 as a regular value.

Then the number of preimages $\widehat{f}(x) = p_0$ is odd, and is congruent to the degree mod two. □

Further Conjectures (corrected page)

It is not hard to see that for any hyperbolic component $\mathcal{H} \subset \mathcal{F}$ and for any $f \in \partial\mathcal{H}$ either

- *there is a critical point in the Julia set* $J(f)$ *, or else*
- f *has an* **indifferent cycle**, *that is a periodic orbit with multiplier satisfying* $|\mu| = 1$ *.*

Conjecture 2:

(a) *If* **every** $f \in \partial\mathcal{H}$ *has an indifferent cycle, then* $\partial\mathcal{H}$ *is semi-algebraic.*

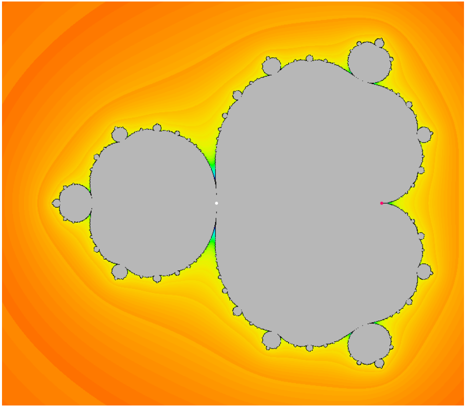
(b) *On the other hand, if some* $f \in \partial\mathcal{H}$ *has no indifferent cycle, then* $\partial\mathcal{H}$ *is a* **fractal set**, *in the sense that its Hausdorff dimension is greater than its topological dimension.*

(c) *Now suppose that* f *has a* **post-critical parabolic cycle,** *which can be perturbed, within the family* $\mathcal{F}$ *, to a parabolic cycle which is* **not** *post-critical. Then* $\partial\mathcal{H}$ *is* **not locally connected.**

Example: the family of maps $f(z) = z^3 + az^2 + \mu z$.

Let $\mathcal{H} \subset \mathbb{C}^2$ be the hyperbolic component centered at

$$a = \mu = 0 \iff f(z) = z^3 .$$

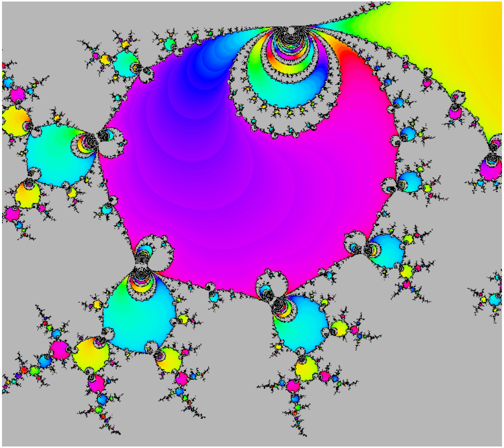


Julia set for the map $f(z) = z^3 + 2z^2 + z$ in $\partial\mathcal{H}$.

Satisfies the conditions of Conjecture 2(c). In particular, the critical point $z = -1$ maps to the parabolic point $z = 0$.

A small perturbation.

Now change the multiplier from $\mu = 1$ to $\mu = e^{.01\,i} \approx 1$.



Magnified image near $z = 0$ for $f(z) = z^3 + 2z^2 + e^{.01\,i} z$.

Simpler Example: the “universal capture component”.

Let $\mathbb{C}_{(z)} \sqcup \mathbb{C}_{(w)}$ be the disjoint union of two copies of $\mathbb{C}$, with coordinates z and w respectively.

Let $f_v : \mathbb{C}_{(z)} \to \mathbb{C}_{(w)}$ be the quadratic map $f_v(z) = z^2 + v$ with critical value $v \in \mathbb{C}_{(w)}$, and let $g_\mu : \mathbb{C}_{(w)} \to \mathbb{C}_{(w)}$ be the quadratic map

$$g_\mu(w) = w^2 + \mu w$$

with a fixed point of multiplier μ at $w = 0$.

Thus we obtain a two parameter family of maps (f_v, g_μ)

$$\mathbb{C}_{(z)} \xrightarrow{f_v} \mathbb{C}_{(w)} \circlearrowleft g_\mu ,$$

from $\mathbb{C}_{(z)} \sqcup \mathbb{C}_{(w)}$ to itself.

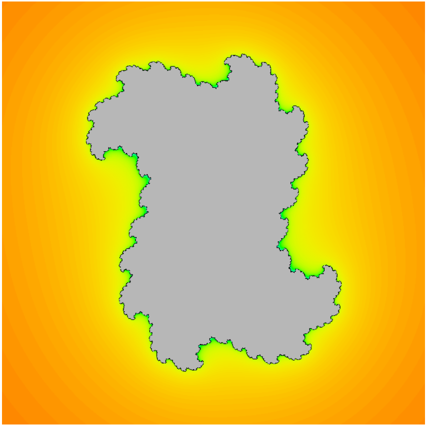
Let $\mathcal{H} \subset \mathbb{C}^2$ be the **hyperbolic component** consisting of all pairs $(v, \mu) \in \mathbb{C}^2$ such that both critical orbits converge to $w = 0$

$$\Longleftrightarrow \qquad |\mu| < 1 \qquad \text{and} \qquad \lim_{n\to\infty} g_\mu^{\circ n}(v) = 0 .$$

Theorem 2. $\overline{\mathcal{H}}$ *is not locally connected.*

The closure of $\mathcal{H}$

The closure $\overline{\mathcal{H}}$ consists of all $(v,\ \mu) \in \mathbb{C}^2$ such that $|\mu| \leq 1$,
and such that v belongs to the ***filled Julia set*** $K(g_\mu)$
$=$ the union of all bounded orbits for g_μ.

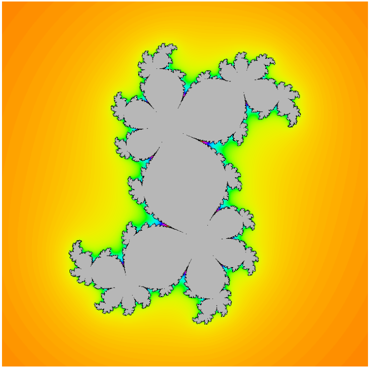


Typical example of $K(g_\mu)$ for the case $|\mu| < 1$.
(This is fractal—Compare Conjecture 2(b).)

The Case $|\mu| = 1$

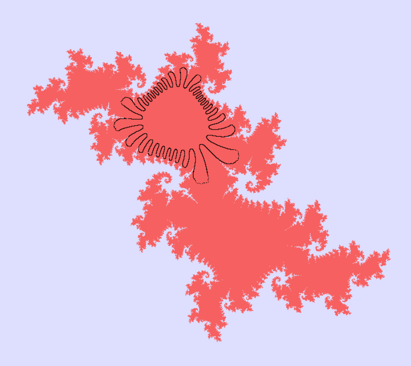
As μ varies around the unit circle, the filled Julia set $K(g_\mu)$ jumps around wildly:

Root of unity case



$\mu = e^{-2\pi i/5}$

For almost every μ

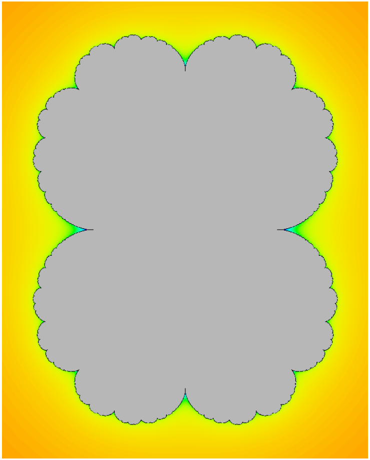


Siegel disk

For **generic** μ (the Cremer case):

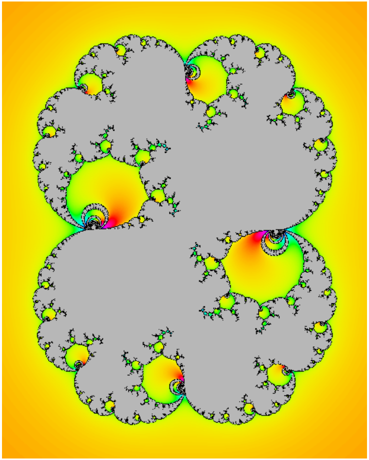
$K(g_\mu)$ is **not locally connected,** and has no interior.

Parabolic Implosion: the fundamental discontinuity.



$\mu = 1$

$z \mapsto z^2 + z$

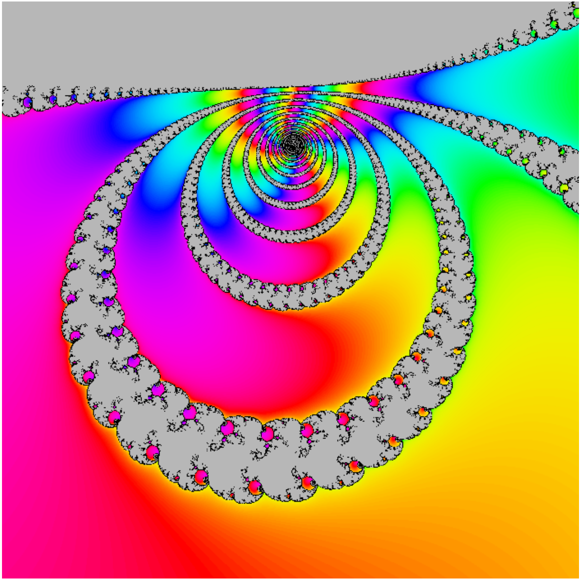


$\mu = e^{.01\,i}$

$z \mapsto z^2 + e^{.01\,i} z$

Under an **arbitrarily small perturbation** of a parabolic map, the basin of infinity **and** the Julia set may explode inwards.

Magnified Julia set for $\mu = e^{.04\, i}$



Foundational Paper:

Pierre Lavaurs, ***Systèmes dynamiques holomorphes: explosion de points périodiques paraboliques,***
Thèse, Université, Paris-Sud, Orsay 1989.

(Widely studied, but never published.)

Consider the family of maps $F_\eta(z) = z^2 + z + \eta^2$, where η is close to zero and $\Re(\eta) > 0$.

Thus F_η has fixed points $\pm i\,\eta$,

with multipliers $\mu = 1 \pm 2\,i\,\eta$.

Now pass to the limit as η tends to zero.

The Limit as $\eta \to 0$, $\eta \neq 0$.

Let $\mathcal{B}$ be the interior of the cauliflower, or in other words the parabolic basin for the map $z \mapsto z^2 + z$.

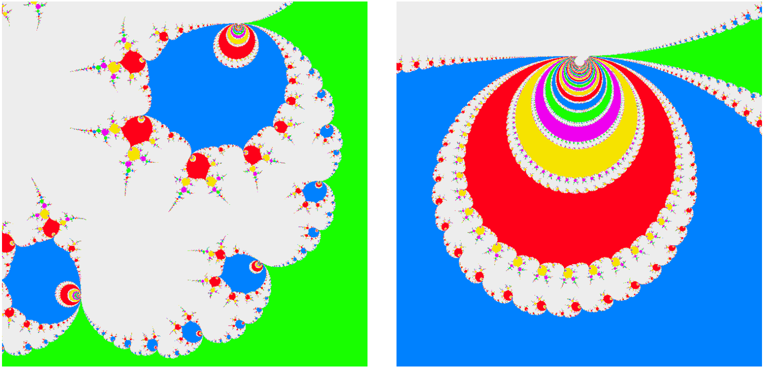
Define the ***phase function*** $\sigma(\eta) = -\frac{\pi}{\eta}$.

Theorem of Lavaurs. *Suppose that a sequence of parameters η_j converges to zero in such a way that the phase $\sigma(\eta_j)$ converges to a limit $\boldsymbol{\sigma}_0$ modulo $\mathbb{Z}$. In other words, suppose that there are integers k_j so that*

$$\lim_{j\to\infty} \left(\sigma(\eta_j) + k_j\right) = \boldsymbol{\sigma}_0 .$$

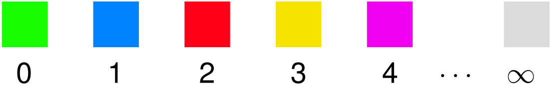
Then the sequence of functions $F_{\eta_j}^{\circ k_j}$ converges locally uniformly on $\mathcal{B}$ to a function $\mathcal{L}_{\boldsymbol{\sigma}_0} : \mathcal{B} \to \mathbb{C}$ which is holomorphic and effectively computable.

Plot of $\mathcal{L}_\sigma : \mathcal{B} \to \mathbb{C}$, for fixed $\sigma_0 = i\,\pi$.



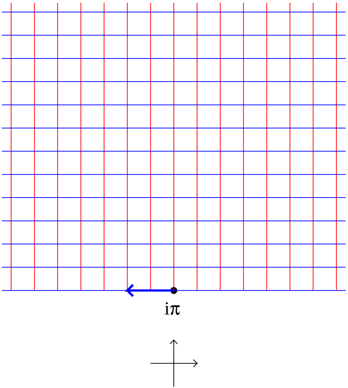
The color indicates the value of the ***escape function***

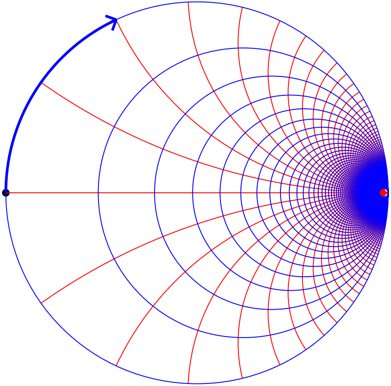
$$\mathrm{esc}(\sigma, z) \;=\; \min\{n\,;\; \mathcal{L}_\sigma^{\circ n}(z) \neq \mathcal{B}\} \qquad \text{(reduced modulo 5).}$$



From the σ-plane to the μ-plane

Lavaurs phase parameter $\sigma(\eta)$ $\longleftrightarrow$ multiplier μ_η

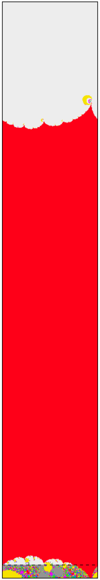




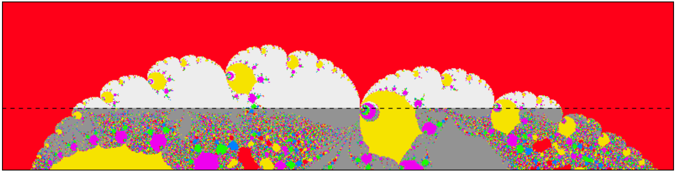
Here $\sigma \ = \ -\pi/\eta$ and $\mu_\eta = 1 + 2\,i\,\eta$.

Thus the half-plane $\Im(\boldsymbol{\sigma}) \geq \pi$ maps conformally onto $\overline{\mathbb{D}} \smallsetminus \{1\}$.

Plot of $\boldsymbol{\sigma} \mapsto \mathrm{esc}(\boldsymbol{\sigma},\, z_0)$ for fixed z_0 .



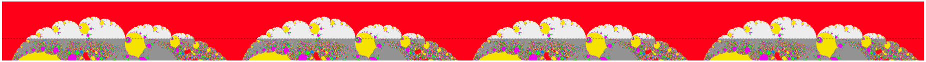
This shows the escape function $\mathrm{esc}(\boldsymbol{\sigma},\, z_0)$ for fixed $z_0 = -.141\, i$ as the Lavaurs parameter $\boldsymbol{\sigma}$ varies over the cylinder $[0, 1] \times [3,\, 9]$.



Detail showing the cylinder
$\mathcal{R} = [-.1,\, .9] \times [3.05,\, 3.3]$.

The analog of the hyperbolic component $\mathcal{H}$ in these coordinates is the set $\mathcal{H}_{\mathrm{Lav}}$ consisting of all $(\boldsymbol{\sigma},\, z) \in (\mathbb{C}/\mathbb{Z}) \times \mathbb{C}$ with $\Im(\boldsymbol{\sigma}) > \pi$ and $\mathrm{esc}(\boldsymbol{\sigma},\, z) = \infty$, **colored near-white.**

LEMMA. For $\boldsymbol{\sigma} \in \partial\mathcal{R}$ and for z close to z_0 , the pair $(\boldsymbol{\sigma},\, z)$ does NOT belong to $\overline{\mathcal{H}}_{\mathrm{Lav}}$.



This shows the escape function $\mathrm{esc}(\boldsymbol{\sigma},\ z_0)$ for fixed

$$z_0 = -.141\,i\,,$$

with $\boldsymbol{\sigma}$ in the rectangle $[-.1,\,3.9] \times [3.05, 3.3]$.

Choose $\boldsymbol{\sigma}_0$ in the white region, and consider the sequence

$$\boldsymbol{\sigma}_0\,, \quad \boldsymbol{\sigma}_0 - 1\,, \quad \boldsymbol{\sigma}_0 - 2\,, \quad \ldots \quad \text{tending to } -\infty\,.$$

Solving the equation

$$\sigma(\eta_k) = -\pi/\eta_k \;=\; \boldsymbol{\sigma}_0 - k$$

we obtain

$$\eta_k \;=\; \pi/(k - \boldsymbol{\sigma}_0) \quad \to \quad 0\,.$$

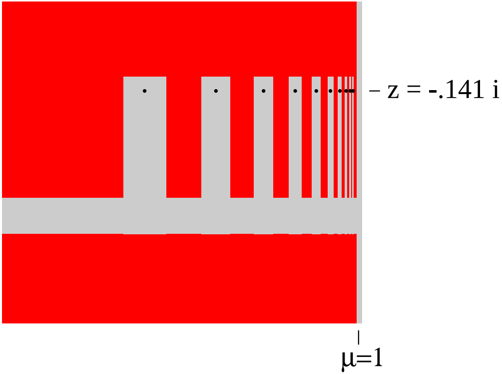
Thus the corresponding quadratic functions

$$f_k(z) \;=\; z^2 + z + \eta_k$$

converge to the Lavaurs map $\mathcal{L}_{\boldsymbol{\sigma}_0}$ on $\mathcal{B}$.

For every $\boldsymbol{\sigma} \in \partial[-.1, .9] \times [3.05, 3.3-]$,
for every η close to zero with $\sigma(\eta) \equiv \boldsymbol{\sigma} \pmod{\mathbb{Z}}$,
and for every z close to z_0, it follows that $(z,\,\mu_\eta) \notin \overline{\mathcal{H}}$.

Conclusion:



We have a sequence of pairs $(z_0\,,\ \mu_{\eta_k}) \ \to\ (z_0,\ 1)$,
with $(z_0\,,\ \mu_{\eta_k}) \in \mathcal{H}$ for large k .

But $(z_0\,,\ \mu_{\eta_k})$ cannot be connected to $(z_0,\ 1)$ within $\overline{\mathcal{H}}$
without changing z_0 by some fixed $\epsilon > 0$,
which is independent of k .

$\Longrightarrow$ $\overline{\mathcal{H}}$ is not locally connected. $\square$