Control Theory and Holomorphic Diffeomorphisms

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To Velimir Jurdjevič on the occasion of his sixtieth birthday.

1 Introduction

This paper surveys some results about holomorphic diffeomorphisms on very symmetric complex manifolds. However, the true underlying objective is to show how this can be achieved using certain ideas which lie at the foundations of control theory.

By *symmetric complex manifolds* we mean those whose holomorphic diffeomorphism groups are infinite dimensional and, even more, as large as they can possibly be. As a complex analyst might expect, the manifolds of greatest interest are Stein. However, much stronger hypotheses than these will be needed. In fact, having so many holomorphic diffeomorphisms is the non-generic situation in complex analysis. This will be explained in more detail below.

In what follows, a central role will be played by complete holomorphic vector fields. While this is to be expected, a new twist in complex analysis is that there are *always* very few complete holomorphic vector fields. (Contrast this with situations where nontrivial compactly supported functions exist.) However, there is a trick, going back to Euler, which allows one to approximate by diffeomorphisms the flow of a holomorphic vector field satisfying certain conditions. One is then lead naturally to the so-called *Density Property*, which we introduced in [17].

One of the simplest ways to exploit the ideas alluded to in the previous paragraph is to study the possible jets of holomorphic diffeomorphisms using these ideas. Here again, ideas from control theory make an appearance, since as we shall see, one wants to look at the orbits of the holomorphic diffeomorphism group in the jet space. With good control over the jets of holomorphic diffeomorphisms, classical techniques from complex dynamics can be used to reveal a lot about the underlying structure of these complex manifolds with very large holomorphic diffeomorphism groups.

The organization of the talk is as follows. We begin in section 2 by explaining the geometry behind the reasons one expects few complex manifolds to have very large holomorphic diffeomorphism groups. The basic obstruction is a generalization of the classical phenomenon of "Schwarz's lemma", i.e., the nondegeneracy of the so called *Kobayashi pseudometric*. We then go on, in section 3, to discuss holomorphic vector fields and Euler's method, which lead naturally to the definition of the *Density Property*, a condition which rigorizes the notion of "largest possible" holomorphic diffeomorphism group. Finally, in section 4 we state results on the jets of holomorphic diffeomorphisms and numerous corollaries which are proved via techniques from complex dynamics. These corollaries reveal much about the underlying complex structure of these manifolds. As we shall see, although the situation where there are largest possible holomorphic diffeomorphism groups resembles in many ways the situation of (real) smooth manifolds, there still arise some rigidities, which again distinguish the complex case.

Since the paper is primarily a survey, proofs are almost always omitted. There are three exceptions. The first is our proof of the Andersén-Lempert theorem (theorem 3 below), which appears in print here for the first time. The second is a sketch of the proof of the general jet theorem (theorem 11 below), which we have chosen to include because it is fundamentally control theoretic. Finally, we prove one of the corollaries, because it gives a sense of the role of dynamical systems in the application of the jet theorems. (The proof we include is based on an old idea, often attributed to König, but the complete proof is due to Rosay-Rudin [13].)

One glaring omission, due only to lack of space, is that of our general notion of shears [19]. This subject is based on the rigid existence criterion for functions which multiply complete holomorphic vector fields to complete holomorphic vector fields, and involves first integrals more directly than the real theory; it would have been nice to include here.

2 Complex analytic considerations

In the complex category it is generally more rare to have diffeomorphisms. As an illustrative example, let's look at the unit disk $\mathbb{D}$. First, for any complex number a with $|a| < 1$, the map $\varphi_a : \zeta \mapsto (a - \zeta)/(1 - \bar{a}\zeta)$ is a holomorphic involution on the unit disc which carries 0 to a. It follows that the unit disc is a homogeneous space, and using the Schwarz lemma one can easily show that the isotropy group of a point is S^1.

In fact, it is possible, without much difficulty, to see that the Schwarz lemma says essentially that every holomorphic diffeomorphism preserves the Poincaré line element $ds = |dz|/(1 - |z|^2)$. There is an analogous construction in a more general context, which we now describe.

Let M be a complex manifold, and let $v \in TM_x$. We define a (generally non-smooth) Finsler pseudometric K_M on M, called the *Kobayashi-Royden* pseudometric, as follows. With $\mathbb{D}_r := \{|\zeta| < r\} \subset \mathbb{C}$, let

$$K_M(v) := \inf \left\{ \frac{1}{r} \; ; \; \exists \, \text{holo} f : \mathbb{D}_r \to M \text{ s.t. } f(0) = x \text{ and } f'(0) = v \right\}.$$

Vaguely, the length of a vector v is one over the radius of the largest possible disc which can be mapped into M such that the origin passes through x with velocity v. We leave it as an exercise for the interested reader to apply the Schwarz lemma to show that when M is the unit disc, the Finsler metric so obtained is the Poincaré line element.

A nontrivial result, due to Royden [14], is that K_M is upper semicontinuous. One can then define a pseudometric ρ_M by

$$\rho_M(a, b) = \inf \left\{ \int_0^1 K_M(\gamma'(t)) dt \; ; \; \gamma([0,1], 0, 1) \to (M, a, b) \text{ smooth} \right\}.$$

If ρ_M is actually a metric, one says M is *Kobayashi hyperbolic*.

Note that the metric ρ_M is invariant under the action of $\text{Diff}_{\mathcal{O}}(M)$. It then follows from general geometric principles that $\text{Diff}_{\mathcal{O}}(M)$ is a locally compact topological group (see, e.g., [11]). This is the main step in proving that $\text{Diff}_{\mathcal{O}}(M)$ is actually a Lie group.

It is a strongly believed piece of folklore that most complex manifolds have Kobayashi pseudometrics with some sort of nondegeneracy. The trouble with proving something like that is that we don't have any idea what *most complex manifolds*

means. However, there are various special situations where the philosophy has been verified. For example, for domains in $\mathbb{C}^n$ there are quite sophisticated results. For compact Riemann surfaces, the uniformization theorem realizes our claim, and for complex surfaces one can witness the Kodaira classification. Recent work of Siu, Demailly and others even states that the generic smooth algebraic variety in $\mathbb{CP}^n$ of sufficiently high degree (depending on n) is hyperbolic.

3 Holomorphic vector fields

In this section we want to study the Lie algebra of $\mathrm{Diff}_{\mathcal{O}}(M)$. In the hyperbolic case, since the latter is a Lie group, we will have the usual Lie theoretic picture provided by the exponential map, but this will not be so in general.

For details on the geometry of holomorphic vector fields, we recommend the appropriate section of chapter 0 in [9].

3.1 Approximating solutions of ODE. The approximation technique we will describe now is due to Euler. Suppose one is given a family $F_t : M \to M$ of (holomorphic) maps on a (complex) manifold M which is $\mathcal{C}^1$ in both variables at once, such that $F_0 = \mathrm{id}_M$, and let

$$X := \left. \frac{d}{dt} \right|_{t=0} F_t.$$

(In practice, one is given X, and constructs the family F_t.) This datum says that, for a very short time, F_t is a good approximation to the flow of X. Euler's idea was to use this approximation repeatedly for shorter and shorter times. He proved that the limit converged to the flow. Precisely, the result is as follows.

Theorem 1. *Let X and F_t be as above, and denote the flow of X by φ_X^t. Then*

$$\lim_{N \to \infty} F_{t/N}^{(N)}(x) \to \varphi_X^t(x),$$

with the limit holding locally uniformly on the subset of $\mathbb{R} \times M$ where either side is defined (the so-called fundamental domain of X).

For a proof, one can see [1] where the family F_t is called an *algorithm* for X.

Two particular algorithms which are of interest to us arise from the following differentiation formulas. Let X and Y be two vector fields. Then

$$\left. \frac{d}{dt} \right|_{t=0} \varphi_X^t \circ \varphi_Y^t = X + Y, \text{ and } \left. \frac{d}{dt} \right|_{t=0^+} \varphi_Y^{-\sqrt{t}} \circ \varphi_X^{-\sqrt{t}} \circ \varphi_Y^{\sqrt{t}} \circ \varphi_X^{\sqrt{t}} = [X, Y].$$

These two algorithms, together with Euler's approximation theorem, lead immediately to the following, key observation.

Observation *If a vector field X lies in a Lie algebra $\mathfrak{g}$ of vector fields which is generated by the complete vector fields on a manifold M, then the time T map of this vector field, if and where it is defined, can be approximated, locally uniformly on its domain of definition, by diffeomorphisms which are time one maps of the complete vector fields in $\mathfrak{g}$.*

We should point out that, while approximating the flows of real vector fields on compact sets by diffeomorphisms is easy (due to the existence of cutoff functions), this is not so for vector fields in Lie algebras which do not admit multiplication by compactly supported functions.

3.2 The density property. In light of what has been developed above, the next definition is quite natural.

Definition 2. *A Lie algebra $\mathfrak{g} \subset \mathcal{X}_{\mathcal{O}}(M)$ of holomorphic vector fields on a complex manifold M is said to have the* density property *if the Lie subalgebra $\mathcal{C}\mathfrak{g} \subset \mathfrak{g}$ generated by complete vector fields is dense in $\mathfrak{g}$ (in the locally uniform topology). If $\mathcal{X}_{\mathcal{O}}(M)$ has the density property, we shall say that M has the density property. Lastly, if (M, ω) is a calibrated complex manifold (meaning ω is a nondegenerate holomorphic top form, or holomorphic volume element, on M, and if $\mathcal{X}_{\mathcal{O}}(M, \omega) := \{X \in \mathcal{X}_{\mathcal{O}}(M) \; ; \; div_{\omega}(X) = 0\}$ has the density property, we say that (M, ω) has the* volume density property.

Of course, for such a definition to be meaningful, one has to have examples. While compact manifolds give examples, they are less interesting, due to the fact that they support only finite dimensional Lie algebras of holomorphic vector fields. A more interesting class to consider is that of Stein manifolds. A complex manifold is called *Stein* if it can be embedded as a closed complex submanifold of $\mathbb{C}^N$ for some N. This condition guarantees existence of a lot of holomorphic functions and vector fields.

Most Stein manifolds do not have the density property. As an example, note that all open Riemann surfaces have infinite dimensional Lie algberas of holomorphic vector fields, but only finite dimensional diffeomorphism groups. (In fact, most of them only have finite diffeomorphism groups.)

The first results in the study of the density property predate the definitions, and were, in fact, the inspiration for the theory. These are the theorems of Andersén [3] and Andersén-Lempert [4].

Theorem 3. *Let $n \geq 2$ and $\omega := dz_1 \wedge ... \wedge dz_n$. Then*
  1. $(\mathbb{C}^n, \omega)$ has the volume density property [3], and
  2. $\mathbb{C}^n$ has the density property [4].

Proof. We restrict to the case $n = 2$, since the higher dimensional case can be obtained by an obvious adaptation of this proof.

1. We will prove that given any polynomial divergence zero vector field, we can write it as a sum of Lie brackets of complete divergence zero vector fields. By linearity, it suffices to consider a vector field of the form

$$X(z_1, z_2) = z_1^m z_2^n \partial_{z_1} + g(z) \partial_{z_2}.$$

Consider the vector field

$$Y(z) = \left[z_1^n \partial_{z_2}, \frac{1}{m+1} z_2^{m+1} \partial_{z_1} \right].$$

Then $X(z) - Y(z) = h(z) \partial_{z_2}$, and since X and Y have zero divergence, so does $X - Y$. Thus h is independent of z_2, and hence $X - Y$ is complete. It follows that $X = Y + (X - Y)$ is completely generated, which proves 1.

2. It suffices to show that if P is any polynomial vector field, then there exists a completely generated vector field Q such that $\mathrm{div}\, Q = \mathrm{div}\, P$, for then $P = Q + (P - Q)$ is completely generated by 1. Moreover, it suffices by linearity to show that there is a completely generated vector field whose divergence is $z_1^m z_2^n$. To this end, notice that

$$\mathrm{div} \left[z_2^n \partial_{z_1}, \frac{1}{m+1} z_1^{m+1} z_2 \partial_{z_2} \right] = z_1^m z_2^n,$$

which completes the proof. $\qquad\qquad\square$

A similar method, together with some basic complex analysis, can be used to prove the following result.

Theorem 4. [17] *Let G be a complex Lie group. Then*
1. *$G \times \mathbb{C}$ has the volume density property.*
2. *If G has the volume density property, then so does $G \times \mathbb{C}^*$.*
3. *If G is Stein and nontrivial, then $G \times \mathbb{C}$ has the density property.*

This theorem gives numerous examples of Stein manifolds having the density property. We note the following.

• If we consider the case $(G, \omega) = (\mathbb{C}^n, dz_1 \wedge ... \wedge dz_n)$, this result recovers the Andersén-Lempert theorem.

• We can apply 2 to $(G, \omega) = ((\mathbb{C}^*)^k, (z_1 \cdot ... \cdot z_k)^{-1} dz_1 \wedge ... \wedge dz_k)$ inductively (it is easy for $k = 1$). It follows that $(\mathbb{C}^*)^k$ has the volume density property for $k \geq 1$.

• Every simply connected Lie group is Stein. In fact, this is the case for most Lie groups. For example, every semisimple Lie group is Stein.

Theorem 5. [15] *Every semisimple Lie group has the density property.*

Finally, a structure result.

Theorem 6. [17] *Let M and N be Stein manifolds.*
1. If M and N have the density property, then so does $M \times N$.
2. If M has the density property, then so do $M \times \mathbb{C}$ and $M \times \mathbb{C}^*$.
3. If $(M \times \mathbb{C}, \omega \wedge dz)$ has the volume density property, then $M \times \mathbb{C}$ has the density property.

It is clear that in order to have complete holomorphic vector fields, a lot of symmetry is needed. One might wonder, based on this and the above results, whether the only possible examples are groups. This turns out not to be the case. In order to give more examples, we need the following definition.

Definition 7. [19] *An EMV manifold is a pair (M, ω), where M is a complex manifold and ω is a holomorphic volume element on M, with the property that for any $V \in \mathcal{X}_{\mathcal{O}}(M)$, compact $K \Subset M$ and $\epsilon > 0$ there are functions $f_1, ..., f_r \in \mathcal{O}(K)$ and divergence zero completely generated vector fields $X_1, ..., X_r$ satisfying*

$$\left\| V - \sum f_j X_j \right\|_K < \epsilon$$

Examples (See [19] for details.)
(1) Every complex Lie group G is EMV with respect to an invariant volume.
(2) Every Stein complex homogeneous space is EMV, again with respect to an invariant volume.
(3) $M := \{(x, y) \in \mathbb{C}^2 \mid xy \neq 1\}$ together with the volume form $(xy-1)^{-1} dx \wedge dy$ is EMV, since the complete vector fields $(xy-1)\partial_x$ and $(xy-1)\partial_y$ parallelize the tangent bundle.
(4) The space
$$\Sigma^3 := \{(a, b, c, d) \in \mathbb{C}^4 \mid a^2 d - bc = 1\},$$
which is a smooth subvariety of $\mathbb{C}^4$ and is also a branched double cover of $SL(2, \mathbb{C})$ is EMV with respect to a certain volume element.

Theorem 8. [19] *If (M, ω) is an EMV manifold, then $(M \times \mathbb{C}, \omega \wedge dz)$ has the volume density property.*

It follows from theorem 6 above that if M is Stein, then $M \times \mathbb{C}$ has the density property. This theorem gives a lot of "non-group" examples of the density property. There are also the following results.

Theorem 9. [19] *The space in example 3 has the volume density property.*

Theorem 10. [15, 16] *The complex quadrics*

$$Q_n := \left\{ (x_0, ..., x_n) \in \mathbb{C}^{n+1} \,\middle|\, \sum_0^n x_i^2 = 1 \right\}$$

have the density and volume density property, the latter with an $SO(n+1, \mathbb{C})$ left-invariant volume element.

Finally, let us conclude this section by mentioning that there are some results about the density property for more general Lie algebras. Due to lack of space, we do not mention these here, but refer the interested reader to [17].

4 JETS

4.1 The jet theorems. Behind all known applications is a key theorem which allows one to realize jets as those of holomorphic diffeomorphisms. To state this theorem, we need to define certain spaces of jets which, a priori, satisfy obvious conditions for being jets of holomorphic diffeomorphisms.

Let M be a complex manifold. To recall, two germs $f, g \in \mathcal{O}(M, M)_{x,y}$ (the subscripts indicate that $f(x) = g(x) = y$) are equivalent if they have the same Taylor expansion to order k, and a k-jet is simply an equivalence class. Let $J^k(M)_{x,y}$ denote the space of k-jets of germs from x to y, and write

$$J^k(M)_{x,*} := \bigcup_{y \in M} J^k(M)_{x,y} \quad \text{and} \quad J^k(M) := \bigcup_{x \in M} J^k(M)_{x,*}.$$

We note that both of these spaces are actually manifolds. Given a map f from a neighborhood U of x in M into M, we denote by $j_x^k(f)$ the induced jet in $J^k(M)_{x,*}$ and by $j^k(f) : U \to J^k(M)$ the map $j^k(f)(x) := j_x^k(f)$.

Definition *Let M be a complex manifold.*

(1) *Let $J^0(M)_{x,y}^\times := J^0(M)_{x,y}$, and for $k \geq 1$ let $J^k(M)_{x,y}^\times$ be the set of all k-jets $[f]$ with the property that $Df(x) : T_x M \to T_y M$ is an isomorphism.*

(2) *Let ω be a holomorphic volume element on M. Then $J^0(M, \omega)_{x,y} := J^0(M)_{x,y}$ and for $k \geq 1$ let $J^k(M, \omega)_{x,y}$ be the set of all k-jets $[f]$ such that the ω-Jacobian determinant J_f of f (defined by $f^*\omega = J_f \omega$) coincides to order k with the constant function $\varphi(x) \equiv 1$.*

The jets in $J^k(M)_{x,y}^\times$ and $J^k(M, \omega)_{x,y}$ might be thought of as jets of maps which satisfy minimal necessary conditions for being holomorphic diffeomorphisms, namely, one point conditions on derivatives.

Let $\mathfrak{g} \subset \mathcal{X}_\mathcal{O}(M)$ be a Lie algebra of holomorphic vector fields.

Definition *The orbit of $\mathfrak{g}$ through $p \in M$, denoted $\mathcal{R}_\mathfrak{g}(p)$, consists of all points $q \in M$ of the form*

$$q = \varphi_{X_N}^{t_N} \circ ... \circ \varphi_{X_1}^{t_1}(p) \tag{1}$$

for some $N \in \mathbb{N}$, $X_1, ...X_N \in \mathfrak{g}$, and $t_1, ..., t_N \in \mathbb{R}$ such that (1) makes sense.

Each $X \in \mathcal{X}_{\mathcal{O}}(M)$ induces a vector field $p_k(X) \in \mathcal{X}_{\mathcal{O}}(J^k(M))$ whose flow is defined by

$$\varphi^t_{p_k(X)}([f]) := [\varphi^t_X \circ f].$$

Clearly p_k maps complete vector fields to complete vector fields. It is not difficult to show that $p_k : \mathcal{X}_{\mathcal{O}}(M) \to \mathcal{X}_{\mathcal{O}}(J^k(M))$ is a Lie algebra isomorphism, and that

$$\left(\varphi^t_{p_k(X)}\right)_* (p_k(Y)) = p_k \left((\varphi^t_X)_* Y\right).$$

Definition *Let $\mathfrak{g}$ be a Lie algebra of holomorphic vector fields on a complex manifold M, and let $k \geq 0$ be an integer. Then*

$$J^k_{\mathfrak{g}}(M)_{x,*} := \mathcal{R}_{p_k(\mathfrak{g})} \left(j^k_x(id_M)\right),$$

and

$$J^k_{\mathfrak{g}}(M) := \bigcup_{x \in M} J^k_{\mathfrak{g}}(M)_{x,*}.$$

We note that when M is Stein, it is easy to show that

$$J^k_{\mathcal{X}_{\mathcal{O}}(M)}(M)_{x,*} = J^k(M)^\times_{x,*} \quad \text{and} \quad J^k_{\mathcal{X}_{\mathcal{O}}(M)}(M)_{x,*} = J^k(M, \omega)_{x,*}.$$

However, this is of course false for a general complex manifold, as for example, a compact manifold would show.

Finally, we let $Aut_{\mathfrak{g}}(M)$ denote the subgroup of $\text{Diff}_{\mathcal{O}}(M)$ generated by time one maps of complete vector fields in $\mathfrak{g}$. The key results are now the following.

Theorem 11. [18] *Let $\mathfrak{g}$ be a Lie algebra of holomorphic vector fields with the density property. Then for each $\gamma \in J^k_{\mathfrak{g}}(M)$ there exists $\Phi \in Aut_{\mathfrak{g}}(M)$ such that*

$$j^k_{\sigma(\gamma)}(\Phi) = \gamma.$$

Here and below, σ and τ are the *source* and *target* maps, respectively.

Theorem 12. [18] *Let M be a connected Stein manifold, and let $K \subset M$ be a compact set.*

(1) *If M has the density property and $\gamma \in J^k(M)^\times$ is a k-jet such that $x := \sigma(\gamma)$ and $\tau(\gamma)$ are not in the $\mathcal{O}(M)$-hull of K, then there exists $\Phi \in \text{Diff}_{\mathcal{O}}(M)$ such that*

$$j^k_x(\Phi) = \gamma,$$

and such that $j^k_z(\Phi)$ is as close to $j^k_z(id)$ as we like for all $z \in K$. Furthermore, we can arrange that $j^k_z(\Phi) = j^k_z(id)$ for z in some finite subset of K.

(2) *If (M, ω) has the volume density property and $\gamma \in J^k(M, \omega)$ is a k-jet such that $x := \sigma(\gamma)$ and $\tau(\gamma)$ are not in the $\mathcal{O}(M)$-hull of K, then there exists $\Phi \in \text{Diff}_{\mathcal{O}}(M)$ with the same properties as in 1, and such that $\Phi^* \omega = \omega$.*

Let us give a sketch of the proof of theorem 11. The first step is to reduce to the case of zero jets; $k = 0$. To this end, note that since $p_k : \mathcal{X}_{\mathcal{O}}(M) \to \mathcal{X}_{\mathcal{O}}(J^k_{\mathfrak{g}}(M))$ is just an invariant way of collecting X and its first k derivatives into a single object, it follows from the Cauchy inequalities that p_k is continuous, and hence $p_k(\mathfrak{g})$ has the density property if and only if $\mathfrak{g}$ does.

Consider next the map associating to each $\Phi \in Aut(M)$ an element $\Phi_\# \in Aut(J^k(M))$ defined by $\Phi_\#[f] = [\Phi \circ f]$. Then

$$\left(Aut_{\mathfrak{g}}(M)\right)_\# = Aut_{p_k(\mathfrak{g})}\left(J^k(M)\right),$$

and we are thus reduced to the case $k = 0$. That is to say, Theorem 11 follows immediately from the following theorem.

Theorem 13. *If a Lie algebra $\mathfrak{g}$ has the density property, then for all $p \in M$, $Aut_{\mathfrak{g}}(M)$ acts transitively on the orbit $\mathcal{R}_{\mathfrak{g}}(p)$.*

Now, given two points on the orbit of $\mathfrak{g}$, the density property easily implies (using Euler's approximation theorem) the existence of a holomorphic diffeomorphism which maps one of these points arbitrarily close to the other. Since by the orbit theorem [10] the orbit is a manifold, one can correct this approximation using an implicit function type argument. A more complete proof can be found in [18].

4.2 Applications. Theorem 3 was quickly applied by many authors to the study of analytic geometry of $\mathbb{C}^n$, $n \geq 2$. Many of the applications in that context are well documented in the surveys [7] and [12], so we shall not discuss them here. Instead, we mention the results on more general (mostly Stein) manifolds with the density and volume density property. These results are corollaries of Theorems 11 and 12. Some of them are just generalizations of similar results in the case of $\mathbb{C}^n$, but others are of interest only in this general context. All of the results of this section can be found in [18].

The Fatou-Bieberbach Phenomenon. The first consequence of Theorem 11 is the following.

Corollary 14. *Let M be a Stein manifold of complex dimension n with the density property. Then there is an open subset of M which is biholomorphic to $\mathbb{C}^n$.*

Proof. Fix $p \in M$, and let $F \in \mathrm{Diff}_{\mathcal{O}}(M)$ be such that $F(p) = p$, and that $DF(p) =: A : TM_p \to TM_p$ has eigenvalues $\lambda_1, ..., \lambda_n$ ($n := \dim_{\mathbb{C}}(M)$) satisfying $|\lambda_1| \geq |\lambda_2| \geq ... \geq |\lambda_n| > |\lambda_1|^2$. Fix a holomorphic diffeomorphism χ from a small neighborhood of p in M to a small neighborhood of 0 in TM_p, and denote by U the basin of attraction to p by F. Then the map $K : U \to TM_p$ given by

$$K := \lim_{k \to \infty} A^{-k} \circ \chi \circ F^{(k)}$$

is a well defined, injective holomorphic map on U, satisfying the functional equation

$$K = A^{-1} \circ K \circ F.$$

(The eigenvalues guarantee the convergence. See [13], theorem 9.1.) Note that K conjugates $F|U$ to a linear map on TM_p, and that K is injective by the Hurwitz principle. It follows from the functional equation that the image of K is invariant by A^{-1}. Since the latter is an expanding linear map, K must be surjective. This completes the proof. □

K conjugates F to the linear map A on the basin of attraction to p.

With little more effort, one can also prove the following result.

Corollary 15. *Let M be an n dimensional Stein manifold with the density property. Then there are infinitely many disjoint domains in M which are biholomorphic to $\mathbb{C}^n$.*

One can also use the so-called "kick out" method of Dixon-Esterle [6] to prove the following result.

Corollary 16. *Let M be a Stein manifold with the density property. Then there exist proper open subsets of M which are biholomorphic to M.*

We note that when $M = \mathbb{C}^n$, this corollary gives another construction of the classical *Fatou-Bieberbach* domains, i.e., proper open subsets of $\mathbb{C}^n$ which are biholomorphic to $\mathbb{C}^n$. (This fact has been exploited in many results of analytic geometry in $\mathbb{C}^n$.) However, these corollaries show that the two methods (dynamical and kick out) might be "different". A natural question is whether every Fatou Bieberbach domain in $\mathbb{C}^n$ is the region of attraction of a holomorphic diffeomorphism. Corollary 16 suggests that the answer might not be very simple.

If we consider now a calibrated Stein manifold (M, ω) with the volume density property, one can show the following.

Corollary 17. *Let (M, ω) be a calibrated Stein manifold with the volume density property. Then there exists a proper open subsets of M which is biholomorphic to M.*

One can also construct nondegenerate maps of $\mathbb{C}^n$ into a calibrated Stein manifold (M, ω) with the volume density property.

Corollary 18. *Let (M, ω) be a calibrated Stein manifold of dimension n having the volume density property. Then there exists a map $h : \mathbb{C}^n \to M$ such that $h^*\omega$ is not identically zero.*

One can also get injective immersions of $\mathbb{C}^{n-1}$ tangent to any given complex hyperplane in TM.

Corollary 19. *Let (M, ω) be a calibrated Stein manifold of dimension n with the volume density property, and let $V_p \subset TM_p$ be a complex hyperplane. Then there is an injective holomorphic immersion $g : \mathbb{C}^{n-1} \to M$ such that $dg_p(\mathbb{C}^{n-1}) = V_p$.*

Finally we have the following proposition.

Proposition 20. *Let (M, ω) be a calibrated Stein manifold with the volume density property, and suppose there exists $F \in Aut(M)$ such that the ω Jacobian determinant J_F of F has modulus different from 1 at some point $p \in M$, then M has an open subset biholomorphic to $\mathbb{C}^n$.*

The idea of the proof is to use holomorphic diffeomorphisms with jets in $J^k(M, \omega)$ to modify F so that p becomes an attracting fixed point, and then apply the same dynamical principle as above.

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Completeness of vector fields. One of the consequences of corollaries 14 and 19 is that on a Stein manifold with the density or volume density property, all bounded plurisubharmonic functions are constant. Then the main theorem of [2] implies the following corollary.

Corollary 21. *Let M be a Stein manifold with the density or volume density property. Then every $\mathbb{R}^+$-complete holomorphic vector field on M is $\mathbb{C}$-complete.*

A holomorphic vector field X is $\mathbb{R}^+$-complete if one can extend the flow of X to all of $\mathbb{R}^+$, and it is called $\mathbb{C}$-complete if X and iX are complete (in the usual sense).

Interpolation results. In this paragraph we note that for manifolds with the density property or volume density property, a given (proper, or closed) complex submanifold can be modified so as to interpolate any given discrete sequence. For the proof of the next result in the case $M = \mathbb{C}^n$ (which can easily be adapted to the more general case stated here) see [7].

Corollary 22. *Let M be a Stein manifold of $\mathbb{C}$-dimension $n \geq 2$ with the density or volume density property, Σ a Stein manifold of $\mathbb{C}$-dimension $r < n$, and $\{\gamma_m; m \geq 1\} \subset J^k(\Sigma, M)$ a sequence of k-jets such that $\{\sigma(\gamma_m)\}$ and $\{\tau(\gamma_m)\}$ are discrete sequences in Σ and M respectively. If Σ admits a proper holomorphic embedding in M, then there exists a proper holomorphic embedding $\varphi : \Sigma \hookrightarrow M$ such that*

$$j^k_{\sigma(\gamma_m)}(\varphi) = \gamma_m.$$

In a recent preprint [20], J. Winkelmann has constructed "non-tame sequences" in any Stein manifold. These can be used, together with corollary 22 to construct non-equivalent embeddings of a given complex manifold Σ into a Stein manifold M with the density or volume density property, provided one such embedding exists. Precisely, one has the following.

Corollary 23. *Let M be a Stein manifold of $\mathbb{C}$-dimension $n \geq 2$ with the density or volume density property, and Σ a Stein manifold of $\mathbb{C}$-dimension $r < n$ such that there exists a proper holomorphic embedding $j : \Sigma \hookrightarrow M$. Then there exists another proper holomorphic embedding $j' : \Sigma \hookrightarrow M$ such that for any $\Phi \in Aut(M)$,*

$$\Phi \circ j(\Sigma) \neq j'(\Sigma).$$

In connection with the last corollary, we should mention that the *analytically inequivalent* embeddings constructed in the proof are all ambiently smoothly isotopic, in the sense that there exists a global smooth diffeomorphism of M which is isotopic to the identity and carries one embedding to the other. Thus the obstructions are not topological. In particular, even in the presence of extreme symmetry, complex rigidity persists.

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