The Density Property For Complex Manifolds and Geometric Structures II

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Introduction

In [V1] we introduced the following definition:

Definition *Let $\mathfrak{g}$ be a Lie algebra of holomorphic vector fields. We say that $\mathfrak{g}$ has the density property if the Lie subalgebra of $\mathfrak{g}$ generated by the complete vector fields in $\mathfrak{g}$ is dense in $\mathfrak{g}$. When the Lie algebra $\mathcal{X}_{\mathcal{O}}(M)$ of all holomorphic vector fields on a complex manifold M has the density property, we say that M has the density property. If (M,ω) is a calibrated complex manifold, i.e., ω is a nondegenerate holomorphic n-form ($n = dim_{\mathbb{C}}M$) on M, we say that (M,ω) has the volume density property if the Lie algebra $\mathcal{X}_{\mathcal{O}}(M,\omega)$ of all holomorphic vector fields with vanishing ω-divergence has the density property.*

Recall that a holomorphic vector field X is complete if for every $x_0 \in M$, the solution of the ODE $\dot{x} = X(x),\ x(0) = x_0$ is defined for all time, and that $div_\omega X := (L_X\omega)/\omega = (d(X\rfloor\omega))/\omega$. Moreover, $\mathcal{X}_{\mathcal{O}}(M)$ is given the locally uniform topology, and any subset is given the subspace topology.

As was pointed out in [V1], one of the consequences of the density property for a Lie algebra $\mathfrak{g}$ is that given any holomorphic vector field in $\mathfrak{g}$, one can approximate its flow by automorphisms. The first objective of the present article is to exploit this fact to obtain a precise and very useful result about jets of automorphisms. We then use this result to prove various corollaries which reveal some of the properties of (mostly Stein) manifolds with the density and volume density property.

Before we can state our main results, we need a few definitions. We denote by $Aut_{\mathfrak{g}}(M)$ the subgroup of $AutM$ generated by time-1 maps of complete vector fields in $\mathfrak{g}$, and by $J^k_{\mathfrak{g}}(M)$ the set of k-jets of local biholomorphisms of the form $\varphi^{t_N}_{X_N} \circ ... \circ \varphi^{t_1}_{X_1}$, where $N \in \mathbb{N}$, $X_1,...,X_N \in \mathfrak{g}$, $\varphi^t_{X_j}$ is the local flow of X_j, and $t_1,...,t_N \in \mathbb{R}$ are such that $\varphi^{t_N}_{X_N} \circ ... \circ \varphi^{t_1}_{X_1}$ makes sense. $J^k(M)^\times$ is the space of k-jets of biholomorphisms of open subsets of M, $J^k(M,\omega)$ is the space of k-jets of biholomorphisms of open subsets of M whose ω-Jacobian is 1 to order k, and $Aut(M,\omega)$ is the group of automorphisms f of M such that $f^*\omega = \omega$. Finally, $j^k_x(F)$ is the k-jet of F at $x \in M$, and $\sigma(\gamma)$ and $\tau(\gamma)$ are the source and target of a jet γ. (Slightly more elaborate definitions are given in section 1.) With this, here are our main results:

Theorem 1 *Let $\mathfrak{g}$ be a Lie algebra of holomorphic vector fields with the density property. Then for each $\gamma \in J^k_{\mathfrak{g}}(M)$ there exists $\Phi \in Aut_{\mathfrak{g}}(M)$ such that*

$$j^k_{\sigma(\gamma)}(\Phi) = \gamma.$$

Theorem 2 *Let M be a connected Stein manifold, and let $K \subset M$ be a compact set.*

(1) *If M has the density property and $\gamma \in J^k(M)^\times$ is a k-jet such that $x := \sigma(\gamma)$ and $\tau(\gamma)$ are not in the $\mathcal{O}(M)$-hull of K, then there exists $\Phi \in Aut(M)$ such*

that

$$j^k_x(\Phi) = \gamma,$$

and such that $j^k_z(\Phi)$ *is as close to* $j^k_z(id)$ *as we like for all* $z \in K$. *Furthermore, we can arrange that* $j^k_z(\Phi) = j^k_z(id)$ *for* z *in some finite subset of* K.

(2) *If* (M, ω) *has the volume density property and* $\gamma \in J^k(M, \omega)$ *is a* k*-jet such that* $x := \sigma(\gamma)$ *and* $\tau(\gamma)$ *are not in the* $\mathcal{O}(M)$*-hull of* K*, then there exists* $\Phi \in Aut(M, \omega)$ *with the same properties as in 1.*

Theorem 1 in the case $\mathfrak{g} = \mathcal{X}_{\mathcal{O}}(\mathbb{C}^n, dz_1 \wedge ... \wedge dz_n)$ was proved by Andersén and Lempert [AL]. Theorem 2 in the case $M = \mathbb{C}^n$ is due to Forstnerič [F1], and some parts of our proof are much the same as his. There are two new ideas here: the first is that we discover a trick which allows us to reduce to the case of zero jets, and makes our proof quite economical; the second is that in the absence of exact formulas provided by shears in $\mathbb{C}^n$, we need a perturbation argument to pass from an approximate version of the theorem to the precise version.

It is natural to ask for examples of complex manifolds (especially Stein manifolds) with the density property. While such manifolds will be rare (see especially section 4 below) there are by now quite a few examples. The first example was of course $\mathbb{C}^n$ and $(\mathbb{C}^n, dz_1 \wedge ... \wedge dz_n)$. The author showed in [V1] that given any complex Lie group G, $G \times \mathbb{C}$ has the volume density property (with respect to the left invariant holomorphic volume element) and that if G is moreover Stein, then $G \times \mathbb{C}$ has the density property. In [V2] it was shown that one can take G to be much more general than a complex Lie group; a so called EMV space, which includes homogeneous spaces but also much more. Finally, in recent work, the author and A. Toth showed that every complex semisimple Lie group has the density property [TV].

The organization of the paper is as follows. Section 1 is devoted to the definition of the objects needed in proving Theorems 1 and 2. Section 2 establishes a lemma which is crucial in the approximation of flows of completely generated vector fields by families of automorphisms. In section 3 we prove Theorems 1 and 2, and finally in section 4 we give a long list of corollaries of our main theorems. In so doing, we hope to demonstrate that these theorems allow one to establish many results about (mainly Stein) manifolds with the density property which have been previously proved in $\mathbb{C}^n$. More importantly, however, Theorems 1 and 2 reveal a lot about the underlying complex structure of manifolds which support the density property in many of its various forms.

Acknowledgments. We wish to thank Laszlo Lempert for his very useful comments on an earlier version of this paper, and Mattias Jonsson for helpful and interesting discussions.

1. Jet spaces associated to Lie algebras

Let M be a complex manifold. To recall, two germs $f, g \in \mathcal{O}(M, M)_{x,y}$ (the subscripts indicate that $f(x) = g(x) = y$) are equivalent if they have the same Taylor expansion to order k, and a k-jet is simply an equivalence class. Let $J^k(M)_{x,y}$ denote the space of k-jets of germs from x to y, and write

$$J^k(M)_{x,*} := \bigcup_{y \in M} J^k(M)_{x,y} \quad \text{and} \quad J^k(M) := \bigcup_{x \in M} J^k(M)_{x,*}.$$

We note that both of these spaces are actually manifolds. Given a map f from a neighborhood U of x in M into M, we denote by $j^k_x(f)$ the induced jet in $J^k(M)_{x,*}$ and by $j^k(f) : U \to J^k(M)$ the map $j^k(f)(x) := j^k_x(f)$.

Definition *Let M be a complex manifold.*

(1) *Let $J^0(M)^\times_{x,y} := J^0(M)_{x,y}$, and for $k \geq 1$ let $J^k(M)^\times_{x,y}$ be the set of all k-jets $[f]$ with the property that $Df(x) : T_xM \to T_yM$ is an isomorphism.*

(2) *Let ω be a holomorphic volume element on M. Then write $J^0(M,\omega)_{x,y} := J^0(M)_{x,y}$ and for $k \geq 1$ let $J^k(M,\omega)_{x,y}$ be the set of all k-jets $[f]$ such that the ω-Jacobian determinant J_f of f (defined by $f^*\omega = J_f\omega$) coincides to order k with the constant function $\varphi(x) \equiv 1$.*

The jets in $J^k(M)^\times_{x,y}$ and $J^k(M,\omega)_{x,y}$ might be thought of as jets of maps which satisfy minimal necessary conditions for being automorphisms, namely, one point conditions on derivatives.

Let $\mathfrak{g} \subset \mathcal{X}_{\mathcal{O}}(M)$ be a Lie algebra of holomorphic vector fields.

Definition *The orbit of $\mathfrak{g}$ through $p \in M$, denoted $\mathcal{R}_{\mathfrak{g}}(p)$, consists of all points $q \in M$ of the form*

$$q = \varphi^{t_N}_{X_N} \circ ... \circ \varphi^{t_1}_{X_1}(p) \tag{1}$$

for some $N \in \mathbb{N}$, $X_1, ...X_N \in \mathfrak{g}$, and $t_1, ..., t_N \in \mathbb{R}$ such that (1) *makes sense.*

Each $X \in \mathcal{X}_{\mathcal{O}}(M)$ induces a vector field $p_k(X) \in \mathcal{X}_{\mathcal{O}}(J^k(M))$ whose flow is defined by

$$\varphi^t_{p_k(X)}([f]) := [\varphi^t_X \circ f].$$

Clearly p_k maps complete vector fields to complete vector fields. It is not difficult to show that $p_k : \mathcal{X}_{\mathcal{O}}(M) \to \mathcal{X}_{\mathcal{O}}(J^k(M))$ is a Lie algebra isomorphism, and that

$$\left(\varphi^t_{p_k(X)}\right)_* (p_k(Y)) = p_k\left((\varphi^t_X)_* Y\right).$$

Definition *Let $\mathfrak{g}$ be a Lie algebra of holomorphic vector fields on a complex manifold M, and let $k \geq 0$ be an integer. Then*

$$J^k_{\mathfrak{g}}(M)_{x,*} := \mathcal{R}_{p_k(\mathfrak{g})}\left(j^k_x(id_M)\right),$$

and

$$J^k_{\mathfrak{g}}(M) := \bigcup_{x \in M} J^k_{\mathfrak{g}}(M)_{x,*}.$$

We note that when M is Stein, it is easy to show that

$$J^k_{\mathcal{X}_{\mathcal{O}}(M)}(M)_{x,*} = J^k(M)^\times_{x,*} \quad \text{and} \quad J^k_{\mathcal{X}_{\mathcal{O}}(M)}(M)_{x,*} = J^k(M,\omega)_{x,*}.$$

However, this is of course false for a general complex manifold, as for example, a compact manifold would show.

2. A Useful Lemma

Lemma 2.1.

(1) *Let X be a holomorphic vector field on a complex manifold Σ, and suppose that X is a finite sum of iterated Lie brackets of complete holomorphic vector fields. Then there exists a family of maps $\{\psi_t \mid t \in \mathbb{R}\} \subset Aut(M)$ such that $(t, x) \to \psi_t(x)$ is $\mathcal{C}^1$, $\psi_0 = id$, and*

$$\left.\frac{d}{dt}\right|_{t=0} \psi_t = X.$$

(2) *Suppose, moreover, that $K \subset\subset M$ is compact, and that $\epsilon > 0$ is such that the flow φ_X of X exists on K for a time $I_\epsilon := [-\epsilon, \epsilon]$. Then for any $\delta > 0$ we can choose ψ_t as in 1 to further satisfy*

$$\sup_{K \times I_\epsilon} dist(\psi_t, \varphi_X^t) < \delta.$$

Proof. Recall (or see [Ar]) that for two vector fields X and Y, one has

$$\varphi_X^t \circ \varphi_Y^t(x) = \varphi_{X+Y}^t(x) + o(t) \tag{a}$$

and

$$\varphi_Y^{-t} \circ \varphi_X^{-s} \circ \varphi_Y^t \circ \varphi_X^s(x) = \varphi_{[X,Y]}^{st}(x) + o(s^2 + t^2) \tag{b}$$

where $o(t)$ and $o(s^2 + t^2)$ hold locally uniformly in $x \in M$. Now let $s(t) := \mathrm{sgn}(t)\sqrt{|t|}$. Setting $\psi_t(x) := \varphi_Y^{-\sqrt{|t|}} \circ \varphi_X^{-s(t)} \circ \varphi_Y^{\sqrt{|t|}} \circ \varphi_X^{s(t)}(x)$ we obtain that

$$\psi_t(x) = \varphi_{[X,Y]}^{s(t)\sqrt{|t|}}(x) + o(s(t)^2 + |t|) = \varphi_{[X,Y]}^t + o(|t|).$$

Since $\psi_t(x)$ is holomorphic in x, one can use the Cauchy integral formula to show that $(x, t) \to \psi_t(x)$ is $\mathcal{C}^1$. Finally, in view of formulas (a) and (b) it suffices to prove the result only for the case of a sum and Lie bracket of two complete vector fields. Hence 1 is proved. To prove 2, one replaces ψ_t obtained in 1 by $\psi_{t/N}^{(N)}$ for N large enough (where the superscript refers to composition) and appeals to standard results in the theory of approximation of solutions to ODE (see, e.g., Theorem 2.1.26 in [AM]). $\square$

3. Proofs

Recall that to a Lie algebra $\mathfrak{g}$ we associated the group $Aut_{\mathfrak{g}}(M)$ of holomorphic automorphisms of M generated by time-1 maps of all complete vector fields in $\mathfrak{g}$.

Our next step is to reduce Theorem 1 to the case of zero jets; $k = 0$. To this end, note that since $p_k : \mathcal{X}_{\mathcal{O}}(M) \to \mathcal{X}_{\mathcal{O}}(J_{\mathfrak{g}}^k(M))$ is just an invariant way of collecting X and its first k derivatives into a single object, it follows from the Cauchy inequalities that p_k is continuous, and hence $p_k(\mathfrak{g})$ has the density property if and only if $\mathfrak{g}$ does.

Consider next the map associating to each $\Phi \in Aut(M)$ an element $\Phi_\# \in Aut(J^k(M))$ defined by

$$\Phi_\#[f] = [\Phi \circ f].$$

Then

$$(Aut_{\mathfrak{g}}(M))_\# = Aut_{p_k(\mathfrak{g})}\left(J^k(M)\right),$$

and we are thus reduced to the case $k = 0$. That is to say, Theorem 1 follows immediately from the following theorem.

Theorem 3.1. *If a Lie algebra $\mathfrak{g}$ has the density property, then for all $p \in M$, $Aut_{\mathfrak{g}}(M)$ acts transitively on the orbit $\mathcal{R}_{\mathfrak{g}}(p)$.*

Proof. Let $q \in \mathcal{R}_{\mathfrak{g}}(p)$, and take $t_1, ..., t_N, X_1, ..., X_N$ such that

$$q = \varphi_{X_N}^{t_N} \circ ... \circ \varphi_{X_1}^{t_1}(p).$$

By part 2 of lemma 2.1, there exist automorphisms $\psi_1, ..., \psi_N \in Aut_{\mathfrak{g}}(M)$ such that $\psi_1 \approx \varphi_{X_1}^{t_1}$ on a neighborhood of p, and $\psi_1 \approx \varphi_{X_j}^{t_j}$ on a neighborhood of $\psi_{i-1} \circ ... \circ \psi_1(p)$. If the approximations are controlled sufficiently carefully, then $\Phi := \psi_N \circ ... \circ \psi_1 \in Aut_{\mathfrak{g}}(M)$ has the property that $q' := \Phi(p) \approx q$. The only thing left to do is to show that one can perturb Φ to $\tilde{\Phi} \in Aut_{\mathfrak{g}}(M)$ so that $\tilde{\Phi}(p) = q$. This is done by the following consequence of the inverse function theorem.

Lemma 3.2. *Let $\mathfrak{g} \subset \mathcal{X}_{\mathcal{O}}(M)$ be a Lie algebra of holomorphic vector fields having the density property, and let $p \in M$. Then there exists $N \in \mathbb{N}$ and a $\mathcal{C}^1$ map $\wp : M \times \mathbb{R}^N \to M$ such that*

(1) *$\wp(\cdot, t) \in Aut_{\mathfrak{g}}(M)$ for each $t \in \mathbb{R}^N$, and*
(2) *$\wp(p, \cdot)$ is a $\mathcal{C}^1$ diffeomorphism of a neighborhood N_p of $0 \in \mathbb{R}^N$ onto a neighborhood $\mathcal{N}_p$ of p in $\mathcal{R}_{\mathfrak{g}}(p)$.*

Proof. By the orbit theorem, $\mathcal{R}_{\mathfrak{g}}(p)$ is a manifold, and by the Hermann-Nagano theorem (see [J] for both of these theorems) there exist $X_1, ..., X_N \in \mathfrak{g}$ such that $\{X_1(p), ..., X_N(p)\}$ is a basis for $T_p(\mathcal{R}_{\mathfrak{g}}(p))$. Since $\mathfrak{g}$ has the density property, we may assume that each X_i is generated (in the Lie algebra sense) by complete vector fields in $\mathfrak{g}$. Hence by lemma 2.1 there exist N families of automorphisms $\psi_1^t, ..., \psi_N^t \in Aut_{\mathfrak{g}}(M)$ such that $\psi_j^0 = id_M$ and $\frac{d}{dt}|_{t=0}\psi_j^t = X_j$ for each $1 \le j \le N$. Then for the mapping $\wp$ defined by $\wp(q, t_1, ..., t_N) := \psi_N^{t_N} \circ ... \circ \psi_1^{t_1}(q)$ one has

$$d_t\wp(p, 0) = (X_1(p), ..., X_N(p)),$$

and the result follows from the implicit function theorem. □

Example 1. *Consider the Lie algebra $\mathfrak{g} = \mathfrak{g}_0^{2,1}$ of holomorphic vector fields in $\mathbb{C}^2$ which vanish on the z_2-axis $\{z_1 = 0\}$. In* [V1] *we showed that $\mathfrak{g}$ has the density property. It can be verified that the space $J_{\mathfrak{g}}^k(\mathbb{C}^2)_{z,*}$ consists of the following jets:*

(1) $(z_1 = 0)$
$(z, w, P_1, P_2, ..., P_k) \in J_{\mathfrak{g}}^k(\mathbb{C}^2)_{z,*} \iff$
$w = z,\ detP_1 \neq 0,\ P_1(0, \zeta_2) = (0, \zeta_2),$ and $P_j(0, \zeta_2) = (0, 0)$ for $2 \le j \le k$.
(2) $(z_1 \neq 0)$
$(z, w, P_1, P_2, ..., P_k) \in J_{\mathfrak{g}}^k(\mathbb{C}^2)_{z,*} \iff$
$w_1 \neq 0,$ and $detP_1 \neq 0$.

As a consequence of Theorem 1, we obtain the following fact:

Given any integer $N \ge 3$, there exists an automorphism $F \in Aut\mathbb{C}^2$ such that $F(0, z_2) = (0, z_2)$, and

$$F(z_1, z_2) = (z_1, z_2)(1 + z_1) + O(|z|^N).$$

This answers a question posed to the author by B. Stensønes.

We turn our attention now to Theorem 2. Since some of the details are similar to those of the proof of Theorem 1 and the rest may be found elsewhere in the literature, we content ourselves with a brief sketch.

Sketch of proof of Theorem 2. The first step is to construct (possibly time dependent) holomorphic vector field whose time-1 map approximately achieves the conclusion of the theorem, but with a biholomorphic map defined only on a neighborhood of $K \cup \{x\}$. To do this, one needs to construct time dependent vector fields on M which behave as needed on x and which are arbitrarily small on K. In the non-calibrated case this is done, using standard facts about holomorphic convexity, as follows. Let $F_t,\ t \in [-\epsilon, 1+\epsilon]$ be a family of holomorphic maps defined in a small neighborhood of x, such that $F_0 = \mathrm{id}$ and $j^k_x(F_1) = \gamma$. Assume further that F_t is defined in a neighborhood of $\hat{K}_{\mathcal{O}(M)}$, where it is the identity map for all t. Let

$$X_t := \frac{dF_t}{dt} \circ (F_t)^{-1}.$$

Consider the "parameter" vector bundle $\pi : TM \times \mathbb{C} \to M \times \mathbb{C}$, where $\pi(v,t) = (x,t)$ whenever $v \in T_xM$. Then X_t defines a section θ of $\pi|U : U \to (TM \times \mathbb{C})|U$ by $\theta(x,t) = (X_t(x), t)$, where U is a neighborhood of a compact subset L of $M \times \mathbb{C}$ defined by

$$L = \bigcup_{t \in [0,1]} K \cup \{F_t(x)\} \times \{t\}.$$

By results of Stolzenberg [S], L is $\mathcal{O}(M \times \mathbb{C})$-convex, and hence θ can be approximated, uniformly on L, by a global section η of π (see, for example, theorem 5.6.2 of [H]). Then $\eta(x,t) = (Y_t(x), t)$, so Y_t gives a global time-dependent vector field on M. By construction and the continuous dependence of solutions of ODE on parameters, the flow G_t of Y_t is defined up to time 1, and $j^k_x(G_1) \approx \gamma$, with the approximation being controllable. For more details in the case where $M = \mathbb{C}^n$ (and which generalize to our case) see, for example, [F1] or [FGS].

In the calibrated case, one also has to deal with the fact that the divergence zero vector fields do not form an analytic subsheaf of $\mathcal{X}_{\mathcal{O}}$. To get around this, one must use the duality provided by ω: every divergence zero vector field X corresponds to a closed $(n-1)$-form $\theta_X := X \rfloor \omega$. Since we need forms which almost vanish on K and are otherwise specified only in a contractible neighborhood, we may restrict to exact forms, which are just $(n-2)$-forms and hence form a coherent sheaf. From here on, one proceeds as in the non-calibrated case.

The next step is to lift the problem to $J^k(M)$ using the map p_k. An approximate version of the theorem (i.e., with 1 and 2 being only roughly true) follows, as in the proof of theorem 3.1, using part 2 of lemma 2.1 and the vector fields constructed above, and we need only to make corrections. To do the latter, we must use a perturbation argument like lemma 3.2, which can be done because we can find enough vector fields which are small on K. The details, though somewhat cumbersome, are straightforward.

4. Corollaries

In this section we prove various corollaries of Theorems 1 and 2. Some of the results are just generalizations of similar results in the case of $\mathbb{C}^n$, but others are of interest only in this general context.

The Fatou-Bieberbach Phenomenon. The first consequence of Theorem 1 is the following.

Corollary 4.1. *Let M be a Stein manifold of complex dimension n with the density property. Then there is an open subset of M which is biholomorphic to $\mathbb{C}^n$.*

The proof of this corollary is as follows. Fix $p \in M$, and choose $\Phi \in Aut(M)$ such that p is an an attracting fixed point for Φ. Such a Φ is guaranteed by Theorem 1. The basin of attraction to p will then be biholomorphic to $\mathbb{C}^n$, as was shown in the appendix of [RR]. (In their paper, J.-P. Rosay and W. Rudin show this result in the case where $M = \mathbb{C}^n$, but their proof generalizes easily; one conjugates Φ on the region of attraction to p to a (contracting) upper triangular map on T_pM, and then the region of attraction is biholomorphic to the tangent space.)

Remark: Note that we have a lot of control over the jets of automorphisms we want. Thus the full generality of the Rosay-Rudin theorem is not needed, since we can choose jets which have no resonances, and thus linearize the automorphism on its basin of attraction. The latter is a classical construction.

In fact, one can get many more results on such "Fatou-Bieberbach" domains. We shall state here only one result.

Corollary 4.2. *Let M be an n dimensional Stein manifold with the density property. Then there are infinitely many disjoint domains in M which are biholomorphic to $\mathbb{C}^n$.*

Sketch of proof. We will construct an injective map of M into itself, which has infinitely many attracting fixed points. Then even though the map may not be onto, the techniques of Rosay and Rudin apply, and the region of attraction to each of the fixed points is such a domain.

To this end, let $\emptyset = K_0 \subset K_1 \subset$ interior $K_2 \subset K_2 \subset$ interior $K_3 \subset K_3 \subset ...$ be an increasing sequence of holomorphically convex compact sets, and let $p_j \in K_j \backslash$interior K_{j-1} $(j \geq 1)$. Theorem 2, applied inductively, gives us a sequence of automorphisms $\{F_j\}$ such that, for each j, F_j has $p_1, ..., p_j$ as attracting fixed points, and such that $F_{j+1} \approx F_j$ on K_j, with equality on $p_1, ..., p_j$. If the approximation on each K_j is good enough, then $F = \lim F_j$ exists and is an injective map from M to M. □

In fact, if in the above sketch we choose another sequence $\{q_j\}$ such that $q_j \in K_j \backslash$(interior $K_{j-1} \cup p_j$), and construct F_j with the additional requirement that $F_j(q_j) = q_{j+1}$, then the limit map F will not be surjective. This is the so-called "kick out method", first introduced by Dickson and Esterle [DE]. We thus obtain a sketch of proof of the following.

Corollary 4.3. *Let M be a Stein manifold with the density property. Then there exist proper open subsets of M which are biholomorphic to M.*

Such subsets of M are also a sort of "Fatou-Bieberbach" domains. We note that when $M = \mathbb{C}^n$, this construction gives a new construction of proper open subsets of $\mathbb{C}^n$ which are biholomorphic to $\mathbb{C}^n$. (This has been exploited in many results of analytic geometry in $\mathbb{C}^n$.) However, these corollaries show that the two methods might be "different". A natural question is whether every Fatou Bieberbach domain in $\mathbb{C}^n$ is the region of attraction of an automorphism. Corollary 4.3 suggests that the answer might not be very simple.

If we consider now a calibrated Stein manifold (M, ω) with the volume density property, the proofs of corollaries 4.1 and 4.2 above break down. However, the same construction, but without the attracting fixed points, can be used to show the following.

Corollary 4.4. *Let (M, ω) be a calibrated Stein manifold with the volume density property. Then there exists a proper open subsets of M which is biholomorphic to M.*

One can also construct nondegenerate maps of $\mathbb{C}^n$ into a calibrated Stein manifold (M, ω) with the volume density property.

Corollary 4.5. *Let (M, ω) be a calibrated Stein manifold of dimension n having the volume density property. Then there exists a map $h : \mathbb{C}^n \to M$ such that $h^*\omega$ is not identically zero.*

Proof. It follows ([V2], Main Theorem 3) that $M \times \mathbb{C}$ has the density property. Hence by corollary 4.1 there is an injective holomorphic map $H : \mathbb{C}^{n+1} \to M \times \mathbb{C}$. The proof is finished by letting h be the restriction of H to the hyperplane which is mapped to T_pM for some p, followed by the projection to M. □

One can also get injective immersions of $\mathbb{C}^{n-1}$ tangent to any given complex hyperplane in TM.

Corollary 4.6. *Let (M, ω) be a calibrated Stein manifold of dimension n with the volume density property, and let $V_p \subset T_pM$ be a complex hyperplane. Then there is an injective holomorphic immersion $g : \mathbb{C}^{n-1} \to M$ such that $dg_p(\mathbb{C}^{n-1}) = V_p$.*

Proof. Let $L : T_pM \to T_pM$ be a linear map which has V_p as a contracting subspace, and whose determinant is 1. Then by Theorem 1 there exists $\Phi \in Aut(M, \omega)$ with $\Phi(p) = p$ and $D\Phi(p) = L$. Then the stable manifold $W_p^s(\Phi)$ associated to Φ at p has tangent space V_p at p. Because Φ is an automorphism, $W_p^s(\Phi)$ is injectively immersed. Now, $\Phi(W_p^s(\Phi)) \subset W_p^s(\Phi)$, p is an attractive fixed point for the restriction of Φ to $W_p^s(\Phi)$, and all of $W_p^s(\Phi)$ is attracted to p by Φ. Hence the theorem of Rosay and Rudin states that $W_p^s(\Phi)$ is in fact biholomorphic to $\mathbb{C}^{n-1}$. □

A very interesting question, first posed to us by J.-P. Rosay, is whether or not there is an open subset of $\mathbb{C}^* \times \mathbb{C}^*$ which is biholomorphic to $\mathbb{C}^2$. In [V1] we showed that $(\mathbb{C}^* \times \mathbb{C}^*, (zw)^{-1} dz \wedge dw)$ has the volume density property, and hence it follows that there is a proper open subset of $\mathbb{C}^* \times \mathbb{C}^*$ which is biholomorphic to $\mathbb{C}^* \times \mathbb{C}^*$.

In connection with the above ideas, we note the following proposition, which in has been known to J.-P. Rosay for a long time.

Proposition 4.7. *Let (M, ω) be a calibrated Stein manifold with the volume density property, and suppose there exists $F \in Aut(M)$ such that the ω Jacobian determinant J_F of F has modulus different from 1 at some point $p \in M$, then M has an open subset biholomorphic to $\mathbb{C}^n$.*

The idea of the proof is to use automorphisms with jets in $J^k(M, \omega)$ to modify F so that p becomes an attractive fixed point, and then apply the same dynamical principle as above.

Particularly noteworthy here is the fact that the possible nonexistence of open copies of $\mathbb{C}^n$ in calibrated Stein manifolds with the volume density property is a form of degenerate hyperbolicity. The last proposition shows how this hyperbolicity (if it exists), as with more conventional hyperbolicity, results in a reduction of the automorphism group.

Completeness of vector fields. One of the consequences of corollaries 4.1 and 4.6 is that on a Stein manifold with the density or volume density property, all bounded plurisubharmonic functions are constant. Then the main theorem of [AFR] implies the following corollary.

Corollary 4.8. *Let M be a Stein manifold with the density or volume density property. Then every $\mathbb{R}^+$-complete holomorphic vector field on M is $\mathbb{C}$-complete.*

Recall that a holomorphic vector field X is $\mathbb{R}^+$-complete (resp. $\mathbb{R}$-complete) if one can extend the flow of X to all of $\mathbb{R}^+$ (resp. $\mathbb{R}$), and that X is $\mathbb{C}$-complete if both X and iX are $\mathbb{R}$-complete.

Interpolation results. In this paragraph we note that for manifolds with the density property or volume density property, a given (proper, or closed) complex submanifold can be modified so as to interpolate any given discrete sequence. For the proof of the next result in the case $M = \mathbb{C}^n$ (which can easily be adapted to the more general case stated here) see [F1].

Corollary 4.9. *Let M be a Stein manifold of $\mathbb{C}$-dimension $n \geq 2$ with the density or volume density property, Σ a Stein manifold of $\mathbb{C}$-dimension $r < n$, and $\{\gamma_m; m \geq 1\} \subset J^k(\Sigma, M)$ a sequence of k-jets such that $\{\sigma(\gamma_m)\}$ and $\{\tau(\gamma_m)\}$ are discrete sequences in Σ and M respectively. If Σ admits a proper holomorphic embedding in M, then there exists a proper holomorphic embedding $\varphi : \Sigma \hookrightarrow M$ such that*

$$j^k_{\sigma(\gamma_m)}(\varphi) = \gamma_m.$$

Remark: It is easily seen that the proof also produces the following fact: Given any pair of discrete sequences $\{e_j\}$ and $\{f_j\}$, there exists an injective holomorphic map $F : M \to M$ such that $F(e_j) = f_j$ for all j. We can even specify jets of such an F at the e_j.

In a recent preprint [W], J. Winkelmann has constructed "non-tame sequences" in any Stein manifold. These can be used, together with corollary 4.9 to construct non-equivalent embeddings of a given complex manifold Σ into a Stein manifold M with the density or volume density property, provided one such embedding exists. Precisely, one has the following.

Corollary 4.10. *Let M be a Stein manifold of $\mathbb{C}$-dimension $n \geq 2$ with the density or volume density property, and Σ a Stein manifold of $\mathbb{C}$-dimension $r < n$ such that there exists a proper holomorphic embedding $j : \Sigma \hookrightarrow M$. Then there exists another proper holomorphic embedding $j' : \Sigma \hookrightarrow M$ such that for any $\Phi \in Aut(M)$,*

$$\Phi \circ j(\Sigma) \neq j'(\Sigma).$$

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