FLOWS ON $\mathbf{C}^2$ WITH POLYNOMIAL TIME ONE MAP

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Abstract. We show that every one parameter holomorphic automorphism group (holomorphic flow) $\{\psi_t : t \in \mathbf{R}\} \subset \mathrm{Aut}\mathbf{C}^2$ on $\mathbf{C}^2$ whose time one map ψ_1 is an affine linear map $E_\lambda(x,y) = (x+1, e^\lambda y)$, $\lambda \in \mathbf{C}$, is conjugate in $\mathrm{Aut}\mathbf{C}^2$ to $\phi_t(x,y) = (x+t, e^{\lambda t}y)$. Together with the results of Ahern and Forstneric [1] this shows that **every holomorphic flow on $\mathbf{C}^2$ with polynomial time one map is conjugate in $\mathrm{Aut}\mathbf{C}^2$ to a polynomial flow with the same time one map.**

&1. A theorem and a corollary.

Let $\mathrm{Aut}\mathbf{C}^2$ be the group of all holomorphic automorphisms of the complex plane $\mathbf{C}^2$ and $\mathcal{G} \subset \mathrm{Aut}\mathbf{C}^2$ the subgroup consisting of all polynomial automorphisms of $\mathbf{C}^2$. We shall denote the complex coordinates on $\mathbf{C}^2$ by $z = (x,y)$ $(x, y \in \mathbf{C})$.

A (real) one parameter subgroup of $\mathrm{Aut}\mathbf{C}^2$ is a family of automorphisms $\{\phi_t : t \in \mathbf{R}\} \subset \mathrm{Aut}\mathbf{C}^2$, depending continuously on $t \in \mathbf{R}$, satisfying $\phi_s \circ \phi_t = \phi_{s+t}$ for all $t, s \in \mathbf{R}$. Every such subgroup is real-analytic in all variables (t, z) and it extends to a complex one parameter subgroup $\{\phi_t : t \in \mathbf{C}\}$ ([3], Corollary 2.2). ϕ_t is the flow of the complete holomorphic vector field $V(z) = \frac{d}{dt}\phi_t(z)|_{t=0}$ (the infinitesimal generator of $\{\phi_t\}$); for this reason such subgroups are also called (holomorphic) *flows* on $\mathbf{C}^2$. A flow $\{\phi_t\} \subset \mathcal{G}$ consisting of polynomial automorphisms is called a *polynomial flow*. The automorphism ϕ_1 is called the *time one map* of the flow.

Two flows ϕ_t, ψ_t are conjugate in $\mathrm{Aut}\mathbf{C}^2$ if there exists an $h \in \mathrm{Aut}\mathbf{C}^2$ such that $\phi_t = h^{-1} \circ \psi_t \circ h$ for all t. By the classification of flows we always refer to their classification up to conjugation.

It is an open problem to classify holomorphic flows on $\mathbf{C}^2$. Polynomial flows on $\mathbf{C}^2$ have been classified in 1977 by M. Suzuki [10] and in 1985 by Bass and Meisters [2]. Recently Ahern and Forstneric [1] classified flows $\{\phi_t\} \subset \mathrm{Aut}\mathbf{C}^2$ whose time one map ϕ_1

is a polynomial automorphism $E \in \mathcal{G}$ of $\mathbf{C}^2$ which is not conjugate to an affine aperiodic map

$$E_\lambda(x, y) = \left(x + 1, e^\lambda y\right), \quad \lambda \in \mathbf{C}. \tag{1}$$

The maps ϕ_t for noninteger values of t need not be polynomial. They showed that every such flow is conjugate in $\mathrm{Aut}\mathbf{C}^2$ to a polynomial flow with the same time one map.

Here we complete this classification by proving

THEOREM. *Every holomorphic flow $\{\psi_t : t \in \mathbf{R}\} \subset \mathrm{Aut}\mathbf{C}^2$ whose time one map ψ_1 is the affine linear automorphism E_λ (1) is conjugate in $\mathrm{Aut}\mathbf{C}^2$ to the flow*

$$\phi_t^\lambda(x, y) = (x + t, e^{\lambda t}y), \quad t \in \mathbf{R}. \tag{2}$$

In particular, every flow with time one map E_0 is conjugate to the flow $\phi_t^0(x, y) = (x + t, y)$.

An elementary proof of this is given in [1] (Theorem 5.1) in the case when the infinitesimal generator V of the flow ψ_t is a polynomial vector field.

It is known [4] that the maps E_λ (1) for distinct values of $e^\lambda \in \mathbf{C}^* = \mathbf{C} \backslash \{0\}$ are not conjugate to each other in the polynomial group $\mathcal{G}$. However, in the larger group $\mathrm{Aut}\mathbf{C}^2$, E_λ is conjugate to E_0 by the automorphism

$$\Phi(x, y) = \left(x, e^{\lambda x}y\right). \tag{3}$$

Moreover, if ϕ_t^λ is the flow (2), then $\Phi^{-1} \circ \phi_t^\lambda \circ \Phi = \phi_t^0$ is the flow $\phi_t^0(x, y) = (x + t, y)$.

Our theorem, together with the results of [1], implies

COROLLARY. *Every holomorphic flow on $\mathbf{C}^2$ whose time one map E is polynomial is conjugate in $\mathrm{Aut}\mathbf{C}^2$ to a polynomial flow with the same time one map E.*

It was already shown in [1] (Corollary 4.3) that *every flow in $\mathrm{Aut}\mathbf{C}^2$ whose time one map E is linear is conjugate to a linear flow with the same time one map E.*

Recall from [2] and [10] that every polynomial flow on $\mathbf{C}^2$ is conjugate in $\mathcal{G}$ to one of the following:

1. $\phi_t(x, y) = \left(e^{\mu t}x, e^{\lambda t}y\right)$, $\lambda, \mu \in \mathbf{C}$ (diagonal linear flows),
2. $\phi_t(x, y) = (x + tf(y), y)$, where f is a nonconstant polynomial (shear flows),
3. $\phi_t(x, y) = \left(e^{n\lambda t}(x + ty^n), e^{\lambda t}y\right)$, $\lambda \in \mathbf{C}$, $n \in \mathbf{Z}_+$.

Taking $n = 0$ in case 3 gives the flow (2).

We now describe the methods which were used in [1] and in this note. Let $\{\phi_t\} \subset \mathrm{Aut}\mathbf{C}^2$ be a flow with a polynomial time one map $E = \phi_1 \in \mathcal{G}$. It is well known [4,5,11] that the group $\mathcal{G}$ is an amalgamated free product of the affine linear group $\mathcal{A} \subset \mathcal{G}$ and the group $\mathcal{E} \subset \mathcal{G}$ consisting of the *elementary* automorphisms

$$(x, y) \mapsto \left(\alpha x + p(y), \beta y + \gamma\right), \tag{4}$$

where $\alpha, \beta \in \mathbf{C}^*$, $\gamma \in \mathbf{C}$, and p is a polynomial. Every $g \in \mathcal{G}$ is conjugate in $\mathcal{G}$ either to an elementary map (4) or to a generalized Henon map [4]. The dynamics of the generalized Henon maps is such that these can not belong to any flow, and hence $E = \phi_1$ is conjugate in $\mathcal{G}$ to an elementary automorphism (4). These were classified up to conjugation in $\mathcal{E}$ by Friedland and Milnor ([4], Theorem 6.5). These *normal forms* are: the diagonal linear maps $(x, y) \to (\alpha x, \beta y)$ $(\alpha,\ \beta \in \mathbf{C}^*)$, the affine maps (1), and the upper triangular maps $(x, y) \mapsto \big(\beta^d(x + q(y)y^d), \beta y\big)$ $(\beta \in \mathbf{C}^*,\ d \in \mathbf{Z}_+)$, where q is a polynomial, $q(0) = 1$, and q is identically one unless β is a root of one. See [4] for the details.

In [1] we found all flows in $\mathrm{Aut}\,\mathbf{C}^2$ whose time one map is a non-periodic normal form $E \in \mathcal{E}$ different from (1). The methods used in [1] were completely elementary, based on the analysis of the functional equation $DE \cdot V = V \circ E$ which relates the time one map $E = \phi_1$ to the infinitesimal generator V of the flow ϕ_t.

If E is a periodic automorphism of $\mathbf{C}^2$ or conjugate to an affine aperiodic map (1), this functional equation does not give sufficient information on V. In this case one can apply the results of M. Suzuki [8, 9, 10] on the existence of meromorphic first integrals of *proper* flows on Stein manifolds, and on the classification of proper flows on $\mathbf{C}^2$. For periodic time one maps E this has already been indicated in [1] (Proposition 3.2). Here we treat the remaining case when the time one map is affine (1).

Suzuki's methods are far from elementary. They depend on deep results of T. Nishino (see the references in [10]) concerning the class of 'algebroid' functions on two dimensional complex manifolds, as well as on the results of Kodaira [6] and Morrow [7] on rationality of certain compactifications of $\mathbf{C}^2$. It would be very desirable to have a simpler proof of the classification theorem for flows whose time one map is periodic or affine aperiodic.

&2. The proof.

Let $\phi_t \in \mathrm{Aut}\,\mathbf{C}^2$ $(t \in \mathbf{R})$ be a flow satisfying $\phi_1 = E_\lambda$ (1). Conjugating ϕ_t by the automorphism (3) we get a flow with time one map $E(x, y) = (x + 1, y)$, and hence it suffices to consider the case when $\phi_1 = E$. Recall that $\{\phi_t\}$ extends to all $t \in \mathbf{C}$ [3].

Let $\pi \colon \mathbf{C}^2 \to \mathbf{C}^* \times \mathbf{C}$ be the projection $\pi(x, y) = \big(e^{2\pi i x}, y\big)$. The relation $\phi_t \circ E = E \circ \phi_t$, which holds for any $t \in \mathbf{C}$, shows that ϕ_t induces a flow ψ_t on $\mathbf{C}^* \times \mathbf{C}$ such that $\pi \circ \phi_t = \psi_t \circ \pi$, and ψ_1 is the identity map on $\mathbf{C}^* \times \mathbf{C}$. Thus $\psi_{t+1} = \psi_t$ for all $t \in \mathbf{C}$. Hence the complex orbit $C_z = \{\psi_t(z) \colon t \in \mathbf{C}\} \subset \mathbf{C}^* \times \mathbf{C}$ of every point $z \in \mathbf{C}^* \times \mathbf{C}$ is isomorphic (as a Riemann surface) to $\mathbf{C}^*$. Such actions of $\mathbf{C}$ are said to be of type $\mathbf{C}^*$.

M. Suzuki proved in [8] (Corollary, p.88) that every action $\psi \colon \mathbf{C} \times M \to M$ of type $\mathbf{C}^*$ on a two dimensional Stein manifold M admits a meromorphic first integral. More precisely, there exists a nonconstant meromorphic function F on M such that $F \circ \psi_t = F$ for all t. Let F be such a first integral of ψ on $\mathbf{C}^* \times \mathbf{C}$. Then $f = F \circ \pi$ is a meromorphic first integral of the action ϕ on $\mathbf{C}^2$.

Suzuki classified all flows on $\mathbf{C}^2$ which admit a meromorphic first integral [10]. (In the earlier paper [8] he also proved that every flow on a two dimensional Stein manifold whose orbits have discrete limit sets admits a meromorphic first integral. Such flows are called *proper*.) Here is the list of conjugacy classes of proper flows on $\mathbf{C}^2$, taken out of [10] (Theorem 4):

(i) $\rho_t(x,y) = \left(e^{n\lambda t}x, e^{m\lambda t}y\right)$, $\lambda \in \mathbf{C}^*$, $m, n \in \mathbf{N} = \{1, 2, 3, \ldots\}$;

(ii) $\rho_t(x,y) = \left(x + tf(y), y\right)$, where f is an entire function on $\mathbf{C}$;

(iii) $\rho_t(x,y) = \left(e^{\lambda(y)t}(x - b(y)) + b(y), y\right)$, where b is a meromorphic function and λ, λb are entire functions on $\mathbf{C}$;

(iv) $\rho_t(x,y) = \left(e^{n\lambda(u)t}x, e^{-m\lambda(u)t}y\right)$, where $m, n \in \mathbf{N}$, $u = x^m y^n$, and λ is an entire function on $\mathbf{C}$;

(v) flows of the form $\alpha^{-1} \circ \rho_t \circ \alpha$, where $\alpha(x,y) = \left(x, x^l y + P_l(x)\right)$, $l \in \mathbf{N}$, P_l is a polynomial of degree $\leq l - 1$ such that $P_l(0) \neq 0$, and ρ_t is a flow of type (iv) above such that $\lambda(z)$ has a zero of order $\geq l/m$ at $z = 0$.

The flow ϕ_t which we are considering has no fixed points on $\mathbf{C}^2$ since its time one map E has no fixed points. We see by inspection that the only flow without fixed points in the above list is of type (ii), with f a nonvanishing entire function on $\mathbf{C}$. If we define $\Psi \in \mathrm{Aut}\,\mathbf{C}^2$ by $\Psi(x,y) = (f(y)x, y)$, then $\phi_t^0 = \Psi^{-1} \circ \rho_t \circ \Psi$ is the flow $\phi_t^0(x,y) = (x + t, y)$. This completes the proof of the Theorem.

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