Four-Manifolds,

Conformal Curvature, &

Differential Topology

Claude LeBrun
Stony Brook University

Pure Mathematics Colloquium,
University of Waterloo, September 12, 2022

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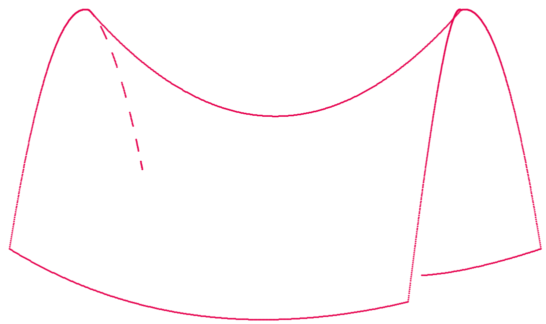

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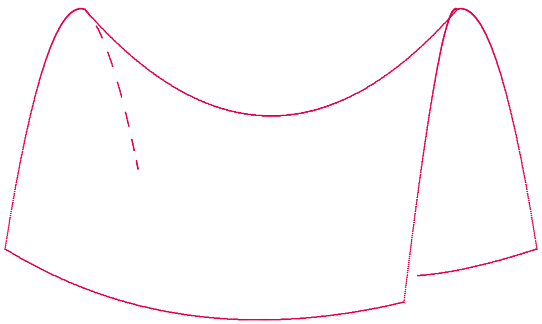

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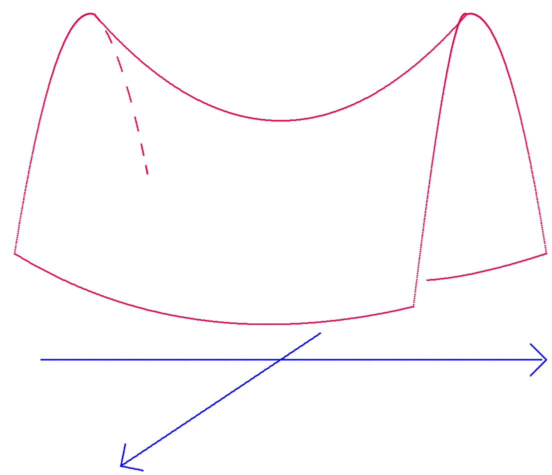

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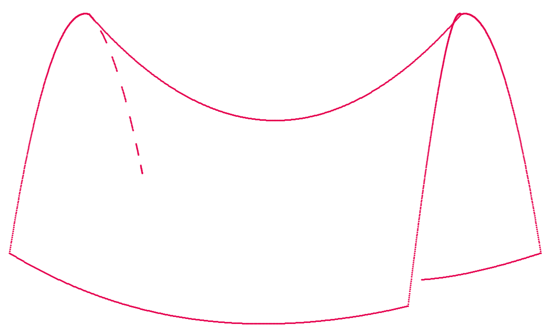

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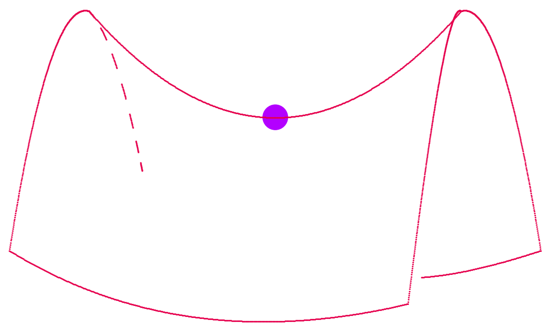

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Lamé (1833): "isothermal coordinates"

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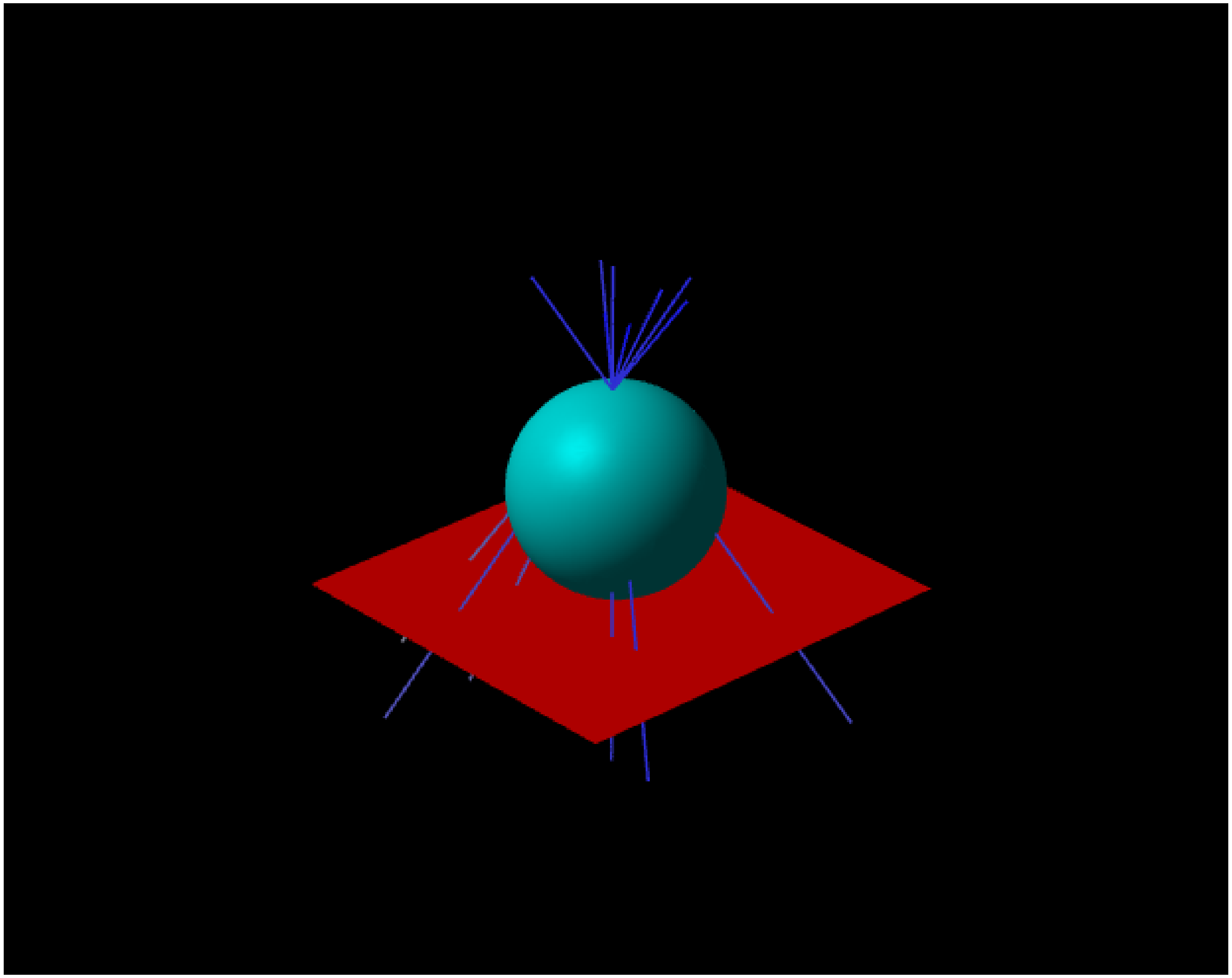
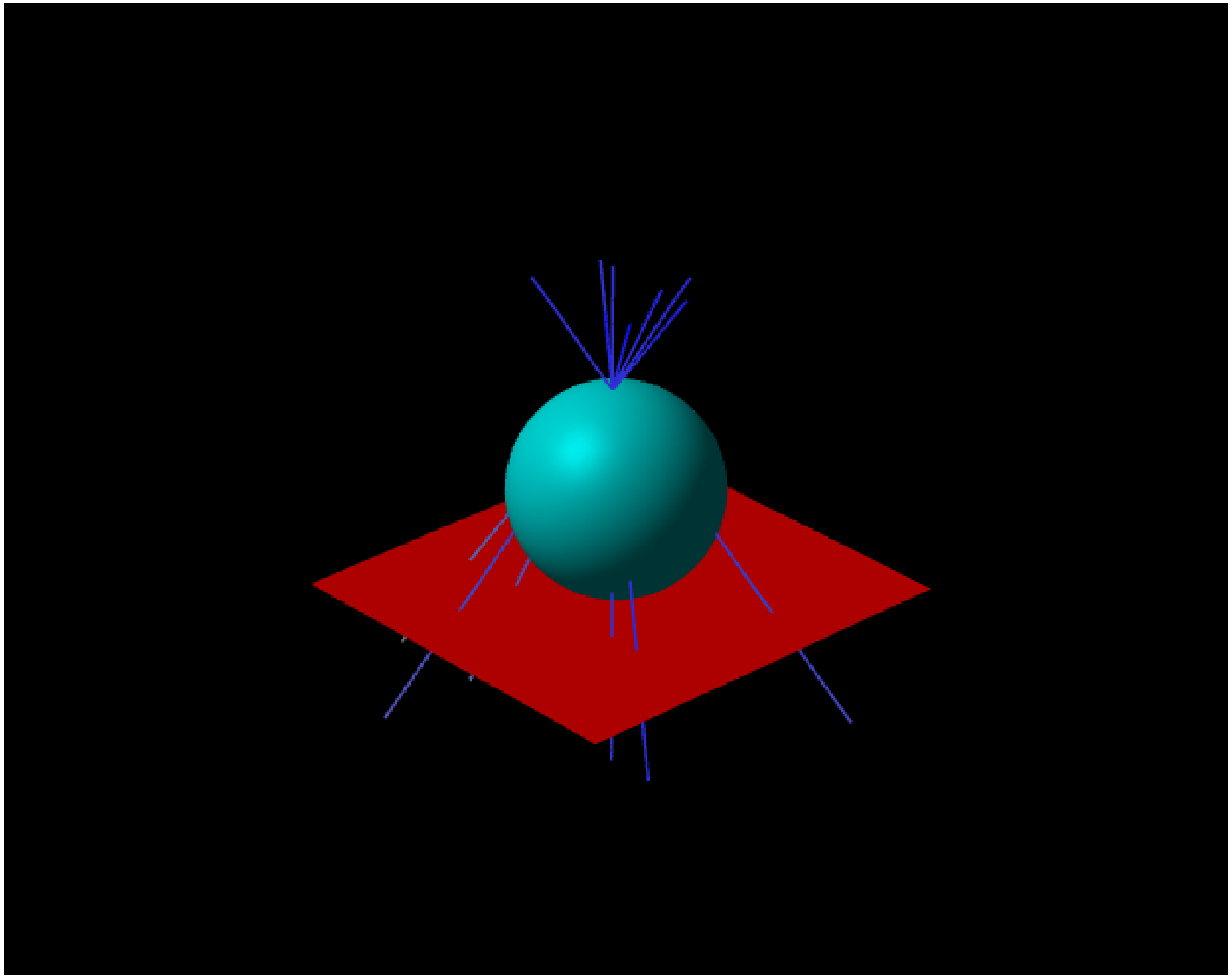
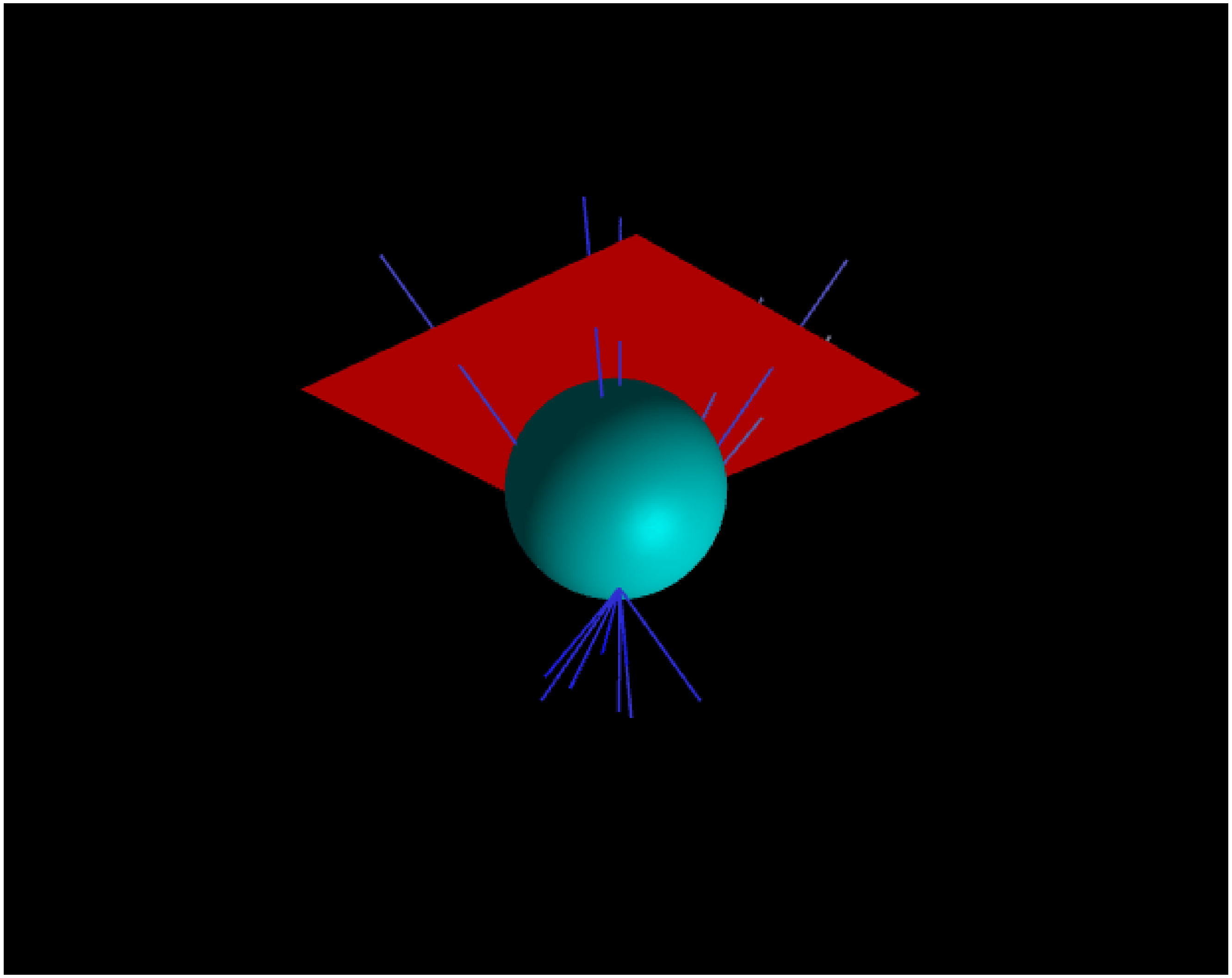
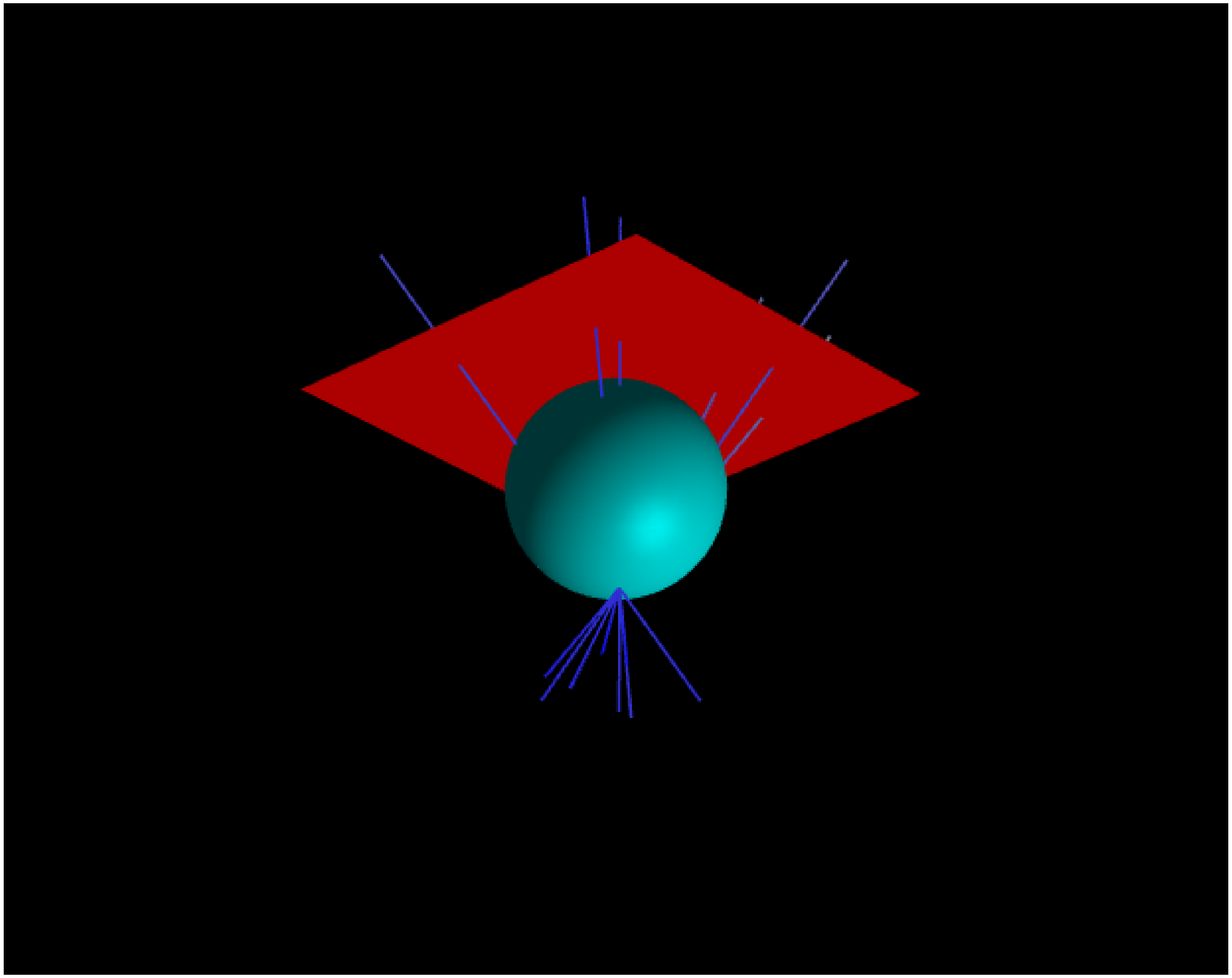
Example. Let (M, g) be the unit 2-sphere in $\mathbb{R}^3$.





Sterographic Projection (from North Pole)





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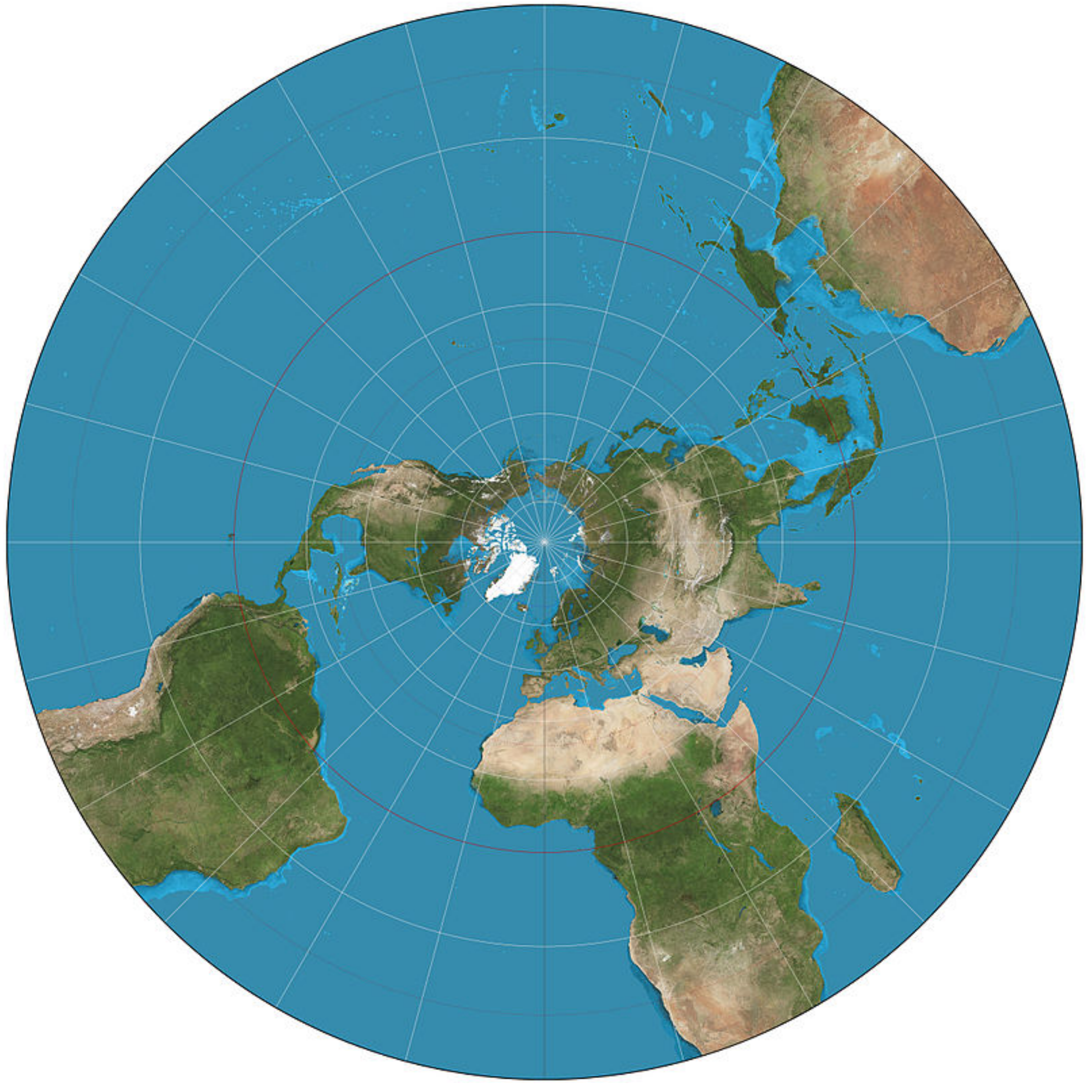

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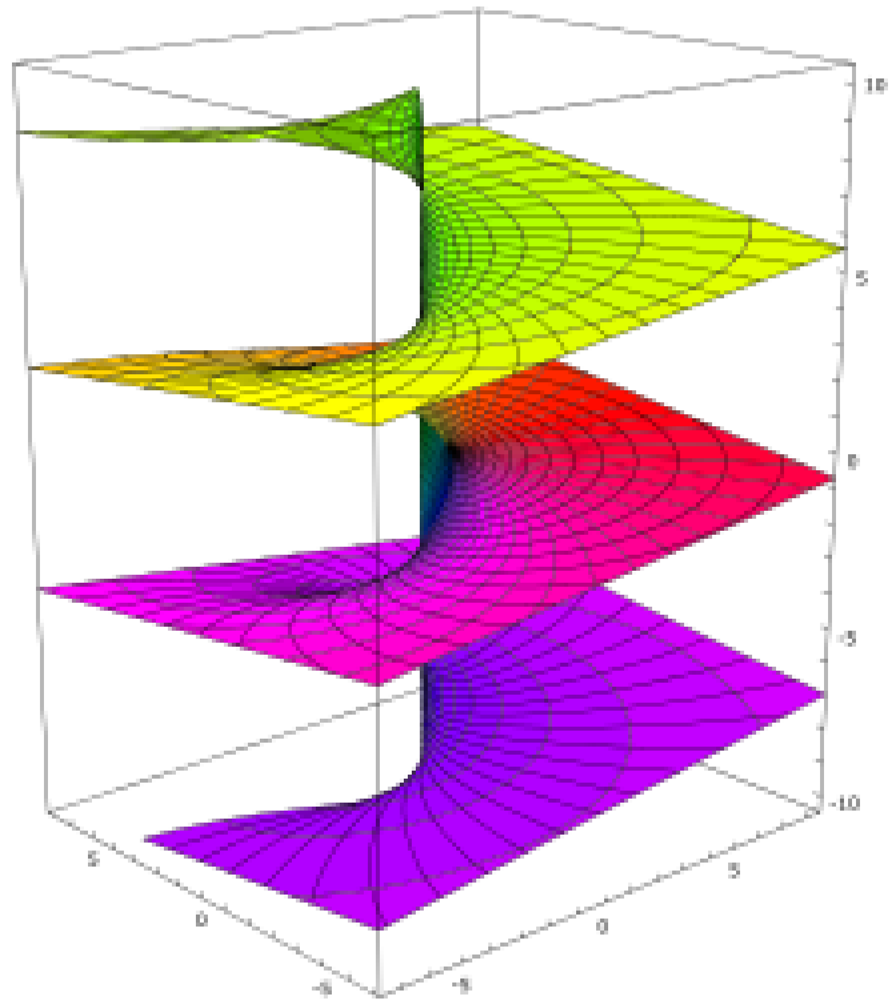

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and then multiplying by $-i$ then yields...

Mercator Projection

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Makes (M, g) into a Kähler manifold.

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To make this precise…

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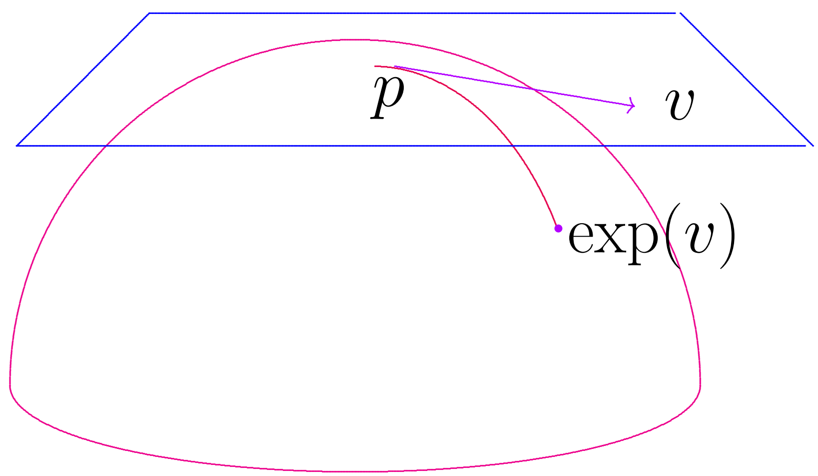

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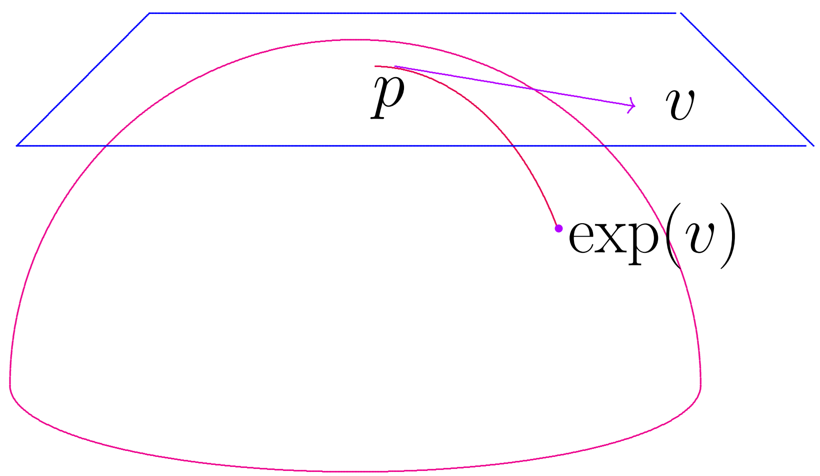

Now choosing $T_pM \xrightarrow{\cong} \mathbb{R}^n$ via some orthonormal basis gives us special coordinates on M.

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Uniquely determined by the above expression for g_{jk} once one also requires Bianchi identities

$$\mathcal{R}_{j\ell km} = -\mathcal{R}_{\ell jkm} = -\mathcal{R}_{j\ell mk}$$

$$\mathcal{R}_{j\ell km} + \mathcal{R}_{jkm\ell} + \mathcal{R}_{jm\ell k} = 0$$

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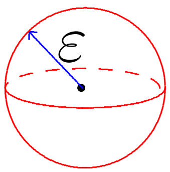
$$\frac{\mathrm{vol}_g(B_\varepsilon(p))}{c_n \varepsilon^n} = 1 - s\, \frac{\varepsilon^2}{6(n+2)} + O(\varepsilon^4)$$



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"…the greatest blunder of my life!"

— A. Einstein, to G. Gamow

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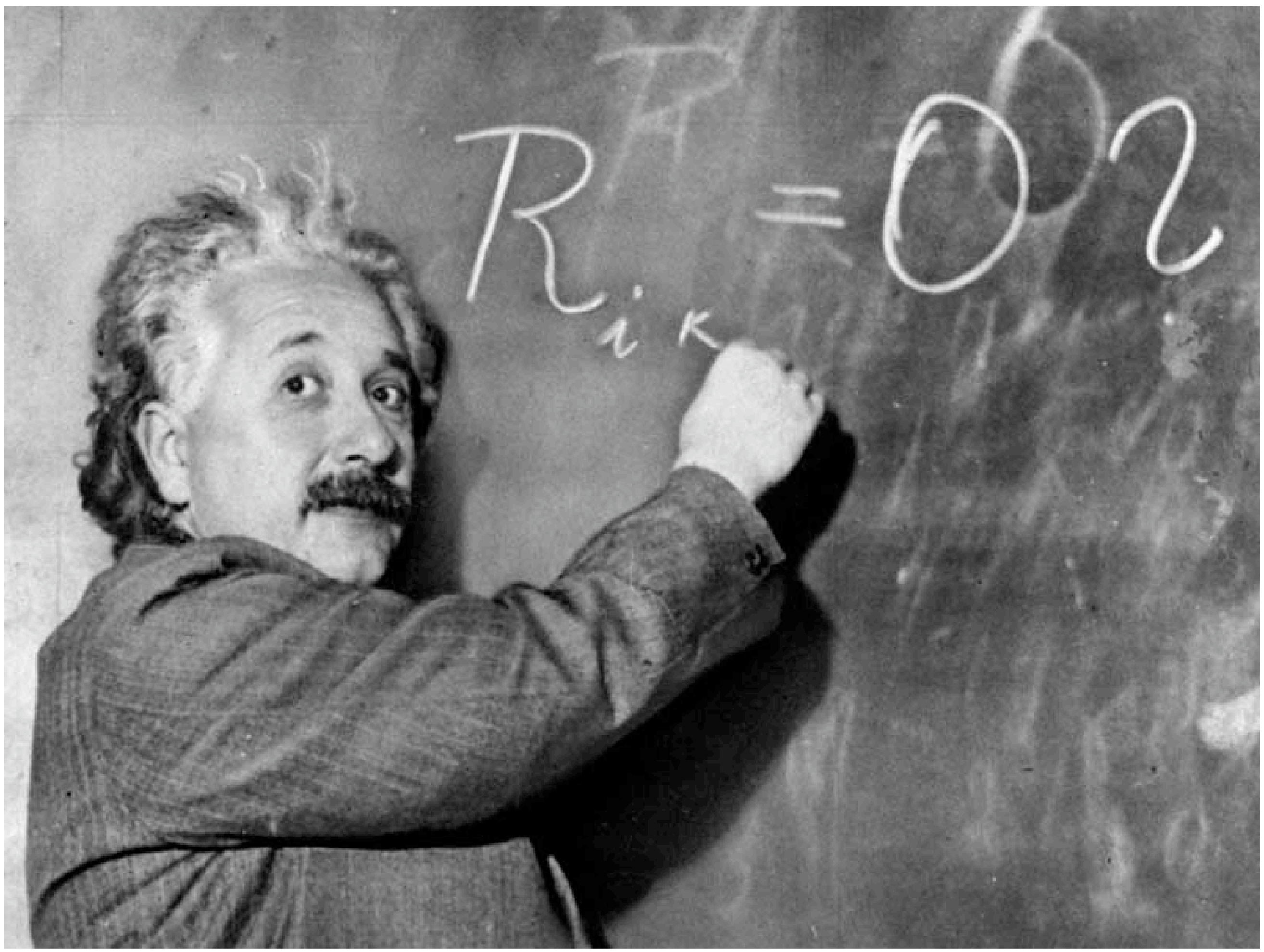

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Riemann $=$ Weyl $\oplus$ trace-free Ricci $\oplus$ scalar

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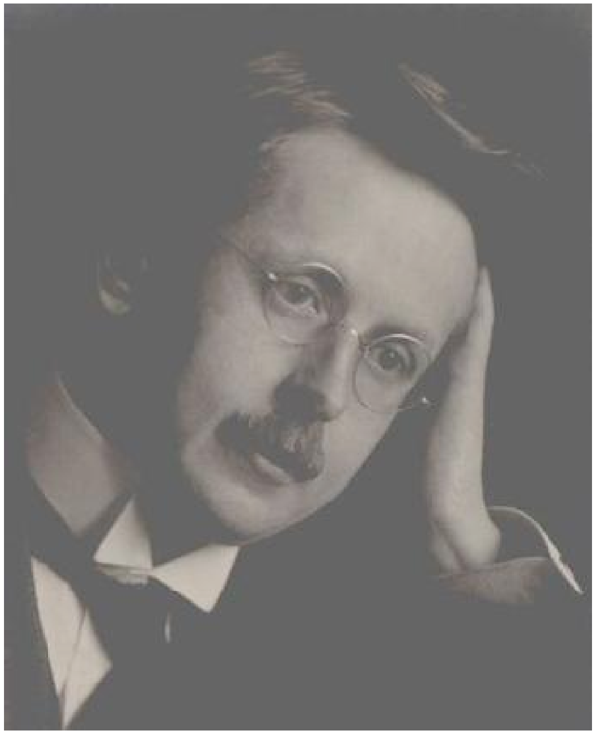

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Cotton tensor $C = \nabla \wedge (\mathring{r} - \frac{s}{12} g)$ obstruction.

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Power $n/2$ is necessary for scale invariance!

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- Do there exist minimizers?

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Of course, conformally Einstein good enough!

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Einstein metrics are usually not critical points.

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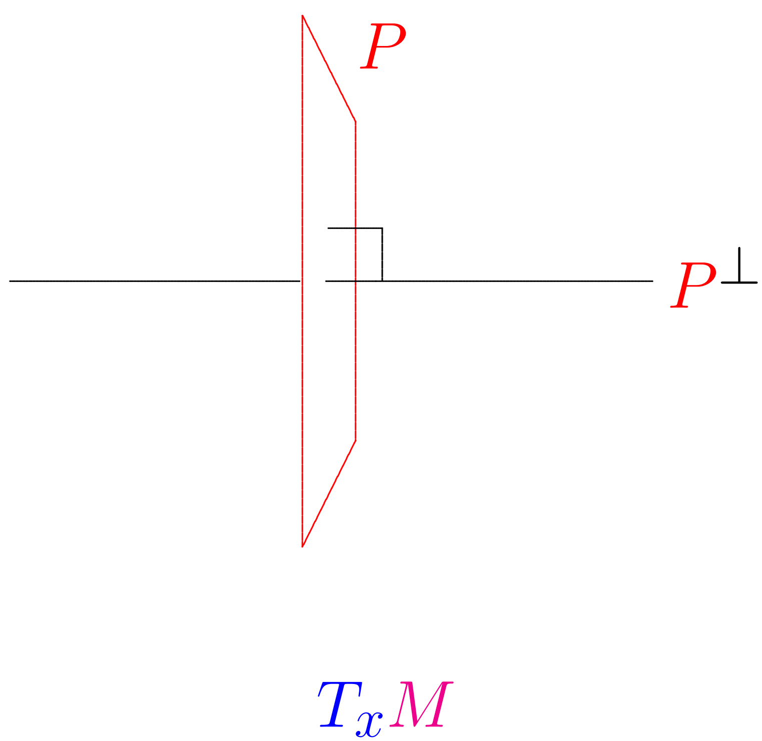

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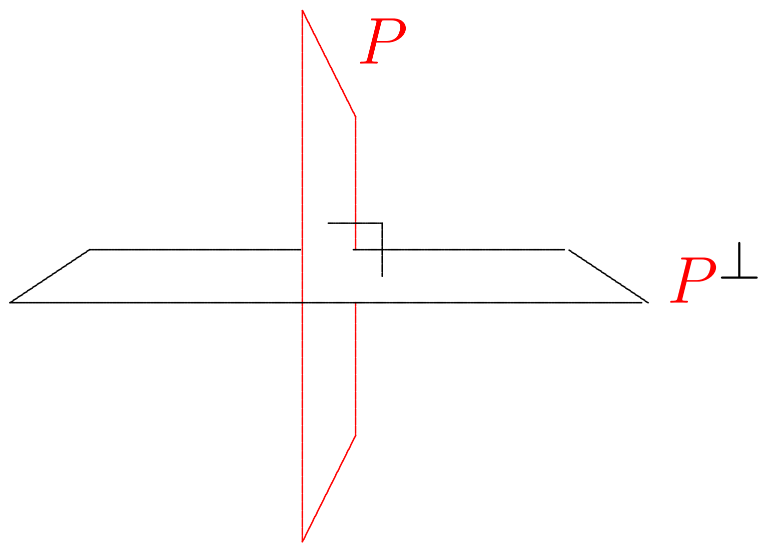

T_xM

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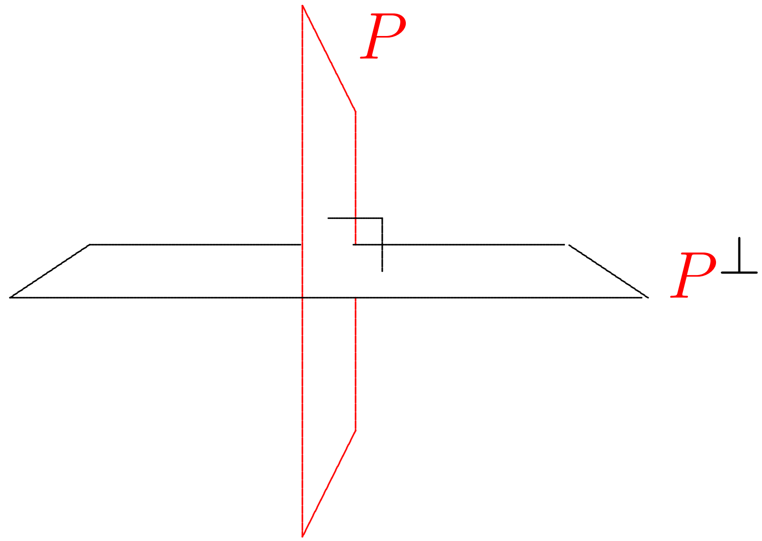

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$$\chi(M) = \frac{1}{8\pi^2}\int_M \left(\frac{s^2}{24} + |W_+|^2 + |W_-|^2 - \frac{|\mathring{r}|^2}{2}\right) d\mu$$

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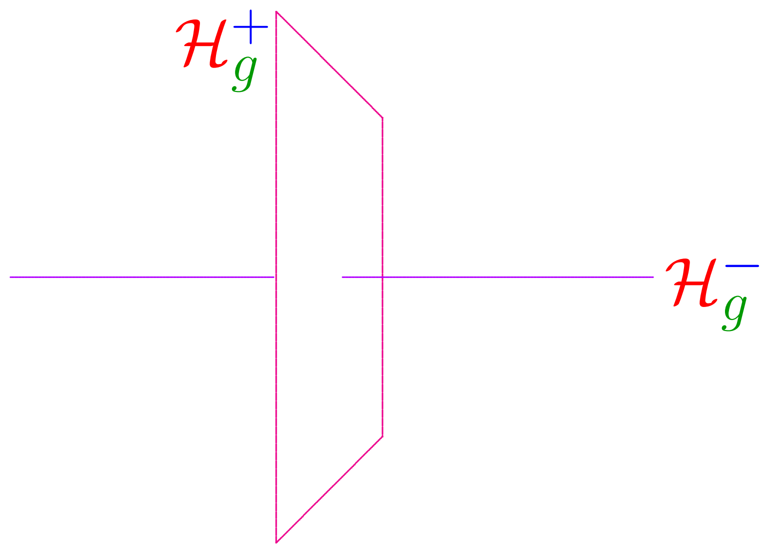

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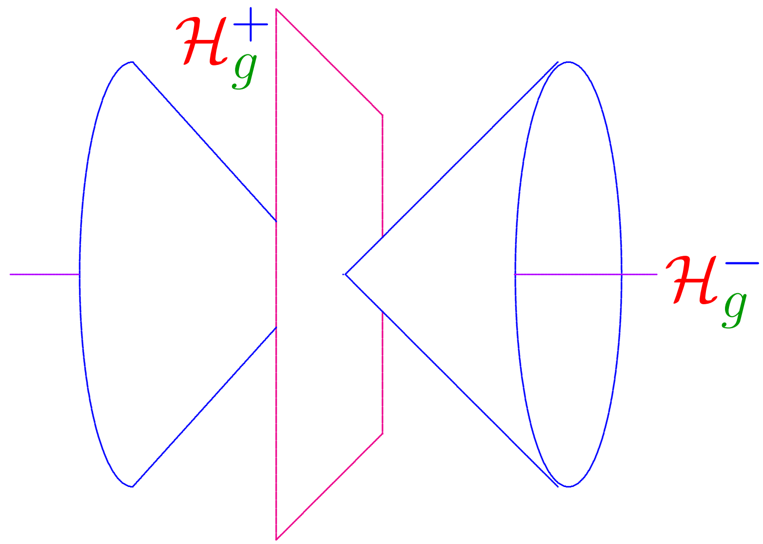

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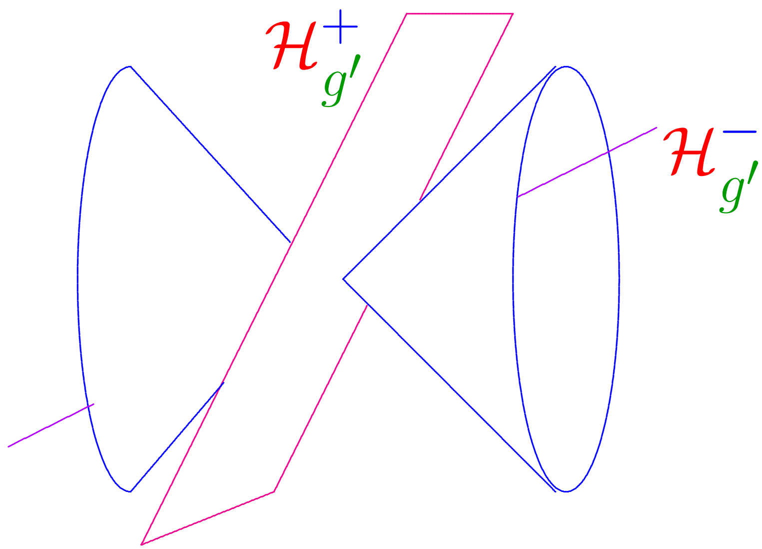

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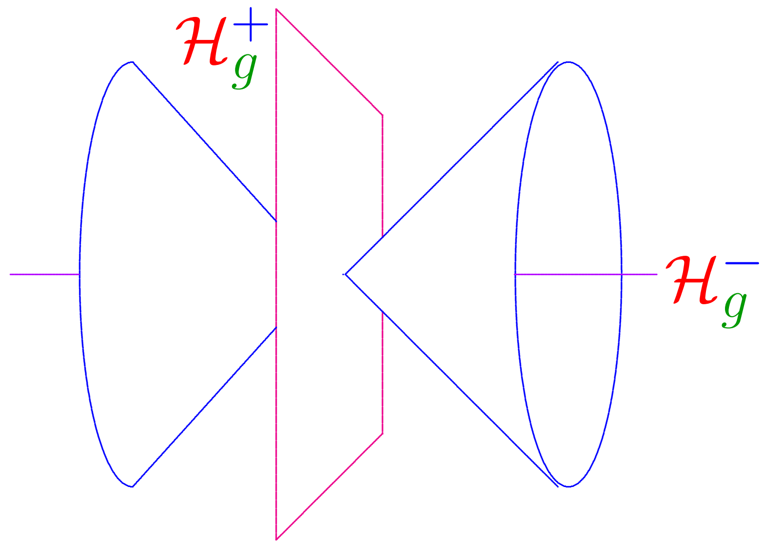

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The subspaces $\mathcal{H}^\pm_g$ are conformally invariant:

Same for g and any $\widehat{g} = u^2 g$.

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self-duality $\longleftrightarrow$ anti-self-duality

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Poon, L, Donaldson-Friedman, Taubes ...

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Proposition (Atiyah-Hitchin-Singer). *The Fubini-Study metric on $\mathbb{CP}_2$ is self-dual. Consequently, minimizes Weyl functional.*

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Theorem (Poon '86). *Up conformal isometry, the Fubini-Study class is the unique self-dual conformal class on* $\mathbb{CP}_2$ *with* $Y([g]) > 0$.

$$Y([g]) = \inf_{\widehat{g}=u^2 g} \frac{\int_M s_{\widehat{g}} \, d\mu_{\widehat{g}}}{\sqrt{\int_M d\mu_{\widehat{g}}}} \; ;$$

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If g has s of fixed sign, agrees with sign of $Y_{[g]}$.

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$$\tau(M) = \frac{1}{12\pi^2} \int_M \left(|W_+|^2 - |W_-|^2 \right) d\mu$$

Theorem (Poon '86). *Up conformal isometry, the Fubini-Study class is the unique self-dual conformal class on $\mathbb{CP}_2$ with $Y([g]) > 0$.*

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Proposition (Atiyah-Hitchin-Singer '78). *The Fubini-Study metric on $\mathbb{CP}_2$ is self-dual. Consequently, minimizes Weyl functional.*

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Kähler means there exists an almost-complex structure J that is invariant under parallel transport with respect to g:

$$\nabla J = 0.$$

Natural Generalization:

Del Pezzo surfaces:

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(M^4, J) for which c_1 is a Kähler class $[\omega]$.

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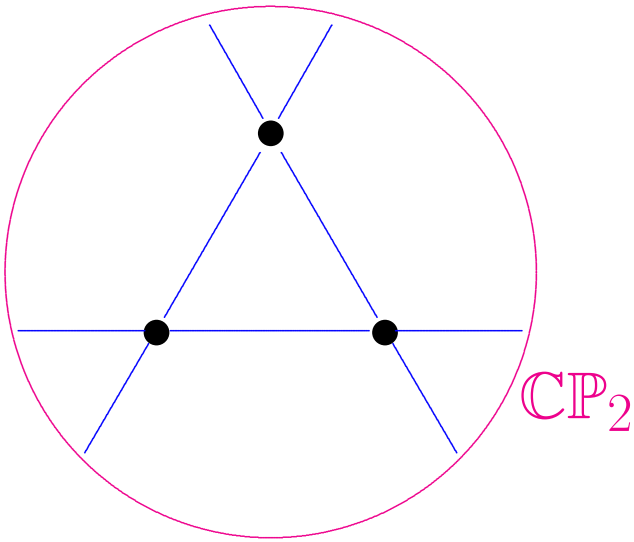

Blowing up:

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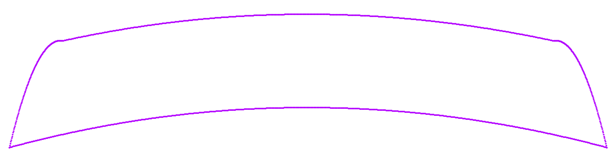

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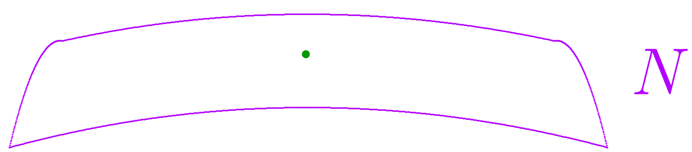

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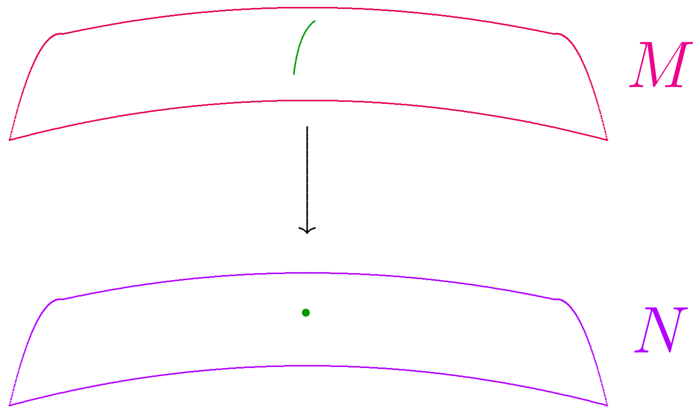

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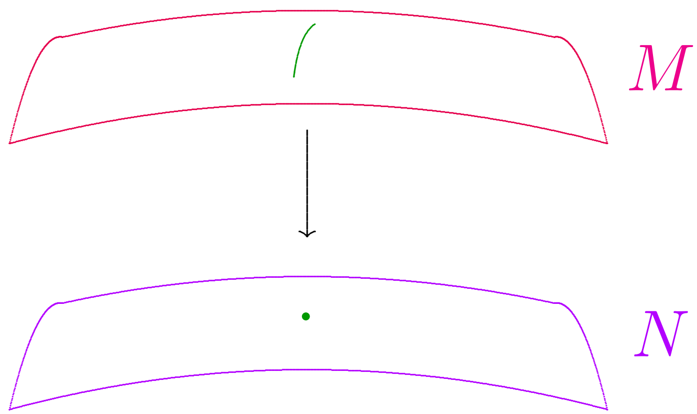
If N is a complex surface, may replace $p \in N$ with $\mathbb{CP}_1$ to obtain blow-up

$$M \approx N \# \overline{\mathbb{CP}}_2$$



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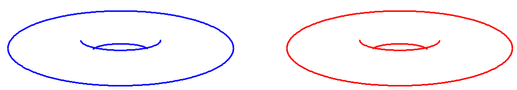

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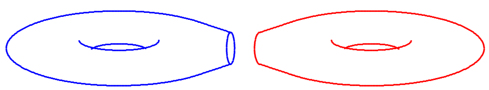

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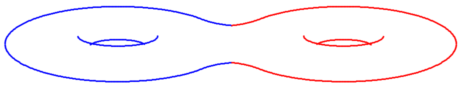

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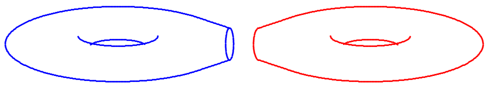

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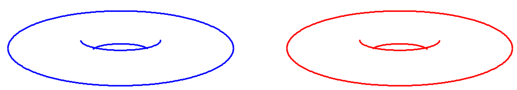

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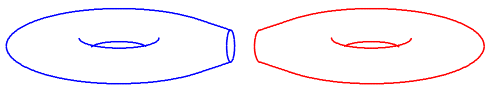

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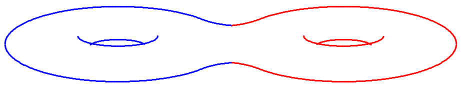

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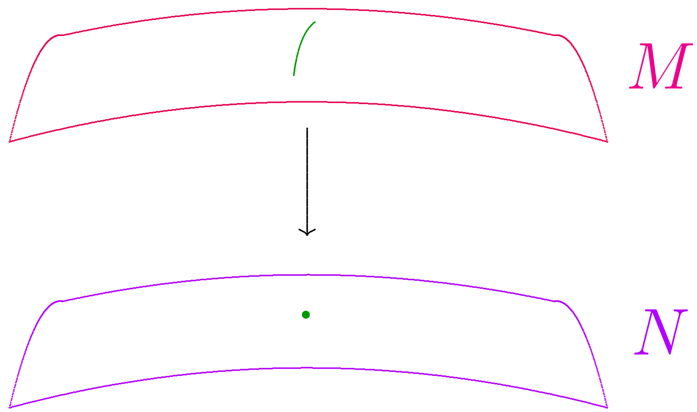

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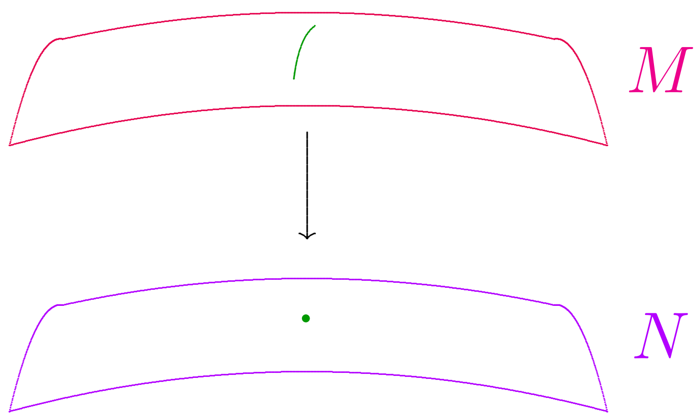

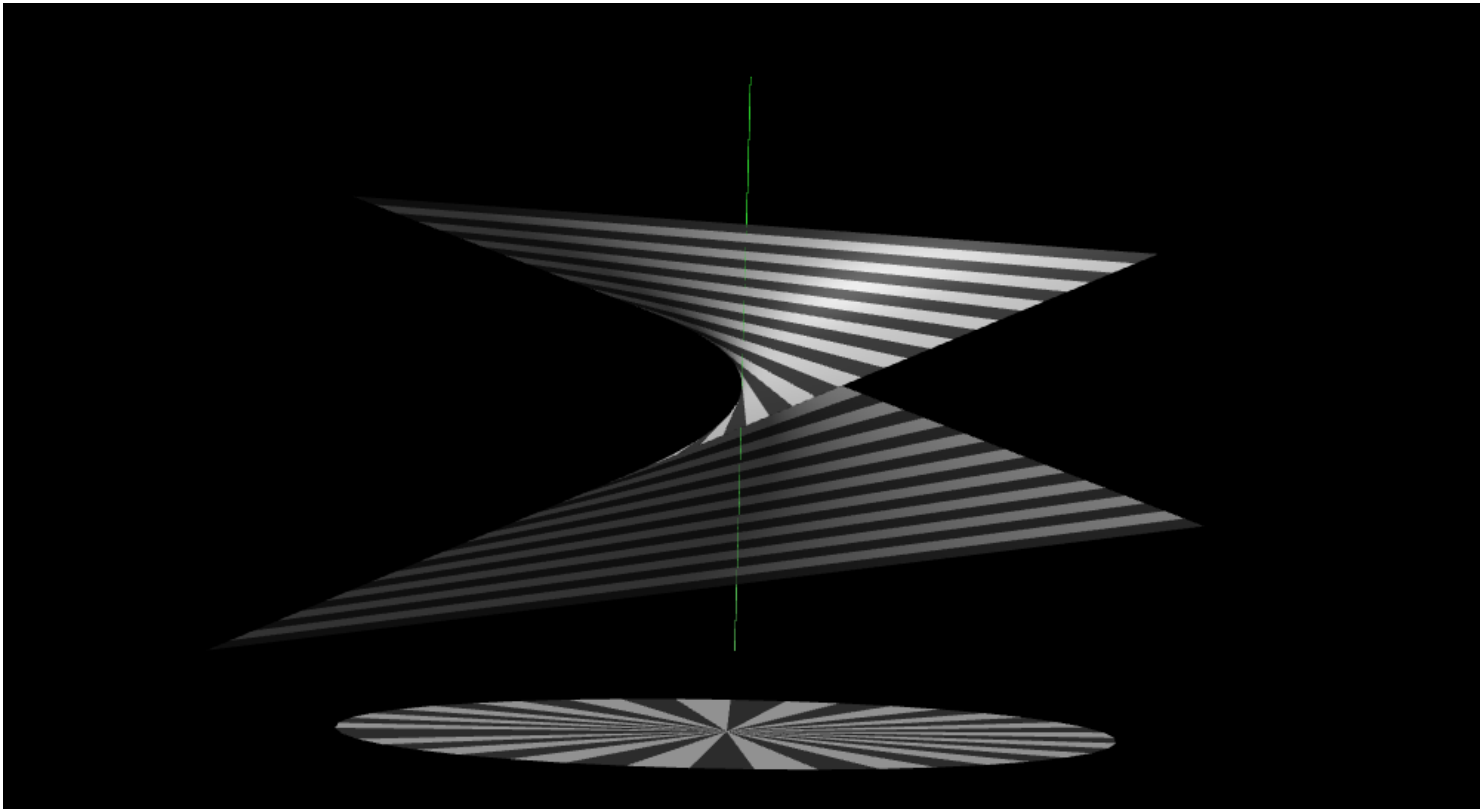

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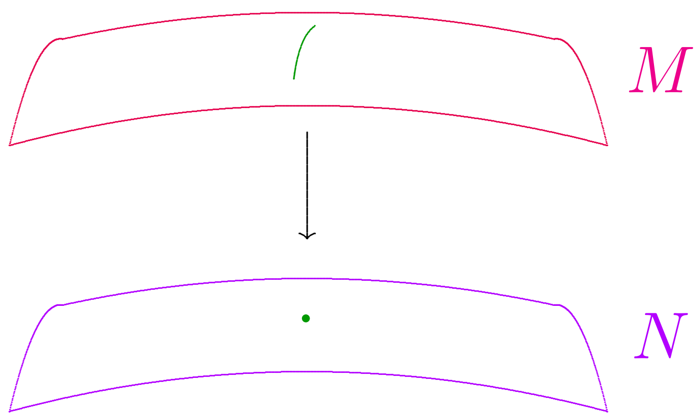

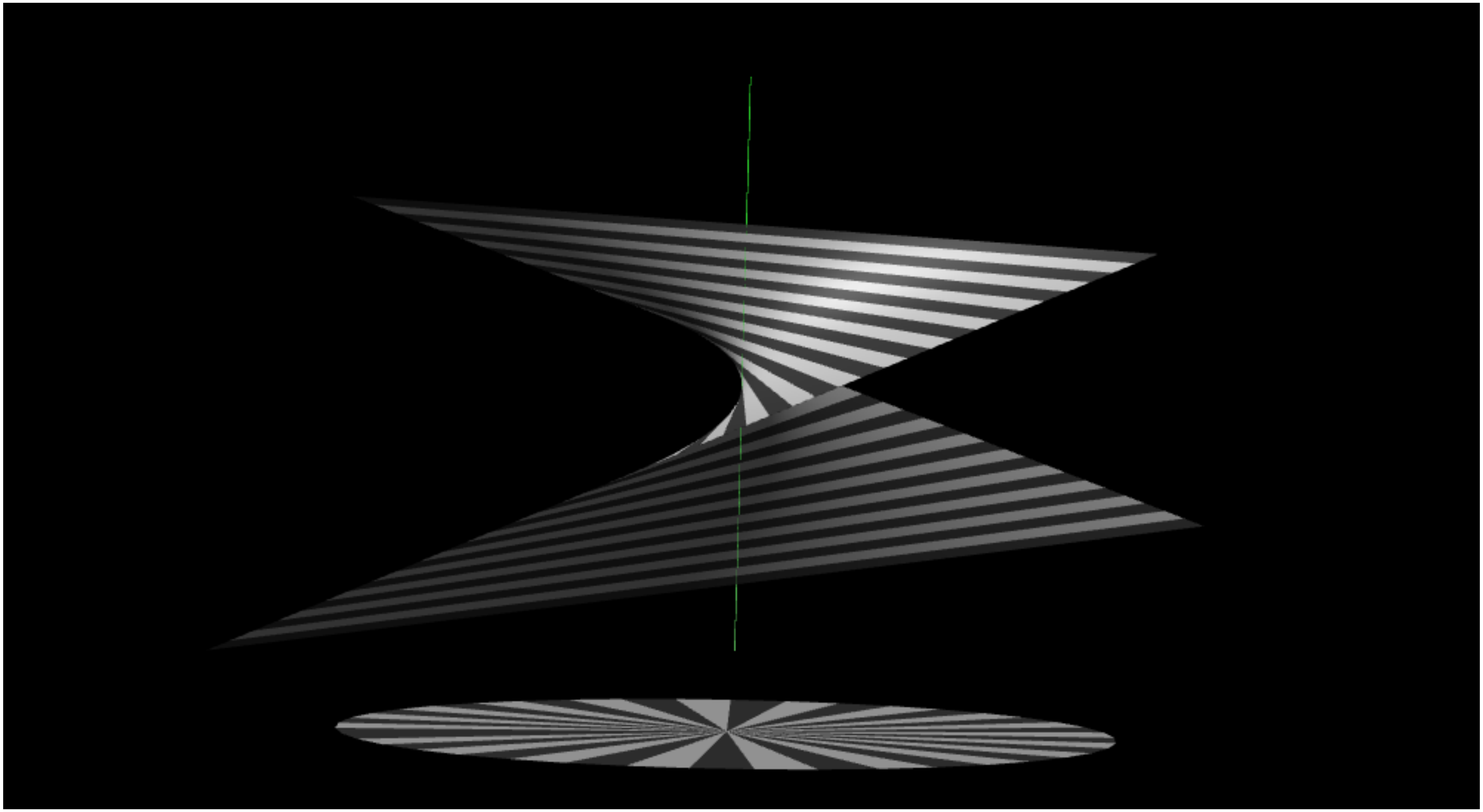

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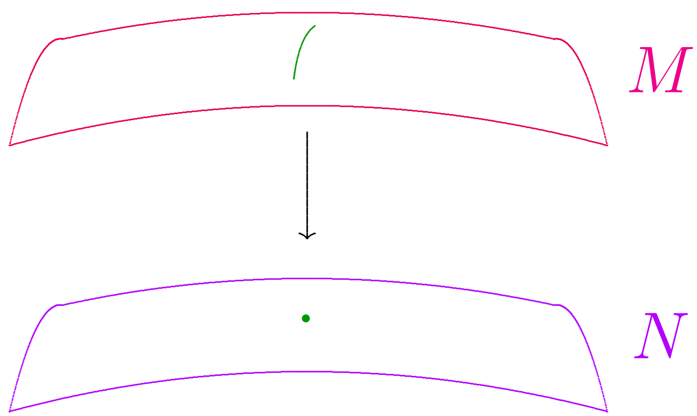

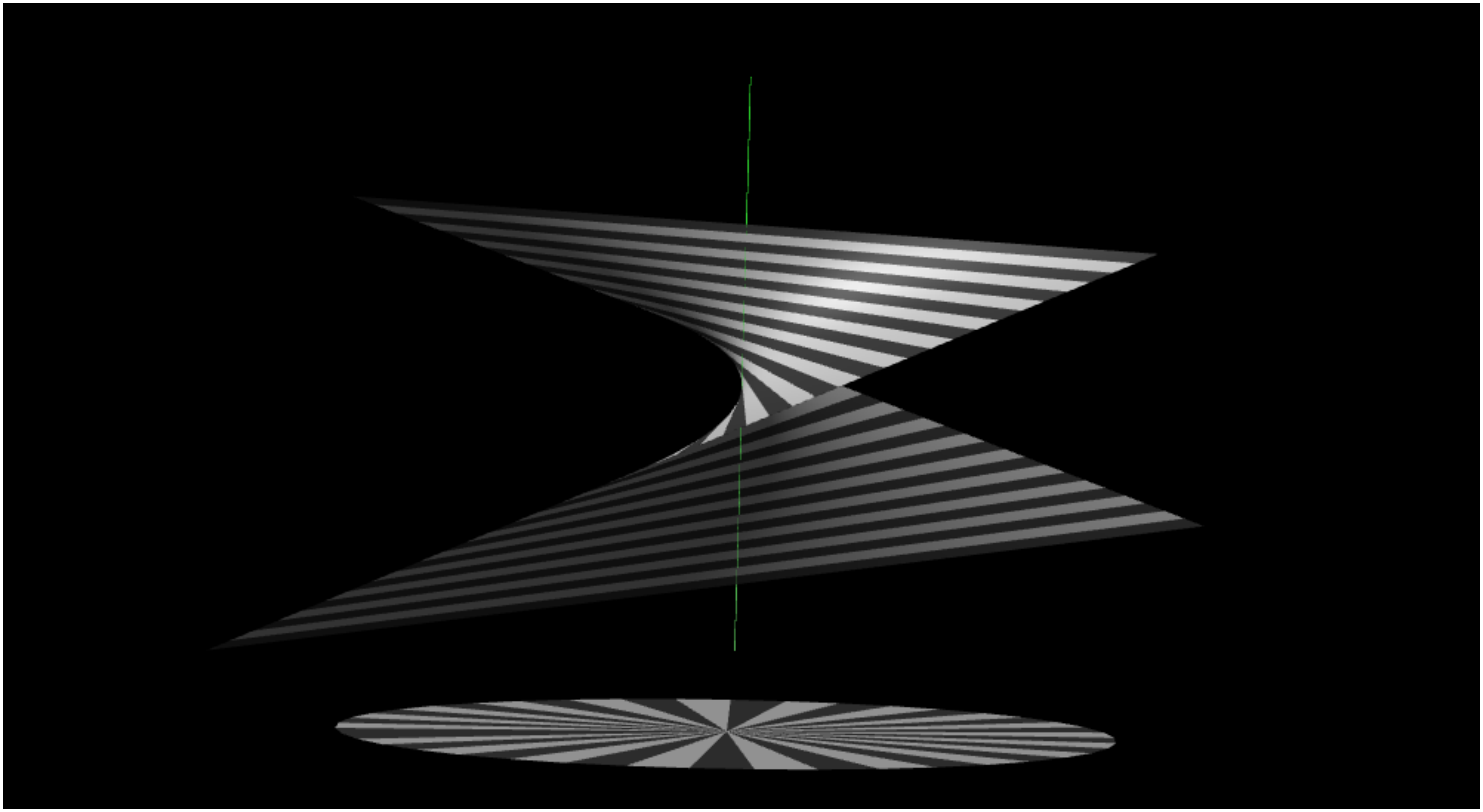

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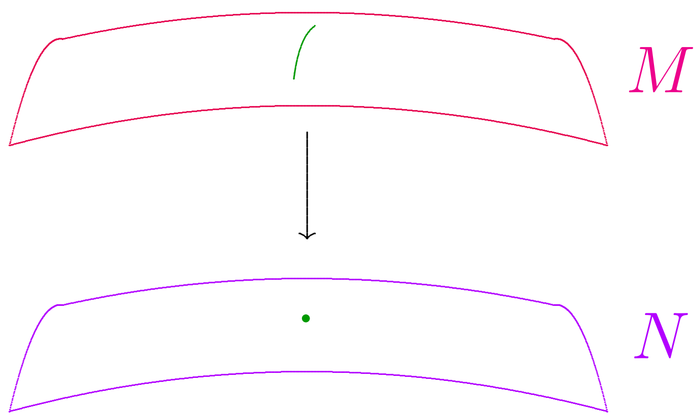

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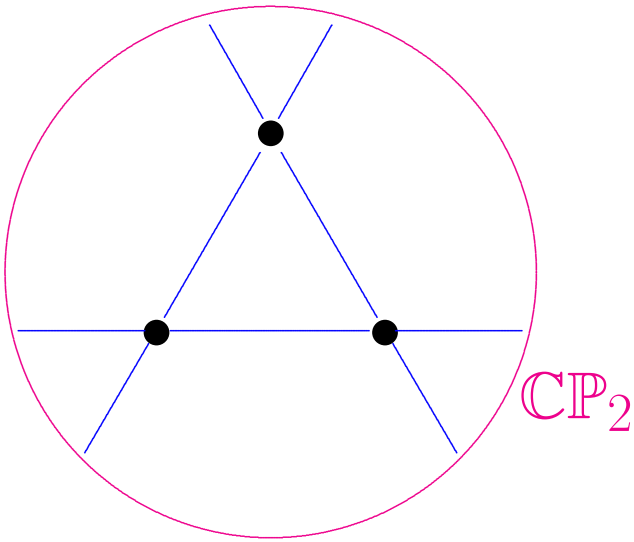

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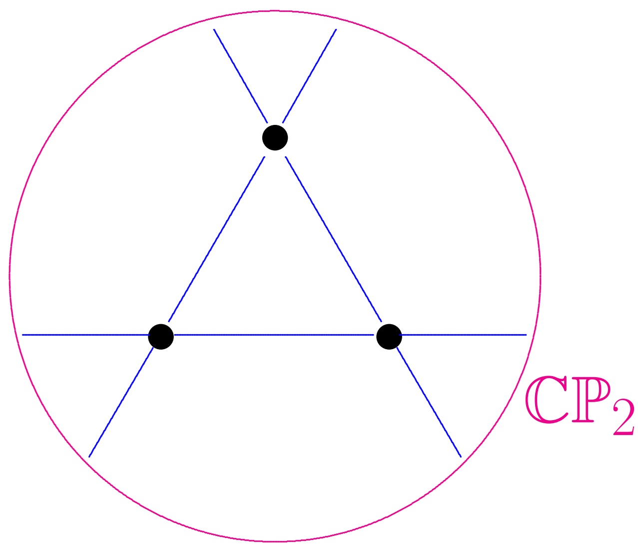

No 3 on a line,

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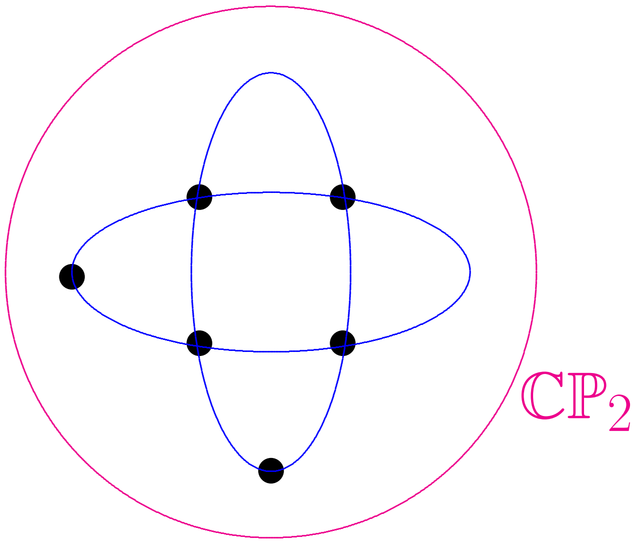

No 3 on a line, no 6 on conic,

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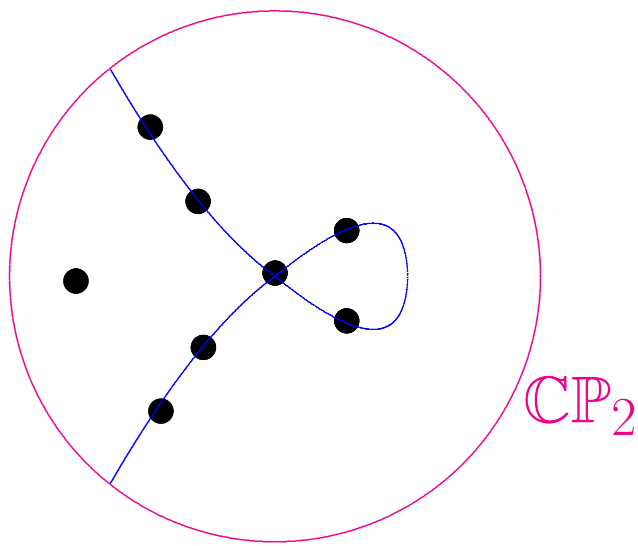

No 3 on a line, no 6 on conic, no 8 on nodal cubic.

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Existence: Page

1978

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Existence: Page-Derdziński,

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Existence: Page-Derdziński, Siu,

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Existence: Page-Derdziński, Siu, Tian-Yau,

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Existence: Page-Derdziński, Siu, Tian-Yau, Tian, Odaka-Spotti-Sun,

2016

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2008

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Uniqueness: Bando-Mabuchi '87

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Diffeotypes: exactly the Del Pezzo surfaces.

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Diffeotypes: exactly the Del Pezzo surfaces.

For known g, can take ω harmonic self-dual 2-form.

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Diffeotypes: exactly the Del Pezzo surfaces.

For known g, can take ω harmonic self-dual 2-form.

But this is not needed in above result.

Osamu Kobayashi '86:

What about $S^2 \times S^2$?

Conjecture (Kobayashi)**.** *The Kähler-Einstein product metric on* $S^2 \times S^2$ *minimizes the Weyl functional* $\mathscr{W}$*.*

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Persuasive partial results.

But problem still not settled!

Theorem (Gursky '98). *Let M be a smooth compact* 4*-manifold with* $b_+(M) \neq 0$.

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i.e. represented by metric with $s > 0$.

$$Y([g]) = \inf_{\widehat{g}=u^2 g} \frac{\int_M s_{\widehat{g}} \, d\mu_{\widehat{g}}}{\sqrt{\int_M d\mu_{\widehat{g}}}} \; ;$$

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But applies in much greater generality.

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But says nothing about $Y([g]) < 0$ realm.

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Hence says nothing about "most" conformal classes.

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$$0 = \frac{1}{2}\Delta|\omega|^2 + |\nabla\omega|^2 - 2W_+(\omega, \omega) + \frac{s}{3}|\omega|^2$$

for self-dual harmonic 2-form ω.

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Open condition in C^2 topology on metrics.

(Harmonic forms depend continuously on metric.)

Theorem (L '15)**.** *Let M be the underlying smooth oriented* 4*-manifold of a del Pezzo surface.*

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This recovers Gursky's inequality — but for a different open set of conformal classes!

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Inequality not limited to the positive Yamabe realm!

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Same technique covers conformally Kähler, Einstein cases among classes with fixed T^2 symmetry.

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Method: Almost-Kähler geometry:

$$3\int_M W_+(\omega,\omega)\ d\mu \geq 4\pi c_1 \bullet [\omega]$$

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This is apparently not an accident!

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What happens there in the Yamabe-negative realm?

Theorem (L '22)**.** *For any sufficiently large integer* m*,*

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Roulleau-Urzúa '15: $\exists$ sequences with $\tau/\chi \to 1/3$.

$\to$ Miyaoka-Yau line! Can choose spin or non-spin!

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Thanks for the invitation!