Einstein Metrics,

Symplectic 4-Manifolds, &

Smooth Topology

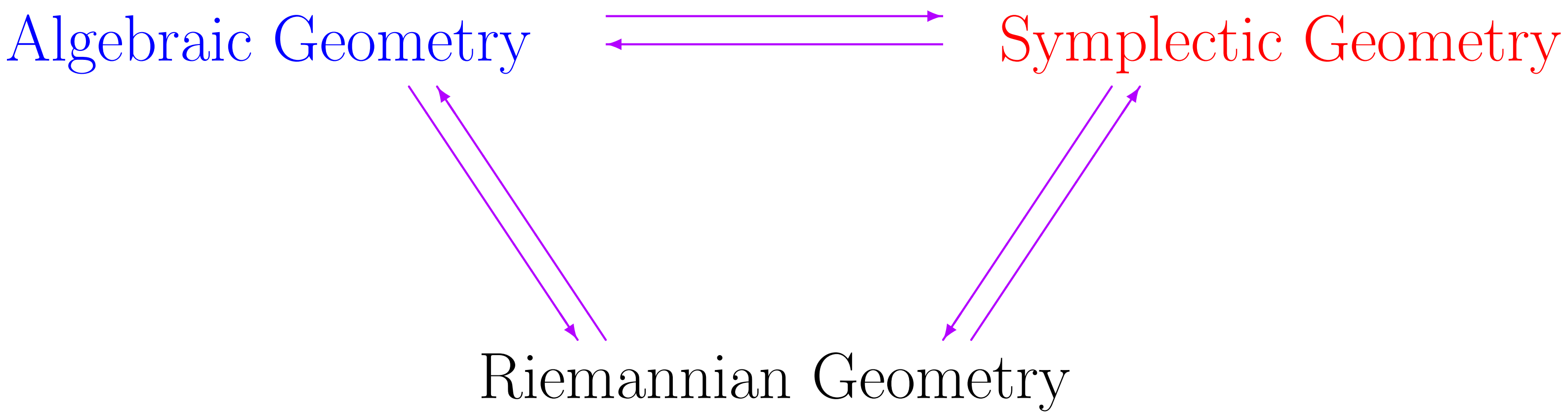
Claude LeBrun
Stony Brook University

Symplectic and Algebraic Geometry
University of Warwick, 2 August, 2017

Theme of this conference:

Algebraic Geometry $\rightleftarrows$ Symplectic Geometry

Theme of this talk:



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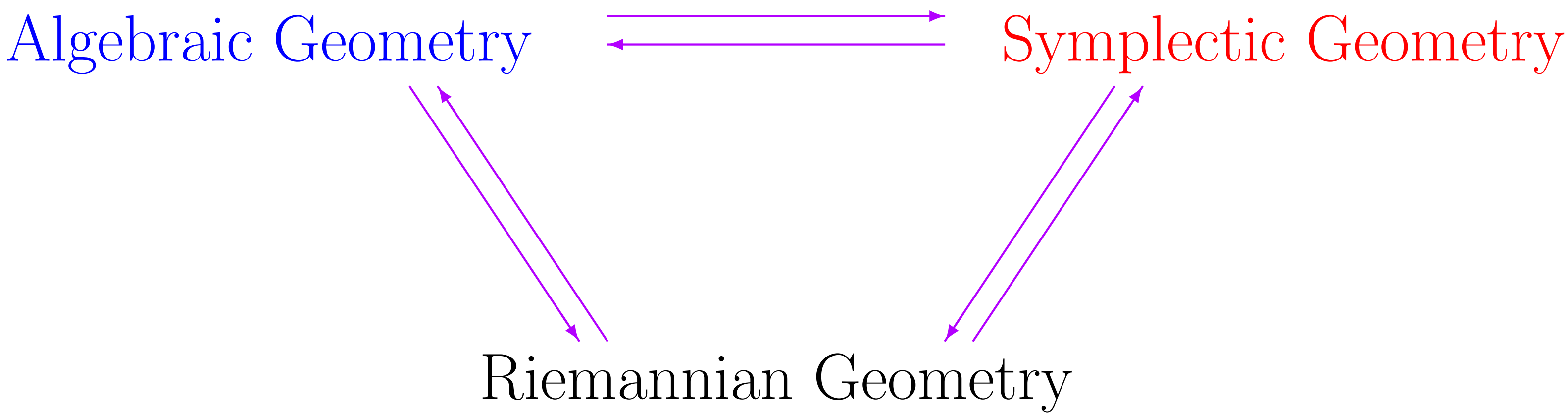

Talk concerns phenomena specific to dimension 4.

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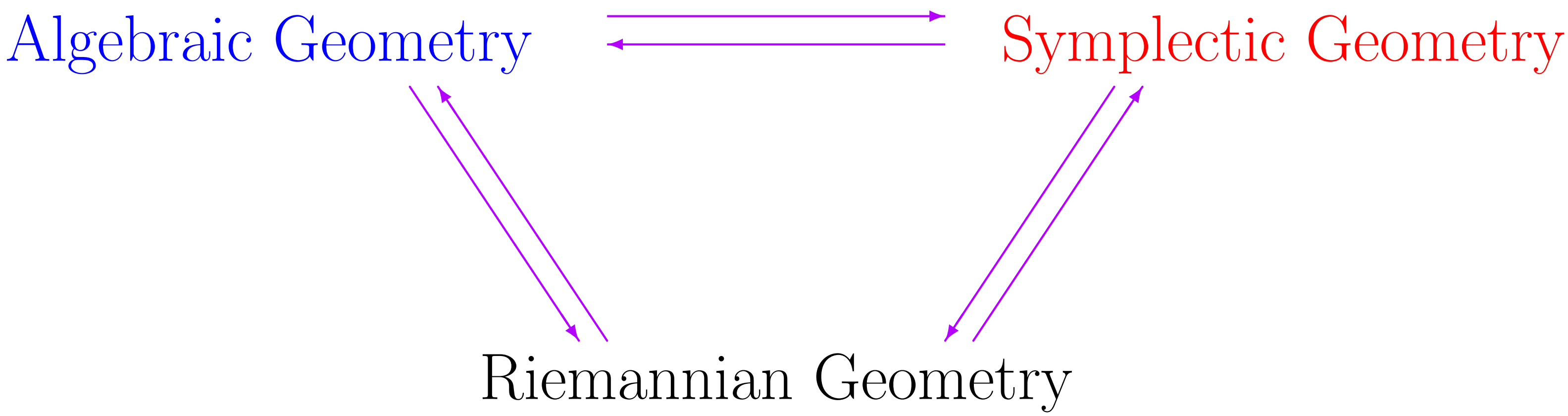

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Higher dimensions are demonstrably different.

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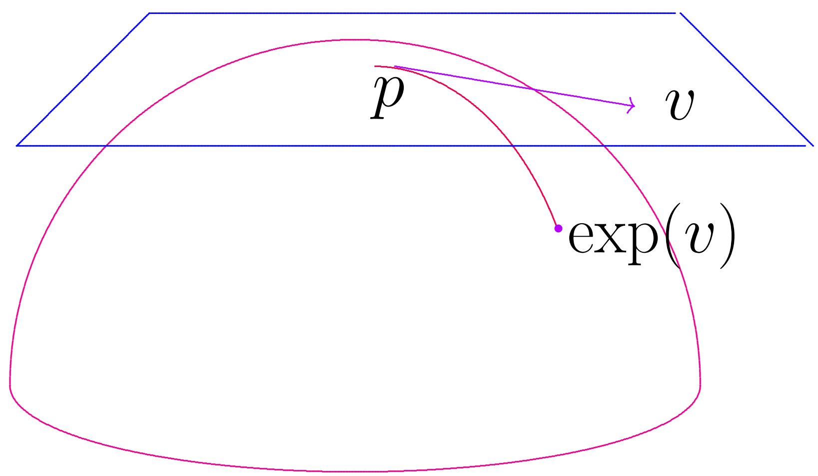

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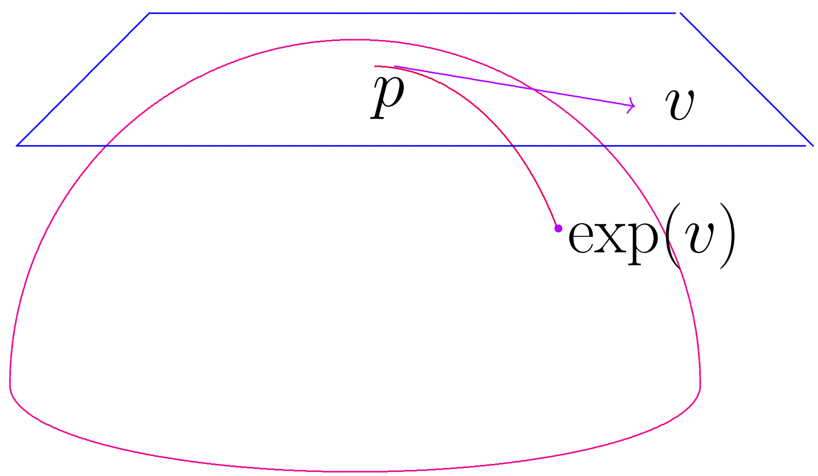

Now choosing $T_pM \xrightarrow{\cong} \mathbb{R}^n$ via some orthonormal basis gives us special coordinates on M.

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"...the greatest blunder of my life!"

— A. Einstein, to G. Gamow

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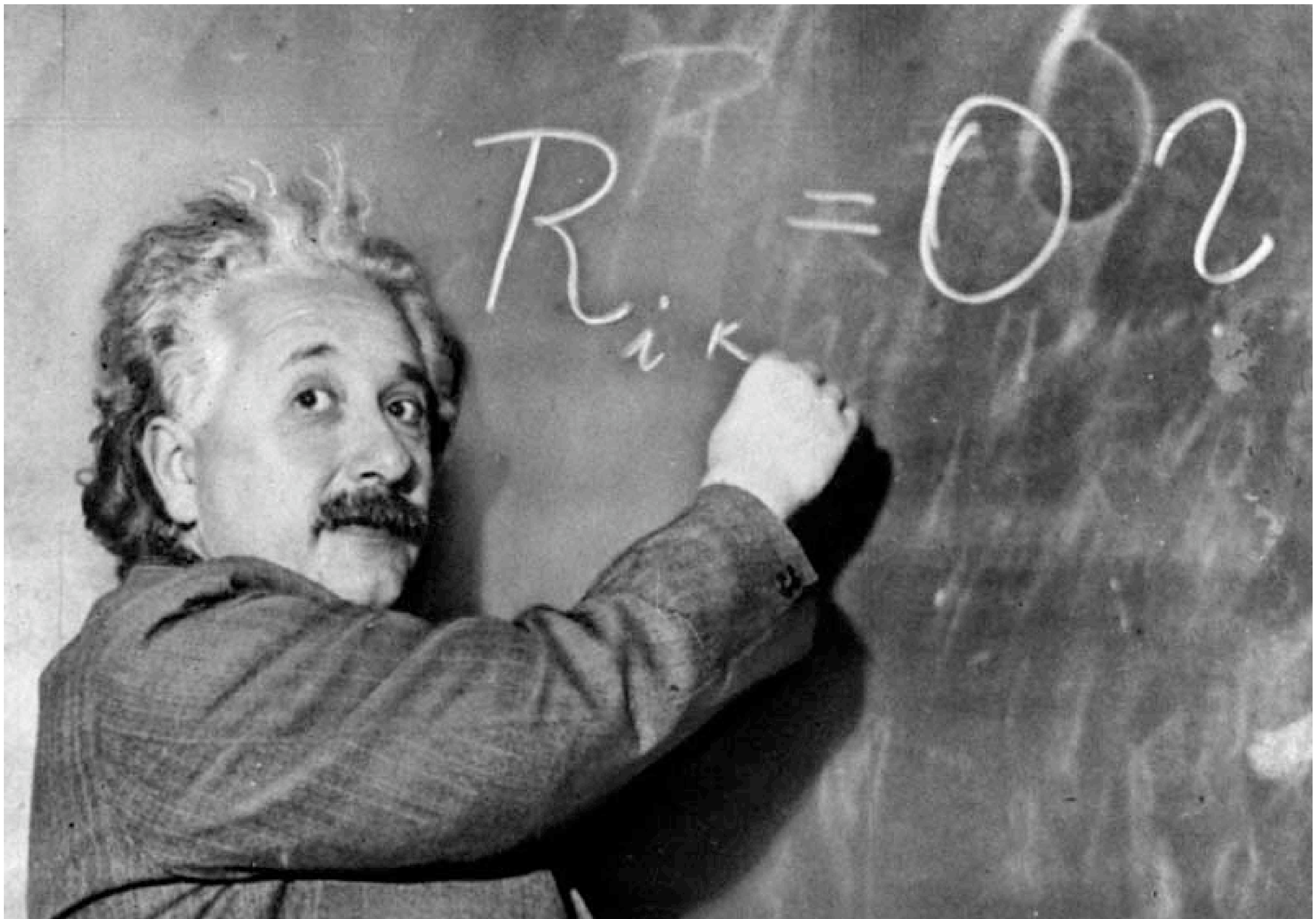

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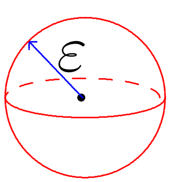

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where $V = \text{Vol}(M, g)$ inserted to make scale-invariant.

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Has $s =$ constant.

Unique up to scale when $s \leq 0$.

"Yamabe metrics"

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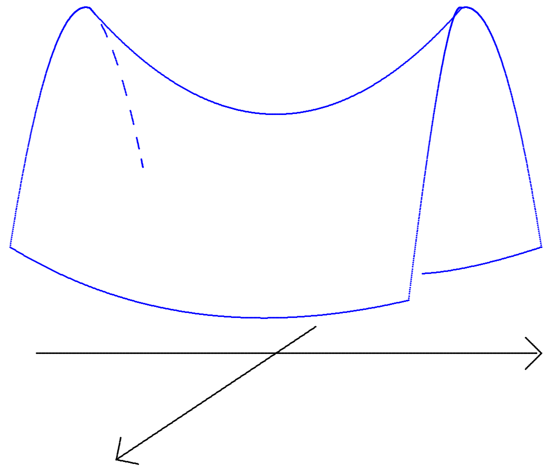

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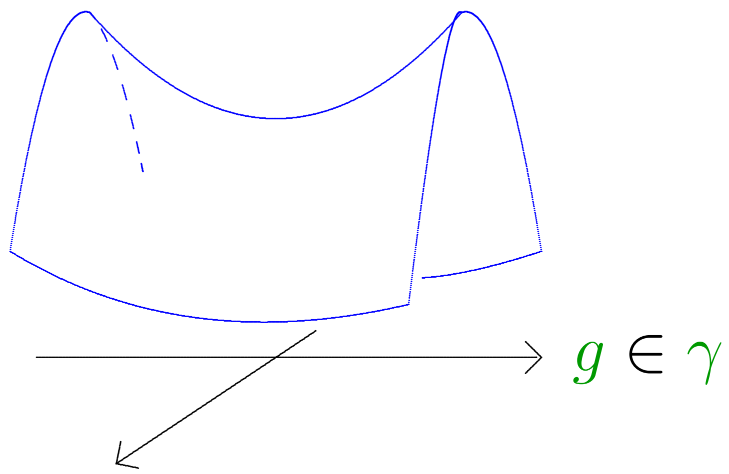

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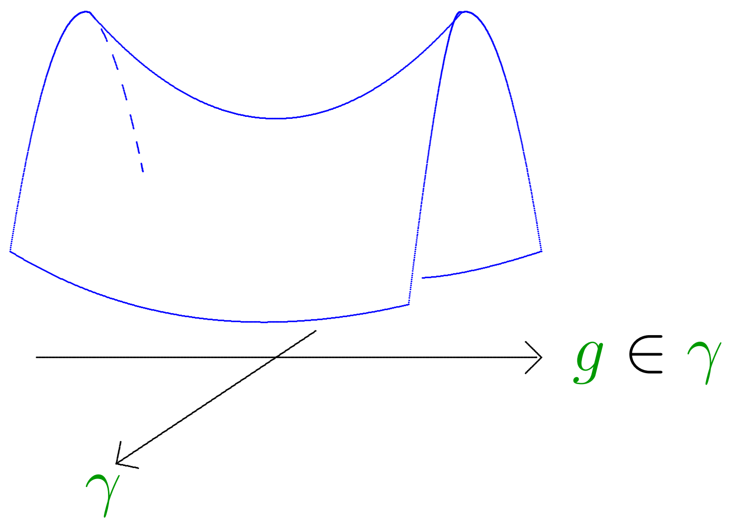

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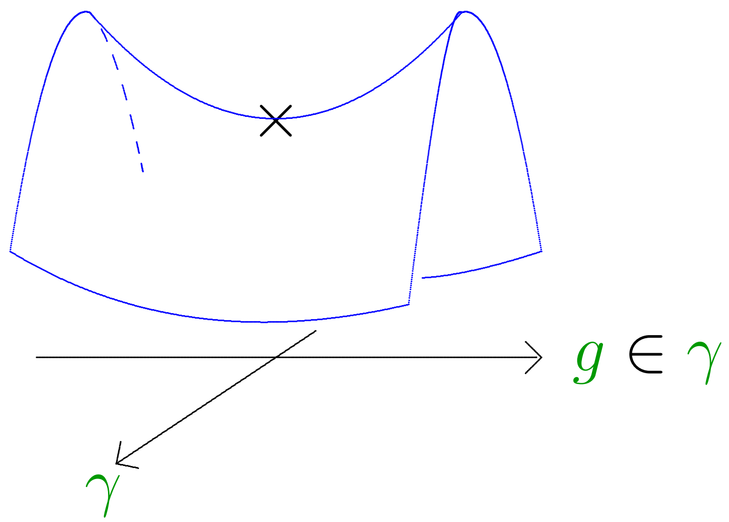

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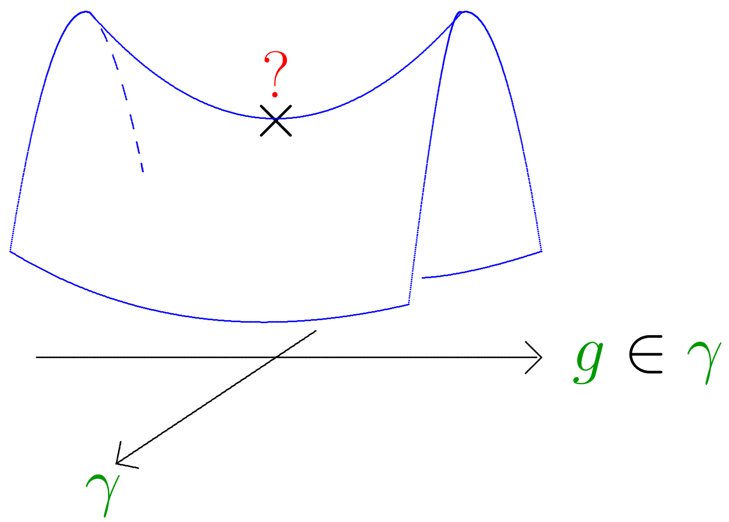

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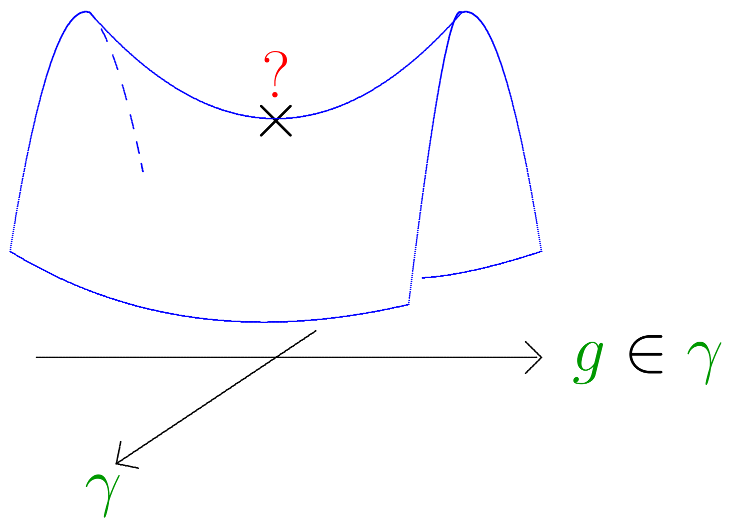

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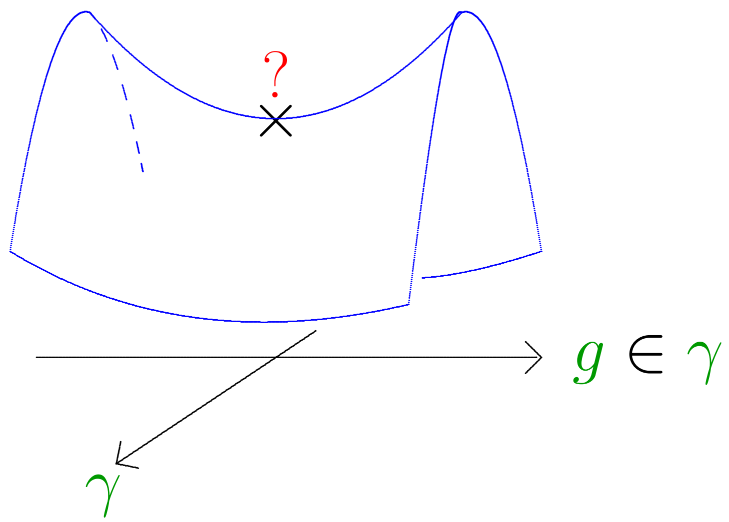

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This is a diffeomorphism invariant of M.

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Theorem (L '99). *Let* (M, J) *be a compact complex surface with* b_1 *even. Then*

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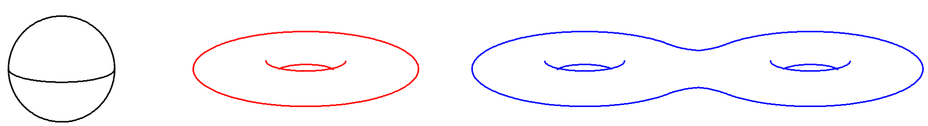
Reminiscent of the situation for complex curves!



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But if $\mathcal{Y}(M^4) = 0$, $\exists g$ with $s \equiv 0 \Longleftrightarrow$ $\mathrm{Kod}(M, J) = 0$ and M minimal.

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By contrast, ...

Theorem (Petean). *Let* M^n *be a simply connected* n*-manifold,* $n \geq 5$*. Then* $\mathcal{Y}(M) \geq 0$*.*

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Builds on:

Theorem (Gromov/Lawson). *Let M^n be a simply connected n-manifold, $n \geq 5$. If M is not spin, then M carries a metric g with $s > 0$. That is,*

$$w_2(TM) \neq 0 \Longrightarrow \mathcal{Y}(M) > 0.$$

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Perelman $\Longrightarrow$ similarly when $n = 3$.

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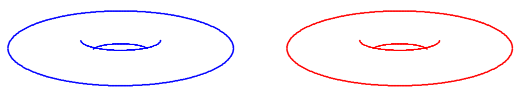

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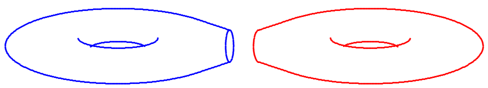

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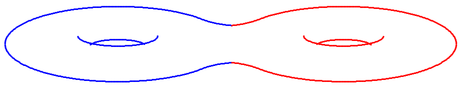

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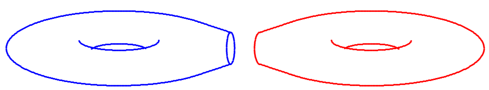

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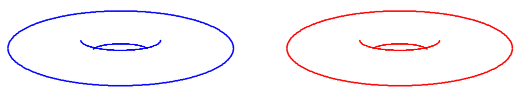

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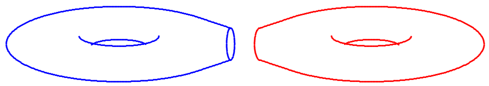

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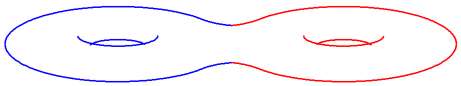

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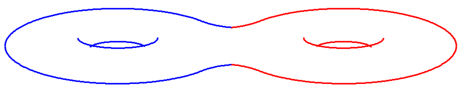

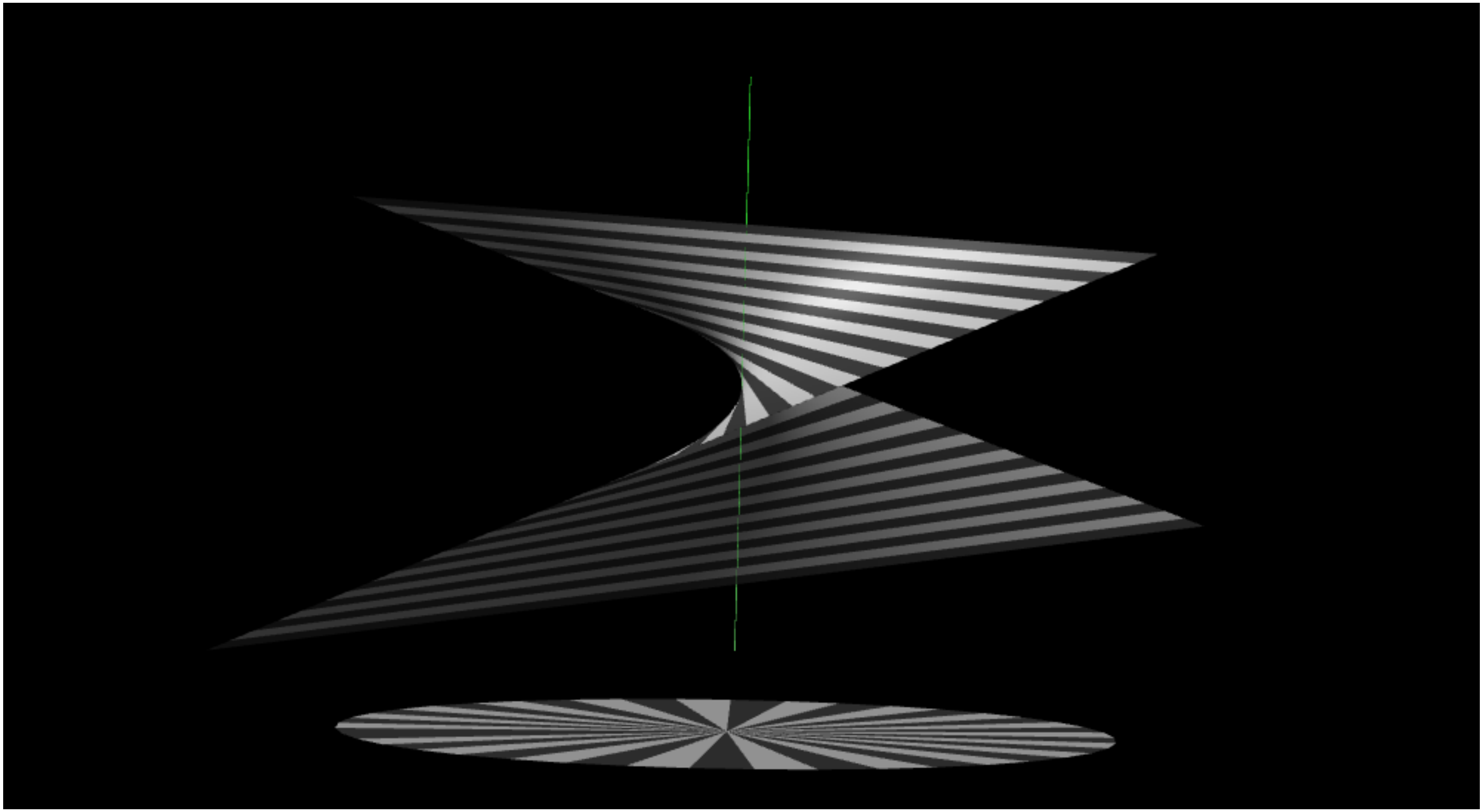

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Ingredients: Seiberg-Witten estimates for s; Kähler-Einstein metrics; gravitational instantons.

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Same definition later rediscovered by Tian-Jun Li, who extended it to define a notion of Kodaira dimension for any symplectic 4-manifold.

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Şuvaina: Equality can occur in non-complex case.

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Key case: $\mathcal{Y}(M) = 0$.

Which 4-manifolds admit Einstein metrics?

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What is the moduli space of such metrics?

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A laboratory for exploring Einstein metrics.

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Del Pezzo surfaces,
K3 surface, Enriques surface,
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No others: Hitchin-Thorpe, Seiberg-Witten, . . .

One tool: Hitchin-Thorpe Inequality:

$$(2\chi + 3\tau)(M) = \frac{1}{4\pi^2}\int_M \left(\frac{s^2}{24} + 2|W_+|^2 - \frac{|\mathring{r}|^2}{2}\right) d\mu_g$$

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In these cases, any Einstein metric is Kähler!

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Again, Hitchin-Thorpe:

${c_1}^2(M) > 0$ unless locally hyper-Kähler.

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Remark. When $m = 2$, such M are necessarily minimal complex surfaces of general type.

Theorem (L '01). *Let X be a minimal symplectic 4-manifold of general type, and let*

$$M = X \# k\overline{\mathbb{CP}}_2.$$

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So being "very" non-minimal is an obstruction.

Theorem (Curvature Estimates). *For any Riemannian metric g on a symplectic manifold M with minimal model X. If $Kod(M, \omega) \neq -\infty$, then g satisfies the following curvature inequalities:*

$$\int_M s^2 d\mu_g \geq 32\pi^2 {c_1}^2(X)$$

$$\int_M \left(s - \sqrt{6}|W_+|\right)^2 d\mu_g \geq 72\pi^2 {c_1}^2(X).$$

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Play off basic classes against each other.

Let us now return to the $\lambda > 0$ case.

Theorem (L '09). *Suppose that M is a smooth compact oriented 4-manifold which admits a symplectic structure ω. Then M also admits an Einstein metric g with $\lambda \geq 0$ if and only if*

$$M \overset{\text{diff}}{\approx} \begin{cases} \mathbb{CP}_2 \# k\overline{\mathbb{CP}}_2, & 0 \leq k \leq 8, \\ S^2 \times S^2, \\ K3, \\ K3/\mathbb{Z}_2, \\ T^4, \\ T^4/\mathbb{Z}_2, T^4/\mathbb{Z}_3, T^4/\mathbb{Z}_4, T^4/\mathbb{Z}_6, \\ T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_2), T^4/(\mathbb{Z}_3 \oplus \mathbb{Z}_3), \textit{ or } T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_4). \end{cases}$$

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Know an Einstein metric on each manifold.

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Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

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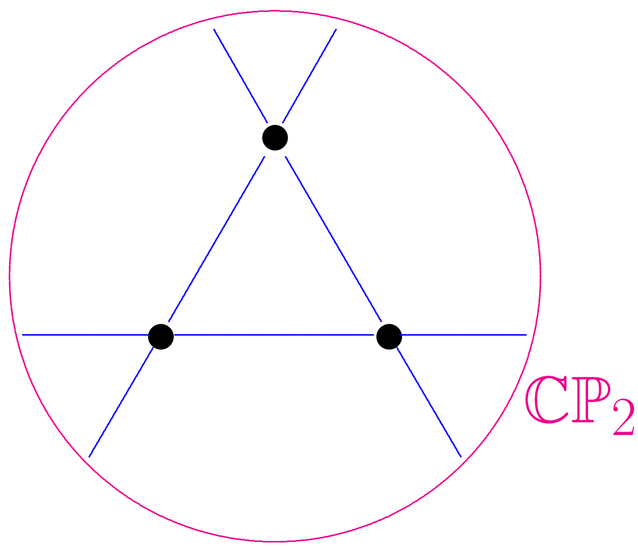

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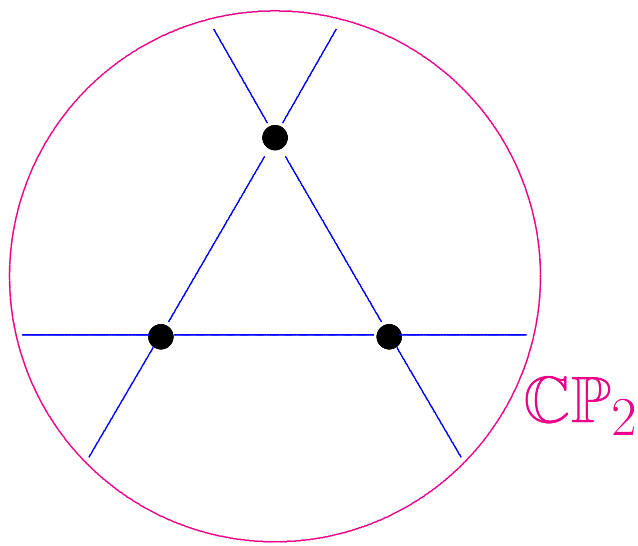

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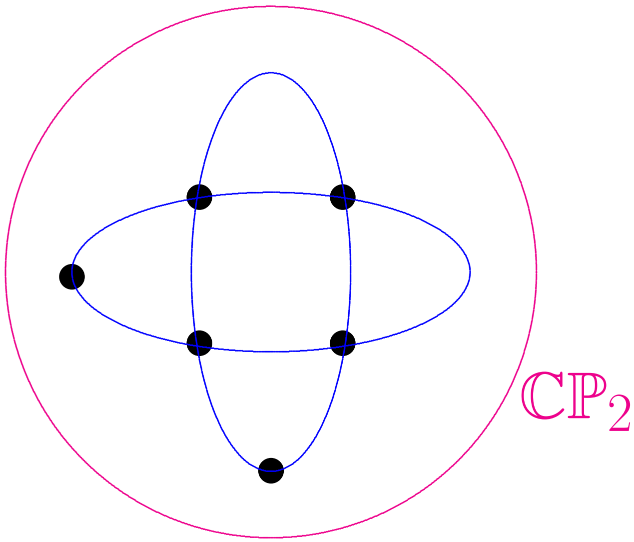

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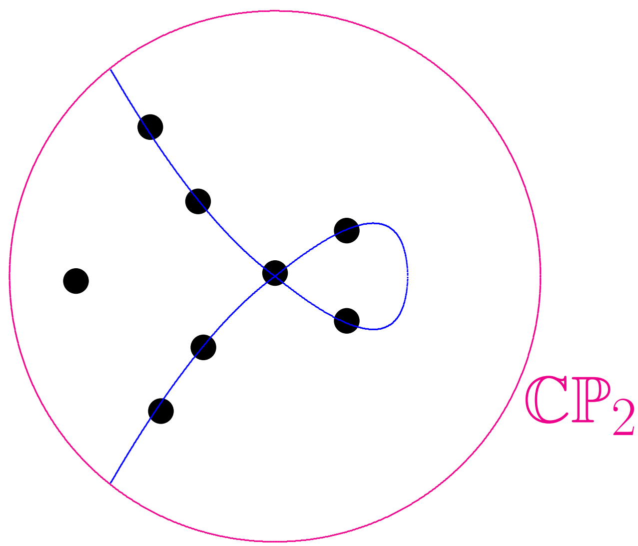

No 3 on a line, no 6 on conic, no 8 on nodal cubic.

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Existence: Tian, Odaka-Spotti-Sun, Chen-L-Weber...

Uniqueness: Bando-Mabuchi, L '12...

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Theorem (L '15)**.**

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Corollary.

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Theorem (L '15). *$\mathscr{E}^+_\omega(M)$ is connected. Moreover, if $b_2(M) \leq 5$, then $\mathscr{E}^+_\omega(M) = \{point\}$.*

Corollary. *$\mathscr{E}^+_\omega(M)$ is exactly one connected component of $\mathscr{E}(M)$.*

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I hope that some of you will be intrigued enough to want to contribute something to the subject!