Curvature in the Balance:

The Weyl Functional &

Scalar Curvature of

4-*Manifolds*

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Special Metrics in Complex Geometry
University of Texas at Dallas. May 19, 2022

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$$\mathcal{R}^{ab}{}_{cd} = W^{ab}{}_{cd} + \frac{4}{n-2}\mathring{r}^{[a}{}_{[c}\delta^{b]}{}_{d]} + \frac{2}{n(n-1)}\, s\, \delta^{a}_{[c}\delta^{b}_{d]}$$

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$W^{a}{}_{bcd}$ unchanged if $g \rightsquigarrow \hat{g} = u^2 g$.

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Proposition. *Assume* $n \geq 4$. *Then* (M^n, g) *locally conformally flat* $\iff W \equiv 0$.

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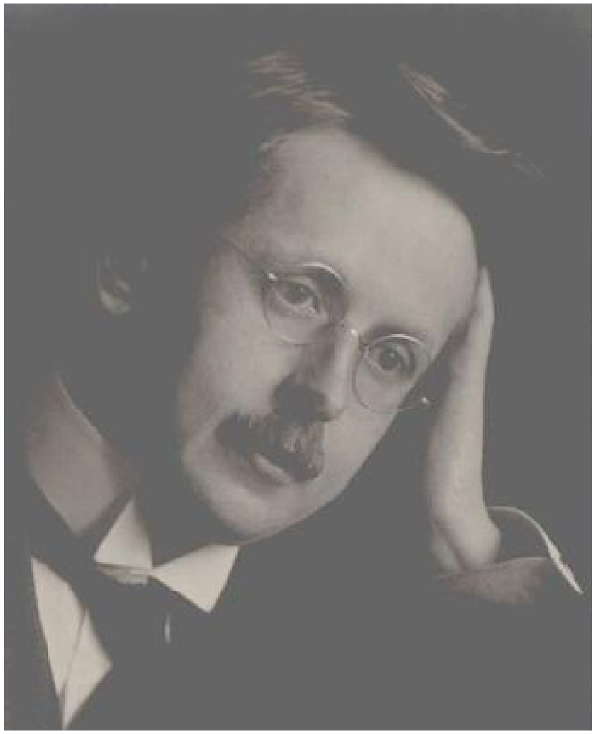

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Ricci-flat $\Longrightarrow$ $W = \mathcal{R}$.

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since, for fixed CY on $K3$, $\quad \mathscr{W}(g) \propto \mathrm{Vol}(\mathbb{T}^{m-4})$.

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Integrals give four scale-invariant functionals.

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However, these are not independent!

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Euler characteristic

$$\chi(M) = \frac{1}{8\pi^2}\int_M \left(\frac{s^2}{24} + |W_+|^2 + |W_-|^2 - \frac{|\mathring{r}|^2}{2}\right) d\mu$$

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Signature

$$\tau(M) = \frac{1}{12\pi^2}\int_M \left(|W_+|^2 - |W_-|^2\right) d\mu$$

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e.g. critical for Weyl functional

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So $\int |W_+|^2 d\mu$ equivalent to Weyl functional.

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Today's theme: How do these compare in size,

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Today's theme: How do these compare in size,

for specific classes of metrics on interesting 4-manifolds?

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$$|W_{+}|^{2} = \frac{s^{2}}{24}$$

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$$\int_M \frac{s^2}{24} d\mu_g = \int_M |W_+|^2 d\mu_g \ .$$

More general Riemannian metrics?

Theorem (Gursky-L '99, Gursky '00). *Let* (M, g) *be a compact oriented Einstein 4-manifold*

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Excluded: Round S^4, Fubini-Study $\overline{\mathbb{CP}}_2$.

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$$\int_M |W_+|^2 d\mu_g \;\geq \int_M \frac{s^2}{24} d\mu_g$$

with equality $\Leftrightarrow g$ *is locally Kähler-Einstein.*

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Excluded: Del Pezzo Surfaces (10 diffeotypes)

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Equivalent to

$$\frac{1}{4\pi^2}\int_M |W_+|^2 d\mu_g \overset{?}{\geq} \frac{1}{3}(2\chi + 3\tau)(M).$$

Since

$$\mathscr{W}([g]) = -12\pi^2\tau(M) + 2\int_M |W_+|^2 d\mu_g$$

this is really a question about $\inf \mathscr{W}$.

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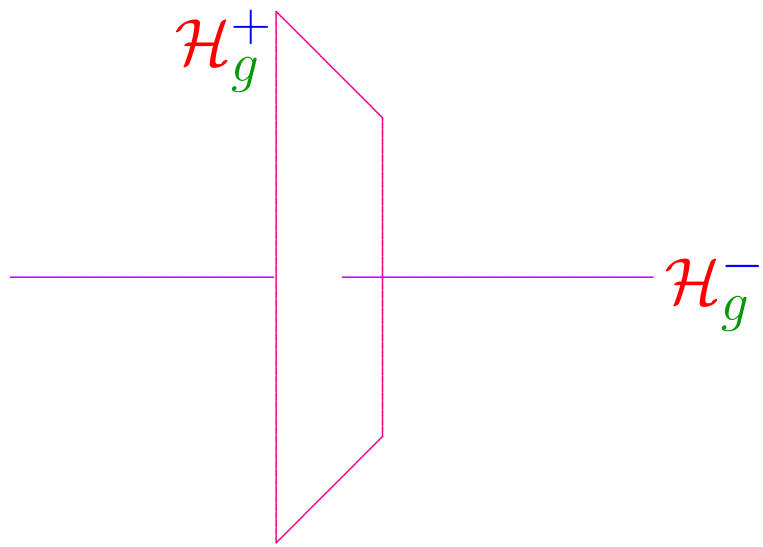

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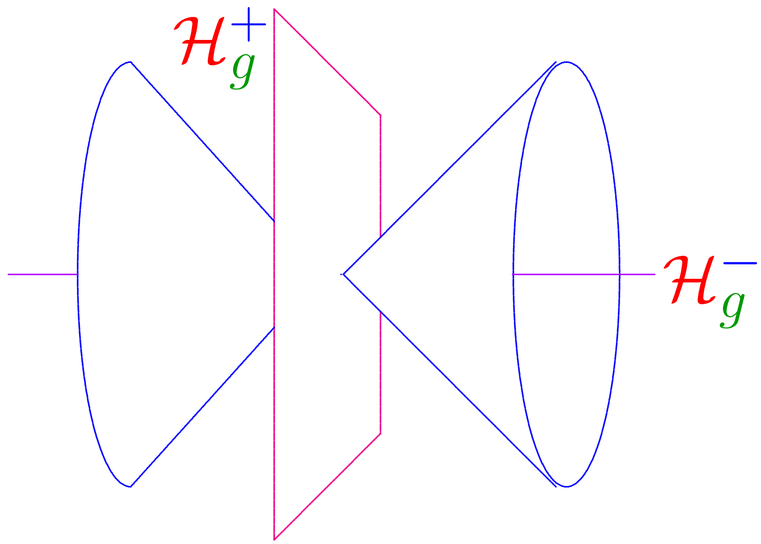

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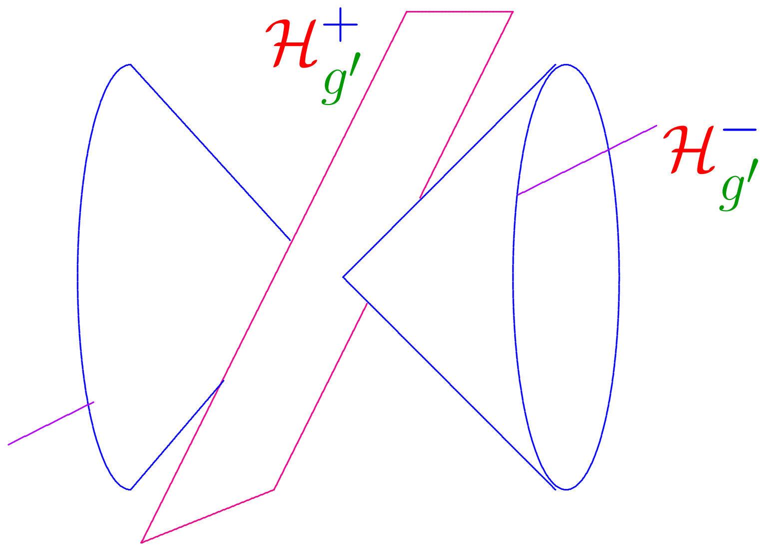

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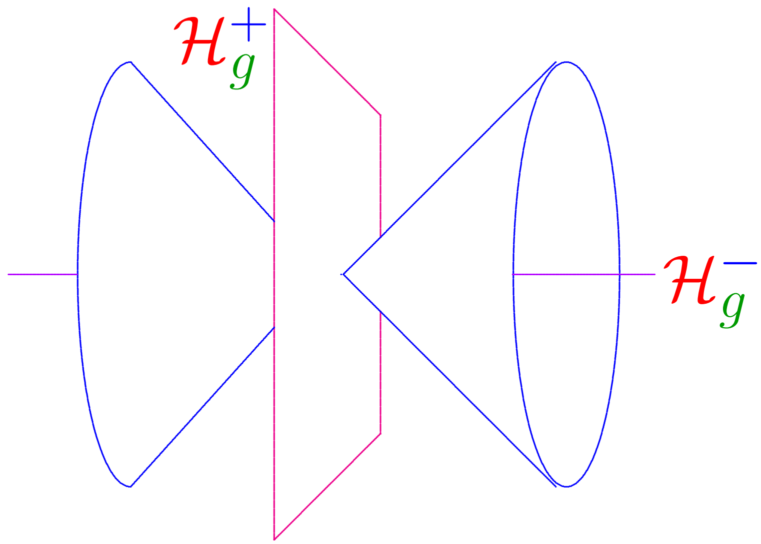

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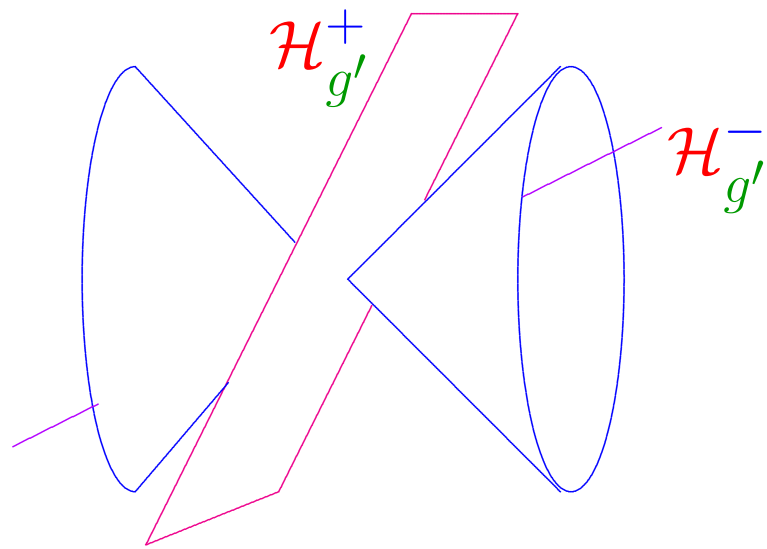

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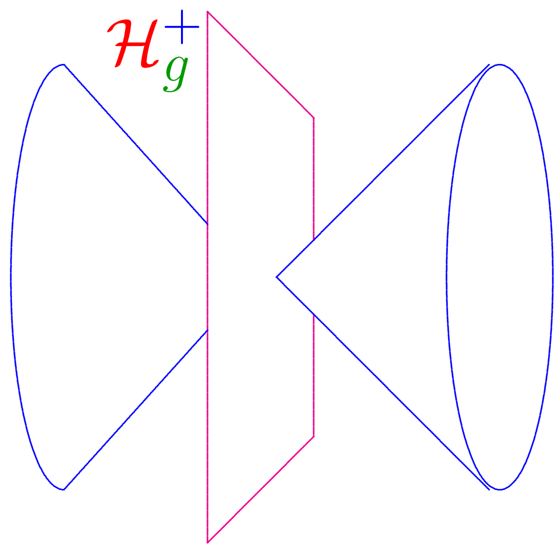

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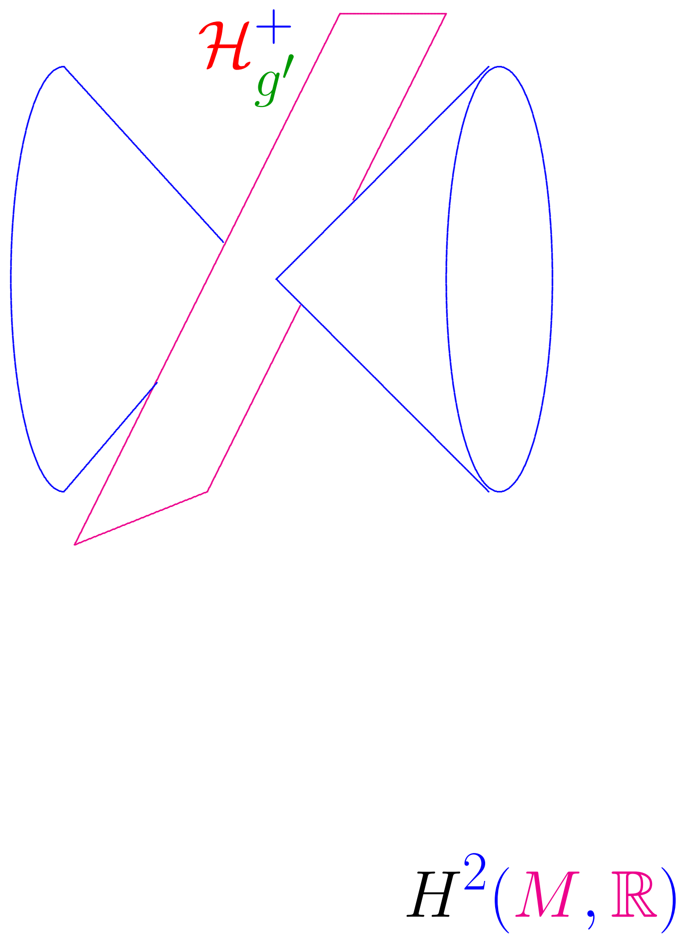
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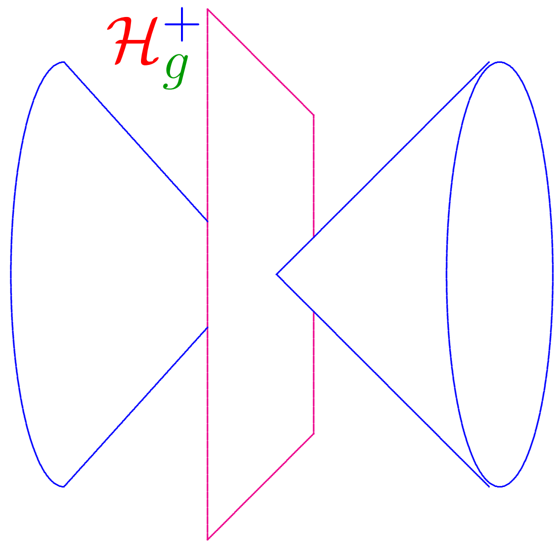

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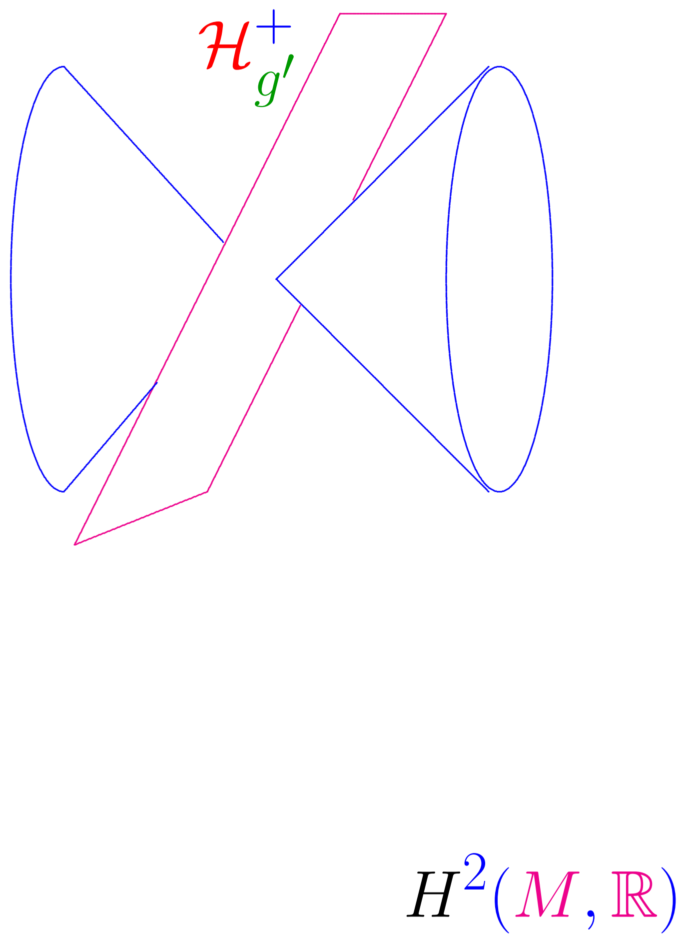
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Proposition (Atiyah-Hitchin-Singer '78)**.** *The Fubini-Study metric on $\mathbb{CP}_2$ is self-dual. Consequently, minimizes Weyl functional.*

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Kuiper '49: $\therefore$ Round S^4! $\Rightarrow\Leftarrow$

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Persuasive partial results.

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Persuasive partial results.

But problem still not settled.

Theorem (Gursky '98)**.** *Let M be a smooth compact 4-manifold with $b_+(M) \neq 0$.*

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But says nothing about $Y([g]) < 0$ realm.

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But says nothing about "most" conformal classes.

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$$0 = \frac{1}{2}\Delta|\omega|^2 + |\nabla\omega|^2 - 2W_+(\omega,\omega) + \frac{s}{3}|\omega|^2$$

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$$\Longrightarrow \exists \widehat{g} = u^2 g \quad \text{s.t.} \quad \widehat{\mathfrak{s}} := \widehat{s} - 2\sqrt{6}\widehat{|W_+|} \leq 0.$$

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Open condition in C^2 topology on metrics.

(Harmonic forms depend continuously on metric.)

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This recovers Gursky's inequality — but for a different open set of conformal classes!

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Inequality not limited to the positive Yamabe realm!

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This is apparently not an accident!

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What happens there in the Yamabe-negative realm?

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admits conformal classes $[g]$ *where the above inequality holds.*

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In proof, we apply this to

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Roulleau-Urzúa '15: $\exists$ sequences with $\tau/\chi \to 1/3$.

$\to$ Miyaoka-Yau line! Can choose spin or non-spin!

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Another new result involving these ideas.

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again with equality $\Leftrightarrow$ (M, g, ω) *is Kähler.*

In particular, any compact almost-Kähler 4*-manifold* (M, g, ω) *with* $\delta W_+ = 0$ *and* $s \geq 0$ *is Kähler.*

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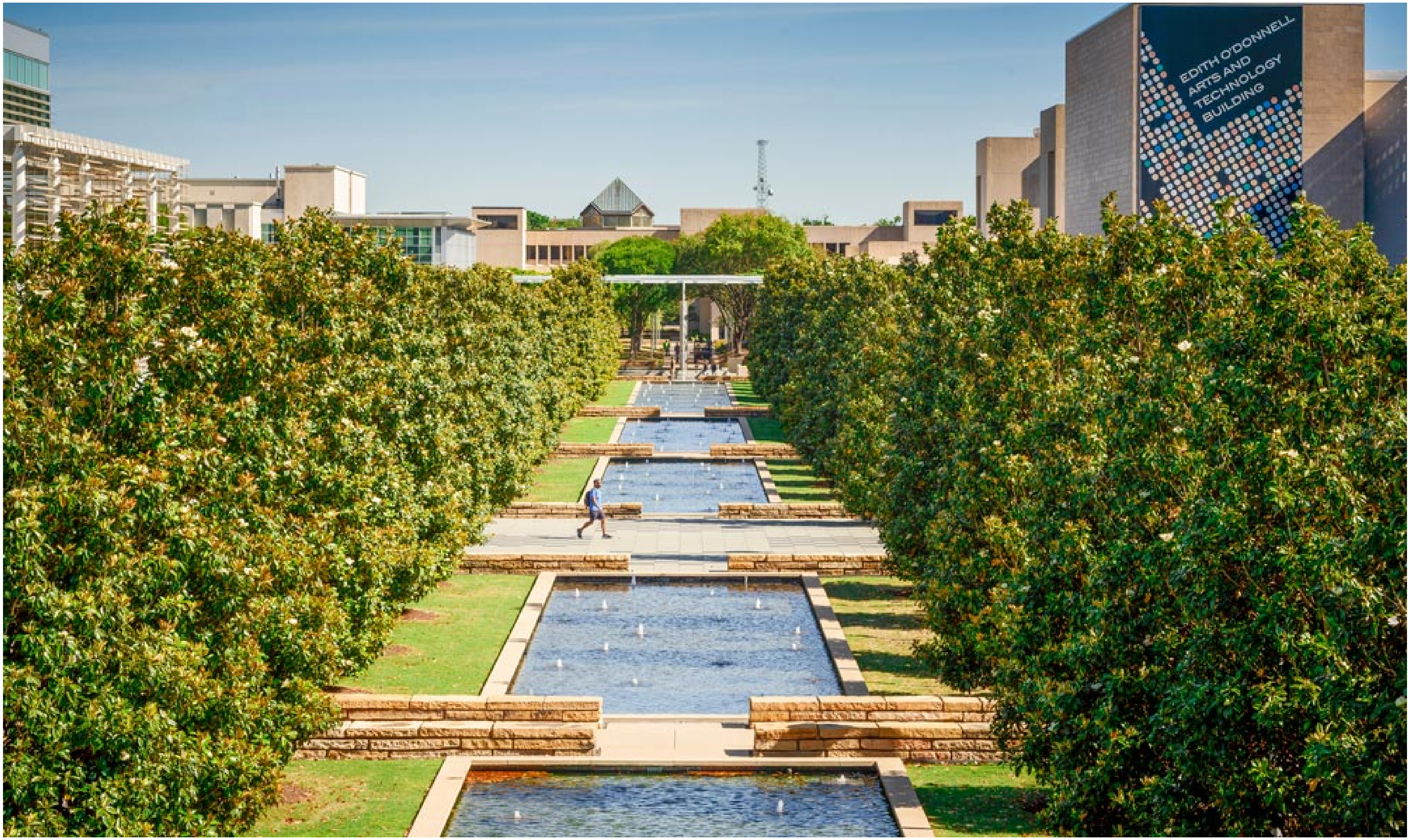

Thanks for the invitation!