Four-Manifolds,

Einstein Metrics, &

Differential Topology

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Stony Brook University

Rademacher Lectures
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Colloquium:

Einstein Metrics
and Geometrization

October 19, 2016
University of Pennsylvania

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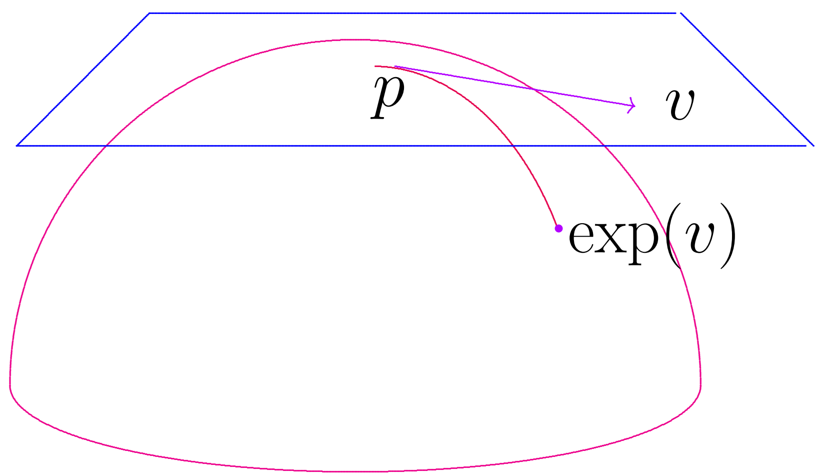

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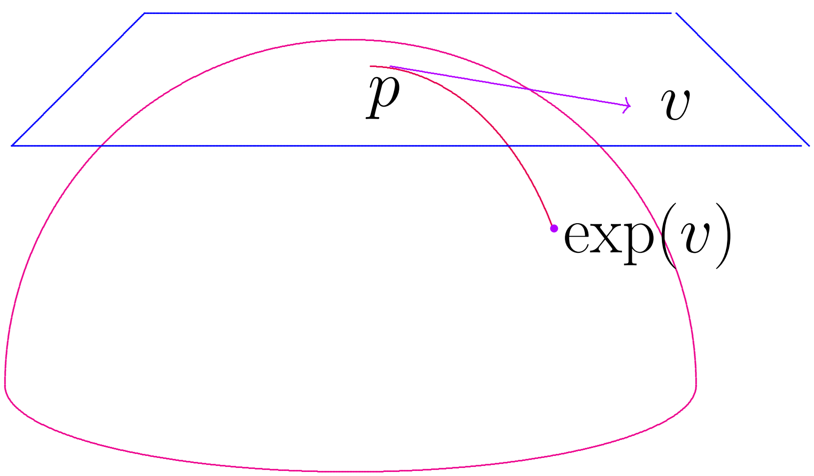

Now choosing $T_pM \xrightarrow{\cong} \mathbb{R}^n$ via some orthonormal basis gives us special coordinates on M.

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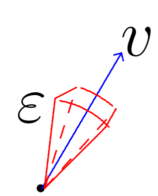
Volume of narrow cone



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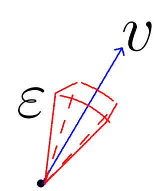
Volume of narrow cone vs. Euclidean expectation:

$$\frac{\text{vol}_g(C_\varepsilon(p, v, \Omega))}{\text{Euclidean answer}} \approx 1 - r(v, v)\, \frac{n\varepsilon^2}{6(n+2)} + O(\varepsilon^3)$$



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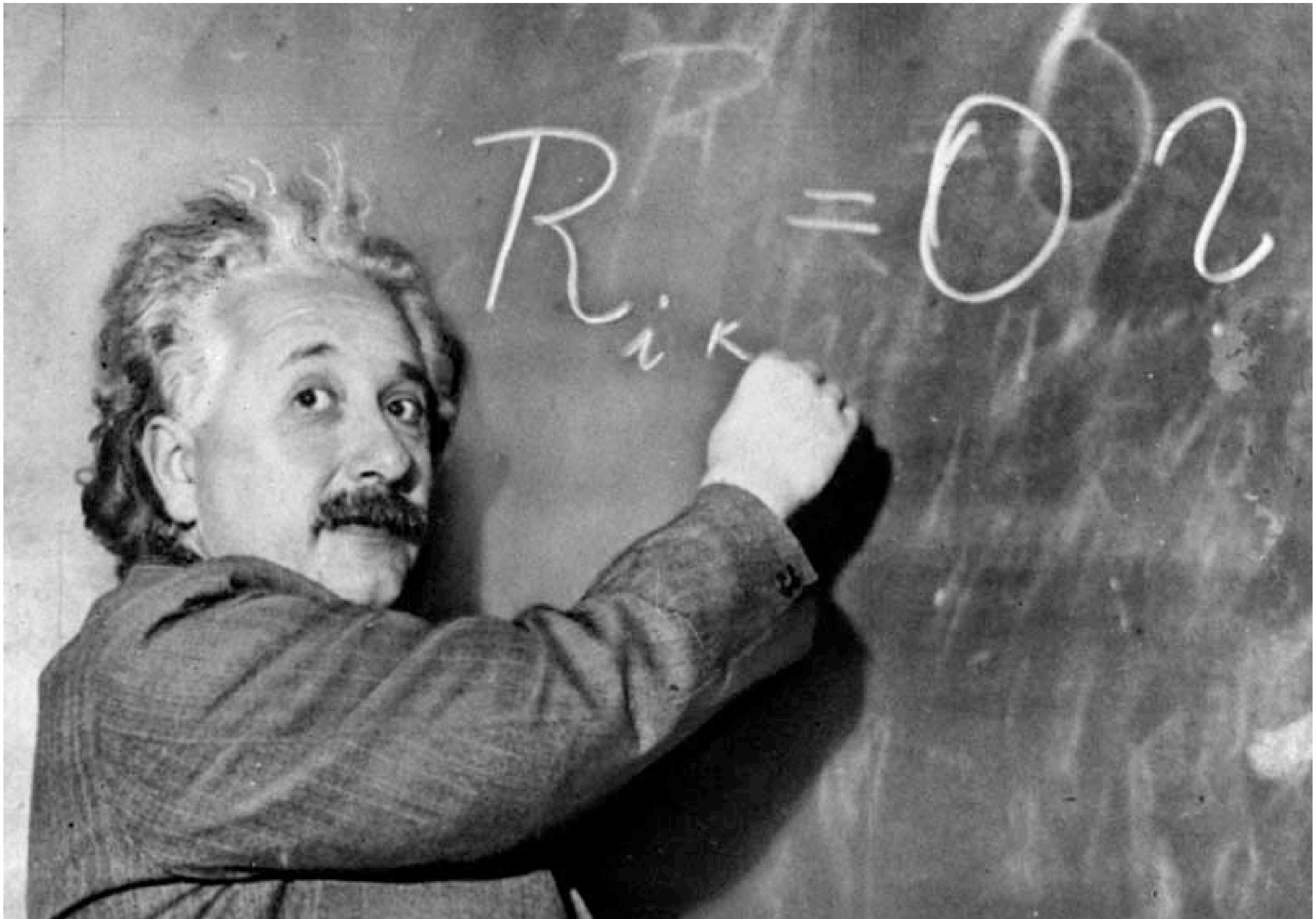

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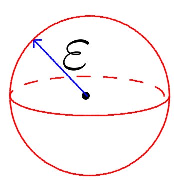
$$\frac{\mathrm{vol}_g(B_\varepsilon(p))}{v_n\ \varepsilon^n} = 1 - s(p)\ \frac{\varepsilon^2}{6(n+2)} + O(\varepsilon^4)$$



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because of "contracted Bianchi identity"

$$\nabla \cdot r = \nabla \frac{s}{2}.$$

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where

$$\mathring{r} := r - \frac{s}{n} g$$

is the trace-free Ricci tensor.

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"Mathematicians are like Frenchmen:
tell them something, they translate it into their own language, and, before you know it, it's something entirely different."

— J.W. von Goethe

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So why are we interested?

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$$\Delta x^j = 0 \Longrightarrow r_{jk} = \frac{1}{2}\Delta g_{jk} + \ell ots.$$

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Yamabe, 1960: New angle on variational problem.

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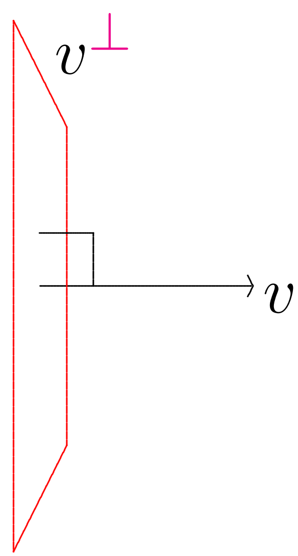

$T_x M$

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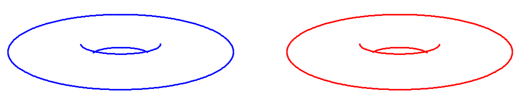

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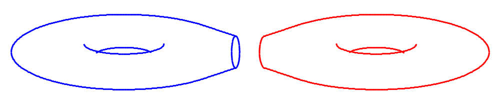

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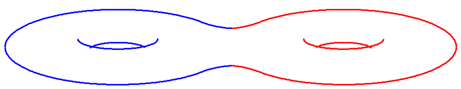

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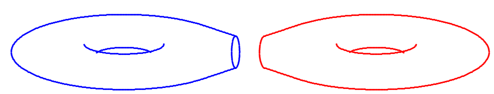

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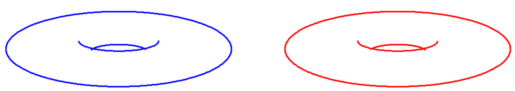

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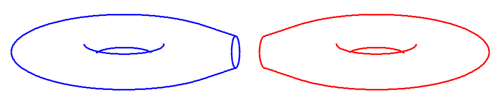

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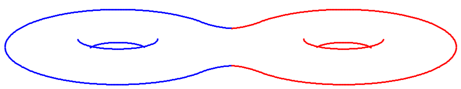

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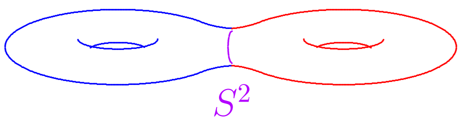

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Ricci flow

$$\frac{\partial}{\partial t} g_{jk} = -2 r_{jk}$$

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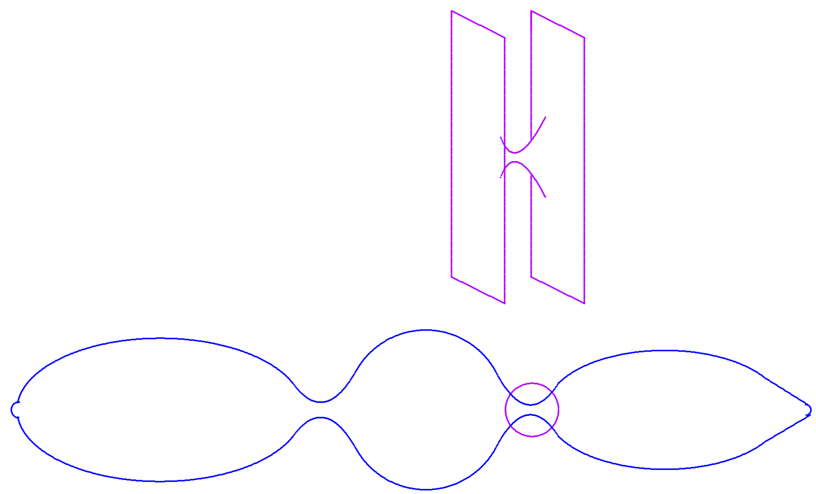

Ricci flow pinches off S^2 necks.

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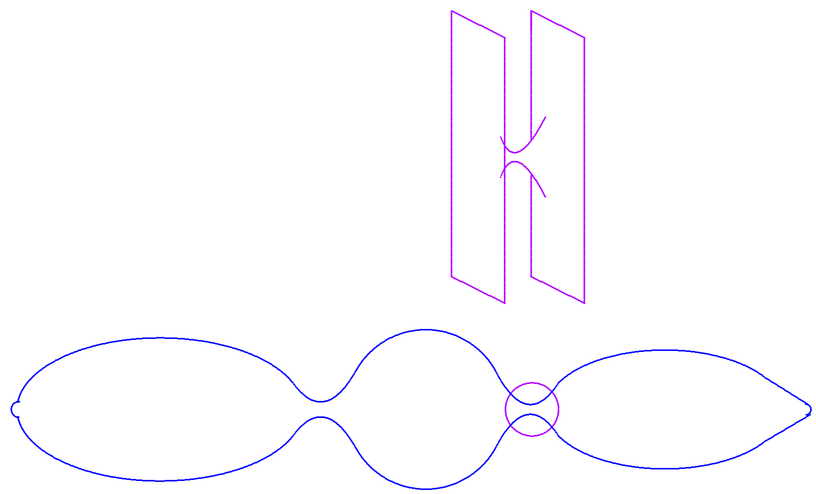
$\Longrightarrow$ If M^3 carries Einstein metric, $\pi_2(M) = 0$.

$\Longrightarrow$ Existence obstructed for connect sums $M^3 \# N^3$.



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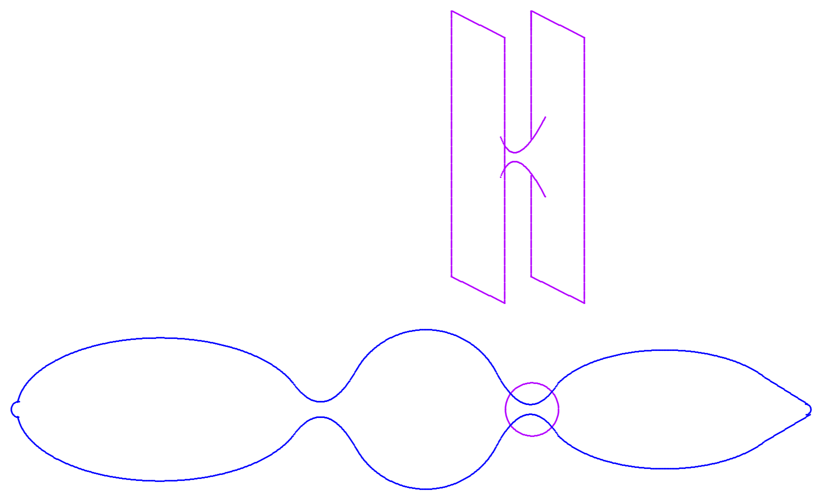

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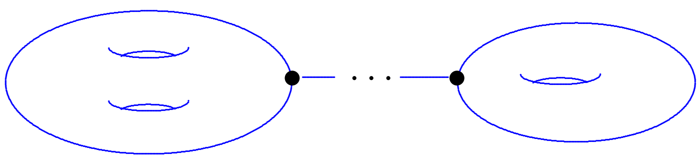

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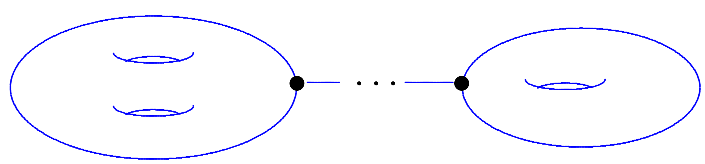

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In fact, typically a point! (Mostow rigidity)

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Similarly: "most" simply-connected spin 5-manifolds.

(Boyer-Galicki, Kollár, Van Coevering)

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$\therefore$ Not a meaningful geometrization of manifold!

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(Terminology to be explained later!)

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By contrast, high-dimensional Einstein metrics too common; have little to do with geometrization.

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$$\star^2 = 1.$$

Λ^+ self-dual 2-forms.
Λ^- anti-self-dual 2-forms.

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s = scalar curvature

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W_- = anti-self-dual Weyl curvature ″

Thus (M^4, g) Einstein $\Longleftrightarrow$

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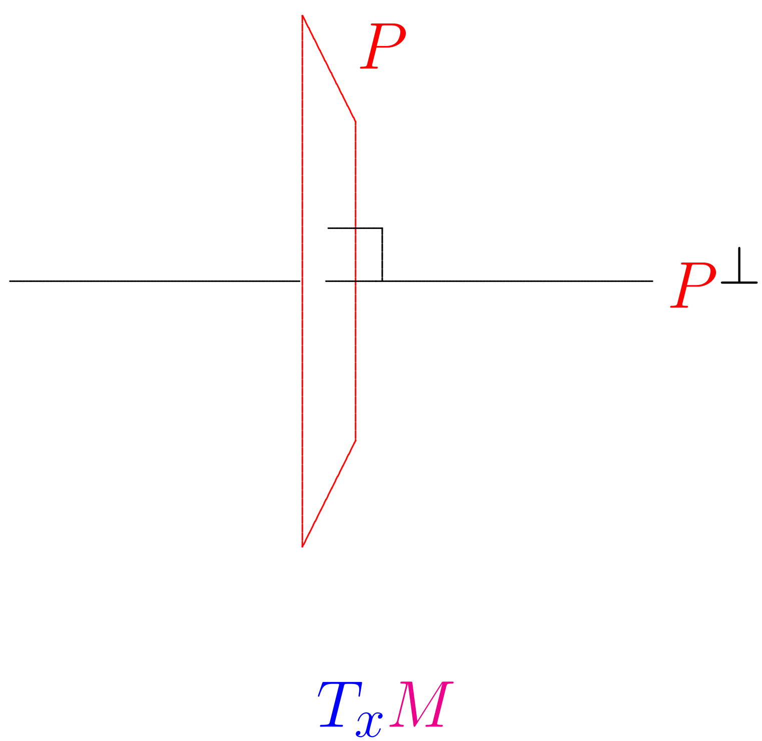

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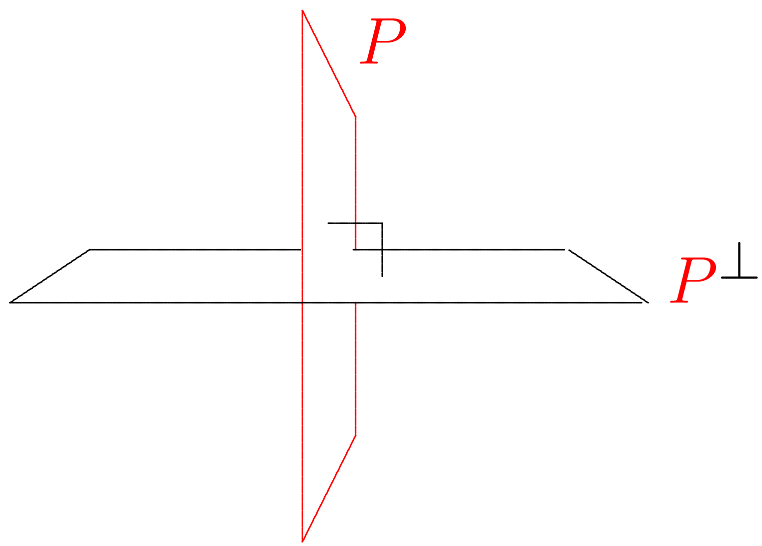

$T_x M$

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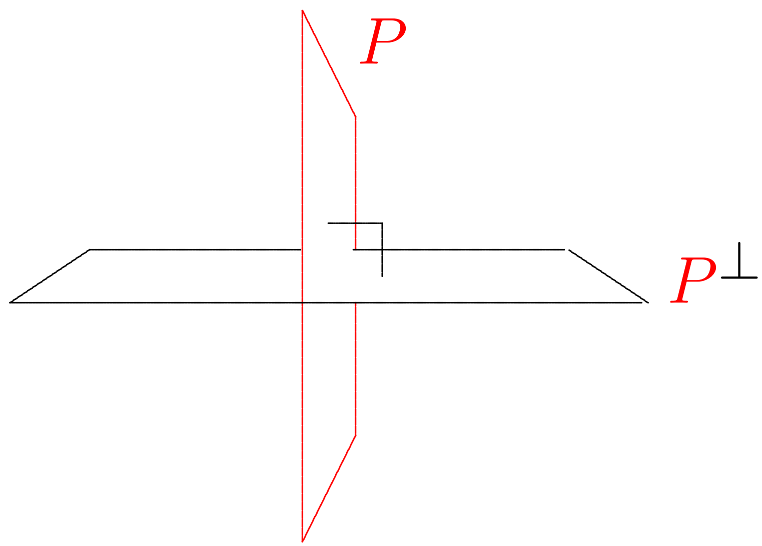

$$K(P) = K(P^{\perp})$$

(M, g) compact oriented Riemannian.

4-dimensional Gauss-Bonnet formula

$$\chi(M) = \frac{1}{8\pi^2} \int_M \left(\frac{s^2}{24} + \qquad \qquad \qquad \qquad \right) d\mu$$

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for Euler-characteristic $\chi(M) = \sum_j (-1)^j b_j(M)$.

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for Euler-characteristic $\chi(M) = 2 - 2b_1 + b_2$.

4-dimensional Hirzebruch signature formula

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For (M^4, g) compact oriented Riemannian,

Euler characteristic

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No diffeomorphism classification currently known!

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- *they have the same Euler characteristic* χ*;*
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$$w_2 = 0 \qquad\qquad w_2 \neq 0$$

Warning: "Exotic differentiable structures!"

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Typically, one homeotype $\longleftrightarrow$ ∞ many diffeotypes.

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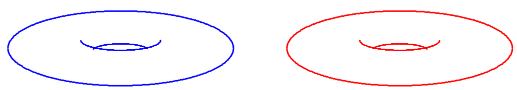

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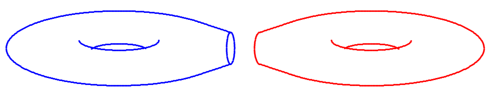

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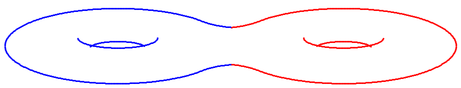

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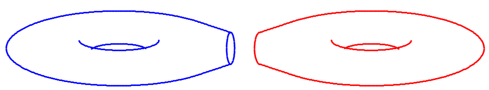

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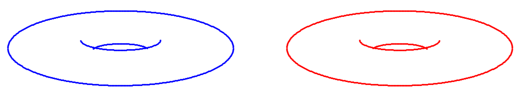

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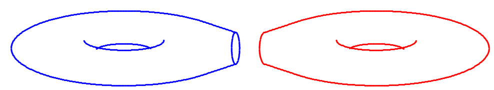

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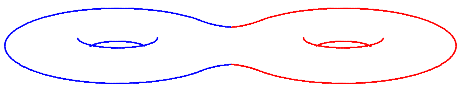

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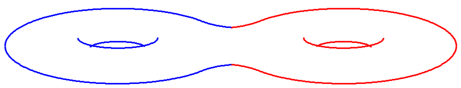

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—André Weil, 1958

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Spin, $\chi = 24$, $\tau = -16$.

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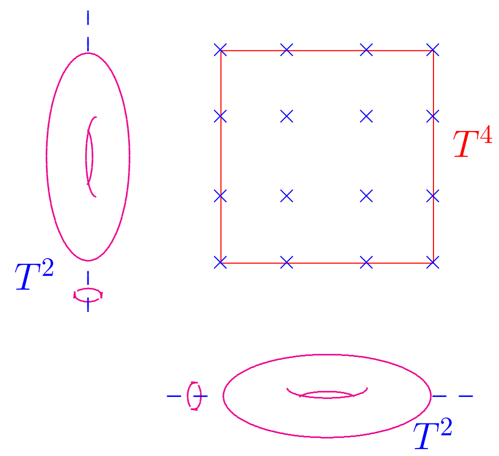

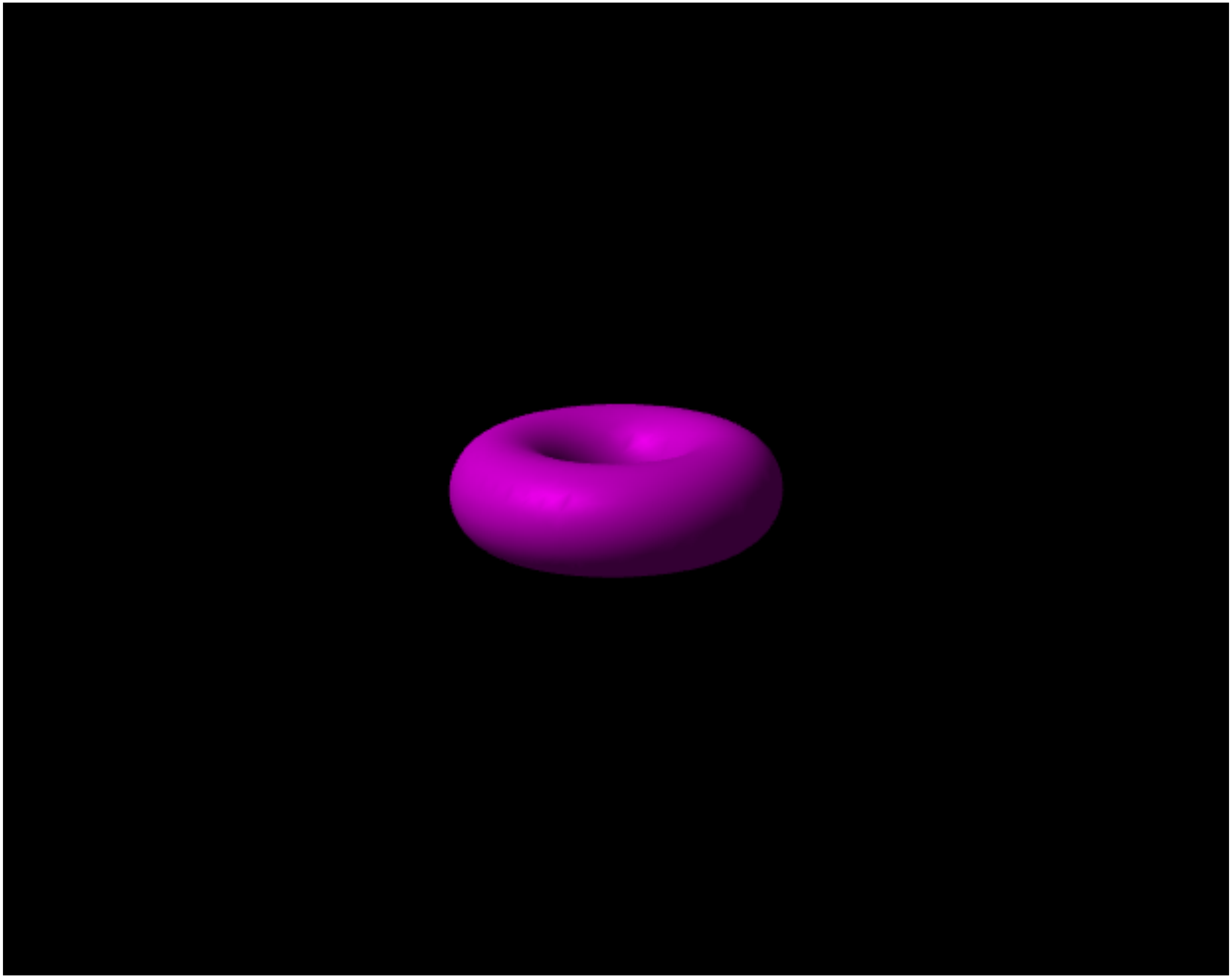

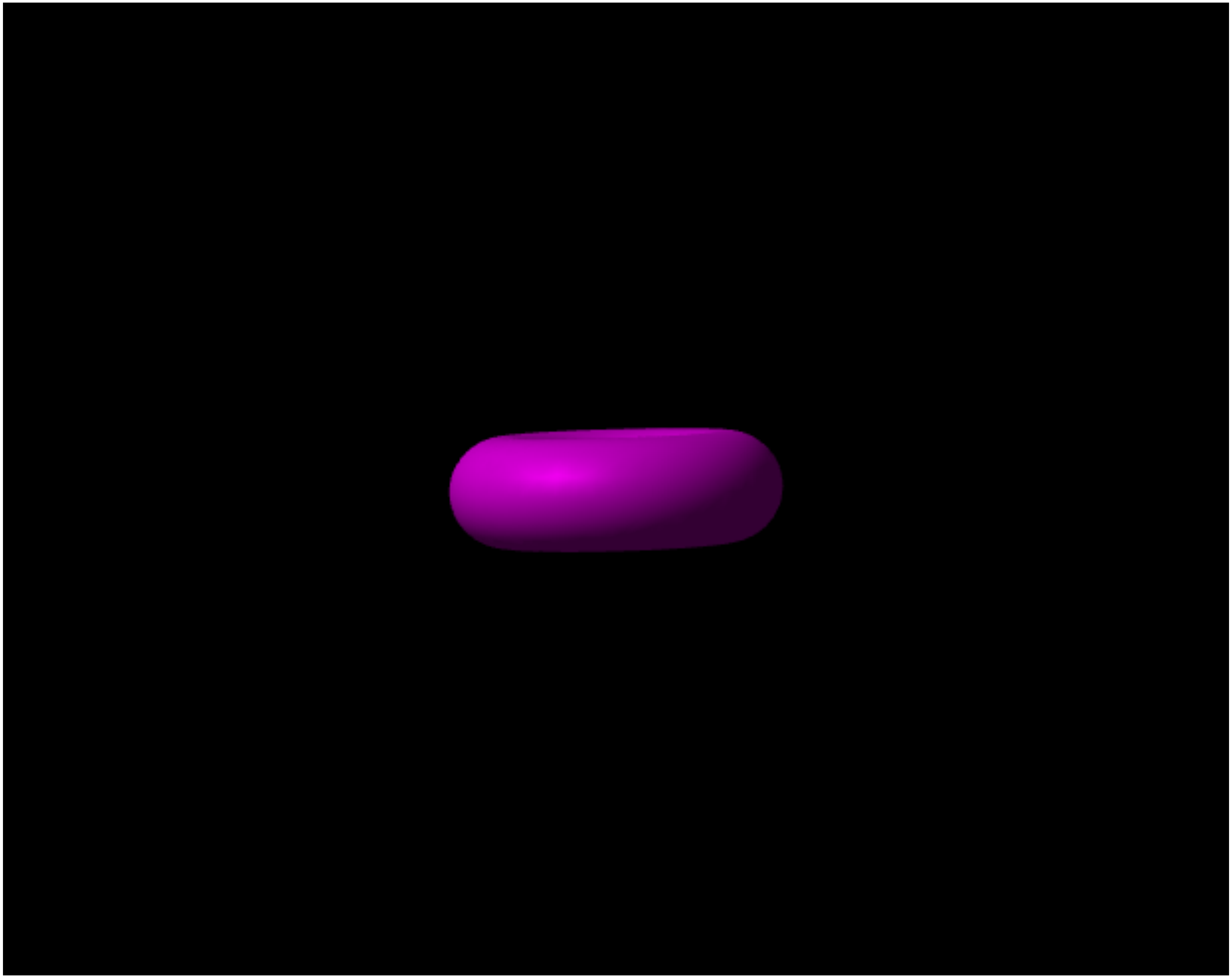

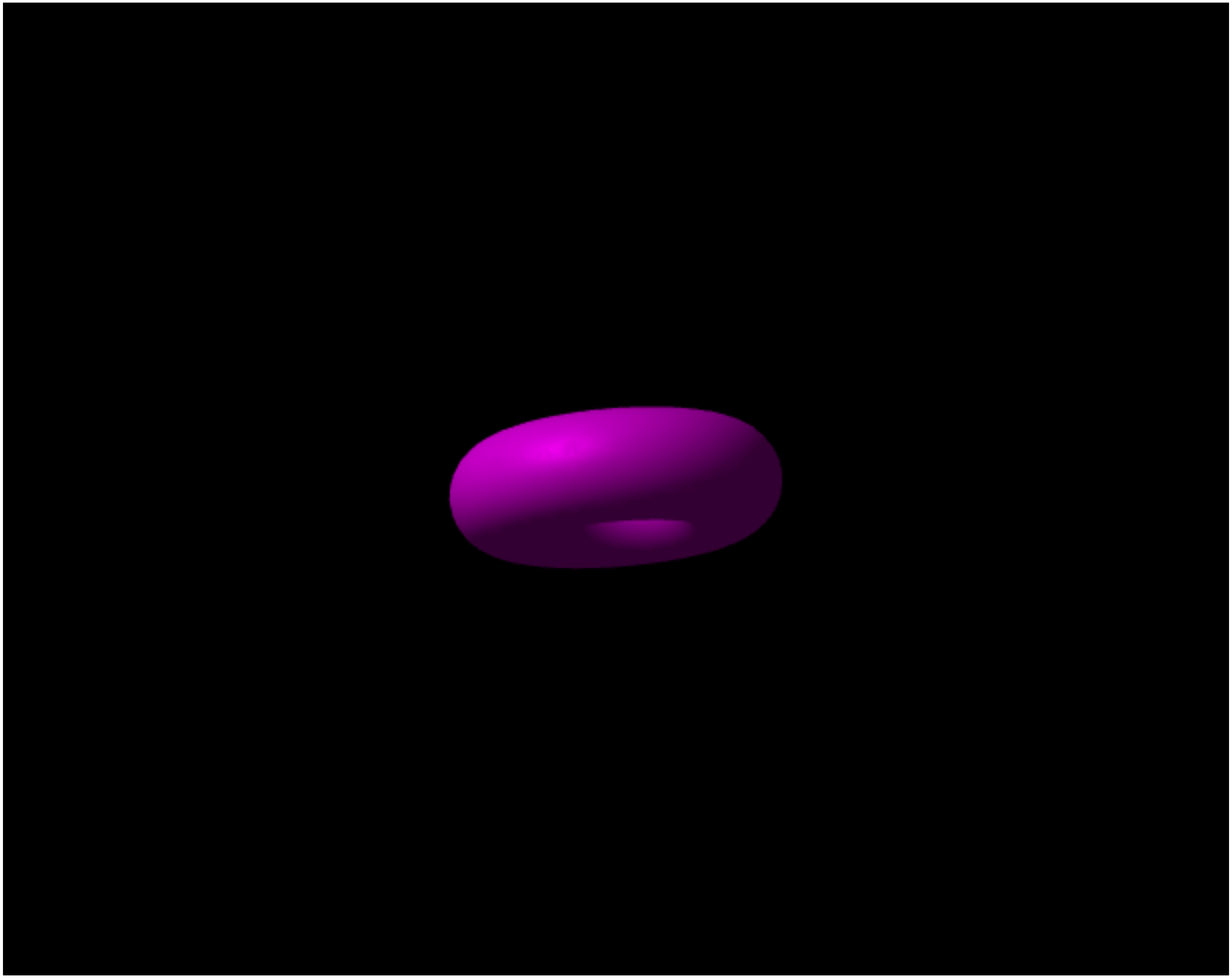

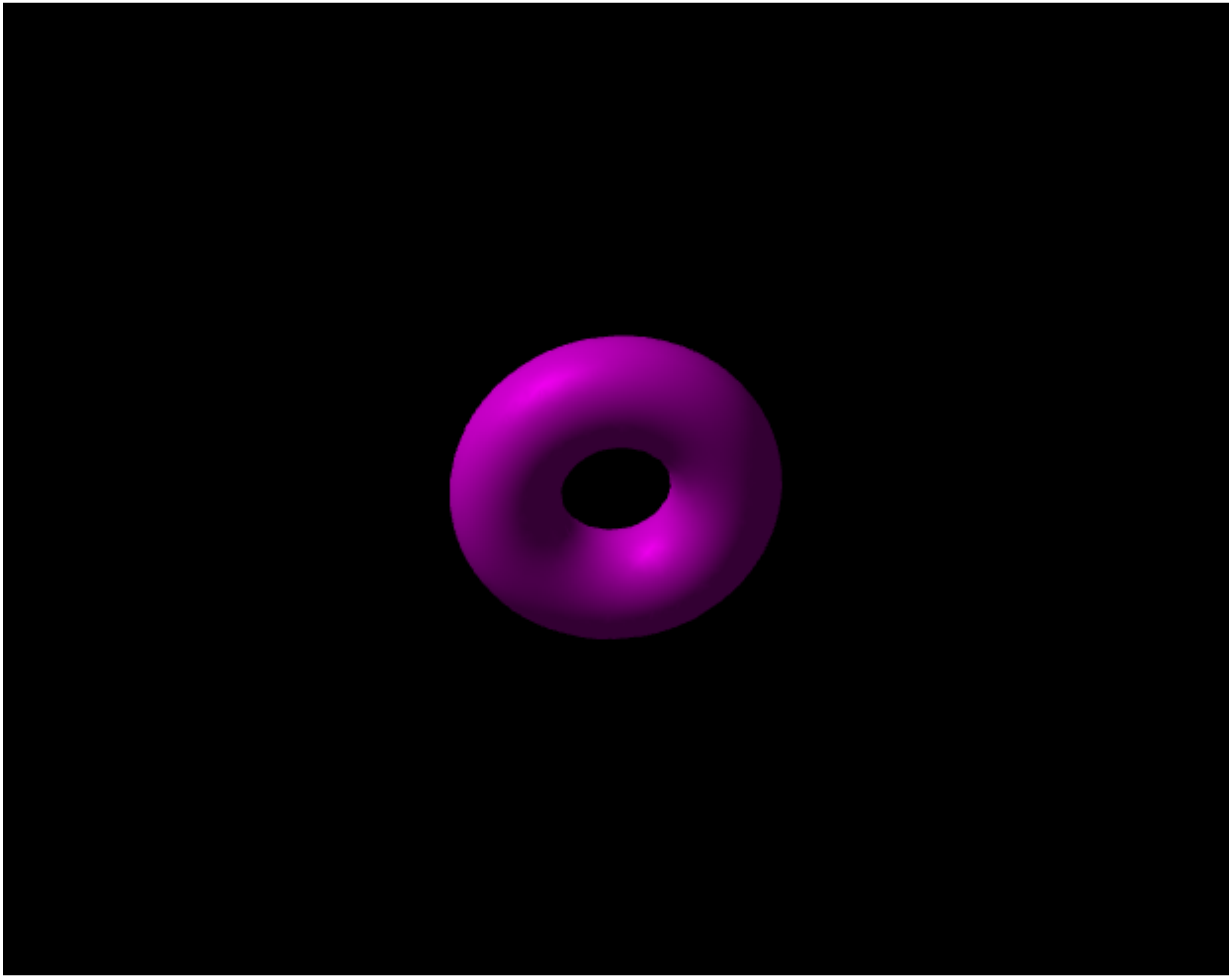

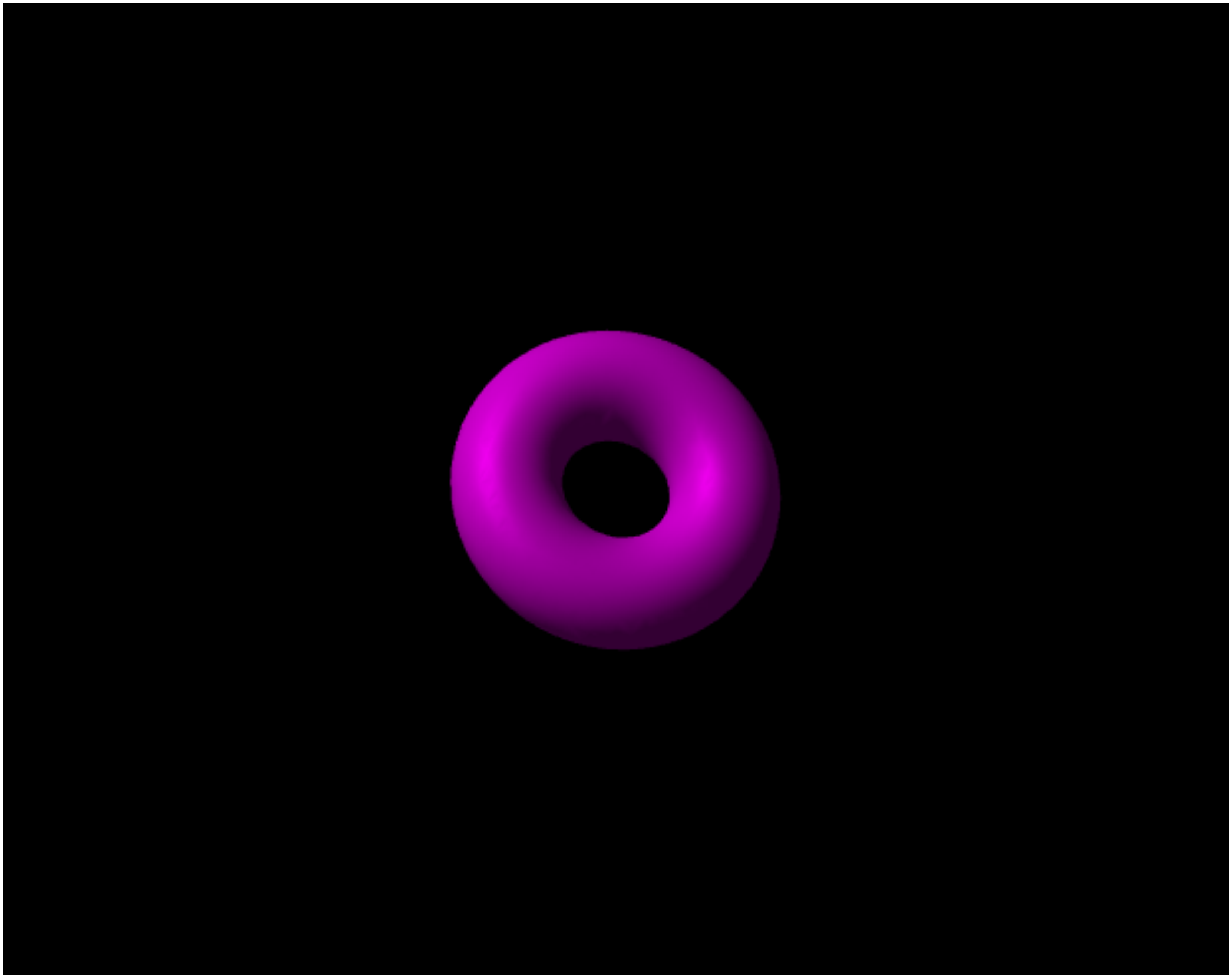

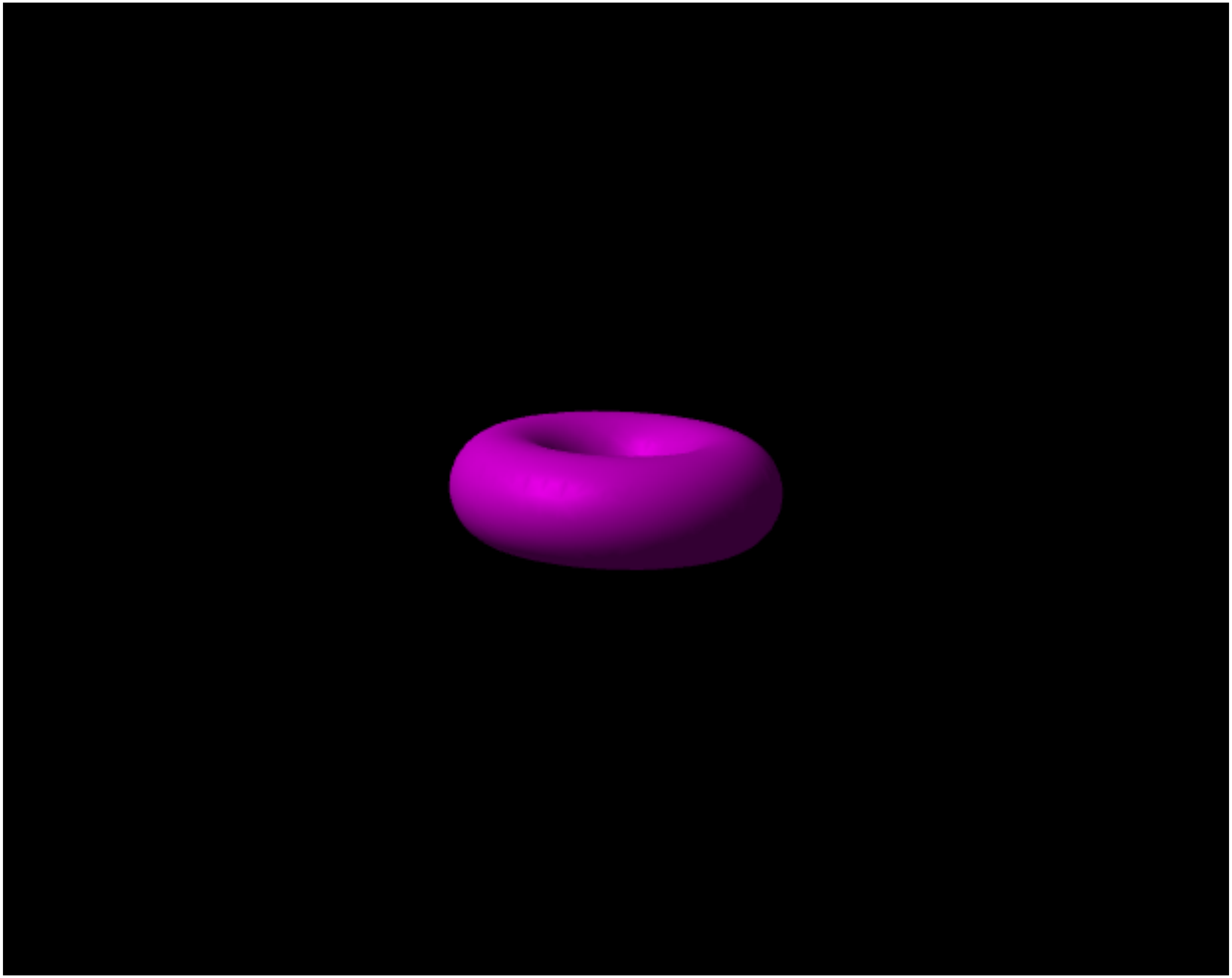

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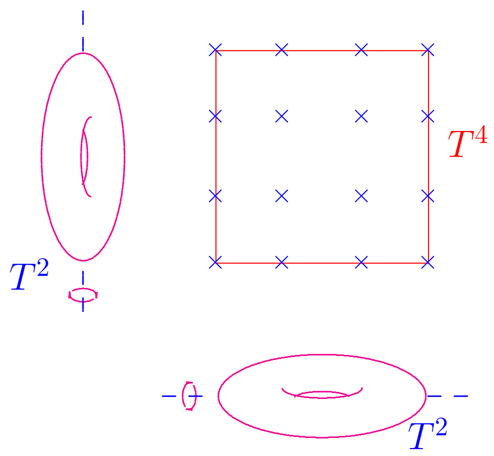

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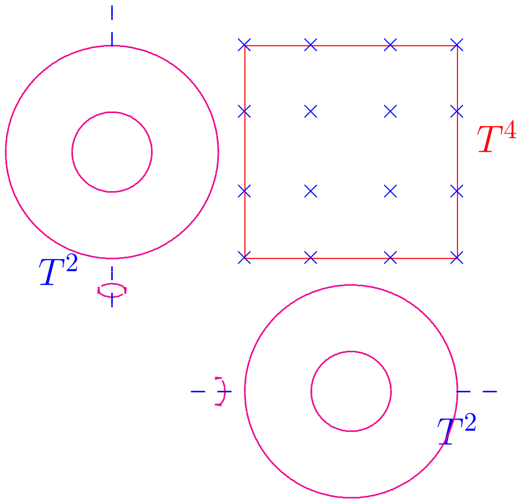

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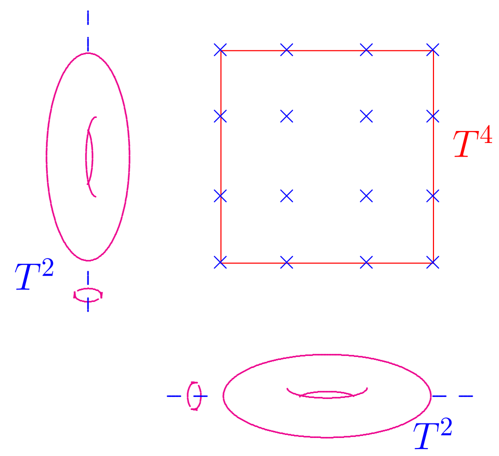

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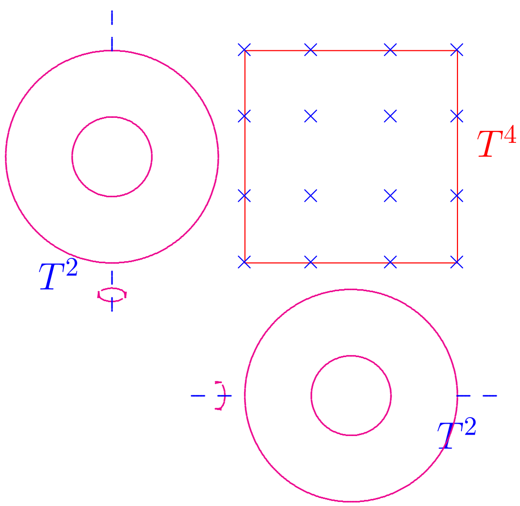

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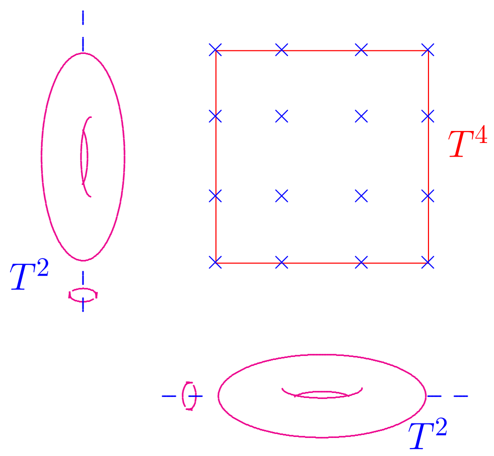

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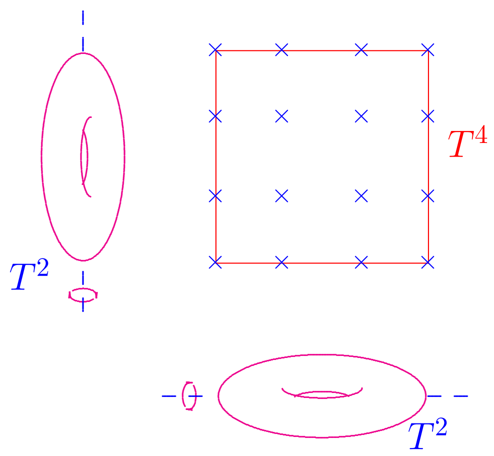

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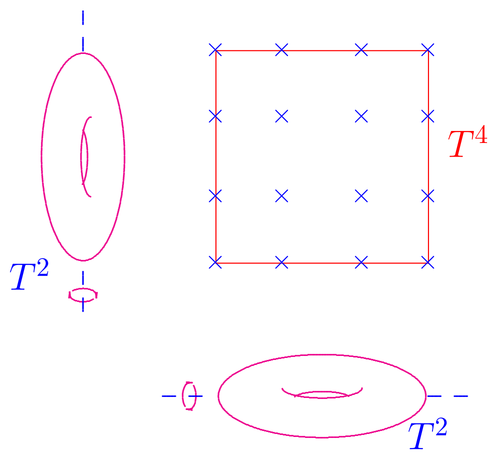

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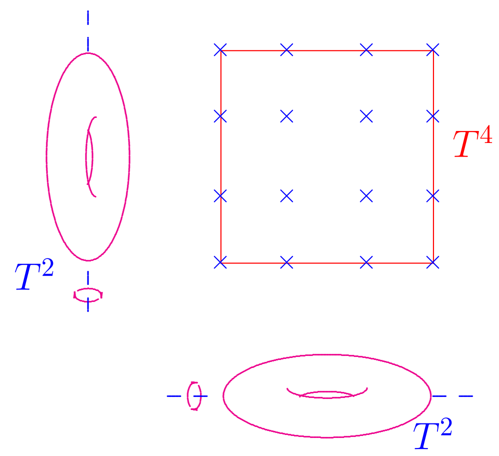

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T^4 = Picard torus of curve of genus 2.

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Certainly true of all examples in these lectures!

Question. *Which smooth compact* 4*-manifolds* M^4 *admit Einstein metrics?*

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Question. *In dimension four, how unique are Einstein metrics, when they exist?*

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However, $\chi(M) > 0$ if M^4 is simply connected…

$$(2\chi + 3\tau)(M) = \frac{1}{4\pi^2}\int_M \left(\frac{s^2}{24} + 2|W_+|^2 - \frac{|\mathring{r}|^2}{2}\right) d\mu_g$$

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M simply connected $\rightsquigarrow$ simply connected examples.

Hitchin-Thorpe Inequality:

$$(2\chi + 3\tau)(M) = \frac{1}{4\pi^2}\int_M \left(\frac{s^2}{24} + 2|W_+|^2 - \frac{|\mathring{r}|^2}{2}\right) d\mu_g$$

Einstein $\Rightarrow$ $$= \frac{1}{4\pi^2}\int_M \left(\frac{s^2}{24} + 2|W_+|^2\right) d\mu_g$$

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Curvature Λ^+ Curvature Λ^-

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(M^n, g): holonomy

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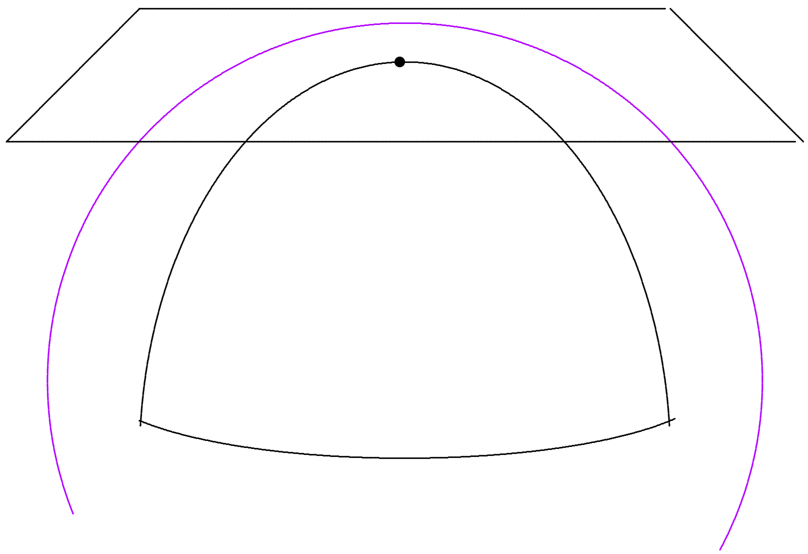
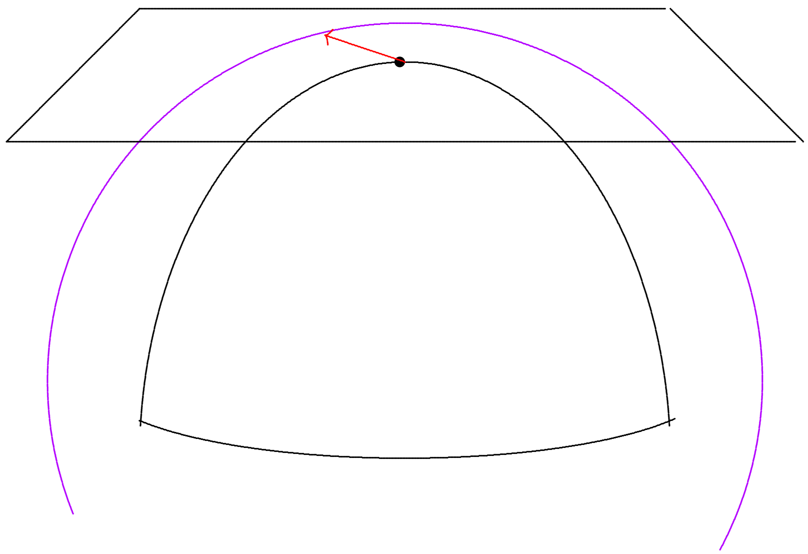


(M^n, g):

holonomy



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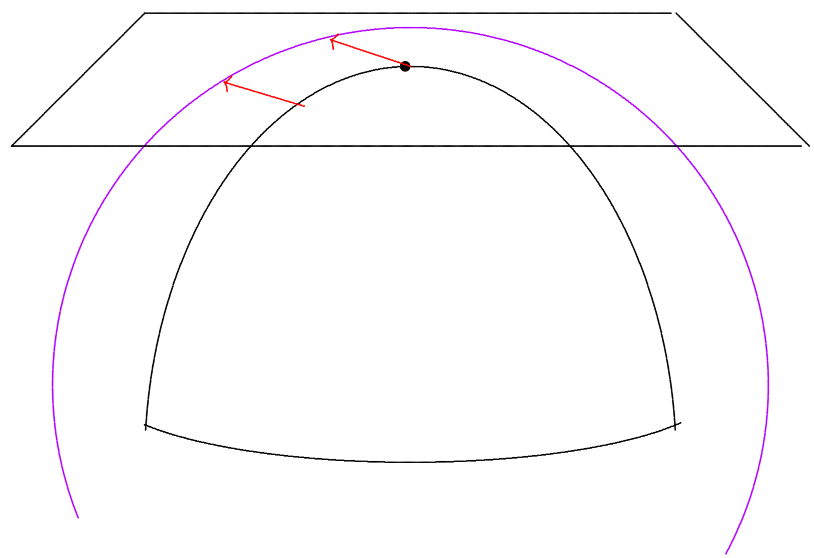

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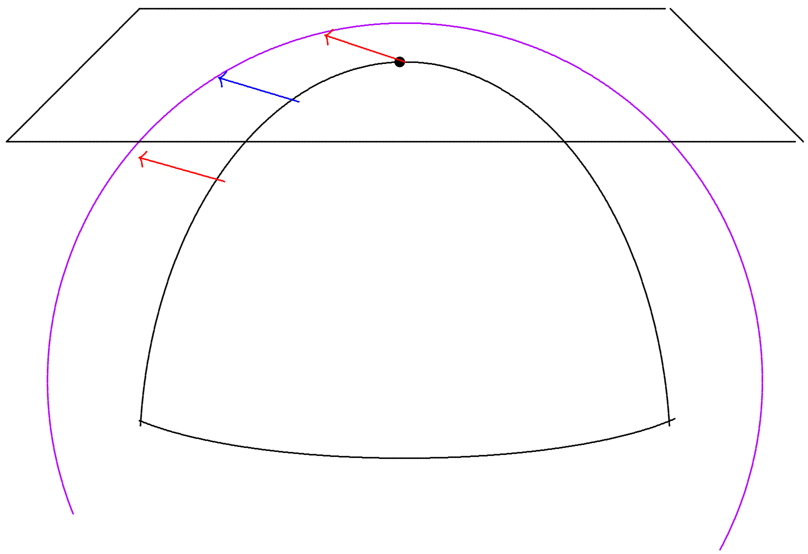

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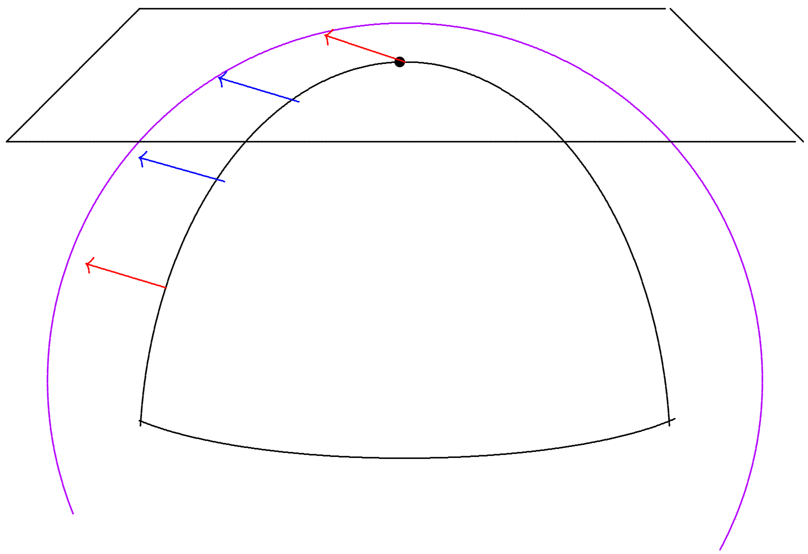

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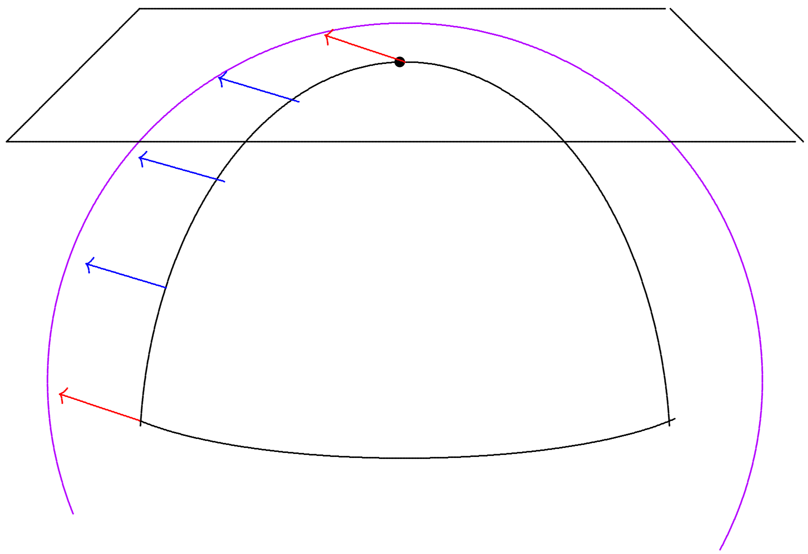

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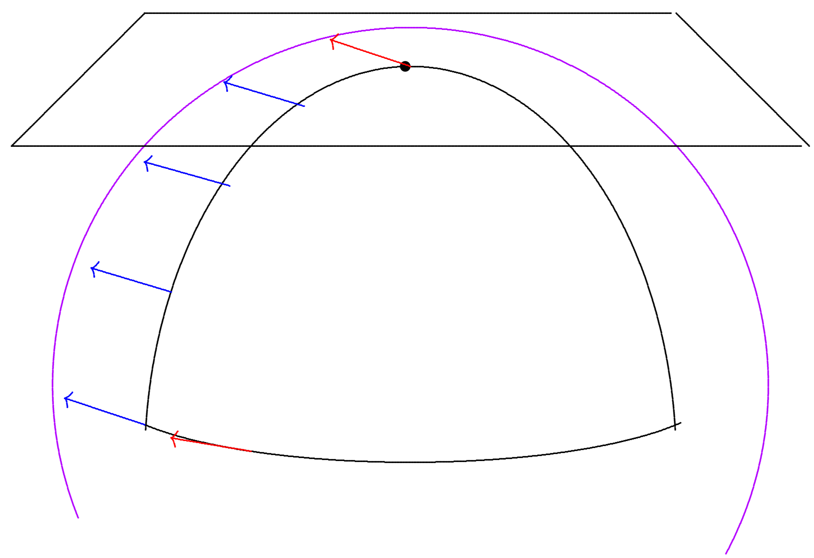

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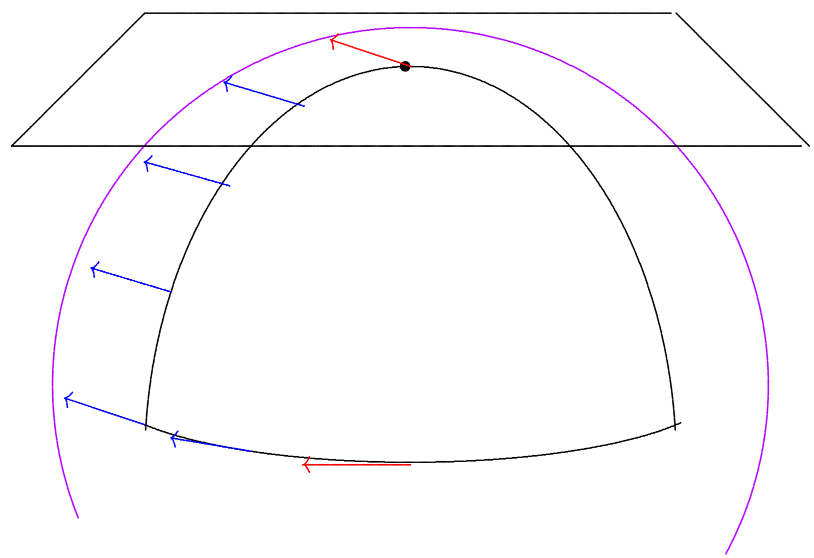

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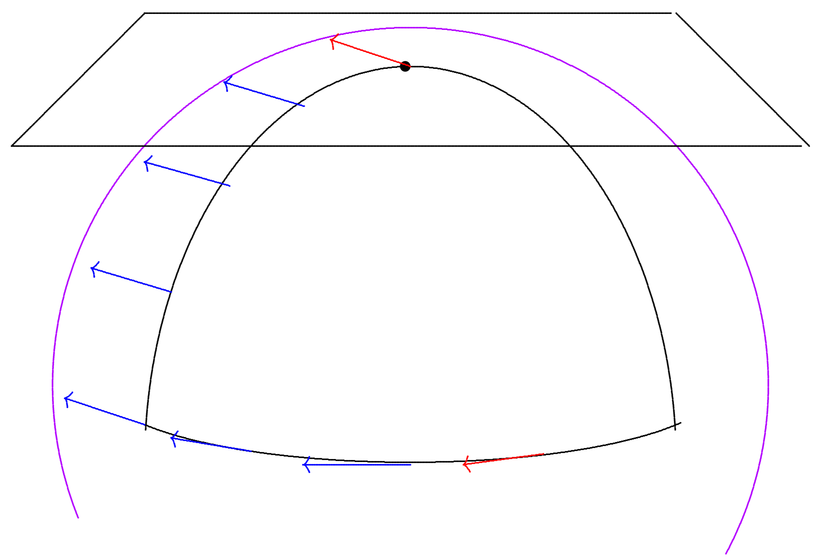

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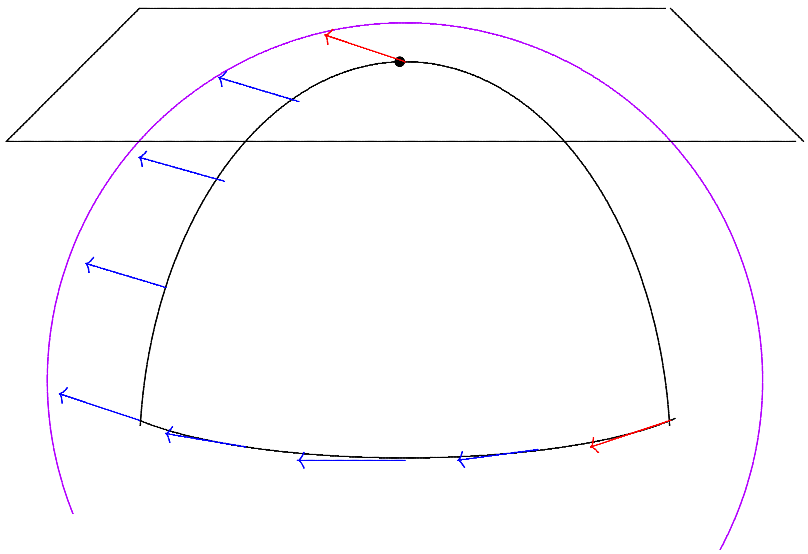

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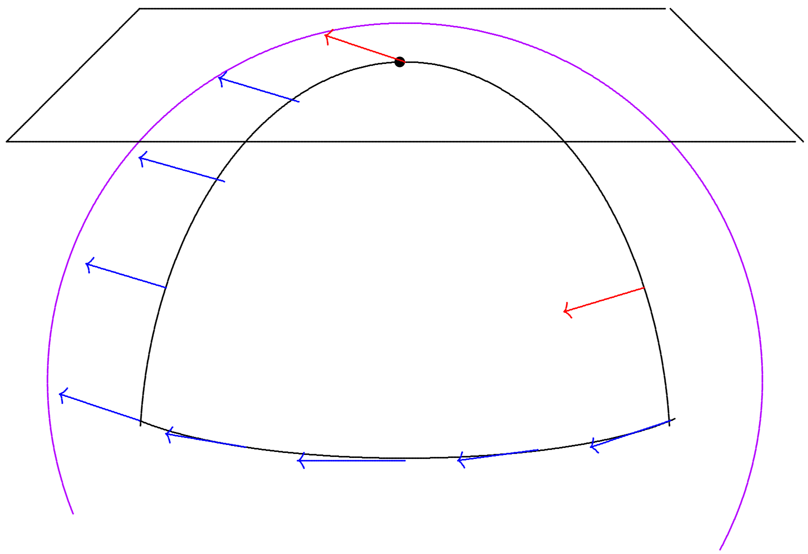

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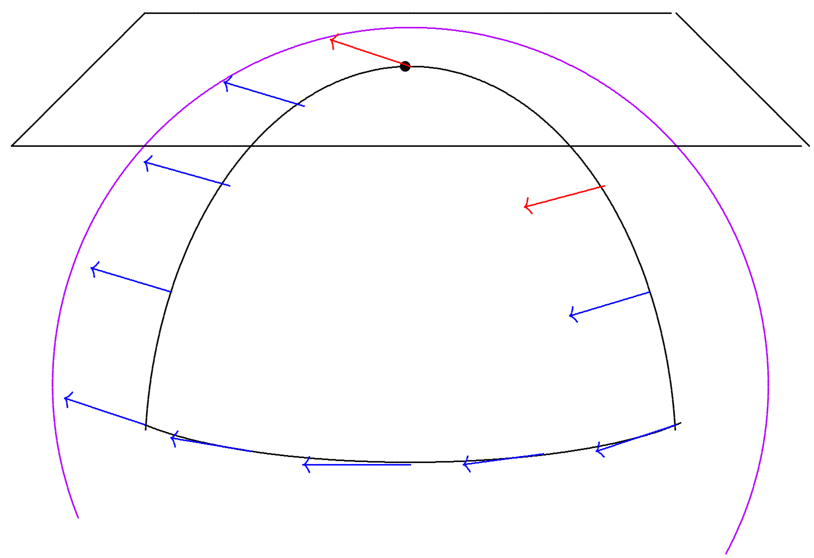

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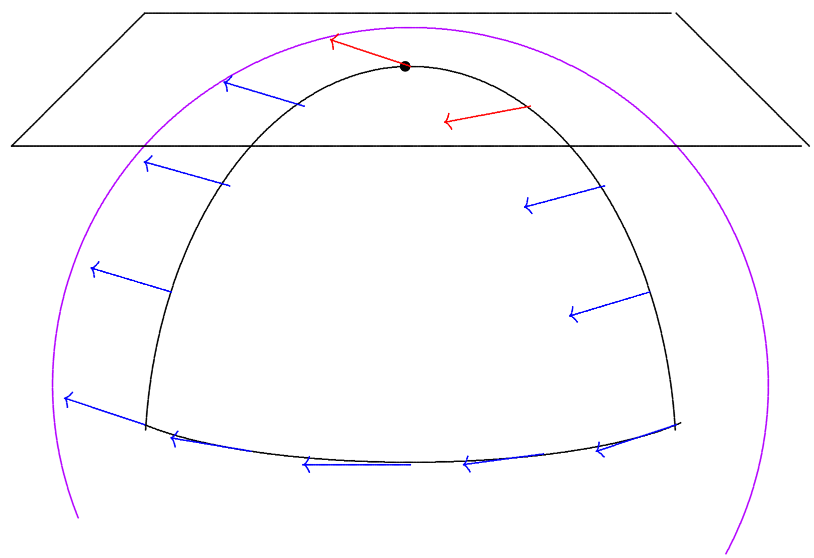

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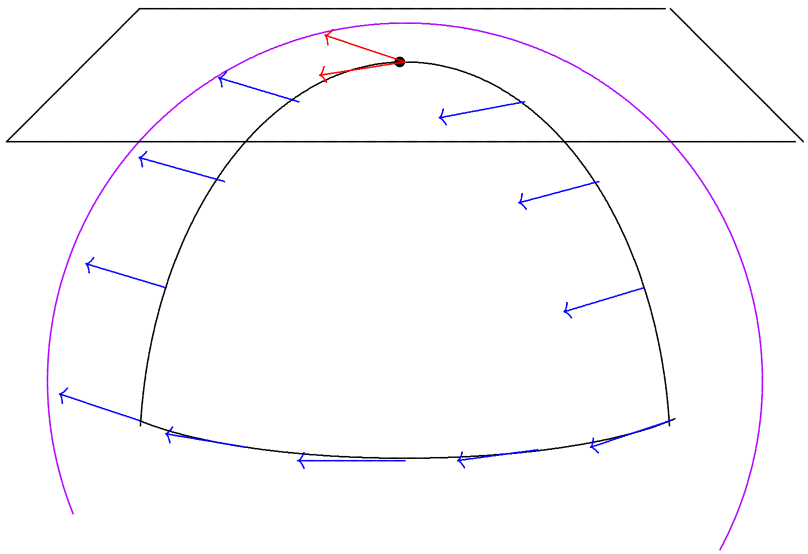

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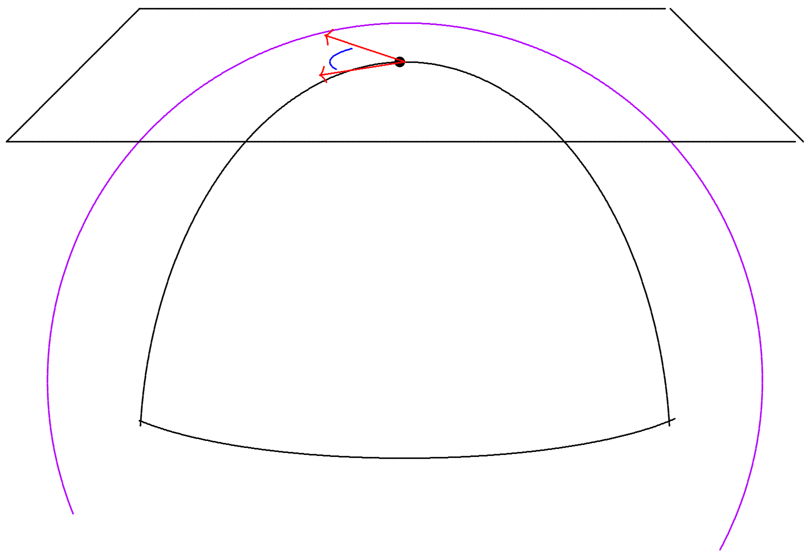
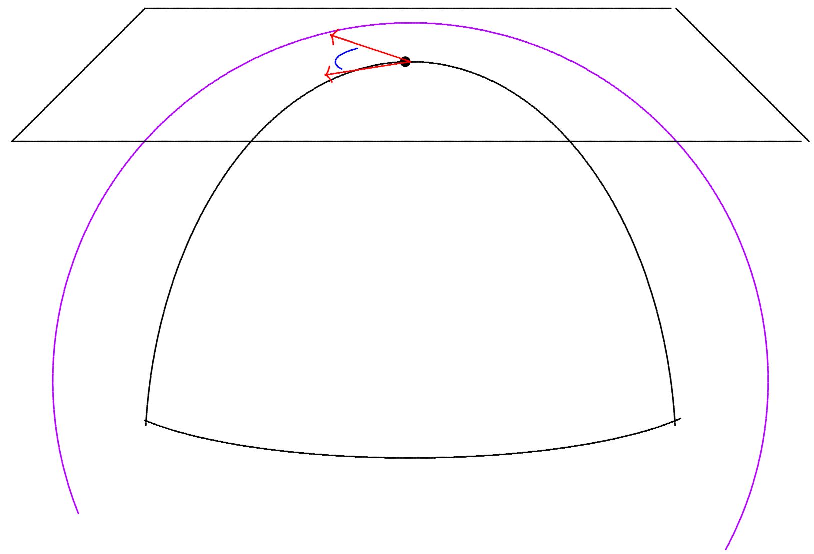
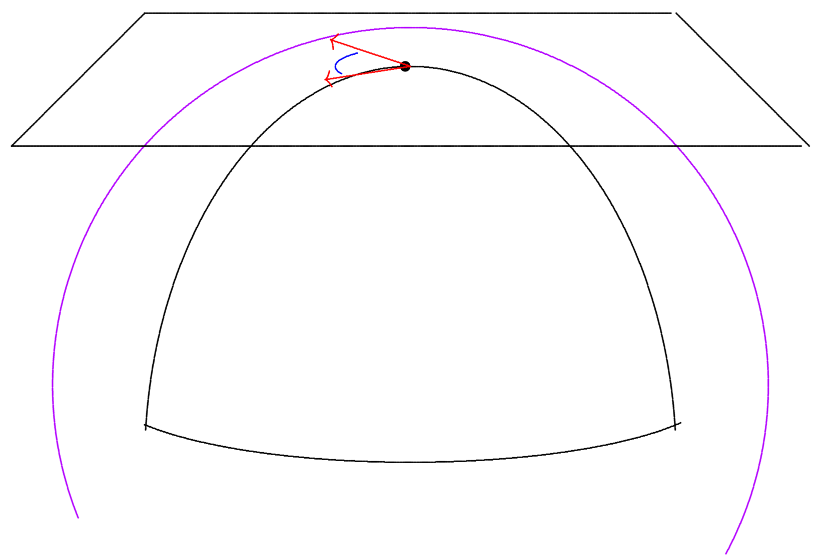


(M^n, g): holonomy $\subset \mathbf{O}(n)$



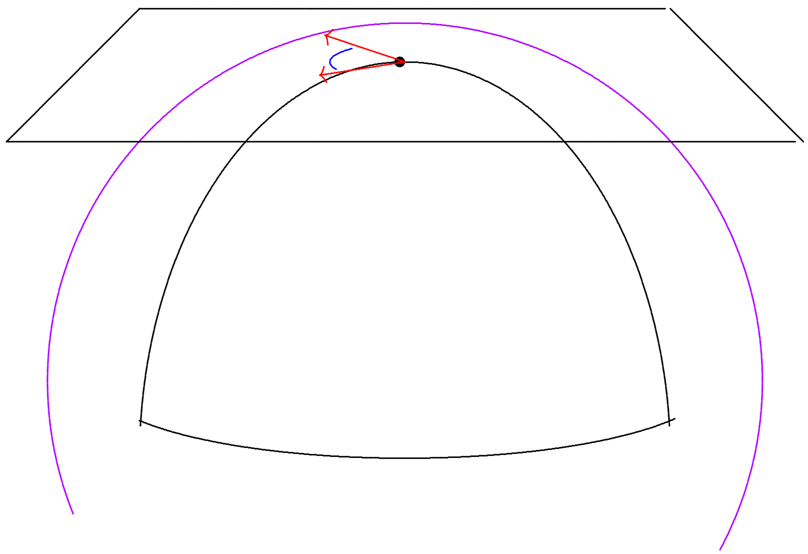
Kähler metrics:

(M^{2m}, g): holonomy



Kähler metrics:

(M^{2m}, g) Kähler $\Longleftrightarrow$ holonomy $\subset \mathbf{U}(m)$



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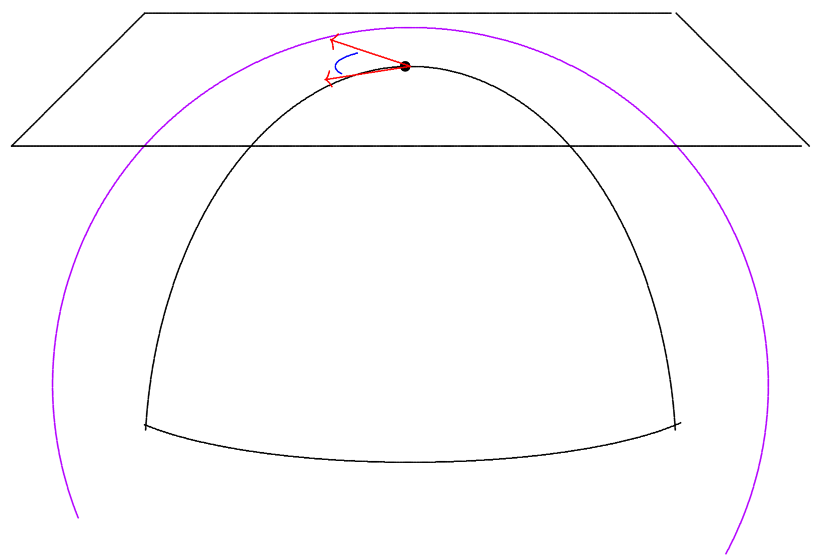

$$\mathbf{U}(m) := \mathbf{O}(2m) \cap \mathbf{GL}(m, \mathbb{C})$$

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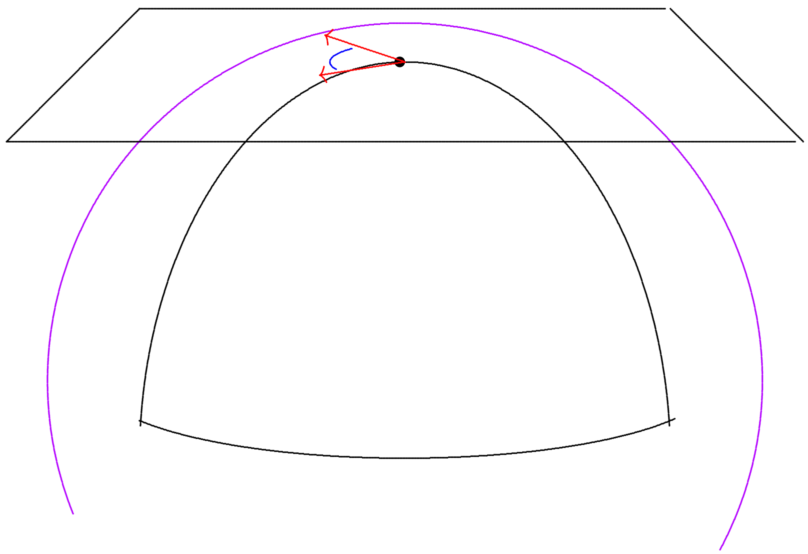

Makes tangent space a complex vector space!

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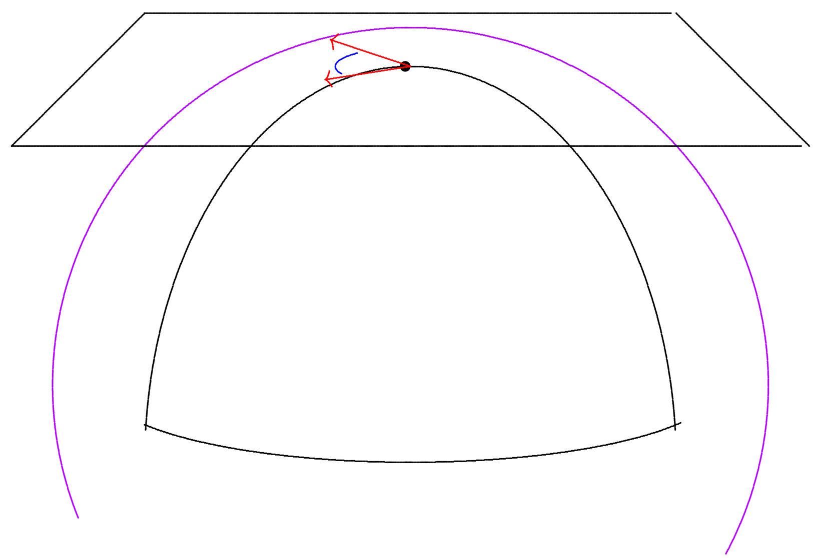

Makes tangent space a complex vector space!

$$J : TM \to TM, \qquad J^2 = -\text{identity}$$

"almost-complex structure"

Kähler metrics:

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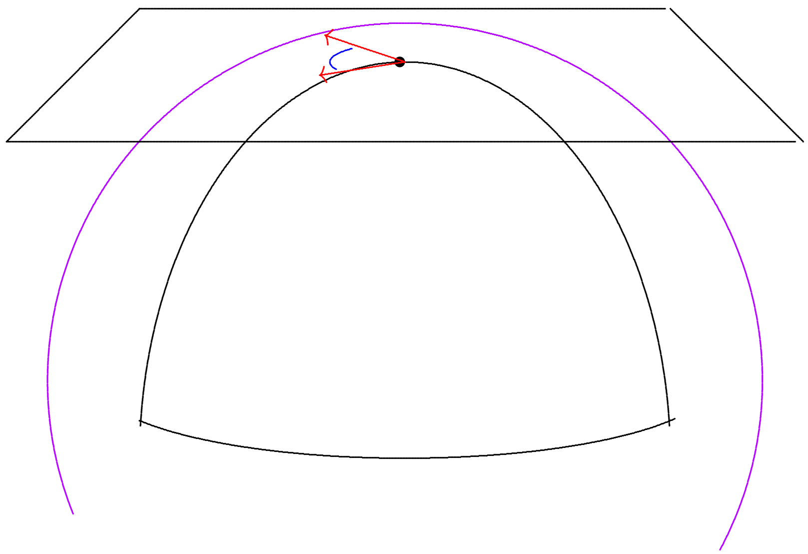

Makes tangent space a complex vector space!

Invariant under parallel transport!

Kähler metrics:

(M^{2m}, g) Kähler $\Longleftrightarrow$ holonomy $\subset \mathbf{U}(m)$

$\Longleftrightarrow$ $\exists$ almost complex-structure J with $\nabla J = 0$ and $g(J\cdot, J\cdot) = g$.

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$$d\omega = 0$$

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$$[\omega] \in H^2(M)$$

"Kähler class"

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$\iff$ In local complex coordinates $(z^1, \ldots, z^m)$,

$$g = -\sum_{j,k=1}^{m} \frac{\partial^2 f}{\partial z^j \partial \bar{z}^k} \left[dz^j \otimes d\bar{z}^k + d\bar{z}^k \otimes dz^j \right]$$

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$$\omega = i \sum_{j,k=1}^{m} \frac{\partial^2 f}{\partial z^j \partial \bar{z}^k}\, dz^j \wedge d\bar{z}^k$$

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Kähler magic:

$$r = -\sum_{j,k=1}^{m} \frac{\partial^2}{\partial z^j \partial \bar{z}^k} \log \det[g_{p\bar{q}}] \left[dz^j \otimes d\bar{z}^k + d\bar{z}^k \otimes dz^j\right]$$

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Kähler magic:

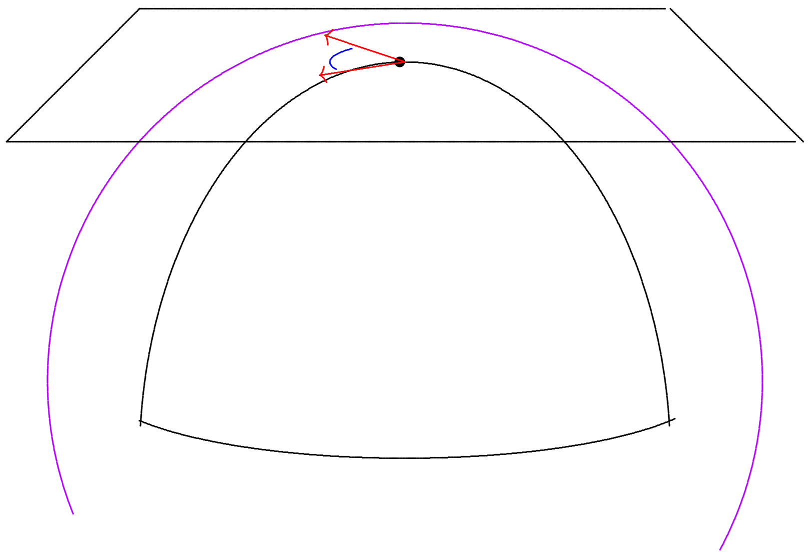
If we define the Ricci form by

$$\rho = r(J\cdot, \cdot)$$

then $i\rho$ is curvature of canonical line bundle $\Lambda^{m,0}$.

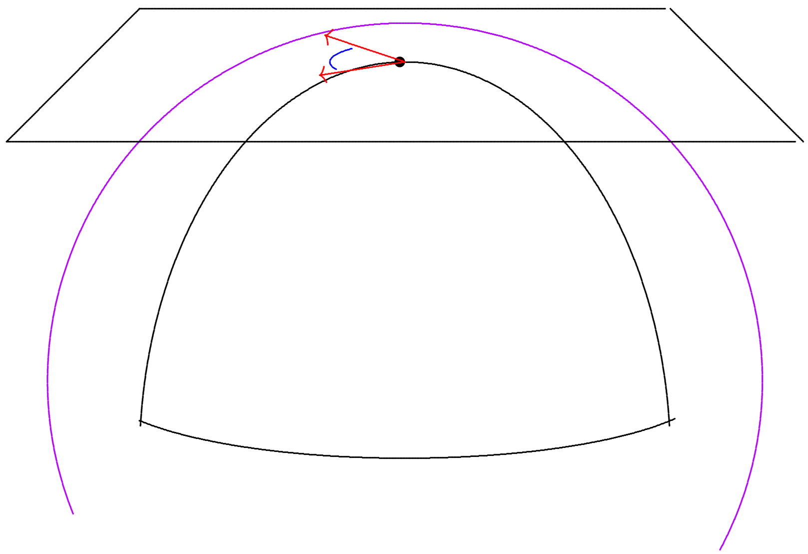
Kähler metrics:

(M^{2m}, g): holonomy



Kähler metrics:

(M^{2m}, g): Ricci-flat Kähler $\Longleftarrow$ holonomy $\subset \mathbf{SU}(m)$



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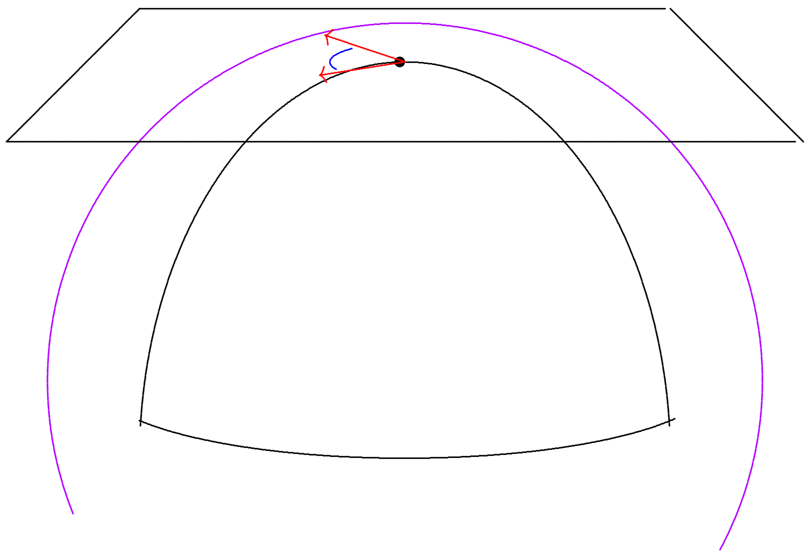
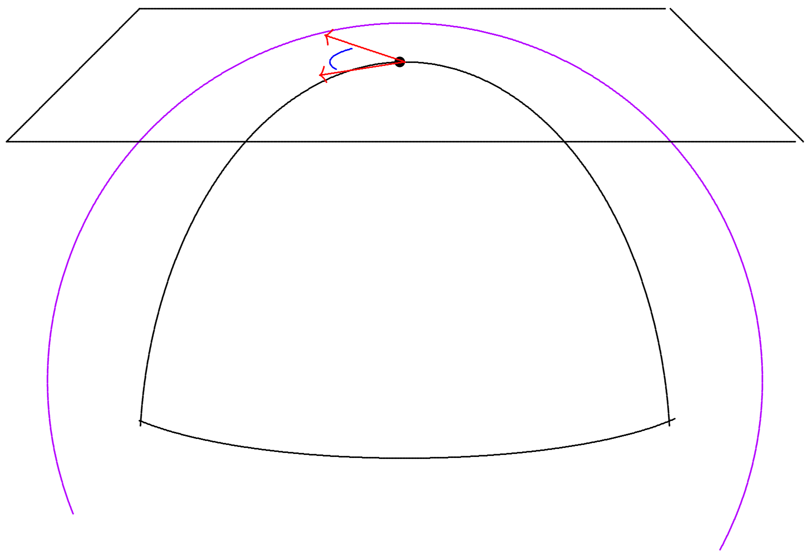


$\mathbf{SU}(m) \subset \mathbf{U}(m)$: $\qquad \{A \mid \det A = 1\}$

Kähler metrics:

(M^{2m}, g): Ricci-flat Kähler $\Longleftarrow$ holonomy $\subset \mathbf{SU}(m)$



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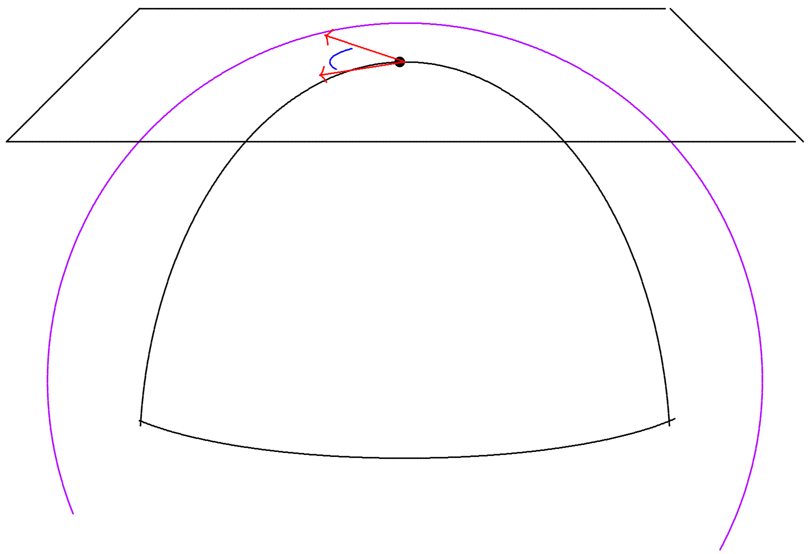
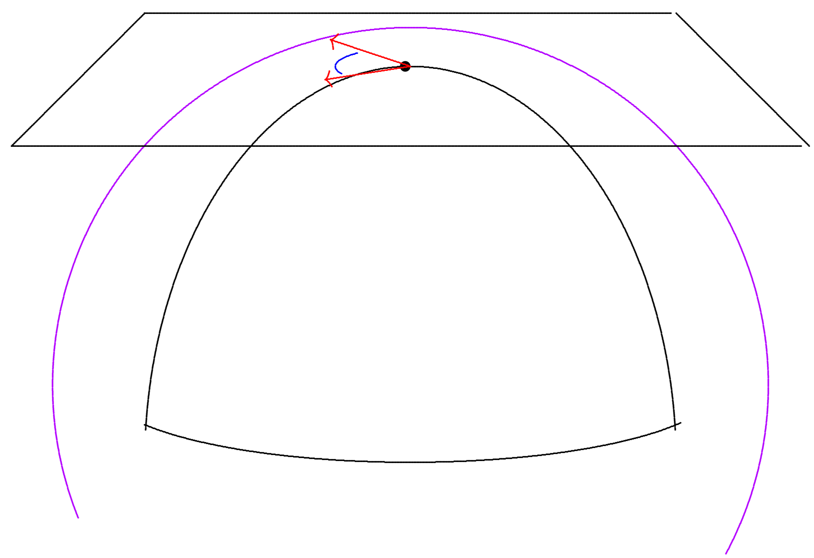


if M is simply connected.

Hyper-Kähler metrics:

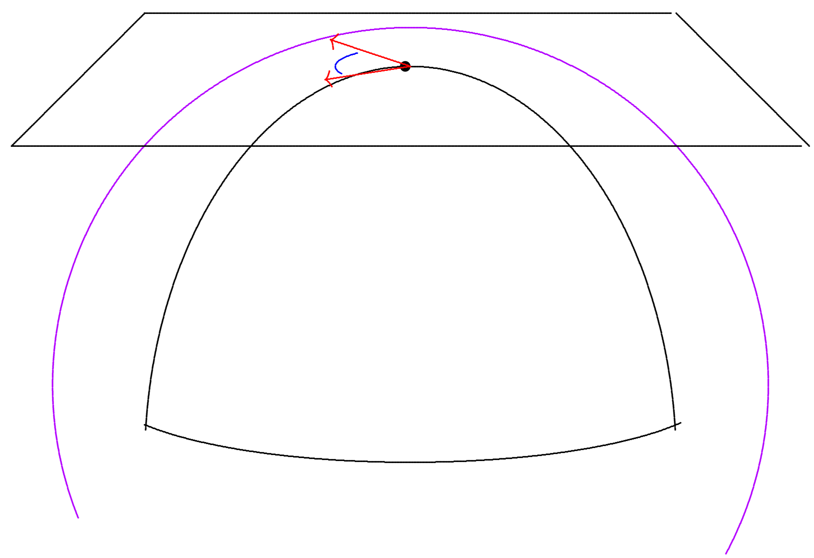
$(M^{4\ell}, g)$

holonomy



Hyper-Kähler metrics:

$(M^{4\ell}, g)$ hyper-Kähler $\Longleftrightarrow$ holonomy $\subset \mathbf{Sp}(\ell)$



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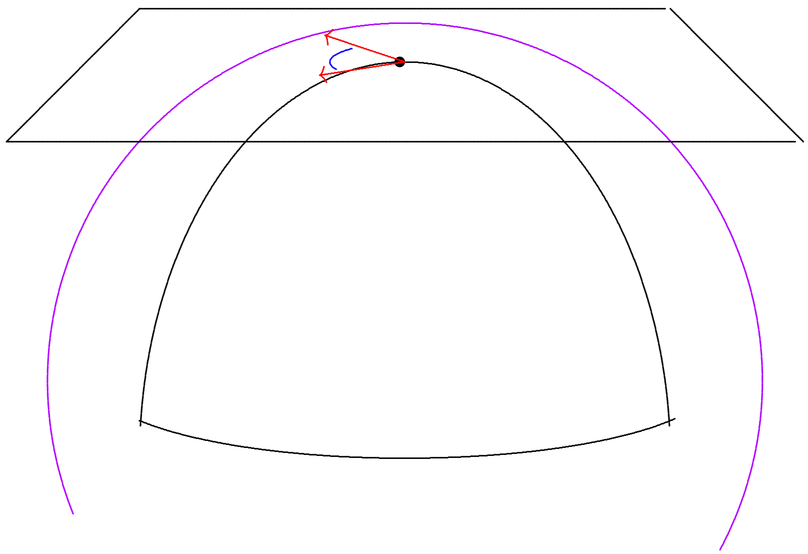

$$\mathbf{Sp}(\ell) := \mathbf{O}(4\ell) \cap \mathbf{GL}(\ell, \mathbb{H})$$

Hyper-Kähler metrics:

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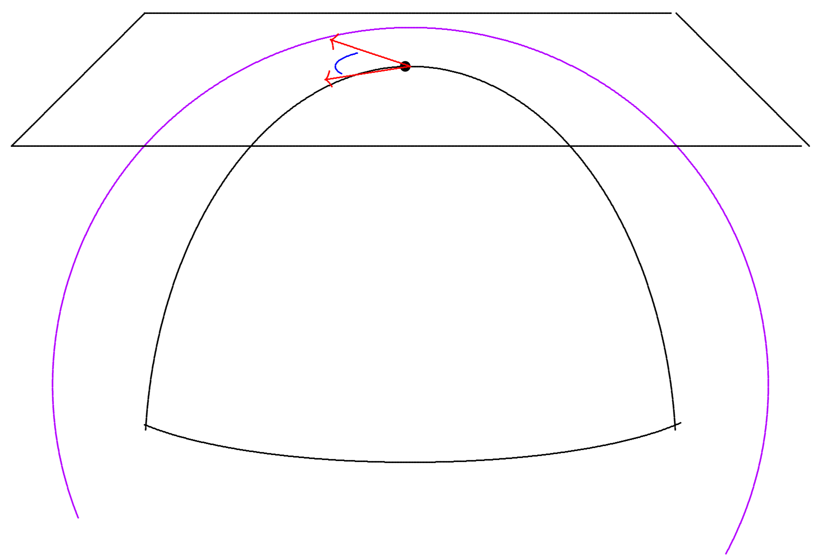

$$\mathbf{Sp}(\ell) \subset \mathbf{SU}(2\ell)$$

Hyper-Kähler metrics:

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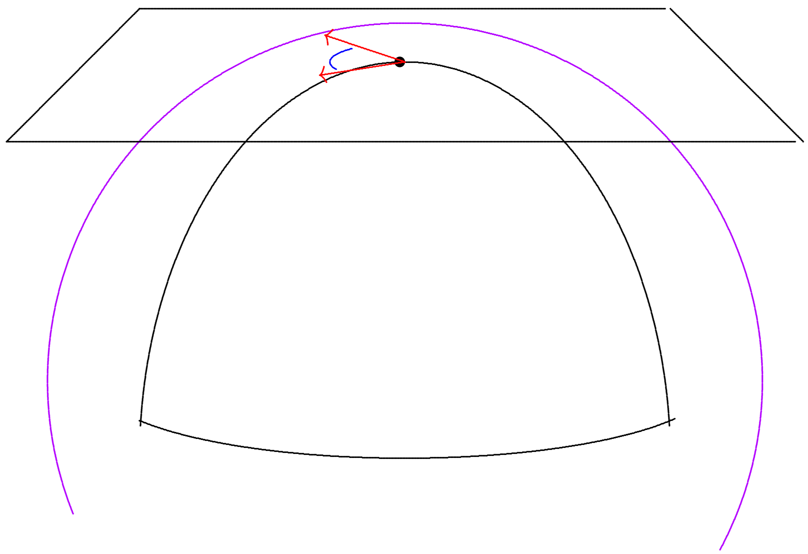

$$\mathbf{Sp}(\ell) \subset \mathbf{SU}(2\ell)$$

in many ways!

Hyper-Kähler metrics:

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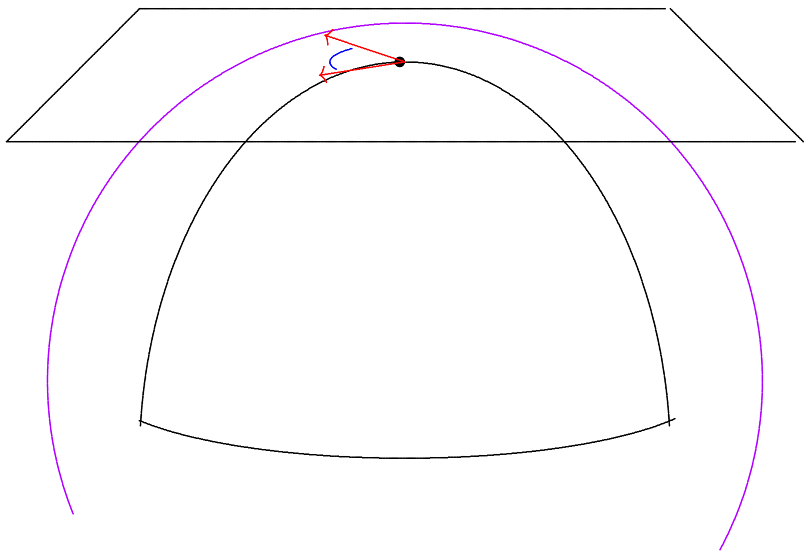

$$\mathbf{Sp}(\ell) \subset \mathbf{SU}(2\ell)$$

in many ways! (For example, permute i, j, k...)

Hyper-Kähler metrics:

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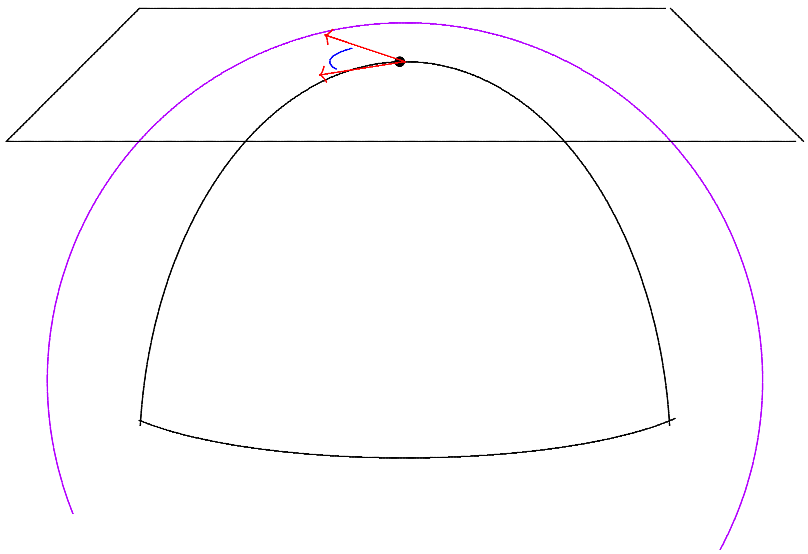


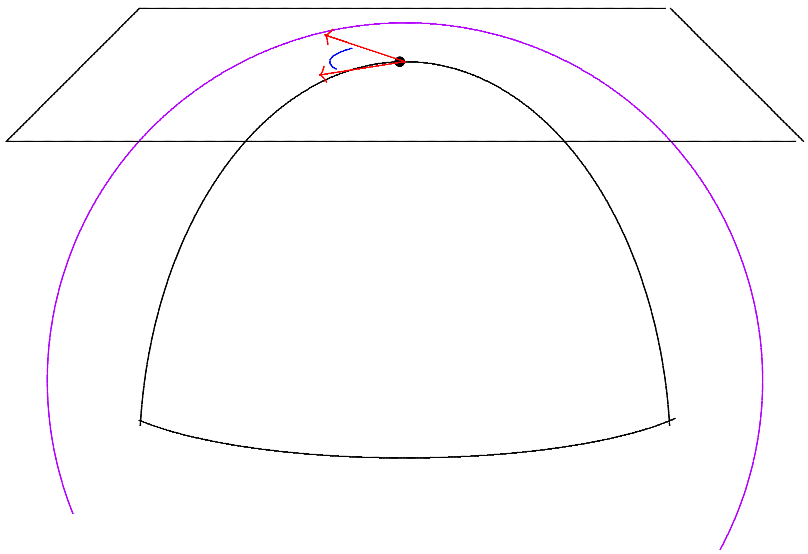
$$\mathbf{Sp}(\ell) \subset \mathbf{SU}(2\ell)$$

Ricci-flat and Kähler,

for many different complex structures!

Hyper-Kähler metrics:

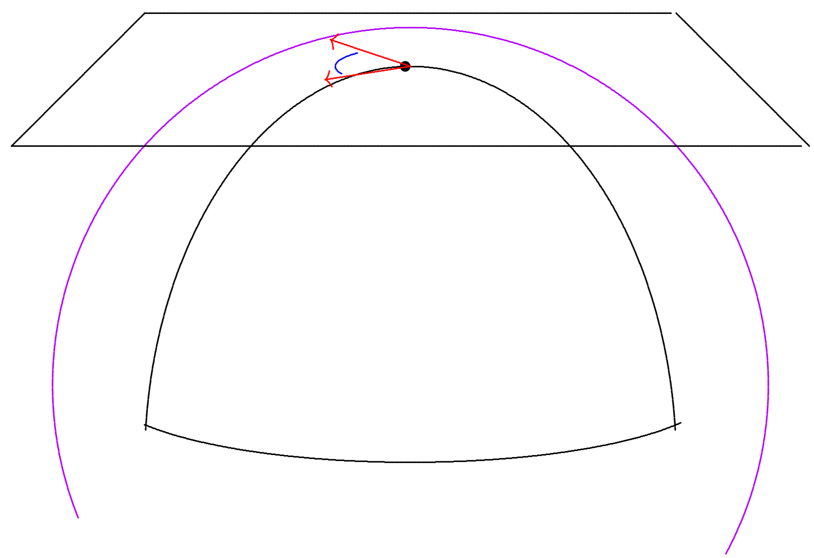
$(M^{4\ell}, g)$ hyper-Kähler $\Longleftrightarrow$ holonomy $\subset \mathbf{Sp}(\ell)$



$\mathbf{Sp}(\ell) \subset \mathbf{SU}(2\ell)$

Hyper-Kähler metrics:

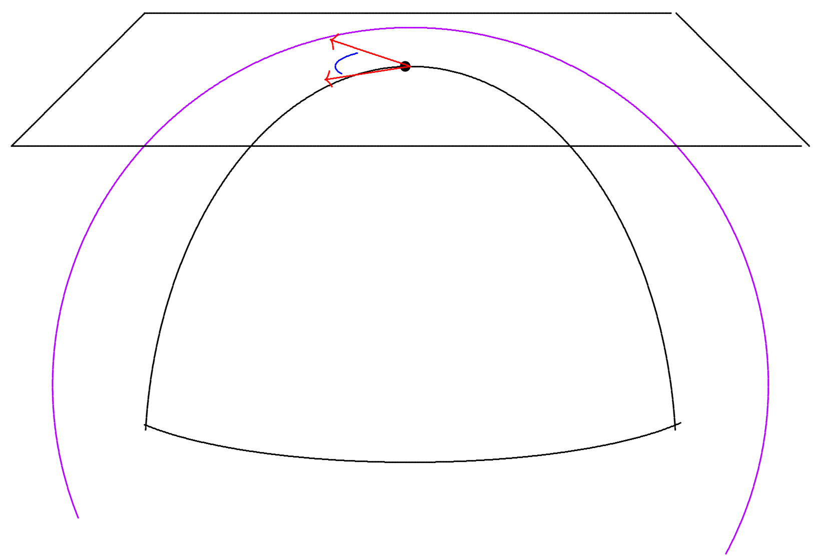
(M^4, g) hyper-Kähler $\Longleftrightarrow$ holonomy $\subset \mathbf{Sp}(1)$



$$\mathbf{Sp}(1) = \mathbf{SU}(2)$$

Hyper-Kähler metrics:

(M^4, g) hyper-Kähler $\Longleftrightarrow$ holonomy $\subset \mathbf{Sp}(1)$



$$\mathbf{Sp}(1) = \mathbf{SU}(2)$$

When (M^4, g) simply connected:

hyper-Kähler $\Longleftrightarrow$ Ricci-flat Kähler $\Longleftrightarrow$ Λ^+ flat.

Hitchin-Thorpe Inequality:

$$(2\chi + 3\tau)(M) = \frac{1}{4\pi^2}\int_M \left(\frac{s^2}{24} + 2|W_+|^2 - \frac{|\mathring{r}|^2}{2}\right) d\mu_g$$

$$\text{Einstein} \Rightarrow \quad = \frac{1}{4\pi^2}\int_M \left(\frac{s^2}{24} + 2|W_+|^2\right) d\mu_g$$

Theorem (Hitchin-Thorpe Inequality)**.** *If smooth compact oriented M^4 admits Einstein g, then*

$$(2\chi + 3\tau)(M) \geq 0,$$

with equality only if Λ^+ is flat on (M, g).

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with equality only if Λ^+ is flat on (M, g).

$= \Longleftrightarrow$ universal cover Ricci-flat Kähler.

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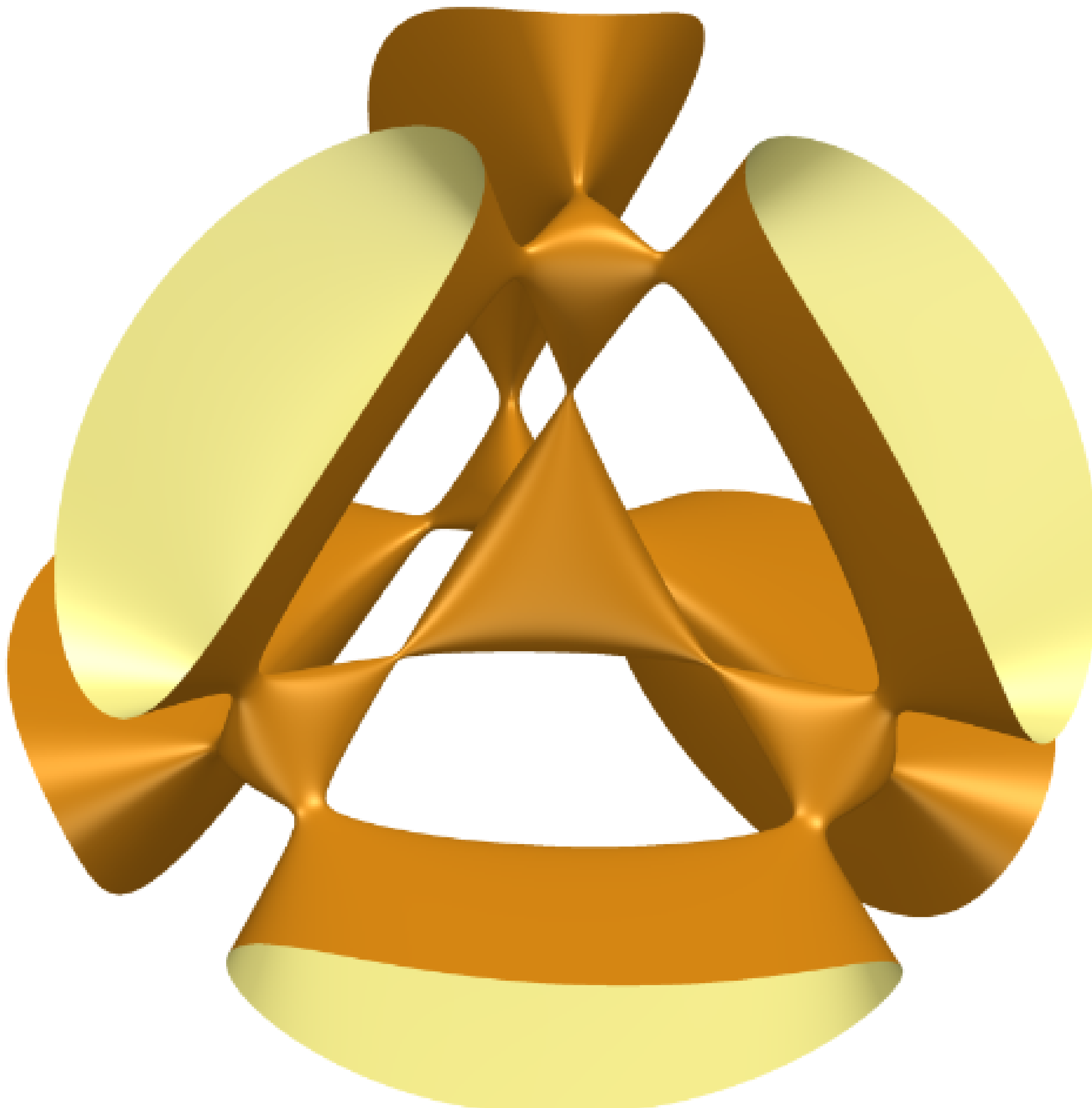

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Indeed, $\exists$ sequences of these $\longrightarrow$ flat orbifold $T^4/\mathbb{Z}_2$.

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$\therefore$ Topological manifold $|K3|$ has infinitely many smooth structures, but only one of these admits Einstein metrics.