Four-Manifolds,

Conformal Curvature, &

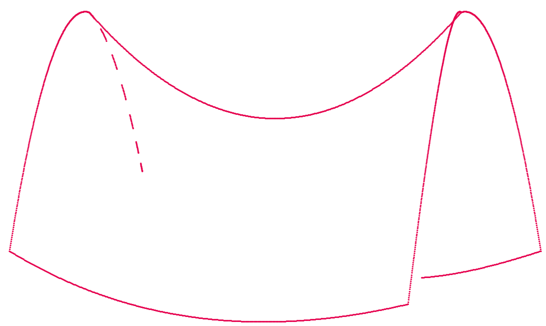
Differential Topology

Claude LeBrun
Stony Brook University

Union College Mathematics Conference,
Schenectady, NY. June 4, 2022

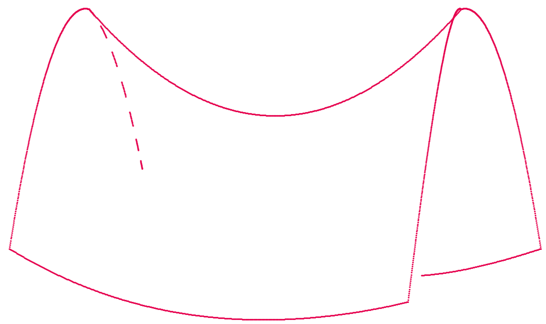
Any Riemannian 2-manifold is conformally flat.

Any Riemannian 2-manifold is conformally flat.



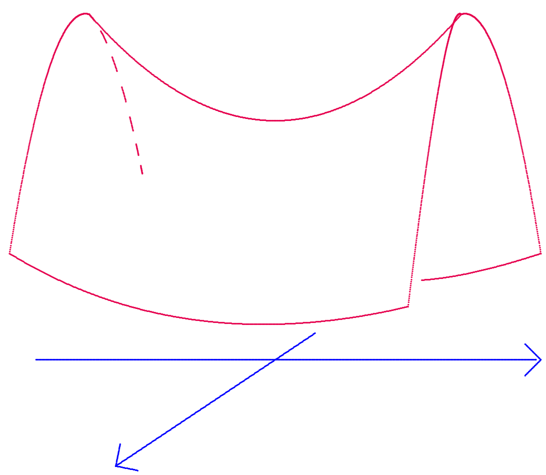
Any Riemannian 2-manifold is conformally flat.

Given (M^2, g)



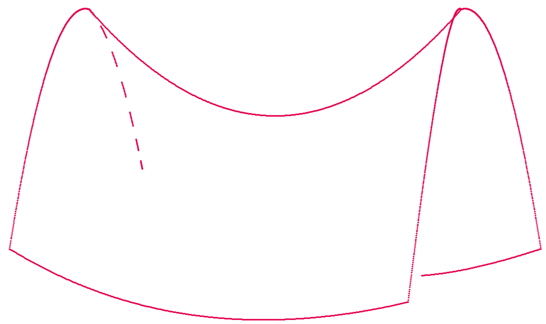
Any Riemannian 2-manifold is conformally flat.

Given (M^2, g)



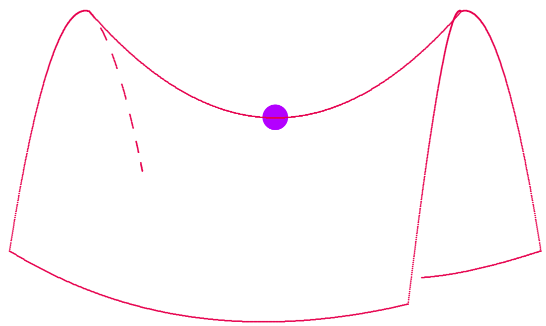
Any Riemannian 2-manifold is conformally flat.

Given (M^2, g)



Any Riemannian 2-manifold is conformally flat.

Given (M^2, g) and any $p \in M$,



Any Riemannian 2-manifold is conformally flat.

Given (M^2, g) and any $p \in M$,

$\exists$ coordinates (x, y) on a neighborhood of p s.t.

Any Riemannian 2-manifold is conformally flat.

Given (M^2, g) and any $p \in M$,

$\exists$ coordinates (x, y) on a neighborhood of p s.t.

$$g = f(x, y)\,[dx \otimes dx + dy \otimes dy]$$

Any Riemannian 2-manifold is conformally flat.

Given (M^2, g) and any $p \in M$,

$\exists$ coordinates (x, y) on a neighborhood of p s.t.

$$g = f(x, y)\left[dx^2 + dy^2\right]$$

Any Riemannian 2-manifold is conformally flat.

Given (M^2, g) and any $p \in M$,

$\exists$ coordinates (x, y) on a neighborhood of p s.t.

$$g = f(x, y)\left[dx^2 + dy^2\right]$$

on this neighborhood, for some $f(x, y) > 0$.

Any Riemannian 2-manifold is conformally flat.

Given (M^2, g) and any $p \in M$,

$\exists$ coordinates (x, y) on a neighborhood of p s.t.

$$g = f(x, y) \left[dx^2 + dy^2\right]$$

on this neighborhood, for some $f(x, y) > 0$.

Gauss (1822)

Any Riemannian 2-manifold is conformally flat.

Given (M^2, g) and any $p \in M$,

$\exists$ coordinates (x, y) on a neighborhood of p s.t.

$$g = f(x, y)\left[dx^2 + dy^2\right]$$

on this neighborhood, for some $f(x, y) > 0$.

Gauss (1822)

Korn, Lichtenstein (1916)

Any Riemannian 2-manifold is conformally flat.

Given (M^2, g) and any $p \in M$,

$\exists$ coordinates (x, y) on a neighborhood of p s.t.

$$g = f(x, y)\left[dx^2 + dy^2\right]$$

on this neighborhood, for some $f(x, y) > 0$.

Any Riemannian 2-manifold is conformally flat.

Given (M^2, g) and any $p \in M$,

$\exists$ coordinates (x, y) on a neighborhood of p s.t.

$$g = f(x, y)\left[dx^2 + dy^2\right]$$

on this neighborhood, for some $f(x, y) > 0$.

Lamé (1833): "isothermal coordinates"

Any Riemannian 2-manifold is conformally flat.

Given (M^2, g) and any $p \in M$,

$\exists$ coordinates (x, y) on a neighborhood of p s.t.

$$g = f(x, y)\left[dx^2 + dy^2\right]$$

on this neighborhood, for some $f(x, y) > 0$.

A better name might be "conformal coordinates".

Any Riemannian 2-manifold is conformally flat.

Given (M^2, g) and any $p \in M$,

$\exists$ coordinates (x, y) on a neighborhood of p s.t.

$$g = f(x, y) \left[dx^2 + dy^2\right]$$

on this neighborhood, for some $f(x, y) > 0$.

A better name might be "conformal coordinates".

Distort distances, but not angles.


UNION
COLLEGE

"Minerva Torus"

Image credit: Davide Cervone, Union College


UNION
COLLEGE

Mercator Projection

Any Riemannian 2-manifold is conformally flat.

Given (M^2, g) and any $p \in M$,

$\exists$ coordinates (x, y) on a neighborhood of p s.t.

$$g = f(x,y)\left[dx^2 + dy^2\right]$$

on this neighborhood, for some $f(x, y) > 0$.

A better name might be "conformal coordinates".

Any Riemannian 2-manifold is conformally flat.

Given (M^2, g) and any $p \in M$,

$\exists$ coordinates (x, y) on a neighborhood of p s.t.

$$g = f(x, y)\left[dx^2 + dy^2\right]$$

on this neighborhood, for some $f(x, y) > 0$.

A better name might be "conformal coordinates".

Oriented case: $z = x + \mathrm{i}y$ complex coordinate.

Any Riemannian 2-manifold is conformally flat.

Given (M^2, g) and any $p \in M$,

$\exists$ coordinates (x, y) on a neighborhood of p s.t.

$$g = f(x, y) \left[dx^2 + dy^2\right]$$

on this neighborhood, for some $f(x, y) > 0$.

A better name might be "conformal coordinates".

Oriented case: $z = x + \mathsf{i}y$ complex coordinate.

Any such z and $\tilde{z}$ related by holomorphic function.

Any Riemannian 2-manifold is conformally flat.

Given (M^2, g) and any $p \in M$,

$\exists$ coordinates (x, y) on a neighborhood of p s.t.

$$g = f(x, y)\left[dx^2 + dy^2\right]$$

on this neighborhood, for some $f(x, y) > 0$.

A better name might be "conformal coordinates".

Oriented case: $z = x + iy$ complex coordinate.

Any such z and $\tilde{z}$ related by holomorphic function.

Makes M into a Riemann surface.

Any Riemannian 2-manifold is conformally flat.

Given (M^2, g) and any $p \in M$,

$\exists$ coordinates (x, y) on a neighborhood of p s.t.

$$g = f(x, y)\left[dx^2 + dy^2\right]$$

on this neighborhood, for some $f(x, y) > 0$.

A better name might be "conformal coordinates".

Oriented case: $z = x + \mathsf{i}y$ complex coordinate.

Any such z and $\tilde{z}$ related by holomorphic function.

Makes (M, g) into a Kähler manifold.

Any Riemannian 2-manifold is conformally flat.

Any Riemannian 2-manifold is conformally flat.

But not true for (M^n, g) of dimension $n \geq 3$!

Any Riemannian 2-manifold is conformally flat.

But not true for (M^n, g) of dimension $n \geq 3$!

Generic g not locally Euclidean times function!

Any Riemannian 2-manifold is conformally flat.

But not true for (M^n, g) of dimension $n \geq 3$!

Generic g not locally Euclidean times function!

Usually, $\nexists$ "isothermal coordinates"...

To make this precise…

Let (M^n, g) be a Riemannian n-manifold, $p \in M$.

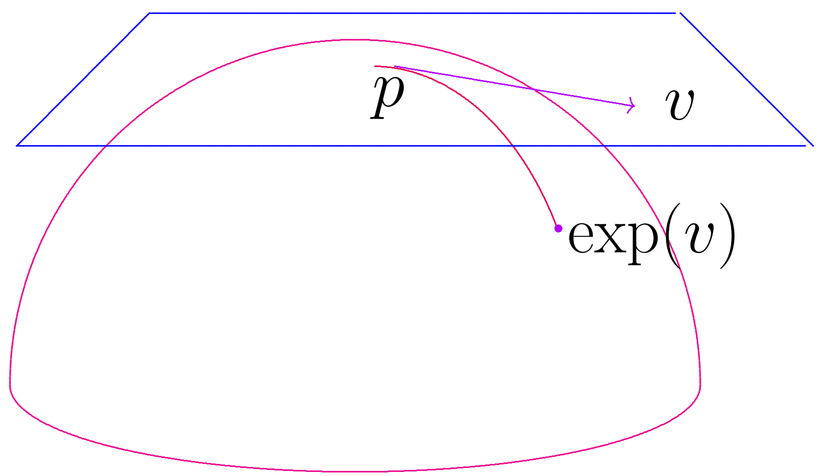
Let (M^n, g) be a Riemannian n-manifold, $p \in M$.
Metric defines locally shortest curves, called *geodesics*.
Following geodesics from p defines a map

$$\exp : T_pM \to M$$

Let (M^n, g) be a Riemannian n-manifold, $p \in M$.
Metric defines locally shortest curves, called *geodesics*.
Following geodesics from p defines a map

$$\exp : T_pM \to M$$

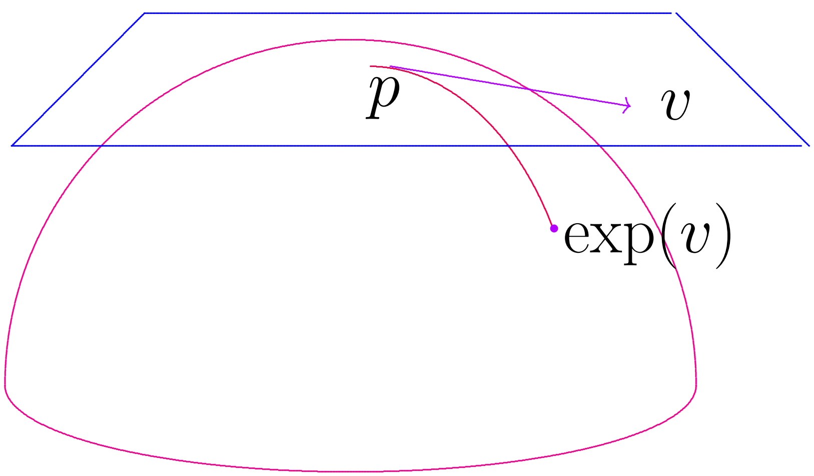
which is a diffeomorphism on a neighborhood of 0:



Let (M^n, g) be a Riemannian n-manifold, $p \in M$.
Metric defines locally shortest curves, called *geodesics*.
Following geodesics from p defines a map

$$\exp : T_pM \to M$$

which is a diffeomorphism on a neighborhood of 0:



Now choosing $T_pM \xrightarrow{\cong} \mathbb{R}^n$ via some orthonormal basis gives us special coordinates on M.

In these “geodesic normal” coordinates,

In these “geodesic normal” coordinates, the metric

$$g = \sum_{j,k=1}^{n} g_{jk} dx^j \otimes dx^k$$

In these "geodesic normal" coordinates, the metric

$$g = g_{jk} \, dx^j \otimes dx^k$$

In these “geodesic normal” coordinates, the metric has components given by

$$g_{jk} =$$

In these "geodesic normal" coordinates, the metric has components given by

$$g_{jk} = \delta_{jk}$$

In these “geodesic normal” coordinates, the metric has components given by

$$g_{jk} = \delta_{jk} - \frac{1}{3}\mathcal{R}_{j\ell km}x^{\ell}x^{m}$$

In these "geodesic normal" coordinates, the metric has components given by

$$g_{jk} = \delta_{jk} - \frac{1}{3}\mathcal{R}_{j\ell km}x^\ell x^m + O(|x|^3)$$

In these “geodesic normal” coordinates, the metric has components given by

$$g_{jk} = \delta_{jk} - \frac{1}{3}\mathcal{R}_{j\ell km}x^{\ell}x^{m} + O(|x|^3)$$

where the $\mathcal{R}_{j\ell km}$ are exactly the components of the Riemann curvature tensor

In these “geodesic normal” coordinates, the metric has components given by

$$g_{jk} = \delta_{jk} - \frac{1}{3}\mathcal{R}_{j\ell km}x^{\ell}x^{m} + O(|x|^3)$$

where the $\mathcal{R}_{j\ell km}$ are exactly the components of the Riemann curvature tensor at the reference point p represented by $\vec{x} = 0$ in these coordinates.

In these “geodesic normal” coordinates, the metric has components given by

$$g_{jk} = \delta_{jk} - \frac{1}{3}\mathcal{R}_{j\ell km} x^\ell x^m + O(|x|^3)$$

where the $\mathcal{R}_{j\ell km}$ are exactly the components of the Riemann curvature tensor at the reference point p represented by $\vec{x} = 0$ in these coordinates.

Uniquely determined by the above expression for g_{jk} once one also requires Bianchi identities

$$\mathcal{R}_{j\ell km} = -\mathcal{R}_{\ell jkm} = -\mathcal{R}_{j\ell mk}$$

$$\mathcal{R}_{j\ell km} + \mathcal{R}_{jkm\ell} + \mathcal{R}_{jm\ell k} = 0$$

In these “geodesic normal” coordinates, the metric has components given by

$$g_{jk} = \delta_{jk} - \frac{1}{3}\mathcal{R}_{j\ell km}x^{\ell}x^{m} + O(|x|^3)$$

where the $\mathcal{R}_{j\ell km}$ are exactly the components of the Riemann curvature tensor at the reference point p represented by $\vec{x} = 0$ in these coordinates.

In these "geodesic normal" coordinates the metric volume measure is given by

$$d\mu_g = \qquad\qquad\qquad\qquad d\mu_{\text{Euclidean}},$$

In these “geodesic normal” coordinates the metric volume measure is given by

$$d\mu_g = \Big[1 - \qquad\qquad\qquad\qquad \Big] d\mu_{\text{Euclidean}},$$

In these “geodesic normal” coordinates the metric volume measure is given by

$$d\mu_g = \left[1 - \frac{1}{6}\, r_{jk}\, x^j x^k + \qquad\qquad \right] d\mu_{\text{Euclidean}},$$

In these “geodesic normal” coordinates the metric volume measure is given by

$$d\mu_g = \left[1 - \frac{1}{6}\, r_{jk}\, x^j x^k + O(|x|^3)\right] d\mu_{\text{Euclidean}},$$

In these "geodesic normal" coordinates the metric volume measure is given by

$$d\mu_g = \left[1 - \tfrac{1}{6}\, r_{jk}\, x^j x^k + O(|x|^3)\right] d\mu_{\text{Euclidean}},$$

where r is the *Ricci tensor*

In these "geodesic normal" coordinates the metric volume measure is given by

$$d\mu_g = \left[1 - \tfrac{1}{6}\, r_{jk}\, x^j x^k + O(|x|^3)\right] d\mu_{\text{Euclidean}},$$

where r is the *Ricci tensor* $r_{jk} = \mathcal{R}^i{}_{jik}$.

Finally, the *scalar curvature* is the trace of Ricci:

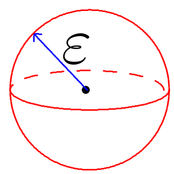
Finally, the *scalar curvature* is the trace of Ricci:

$$s = r^{j}_{\ j} = \mathcal{R}^{ij}{}_{ij}.$$

Finally, the *scalar curvature* is the trace of Ricci:

$$s = r_j^{\ j} = \mathcal{R}^{ij}{}_{ij}.$$

$$\frac{\mathrm{vol}_g(B_\varepsilon(p))}{c_n \varepsilon^n} = 1 - s\,\frac{\varepsilon^2}{6(n+2)} + O(\varepsilon^4)$$



Definition. *A Riemannian metric* g *is said to be* Einstein *if it has constant Ricci curvature*

Definition. *A Riemannian metric* g *is said to be* Einstein *if it has constant Ricci curvature — i.e.*

$$r = \lambda g$$

for some constant $\lambda \in \mathbb{R}$.

Definition. *A Riemannian metric g is said to be* Einstein *if it has constant Ricci curvature — i.e.*

$$r = \lambda g$$

for some constant $\lambda \in \mathbb{R}$.

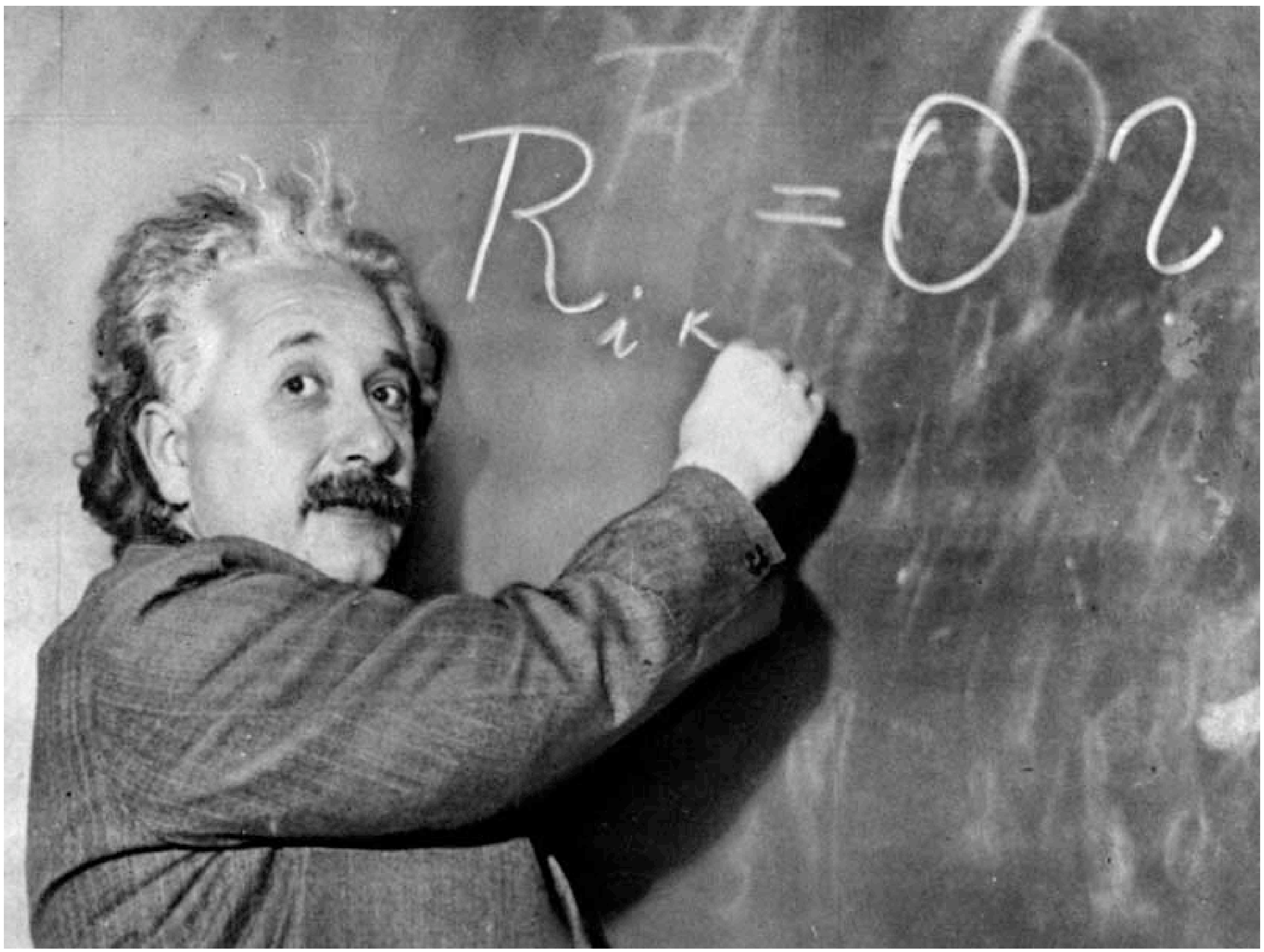
"…the greatest blunder of my life!"

— A. Einstein, to G. Gamow

Definition. *A Riemannian metric g is said to be* Einstein *if it has constant Ricci curvature — i.e.*

$$r = \lambda g$$

for some constant $\lambda \in \mathbb{R}$.



Definition. *A Riemannian metric* g *is said to be* Einstein *if it has constant Ricci curvature — i.e.*

$$r = \lambda g$$

for some constant $\lambda \in \mathbb{R}$.

As punishment ...

Definition. *A Riemannian metric* g *is said to be* Einstein *if it has constant Ricci curvature — i.e.*

$$r = \lambda g$$

for some constant $\lambda \in \mathbb{R}$.

λ called Einstein constant.

Definition. *A Riemannian metric* g *is said to be* Einstein *if it has constant Ricci curvature — i.e.*

$$r = \lambda g$$

for some constant $\lambda \in \mathbb{R}$.

λ called Einstein constant.

Has same sign as the *scalar curvature*

$$s = r_j^{\ j} = \mathcal{R}^{ij}{}_{ij}.$$

Definition. *A Riemannian metric* g *is said to be* Einstein *if it has constant Ricci curvature — i.e.*

$$r = \lambda g$$

for some constant $\lambda \in \mathbb{R}$.

λ called Einstein constant.

Has same sign as the *scalar curvature*

$$s = r_j^{\ j} = \mathcal{R}^{ij}{}_{ij}.$$

$$s = n\lambda$$

Any Riemannian 2-manifold is conformally flat.

But not true for (M^n, g) of dimension $n \geq 3$!

Generic g not locally Euclidean times function!

Usually, $\nexists$ "isothermal coordinates"...

On Riemannian n-manifold (M, g),

On Riemannian n-manifold (M, g), $n \geq 3$,

$$\text{Riemann} = \text{Weyl} \oplus \text{Ricci}$$

On Riemannian n-manifold (M, g), $n \geq 3$,

Riemann = Weyl $\oplus$ trace-free Ricci $\oplus$ scalar

On Riemannian n-manifold (M, g), $n \geq 3$,

$$\mathcal{R}^{ab}{}_{cd} = W^{ab}{}_{cd} + \frac{4}{n-2}\mathring{r}^{[a}{}_{[c}\delta^{b]}{}_{d]} + \frac{2}{n(n-1)}\, s\, \delta^{a}_{[c}\delta^{b}_{d]}$$

On Riemannian n-manifold (M, g),

$$\mathcal{R}^{ab}{}_{cd} = W^{ab}{}_{cd} + \frac{4}{n-2}\mathring{r}^{[a}_{[c}\delta^{b]}_{d]} + \frac{2}{n(n-1)}\, s\, \delta^{a}_{[c}\delta^{b}_{d]}$$

where

s = scalar curvature

$\mathring{r}$ = trace-free Ricci curvature

W = Weyl curvature

On Riemannian n-manifold (M, g),

$$\mathcal{R}^{ab}{}_{cd} = W^{ab}{}_{cd} + \frac{4}{n-2}\mathring{r}^{[a}_{[c}\delta^{b]}_{d]} + \frac{2}{n(n-1)}\, s\, \delta^{a}_{[c}\delta^{b}_{d]}$$

where

s = scalar curvature

$\mathring{r}$ = trace-free Ricci curvature

W = Weyl curvature *(conformally invariant)*

On Riemannian n-manifold (M, g),

$$\mathcal{R}^{ab}{}_{cd} = W^{ab}{}_{cd} + \frac{4}{n-2}\mathring{r}^{[a}_{[c}\delta^{b]}_{d]} + \frac{2}{n(n-1)}\ s\ \delta^{a}_{[c}\delta^{b}_{d]}$$

where

s = scalar curvature

$\mathring{r}$ = trace-free Ricci curvature

W = Weyl curvature *(conformally invariant)*

$W^{a}{}_{bcd}$ unchanged if $g \rightsquigarrow \hat{g} = u^2 g$.

On Riemannian n-manifold (M, g),

$$\mathcal{R}^{ab}{}_{cd} = W^{ab}{}_{cd} + \frac{4}{n-2}\mathring{r}^{[a}_{[c}\delta^{b]}_{d]} + \frac{2}{n(n-1)}\, s\ \delta^{a}_{[c}\delta^{b}_{d]}$$

where

$$s = \text{scalar curvature}$$

$$\mathring{r} = \text{trace-free Ricci curvature}$$

$$W = \text{Weyl curvature } \textit{(conformally invariant)}$$

$W^{a}{}_{bcd}$ unchanged if $g \rightsquigarrow \hat{g} = u^2 g$.

But $\exists u$ such that $\hat{\mathring{r}} = 0$ at any given $p \in M$.

On Riemannian n-manifold (M, g),

$$\mathcal{R}^{ab}{}_{cd} = W^{ab}{}_{cd} + \frac{4}{n-2}\mathring{r}^{[a}_{[c}\delta^{b]}_{d]} + \frac{2}{n(n-1)}\, s\, \delta^{a}_{[c}\delta^{b}_{d]}$$

where

s = scalar curvature

$\mathring{r}$ = trace-free Ricci curvature

W = Weyl curvature *(conformally invariant)*

$W^{a}{}_{bcd}$ unchanged if $g \rightsquigarrow \hat{g} = u^2 g$.

On Riemannian n-manifold (M, g),

$$\mathcal{R}^{ab}{}_{cd} = W^{ab}{}_{cd} + \frac{4}{n-2}\mathring{r}^{[a}_{[c}\delta^{b]}_{d]} + \frac{2}{n(n-1)}\ s\ \delta^{a}_{[c}\delta^{b}_{d]}$$

where

$$\begin{aligned} s &= \text{scalar curvature} \\ \mathring{r} &= \text{trace-free Ricci curvature} \\ W &= \text{Weyl curvature } \textit{(conformally invariant)} \end{aligned}$$

Proposition. *Assume* $n \geq 4$. *Then* (M^n, g) *locally conformally flat* $\Longleftrightarrow W \equiv 0$.

On Riemannian n-manifold (M, g),

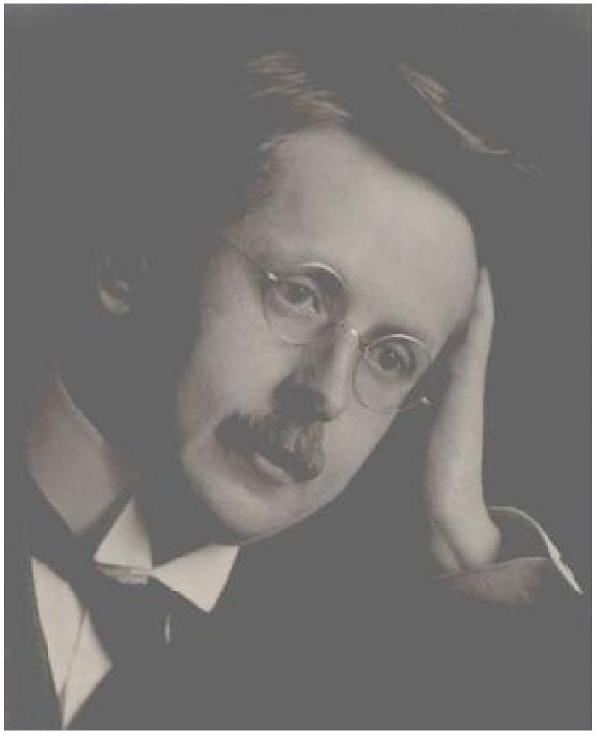
$$\mathcal{R}^{ab}{}_{cd} = W^{ab}{}_{cd} + \frac{4}{n-2}\mathring{r}^{[a}_{[c}\delta^{b]}_{d]} + \frac{2}{n(n-1)}\, s\, \delta^{a}_{[c}\delta^{b}_{d]}$$

where

s = scalar curvature

$\mathring{r}$ = trace-free Ricci curvature

W = Weyl curvature *(conformally invariant)*



On Riemannian n-manifold (M, g),

$$\mathcal{R}^{ab}{}_{cd} = W^{ab}{}_{cd} + \frac{4}{n-2} \mathring{r}^{[a}_{[c} \delta^{b]}_{d]} + \frac{2}{n(n-1)} \, s \, \delta^{a}_{[c} \delta^{b}_{d]}$$

where

s = scalar curvature

$\mathring{r}$ = trace-free Ricci curvature

W = Weyl curvature *(conformally invariant)*

Warning: When $n = 3$, story is different.

On Riemannian n-manifold (M, g),

$$\mathcal{R}^{ab}{}_{cd} = W^{ab}{}_{cd} + \frac{4}{n-2}\mathring{r}^{[a}_{[c}\delta^{b]}_{d]} + \frac{2}{n(n-1)}\, s\, \delta^{a}_{[c}\delta^{b}_{d]}$$

where

s = scalar curvature

$\mathring{r}$ = trace-free Ricci curvature

W = Weyl curvature *(conformally invariant)*

Warning: When $n = 3$, story is different.

W always 0, but g usually not conformally flat.

On Riemannian n-manifold (M, g),

$$\mathcal{R}^{ab}{}_{cd} = W^{ab}{}_{cd} + \frac{4}{n-2}\mathring{r}^{[a}_{[c}\delta^{b]}_{d]} + \frac{2}{n(n-1)}\, s\, \delta^{a}_{[c}\delta^{b}_{d]}$$

where

s = scalar curvature

$\mathring{r}$ = trace-free Ricci curvature

W = Weyl curvature *(conformally invariant)*

Warning: When $n = 3$, story is different.

W always 0, but g usually not conformally flat.

Cotton tensor $C = \nabla \wedge (\mathring{r} - \frac{s}{12}g)$ obstruction.

On Riemannian n-manifold (M, g),

$$\mathcal{R}^{ab}{}_{cd} = W^{ab}{}_{cd} + \frac{4}{n-2}\mathring{r}^{[a}_{[c}\delta^{b]}_{d]} + \frac{2}{n(n-1)}\; s\; \delta^{a}_{[c}\delta^{b}_{d]}$$

where

s = scalar curvature

$\mathring{r}$ = trace-free Ricci curvature

W = Weyl curvature *(conformally invariant)*

Proposition. *Assume* $n \geq 4$. *Then* (M^n, g) *locally conformally flat* $\Longleftrightarrow W \equiv 0$.

For metrics on fixed M^n,

For metrics on fixed M^n, compact, $\partial M = \varnothing$,

For metrics on fixed M^n, Weyl functional

For metrics on fixed M^n, Weyl functional

$$\mathscr{W} : \{\text{metrics}\} \longrightarrow \mathbb{R}$$

For metrics on fixed M^n, Weyl functional

$$\mathscr{W} : \mathcal{G}_M \longrightarrow \mathbb{R}$$

For metrics on fixed M^n, Weyl functional

$$\mathscr{W}(g) = \int_M |W_g|^{n/2} d\mu_g$$

For metrics on fixed M^n, Weyl functional

$$\mathscr{W}(g) = \int_M |W_g|^{n/2} d\mu_g$$

only depends on the conformal class

For metrics on fixed M^n, Weyl functional

$$\mathscr{W}([g]) = \int_M |W_g|^{n/2} d\mu_g$$

only depends on the conformal class

$$[g] = \{u^2 g \mid u : M \xrightarrow{C^\infty} \mathbb{R}^+\}.$$

For metrics on fixed M^n, Weyl functional

$$\mathscr{W}([g]) = \int_M |W_g|^{n/2} d\mu_g$$

only depends on the conformal class

$$[g] = \{u^2 g \mid u : M \xrightarrow{C^\infty} \mathbb{R}^+\}.$$

Power $n/2$ is necessary for scale invariance!

For metrics on fixed M^n, Weyl functional

$$\mathscr{W}([g]) = \int_M |W_g|^{n/2} d\mu_g$$

only depends on the conformal class

$$[g] = \{u^2 g \mid u : M \xrightarrow{C^\infty} \mathbb{R}^+\}.$$

For metrics on fixed M^n, Weyl functional

$$\mathscr{W}([g]) = \int_M |W_g|^{n/2} d\mu_g$$

only depends on the conformal class

$$[g] = \{u^2 g \mid u : M \xrightarrow{C^\infty} \mathbb{R}^+\}.$$

$$\mathscr{W} : \mathcal{G}_M/(C^\infty)^+ \longrightarrow \mathbb{R}$$

For metrics on fixed M^n, Weyl functional

$$\mathscr{W}([g]) = \int_M |W_g|^{n/2} d\mu_g$$

only depends on the conformal class

$$[g] = \{u^2 g \mid u : M \xrightarrow{C^\infty} \mathbb{R}^+\}.$$

Measures deviation $[g]$ from conformal flatness.

For metrics on fixed M^n, Weyl functional

$$\mathscr{W}([g]) = \int_M |W_g|^{n/2} d\mu_g$$

only depends on the conformal class

$$[g] = \{u^2 g \mid u : M \xrightarrow{C^\infty} \mathbb{R}^+\}.$$

Measures deviation $[g]$ from conformal flatness.

Basic problems: For given smooth compact M,

For metrics on fixed M^n, Weyl functional

$$\mathscr{W}([g]) = \int_M |W_g|^{n/2} d\mu_g$$

only depends on the conformal class

$$[g] = \{u^2 g \mid u : M \xrightarrow{C^\infty} \mathbb{R}^+\}.$$

Measures deviation $[g]$ from conformal flatness.

Basic problems: For given smooth compact M,

- What is $\inf \mathscr{W}$?

For metrics on fixed M^n, Weyl functional

$$\mathscr{W}([g]) = \int_M |W_g|^{n/2} d\mu_g$$

only depends on the conformal class

$$[g] = \{u^2 g \mid u : M \xrightarrow{C^\infty} \mathbb{R}^+\}.$$

Measures deviation $[g]$ from conformal flatness.

Basic problems: For given smooth compact M,

- What is $\inf \mathscr{W}$?
- Do there exist minimizers?

Dimension Four is Exceptional

Dimension Four is Exceptional

For M^4,

Dimension Four is Exceptional

For M^4,

$$\mathscr{W}([g]) = \int_M |W_g|^2 d\mu_g$$

Dimension Four is Exceptional

For M^4,

$$\mathscr{W}([g]) = \int_M |W_g|^2 d\mu_g$$

Euler-Lagrange equations $B = 0$ elliptic mod gauge.

Dimension Four is Exceptional

For M^4,

$$\mathscr{W}([g]) = \int_M |W_g|^2 d\mu_g$$

Euler-Lagrange equations $B = 0$ elliptic mod gauge.

Here

$$B_{ab} := (\nabla^c \nabla^d + \frac{1}{2} \mathring{r}^{cd}) W_{acbd}$$

Dimension Four is Exceptional

For M^4,

$$\mathscr{W}([g]) = \int_M |W_g|^2 d\mu_g$$

Euler-Lagrange equations $B = 0$ elliptic mod gauge.

Here

$$B_{ab} := (\nabla^c \nabla^d + \frac{1}{2} \mathring{r}^{cd}) W_{acbd}$$

called Bach tensor.

Dimension Four is Exceptional

For M^4,

$$\mathscr{W}([g]) = \int_M |W_g|^2 d\mu_g$$

Euler-Lagrange equations $B = 0$ elliptic mod gauge.

Here

$$B_{ab} := (\nabla^c \nabla^d + \frac{1}{2} \mathring{r}^{cd}) W_{acbd}$$

called Bach tensor.

Solutions called Bach-flat metrics.

Dimension Four is Exceptional

For M^4,

$$\mathscr{W}([g]) = \int_M |W_g|^2 d\mu_g$$

Euler-Lagrange equations $B = 0$ elliptic mod gauge.

Here

$$B_{ab} := (\nabla^c \nabla^d + \frac{1}{2} \mathring{r}^{cd}) W_{acbd}$$

called Bach tensor.

Solutions called Bach-flat metrics.

Bianchi $\Longrightarrow$ Any Einstein (M^4, g) is Bach-flat.

Dimension Four is Exceptional

For M^4,

$$\mathscr{W}([g]) = \int_M |W_g|^2 d\mu_g$$

Euler-Lagrange equations $B = 0$ elliptic mod gauge.

Here

$$B_{ab} := (\nabla^c\nabla^d + \frac{1}{2}\mathring{r}^{cd})W_{acbd}$$

called Bach tensor.

Solutions called Bach-flat metrics.

Bianchi $\Longrightarrow$ Any Einstein (M^4, g) is Bach-flat.

Of course, conformally Einstein good enough!

By contrast:

By contrast:

For M^n,

$$\mathscr{W}([g]) = \int_M |W_g|^{n/2} d\mu_g$$

By contrast:

For M^n,

$$\mathscr{W}([g]) = \int_M |W_g|^{n/2} d\mu_g$$

has degenerate Euler-Lagrange equation

$$|W_g|^{(n-4)/2}(\nabla\nabla\cdot W + \cdots) = 0$$

when $n > 4$.

By contrast:

For M^n,

$$\mathscr{W}([g]) = \int_M |W_g|^{n/2} d\mu_g$$

has degenerate Euler-Lagrange equation

$$|W_g|^{(n-4)/2}(\nabla\nabla\cdot W + \cdots) = 0$$

when $n > 4$.

Einstein metrics are usually not critical points.

What's So Special About Dimension Four?

What's So Special About Dimension Four?

The Lie group $SO(4)$ is *not simple*:

What's So Special About Dimension Four?

The Lie group $SO(4)$ is *not simple*:

$$\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3).$$

What's So Special About Dimension Four?

The Lie group $SO(4)$ is *not simple*:

$$\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3).$$

On oriented (M^4, g),

What's So Special About Dimension Four?

The Lie group $SO(4)$ is *not simple*:

$$\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3).$$

On oriented (M^4, g), $\Longrightarrow$

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^-$$

What's So Special About Dimension Four?

The Lie group $SO(4)$ is *not simple*:

$$\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3).$$

On oriented (M^4, g), $\Longrightarrow$

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^-$$

where $\Lambda^\pm$ are (± 1)-eigenspaces of

$$\star : \Lambda^2 \to \Lambda^2,$$

$$\star^2 = 1.$$

What's So Special About Dimension Four?

The Lie group $SO(4)$ is *not simple*:

$$\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3).$$

On oriented (M^4, g), $\Longrightarrow$

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^-$$

where $\Lambda^\pm$ are (± 1)-eigenspaces of

$$\star : \Lambda^2 \to \Lambda^2,$$

$$\star^2 = 1.$$

Λ^+ self-dual 2-forms.

What's So Special About Dimension Four?

The Lie group $SO(4)$ is *not simple*:

$$\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3).$$

On oriented (M^4, g), $\Longrightarrow$

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^-$$

where $\Lambda^\pm$ are (± 1)-eigenspaces of

$$\star : \Lambda^2 \to \Lambda^2,$$

$$\star^2 = 1.$$

Λ^+ self-dual 2-forms.

Λ^- anti-self-dual 2-forms.

Riemann curvature of g

$$\mathcal{R} : \Lambda^2 \to \Lambda^2$$

Riemann curvature of g

$$\mathcal{R} : \Lambda^2 \to \Lambda^2$$

splits into 4 irreducible pieces:

Riemann curvature of g

$$\mathcal{R} : \Lambda^2 \to \Lambda^2$$

splits into 4 irreducible pieces:

$$\mathcal{R} = \left(\begin{array}{c|c} W_+ + \frac{s}{12} & \mathring{r} \\ \hline \mathring{r} & W_- + \frac{s}{12} \end{array} \right)$$

Riemann curvature of g

$$\mathcal{R} : \Lambda^2 \to \Lambda^2$$

splits into 4 irreducible pieces:

	Λ^{+*}	Λ^{-*}
Λ^{+}	$W_{+} + \frac{s}{12}$	$\mathring{r}$
Λ^{-}	$\mathring{r}$	$W_{-} + \frac{s}{12}$

Riemann curvature of g

$$\mathcal{R} : \Lambda^2 \to \Lambda^2$$

splits into 4 irreducible pieces:

	Λ^{+*}	Λ^{-*}
Λ^+	$W_+ + \frac{s}{12}$	$\mathring{r}$
Λ^-	$\mathring{r}$	$W_- + \frac{s}{12}$

where

s = scalar curvature

$\mathring{r}$ = trace-free Ricci curvature

W_+ = self-dual Weyl curvature

W_- = anti-self-dual Weyl curvature

Riemann curvature of g

$$\mathcal{R} : \Lambda^2 \to \Lambda^2$$

splits into 4 irreducible pieces:

	Λ^{+*}	Λ^{-*}
Λ^{+}	$W_+ + \frac{s}{12}$	$\mathring{r}$
Λ^{-}	$\mathring{r}$	$W_- + \frac{s}{12}$

where

s = scalar curvature
$\mathring{r}$ = trace-free Ricci curvature
W_+ = self-dual Weyl curvature *(conformally invariant)*
W_- = anti-self-dual Weyl curvature ″

Thus (M^4, g) Einstein $\Longleftrightarrow$

Thus (M^4, g) Einstein $\Longleftrightarrow$

$$\mathcal{R} : \Lambda^2 \to \Lambda^2$$

Thus (M^4, g) Einstein $\Longleftrightarrow$

$$\mathcal{R} : \Lambda^2 \to \Lambda^2$$

commutes with

Thus (M^4, g) Einstein $\Longleftrightarrow$

$$\mathcal{R} : \Lambda^2 \to \Lambda^2$$

commutes with

$$\star : \Lambda^2 \to \Lambda^2 :$$

Thus (M^4, g) Einstein $\Longleftrightarrow$

$$\mathcal{R} : \Lambda^2 \to \Lambda^2$$

commutes with

$$\star : \Lambda^2 \to \Lambda^2 :$$

$$\mathcal{R} = \left(\begin{array}{c|c} W_+ + \frac{s}{12} & \mathring{r} \\ \hline \mathring{r} & W_- + \frac{s}{12} \end{array} \right).$$

Thus (M^4, g) Einstein $\Longleftrightarrow$

$$\mathcal{R} : \Lambda^2 \to \Lambda^2$$

commutes with

$$\star : \Lambda^2 \to \Lambda^2 :$$

$$\mathcal{R} = \left(\begin{array}{c|c} W_+ + \frac{s}{12} & 0 \\ \hline 0 & W_- + \frac{s}{12} \end{array} \right).$$

Thus (M^4, g) Einstein $\Longleftrightarrow$

$$\mathcal{R} : \Lambda^2 \to \Lambda^2$$

commutes with

$$\star : \Lambda^2 \to \Lambda^2 :$$

$$\mathcal{R} = \left(\begin{array}{c|c} W_+ + \frac{s}{12} & \mathring{r} \\ \hline \mathring{r} & W_- + \frac{s}{12} \end{array} \right).$$

Thus (M^4, g) Einstein $\Longleftrightarrow$

$$\mathcal{R} : \Lambda^2 \to \Lambda^2$$

commutes with

$$\star : \Lambda^2 \to \Lambda^2 :$$

$$\mathcal{R} = \left(\begin{array}{c|c} W_+ + \frac{s}{12} & 0 \\ \hline 0 & W_- + \frac{s}{12} \end{array} \right).$$

Thus (M^4, g) Einstein $\Longleftrightarrow$

$$\mathcal{R} : \Lambda^2 \to \Lambda^2$$

commutes with

$$\star : \Lambda^2 \to \Lambda^2 :$$

$$\mathcal{R} = \left(\begin{array}{c|c} W_+ + \frac{s}{12} & \mathring{r} \\ \hline \mathring{r} & W_- + \frac{s}{12} \end{array} \right).$$

Thus (M^4, g) Einstein $\Longleftrightarrow$

$$\mathcal{R} : \Lambda^2 \to \Lambda^2$$

commutes with

$$\star : \Lambda^2 \to \Lambda^2 :$$

$$\mathcal{R} = \left(\begin{array}{c|c} W_+ + \frac{s}{12} & 0 \\ \hline 0 & W_- + \frac{s}{12} \end{array} \right).$$

Thus (M^4, g) Einstein $\Longleftrightarrow$

$$\mathcal{R} : \Lambda^2 \to \Lambda^2$$

commutes with

$$\star : \Lambda^2 \to \Lambda^2 :$$

$$\mathcal{R} = \left(\begin{array}{c|c} W_+ + \frac{s}{12} & \mathring{r} \\ \hline \mathring{r} & W_- + \frac{s}{12} \end{array} \right).$$

Thus (M^4, g) Einstein $\Longleftrightarrow$

$$\mathcal{R} : \Lambda^2 \to \Lambda^2$$

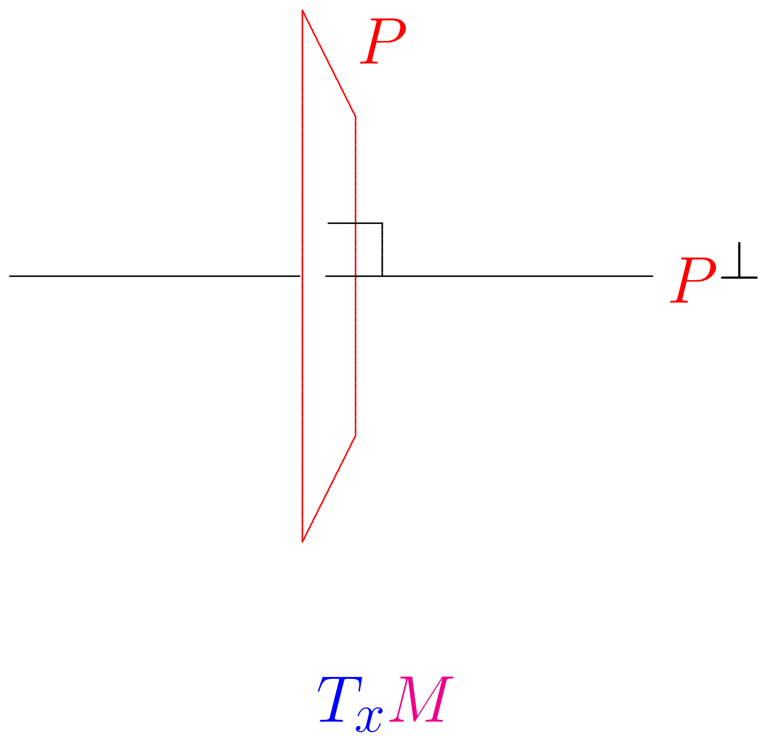
commutes with

$$\star : \Lambda^2 \to \Lambda^2 :$$

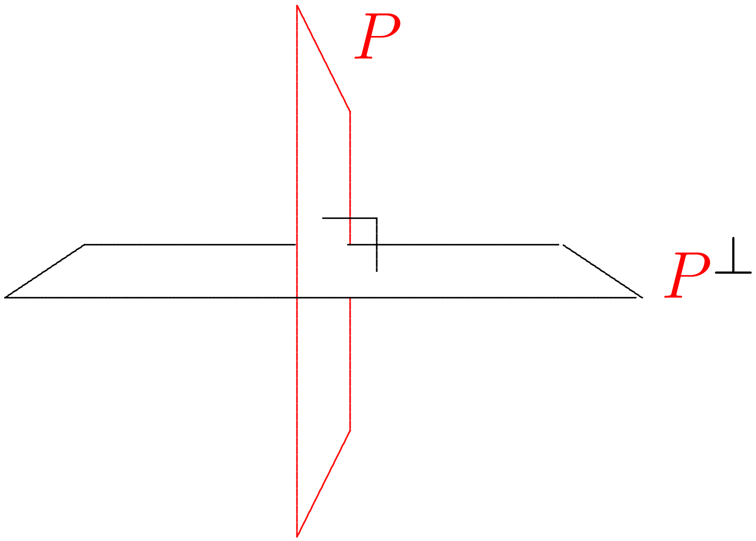
$$\mathcal{R} = \left(\begin{array}{c|c} W_+ + \frac{s}{12} & 0 \\ \hline 0 & W_- + \frac{s}{12} \end{array} \right).$$

Corollary. *A Riemannian 4-manifold* (M, g) *is Einstein* $\Longleftrightarrow$ *sectional curvatures are equal for any pair of perpendicular 2-planes.*

Corollary. *A Riemannian 4-manifold* (M, g) *is Einstein* $\Longleftrightarrow$ *sectional curvatures are equal for any pair of perpendicular 2-planes.*

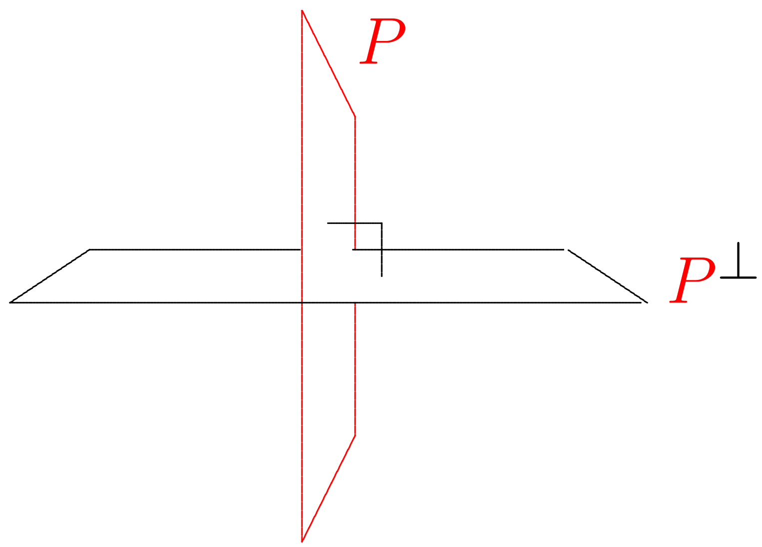


Corollary. *A Riemannian 4-manifold* (M, g) *is Einstein* $\Longleftrightarrow$ *sectional curvatures are equal for any pair of perpendicular 2-planes.*



T_xM

Corollary. *A Riemannian 4-manifold* (M, g) *is Einstein* $\Longleftrightarrow$ *sectional curvatures are equal for any pair of perpendicular 2-planes.*



$$K(P) = K(P^{\perp})$$

For (M^4, g) compact oriented Riemannian,

For (M^4, g) compact oriented Riemannian,

Euler characteristic

$$\chi(M) = \frac{1}{8\pi^2}\int_M \left(\frac{s^2}{24} + |W_+|^2 + |W_-|^2 - \frac{|\mathring{r}|^2}{2}\right) d\mu$$

For (M^4, g) compact oriented Riemannian,

Euler characteristic

$$\chi(M) = \frac{1}{8\pi^2}\int_M \left(\frac{s^2}{24} + |W_+|^2 + |W_-|^2 - \frac{|\mathring{r}|^2}{2}\right) d\mu$$

Signature

$$\tau(M) = \frac{1}{12\pi^2}\int_M \left(|W_+|^2 - |W_-|^2\right) d\mu$$

Here $\tau(M) = b_+(M) - b_-(M)$ defined in terms of intersection pairing

Here $\tau(M) = b_+(M) - b_-(M)$ defined in terms of intersection pairing

$$\begin{aligned} H^2(M,\mathbb{R}) \times H^2(M,\mathbb{R}) &\longrightarrow \quad \mathbb{R} \\ (\ [\varphi]\ ,\ [\psi]\ ) \qquad &\longmapsto \int_M \varphi \wedge \psi \end{aligned}$$

Here $\tau(M) = b_+(M) - b_-(M)$ defined in terms of intersection pairing

$$\begin{array}{ccc} H^2(M,\mathbb{R}) \times H^2(M,\mathbb{R}) & \longrightarrow & \mathbb{R} \\ (\ [\varphi]\ ,\ [\psi]\ ) & \longmapsto & \int_M \varphi \wedge \psi \end{array}$$

Diagonalize:

Here $\tau(M) = b_+(M) - b_-(M)$ defined in terms of intersection pairing

$$\begin{aligned} H^2(M,\mathbb{R}) \times H^2(M,\mathbb{R}) &\longrightarrow \quad \mathbb{R} \\ (\ [\varphi]\ ,\ [\psi]\ ) \quad &\longmapsto \int_M \varphi \wedge \psi \end{aligned}$$

Diagonalize:

$$\begin{bmatrix} +1 & & & & & \\ & \ddots & & & & \\ & & +1 & & & \\ & & & -1 & & \\ & & & & \ddots & \\ & & & & & -1 \end{bmatrix}.$$

Here $\tau(M) = b_+(M) - b_-(M)$ defined in terms of intersection pairing

$$\begin{aligned} H^2(M,\mathbb{R}) \times H^2(M,\mathbb{R}) &\longrightarrow \mathbb{R} \\ (\,[\varphi]\,,\,[\psi]\,) &\longmapsto \int_M \varphi \wedge \psi \end{aligned}$$

Diagonalize:

$$\begin{bmatrix} \underbrace{\begin{matrix} +1 & & \\ & \ddots & \\ & & +1 \end{matrix}}_{b_+(M)} & \\ & b_-(M)\left\{\begin{matrix} -1 & & \\ & \ddots & \\ & & -1 \end{matrix}\right. \end{bmatrix}.$$

Hodge theory:

$$H^2(M, \mathbb{R}) = \{\varphi \in \Gamma(\Lambda^2) \mid d\varphi = 0,\ d \star \varphi = 0\}.$$

Hodge theory:

$$H^2(M, \mathbb{R}) = \{\varphi \in \Gamma(\Lambda^2) \mid d\varphi = 0, \; d \star \varphi = 0\}.$$

Since $\star$ is involution of RHS, $\Longrightarrow$

$$H^2(M, \mathbb{R}) = \mathcal{H}^+_g \oplus \mathcal{H}^-_g,$$

Hodge theory:

$$H^2(M,\mathbb{R}) = \{\varphi \in \Gamma(\Lambda^2) \mid d\varphi = 0,\ d\star\varphi = 0\}.$$

Since $\star$ is involution of RHS, $\Longrightarrow$

$$H^2(M,\mathbb{R}) = \mathcal{H}^+_g \oplus \mathcal{H}^-_g,$$

where

$$\mathcal{H}^\pm_g = \{\varphi \in \Gamma(\Lambda^\pm) \mid d\varphi = 0\}$$

self-dual & anti-self-dual harmonic forms.

Hodge theory:

$$H^2(M,\mathbb{R}) = \{\varphi \in \Gamma(\Lambda^2) \mid d\varphi = 0,\ d\star\varphi = 0\}.$$

Since $\star$ is involution of RHS, $\Longrightarrow$

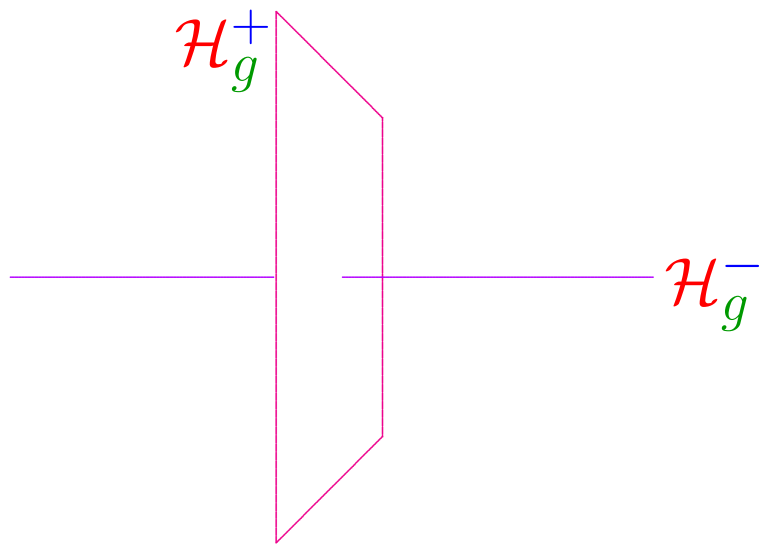
$$H^2(M,\mathbb{R}) = \mathcal{H}^+_g \oplus \mathcal{H}^-_g,$$

where

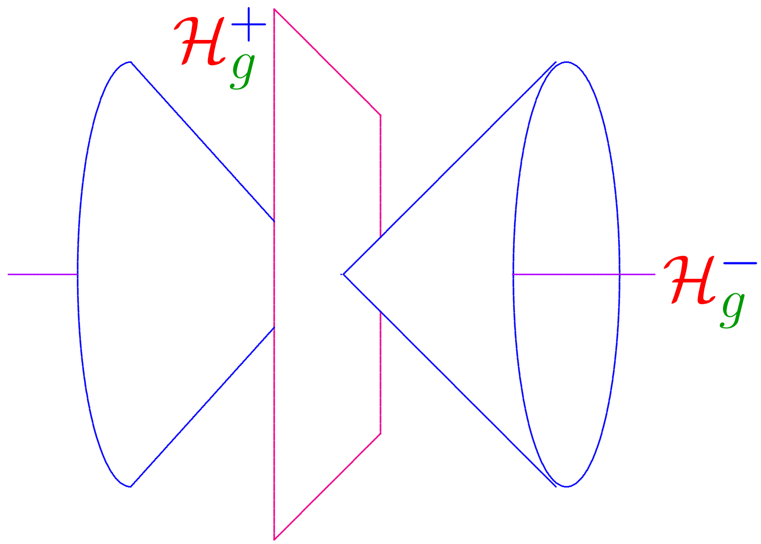
$$\mathcal{H}^\pm_g = \{\varphi \in \Gamma(\Lambda^\pm) \mid d\varphi = 0\}$$

self-dual & anti-self-dual harmonic forms. Then

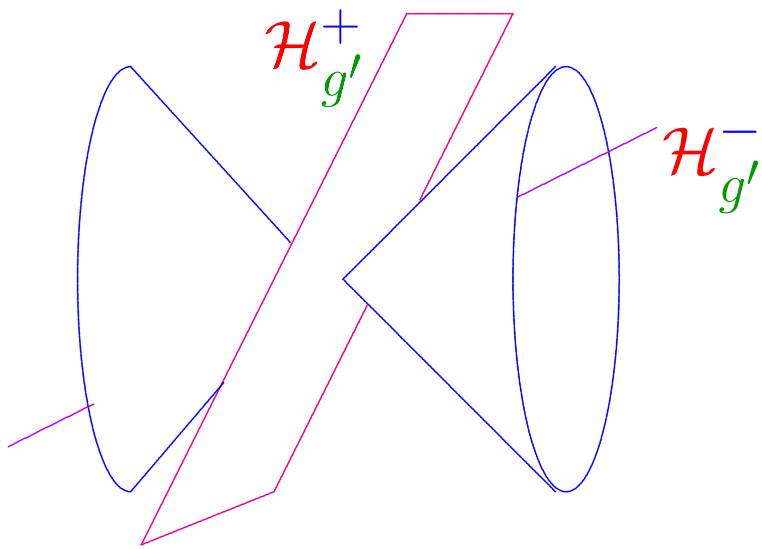
$$b_\pm(M) = \dim \mathcal{H}^\pm_g.$$



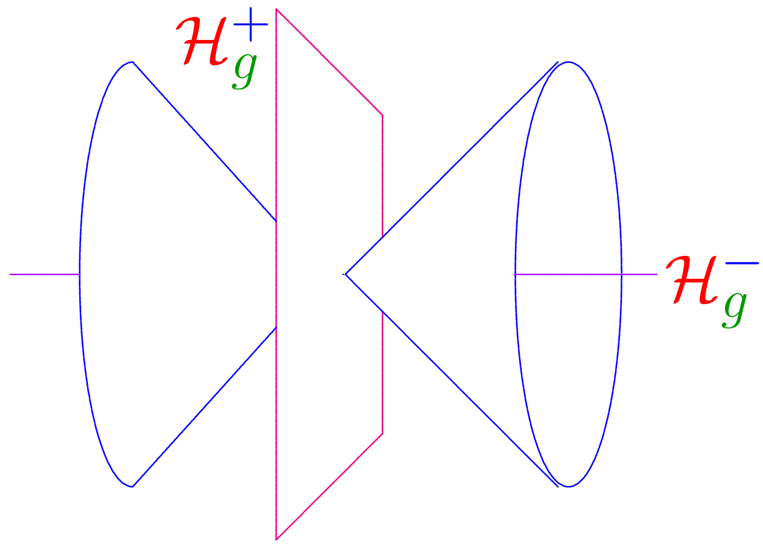
$$H^2(M, \mathbb{R})$$



$$\{a \mid a \cdot a = 0\} \subset H^2(M, \mathbb{R})$$



$$\{a \mid a \cdot a = 0\} \subset H^2(M, \mathbb{R})$$



$$\{a \mid a \cdot a = 0\} \subset H^2(M, \mathbb{R})$$

Hodge theory:

$$H^2(M,\mathbb{R}) = \{\varphi \in \Gamma(\Lambda^2) \mid d\varphi = 0,\ d\star\varphi = 0\}.$$

Since $\star$ is involution of RHS, $\Longrightarrow$

$$H^2(M,\mathbb{R}) = \mathcal{H}^+_g \oplus \mathcal{H}^-_g,$$

where

$$\mathcal{H}^\pm_g = \{\varphi \in \Gamma(\Lambda^\pm) \mid d\varphi = 0\}$$

self-dual & anti-self-dual harmonic forms. Then

$$b_\pm(M) = \dim \mathcal{H}^\pm_g.$$

Hodge theory:

$$H^2(M,\mathbb{R}) = \{\varphi \in \Gamma(\Lambda^2) \mid d\varphi = 0,\ d \star \varphi = 0\}.$$

Since $\star$ is involution of RHS, $\Longrightarrow$

$$H^2(M,\mathbb{R}) = \mathcal{H}^+_g \oplus \mathcal{H}^-_g,$$

where

$$\mathcal{H}^\pm_g = \{\varphi \in \Gamma(\Lambda^\pm) \mid d\varphi = 0\}$$

self-dual & anti-self-dual harmonic forms. Then

$$b_\pm(M) = \dim \mathcal{H}^\pm_g.$$

The subspaces $\mathcal{H}^\pm_g$ are conformally invariant:
Same for g and any $\widehat{g} = u^2 g$.

For (M^4, g) compact oriented Riemannian,

Signature

$$\tau(M) = \frac{1}{3}\langle p_1(TM), [M]\rangle$$

For (M^4, g) compact oriented Riemannian,

Signature

$$\tau(M) = \frac{1}{12\pi^2}\int_M \left(|W_+|^2 - |W_-|^2\right) d\mu$$

For (M^4, g) compact oriented Riemannian,

Signature

$$\tau(M) = \frac{1}{12\pi^2} \int_M \left(|W_+|^2 - |W_-|^2 \right) d\mu$$

$$\mathscr{W}([g]) = \int_M |W_g|^2 d\mu_g$$

For (M^4, g) compact oriented Riemannian,

Signature

$$\tau(M) = \frac{1}{12\pi^2}\int_M \left(|W_+|^2 - |W_-|^2\right) d\mu$$

$$\mathscr{W}([g]) \geq 12\pi^2\tau(M)$$

For (M^4, g) compact oriented Riemannian,

Signature

$$\tau(M) = \frac{1}{12\pi^2}\int_M \left(|W_+|^2 - |W_-|^2\right) d\mu$$

$$\mathscr{W}([g]) \geq 12\pi^2 \tau(M)$$

with $= \iff W_- \equiv 0.$

For (M^4, g) compact oriented Riemannian,

Signature

$$\tau(M) = \frac{1}{12\pi^2}\int_M \left(|W_+|^2 - |W_-|^2\right) d\mu$$

$$\mathscr{W}([g]) \geq 12\pi^2\tau(M)$$

with $= \iff W_- \equiv 0.$ "self-dual"

For (M^4, g) compact oriented Riemannian,

Signature

$$\tau(M) = \frac{1}{12\pi^2}\int_M \left(|W_+|^2 - |W_-|^2\right) d\mu$$

$$\mathscr{W}([g]) \geq -12\pi^2\tau(M)$$

For (M^4, g) compact oriented Riemannian,

Signature

$$\tau(M) = \frac{1}{12\pi^2}\int_M \left(|W_+|^2 - |W_-|^2\right) d\mu$$

$$\mathscr{W}([g]) \geq -12\pi^2\tau(M)$$

$$\text{with} = \iff W_+ \equiv 0.$$

For (M^4, g) compact oriented Riemannian,

Signature

$$\tau(M) = \frac{1}{12\pi^2}\int_M \left(|W_+|^2 - |W_-|^2\right) d\mu$$

$$\mathscr{W}([g]) \geq -12\pi^2\tau(M)$$

with $= \iff W_+ \equiv 0.$ "anti-self-dual"

For (M^4, g) compact oriented Riemannian,

Signature

$$\tau(M) = \frac{1}{12\pi^2}\int_M \left(|W_+|^2 - |W_-|^2\right) d\mu$$

$$\mathscr{W}([g]) \geq -12\pi^2\tau(M)$$

with $= \iff W_+ \equiv 0$. "anti-self-dual"

Reversing orientation $\rightsquigarrow$

self-duality $\longleftrightarrow$ anti-self-duality

For (M^4, g) compact oriented Riemannian,

Signature

$$\tau(M) = \frac{1}{12\pi^2} \int_M \left(|W_+|^2 - |W_-|^2 \right) d\mu$$

$$\mathscr{W}([g]) \geq -12\pi^2 \tau(M)$$

with $= \iff W_+ \equiv 0$. "anti-self-dual"

1985-1995: $\exists$ self-dual/anti-self g on many M^4.

For (M^4, g) compact oriented Riemannian,

Signature

$$\tau(M) = \frac{1}{12\pi^2}\int_M \left(|W_+|^2 - |W_-|^2\right) d\mu$$

$$\mathscr{W}([g]) \geq -12\pi^2\tau(M)$$

with $= \iff W_+ \equiv 0$. "anti-self-dual"

1985-1995: $\exists$ self-dual/anti-self g on many M^4.

Poon, L, Donaldson-Friedman, Taubes ...

For (M^4, g) compact oriented Riemannian,

Signature

$$\tau(M) = \frac{1}{12\pi^2}\int_M \left(|W_+|^2 - |W_-|^2\right) d\mu$$

For (M^4, g) compact oriented Riemannian,

Signature

$$\tau(M) = \frac{1}{12\pi^2}\int_M \left(|W_+|^2 - |W_-|^2\right) d\mu$$

Proposition (Atiyah-Hitchin-Singer). *The Fubini-Study metric on $\mathbb{CP}_2$ is self-dual. Consequently, minimizes Weyl functional.*

For (M^4, g) compact oriented Riemannian,

Signature

$$\tau(M) = \frac{1}{12\pi^2}\int_M \left(|W_+|^2 - |W_-|^2\right) d\mu$$

Theorem (Poon '86). *Up conformal isometry, the Fubini-Study class is the unique self-dual conformal class on* $\mathbb{CP}_2$ *with* $Y([g]) > 0$.

$$Y([g]) = \inf_{\widehat{g}=u^2 g} \frac{\int_M s_{\widehat{g}} \, d\mu_{\widehat{g}}}{\sqrt{\int_M d\mu_{\widehat{g}}}} \; ;$$

$$Y([g]) = \inf_{\widehat{g}=u^2 g} \frac{\int_M s_{\widehat{g}} \, d\mu_{\widehat{g}}}{\sqrt{\int_M d\mu_{\widehat{g}}}} \; ;$$

If g has s of fixed sign, agrees with sign of $Y_{[g]}$.

For (M^4, g) compact oriented Riemannian,

Signature

$$\tau(M) = \frac{1}{12\pi^2}\int_M \left(|W_+|^2 - |W_-|^2\right) d\mu$$

Theorem (Poon '86). *Up conformal isometry, the Fubini-Study class is the unique self-dual conformal class on* $\mathbb{CP}_2$ *with* $Y([g]) > 0$.

For (M^4, g) compact oriented Riemannian,

Signature

$$\tau(M) = \frac{1}{12\pi^2}\int_M \left(|W_+|^2 - |W_-|^2\right) d\mu$$

Proposition (Atiyah-Hitchin-Singer '78). *The Fubini-Study metric on $\mathbb{CP}_2$ is self-dual. Consequently, minimizes Weyl functional.*

Osamu Kobayashi '86:

Osamu Kobayashi ’86:

What about $S^2 \times S^2$?

Osamu Kobayashi '86:

What about $S^2 \times S^2$?

No self-dual metric!

Osamu Kobayashi '86:

What about $S^2 \times S^2$?

No self-dual metric!

Would be conformally flat, because $\tau = 0$.

$$\tau(M) = \frac{1}{12\pi^2} \int_M \left(|W_+|^2 - |W_-|^2 \right) d\mu$$

Osamu Kobayashi '86:

What about $S^2 \times S^2$?

No self-dual metric!

Would be conformally flat, because $\tau = 0$.

Also $\pi_1 = 0$.

Osamu Kobayashi '86:

What about $S^2 \times S^2$?

No self-dual metric!

Would be conformally flat, because $\tau = 0$.

Also $\pi_1 = 0$.

Kuiper '49: $\therefore$ Round S^4! $\Rightarrow\Leftarrow$

Osamu Kobayashi '86:

What about $S^2 \times S^2$?

Osamu Kobayashi '86:

What about $S^2 \times S^2$?

Conjecture (Kobayashi). *The Kähler-Einstein product metric on* $S^2 \times S^2$ *minimizes the Weyl functional* $\mathscr{W}$.

Osamu Kobayashi '86:

What about $S^2 \times S^2$?

Conjecture (Kobayashi)**.** *The Kähler-Einstein product metric on* $S^2 \times S^2$ *minimizes the Weyl functional* $\mathscr{W}$ *.*

Gave weak evidence:

Osamu Kobayashi '86:

What about $S^2 \times S^2$?

Conjecture (Kobayashi)**.** *The Kähler-Einstein product metric on* $S^2 \times S^2$ *minimizes the Weyl functional* $\mathscr{W}$.

Gave weak evidence:

Local minimum.

Osamu Kobayashi '86:

What about $S^2 \times S^2$?

Conjecture (Kobayashi)**.** *The Kähler-Einstein product metric on* $S^2 \times S^2$ *minimizes the Weyl functional* $\mathscr{W}$*.*

Osamu Kobayashi '86:

What about $S^2 \times S^2$?

Conjecture (Kobayashi)**.** *The Kähler-Einstein product metric on* $S^2 \times S^2$ *minimizes the Weyl functional* $\mathscr{W}$*.*

Commonality between $\mathbb{CP}_2$ and $S^2 \times S^2$?

Osamu Kobayashi '86:

What about $S^2 \times S^2$?

Conjecture (Kobayashi). *The Kähler-Einstein product metric on* $S^2 \times S^2$ *minimizes the Weyl functional* $\mathscr{W}$.

Commonality between $\mathbb{CP}_2$ and $S^2 \times S^2$?

Kähler-Einstein, with $\lambda > 0$.

Osamu Kobayashi '86:

What about $S^2 \times S^2$?

Conjecture (Kobayashi)**.** *The Kähler-Einstein product metric on $S^2 \times S^2$ minimizes the Weyl functional $\mathscr{W}$.*

Commonality between $\mathbb{CP}_2$ and $S^2 \times S^2$?

Kähler-Einstein, with $\lambda > 0$.

Kähler-Einstein means Einstein, and also Kähler.

Osamu Kobayashi '86:

What about $S^2 \times S^2$?

Conjecture (Kobayashi). *The Kähler-Einstein product metric on $S^2 \times S^2$ minimizes the Weyl functional $\mathscr{W}$.*

Commonality between $\mathbb{CP}_2$ and $S^2 \times S^2$?

Kähler-Einstein, with $\lambda > 0$.

Kähler-Einstein means Einstein, and also Kähler.

Kähler means there exists an almost-complex structure J that is invariant under parallel transport with respect to g:

$$\nabla J = 0.$$

Natural Generalization:

Del Pezzo surfaces:

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points,
in general position,

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$,
in general position,

Del Pezzo surfaces:

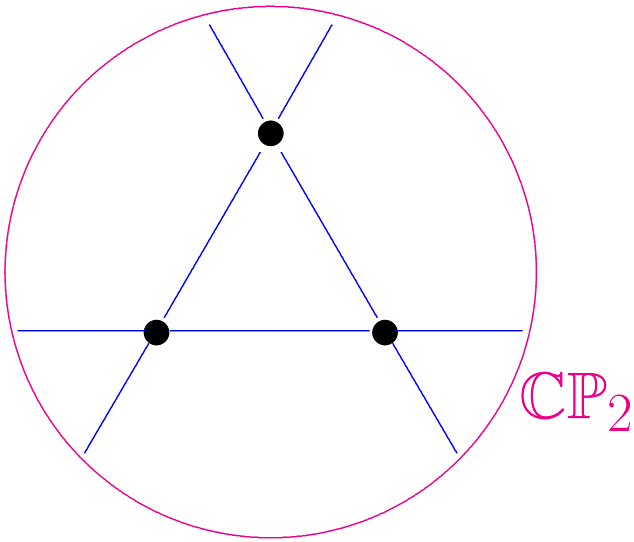
(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.



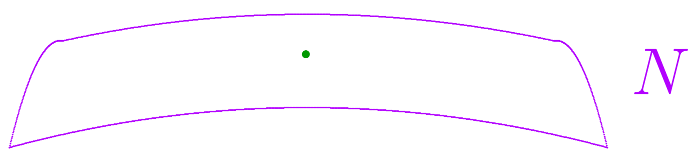
Blowing up:

Blowing up:

If N is a complex surface,

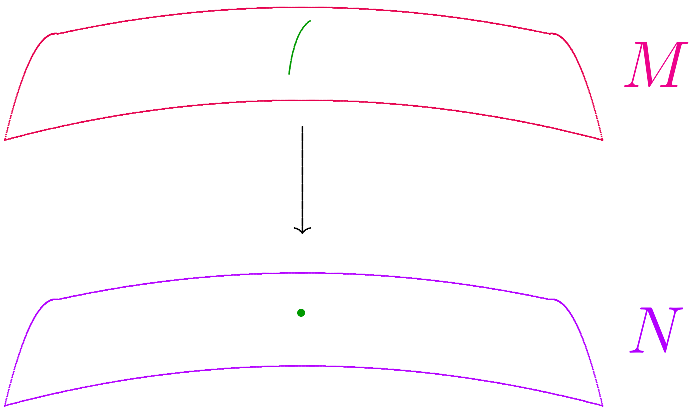
Blowing up:

If N is a complex surface, may replace $p \in N$



Blowing up:

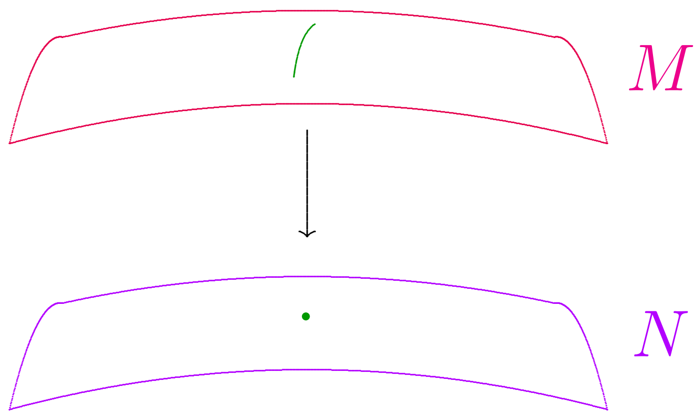
If N is a complex surface, may replace $p \in N$ with $\mathbb{CP}_1$



Blowing up:

If N is a complex surface, may replace $p \in N$ with $\mathbb{CP}_1$ to obtain blow-up

$$M \approx N \# \overline{\mathbb{CP}}_2$$



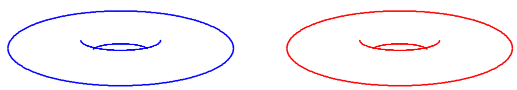
Conventions:

$\overline{\mathbb{CP}}_2$ = reverse oriented $\mathbb{CP}_2$.

Conventions:

$\overline{\mathbb{CP}_2}$ = reverse oriented $\mathbb{CP}_2$.

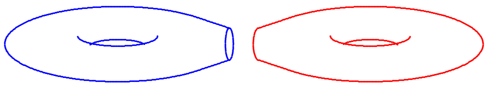
Connected sum $\#$:



Conventions:

$\overline{\mathbb{CP}}_2$ = reverse oriented $\mathbb{CP}_2$.

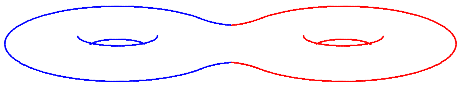
Connected sum #:



Conventions:

$\overline{\mathbb{CP}_2}$ = reverse oriented $\mathbb{CP}_2$.

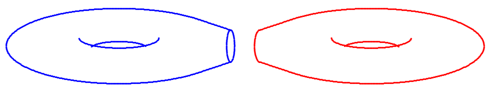
Connected sum #:



Conventions:

$\overline{\mathbb{CP}}_2$ = reverse oriented $\mathbb{CP}_2$.

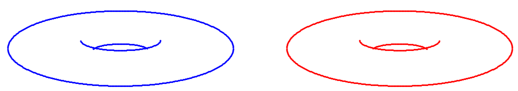
Connected sum $\#$:



Conventions:

$\overline{\mathbb{CP}_2}$ = reverse oriented $\mathbb{CP}_2$.

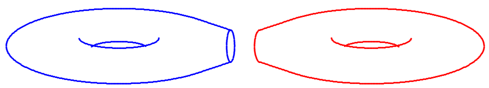
Connected sum #:



Conventions:

$\overline{\mathbb{CP}_2}$ = reverse oriented $\mathbb{CP}_2$.

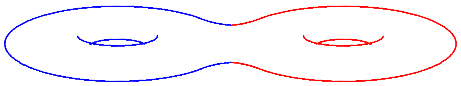
Connected sum $\#$:



Conventions:

$\overline{\mathbb{CP}_2}$ = reverse oriented $\mathbb{CP}_2$.

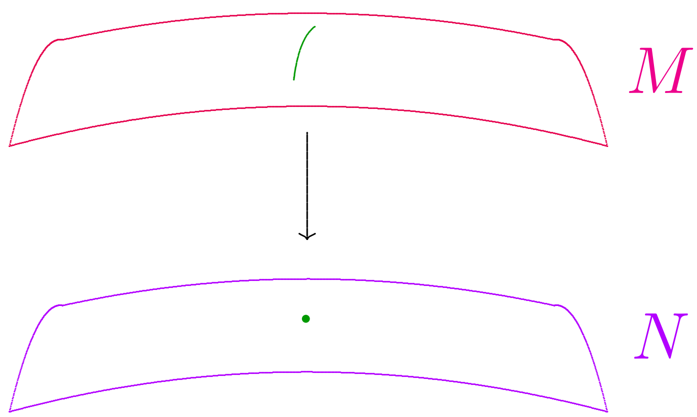
Connected sum $\#$:



Blowing up:

If N is a complex surface, may replace $p \in N$ with $\mathbb{CP}_1$ to obtain blow-up

$$M \approx N \# \overline{\mathbb{CP}}_2$$

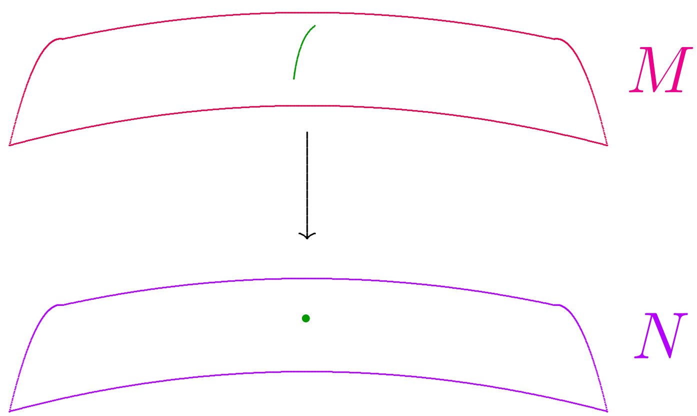


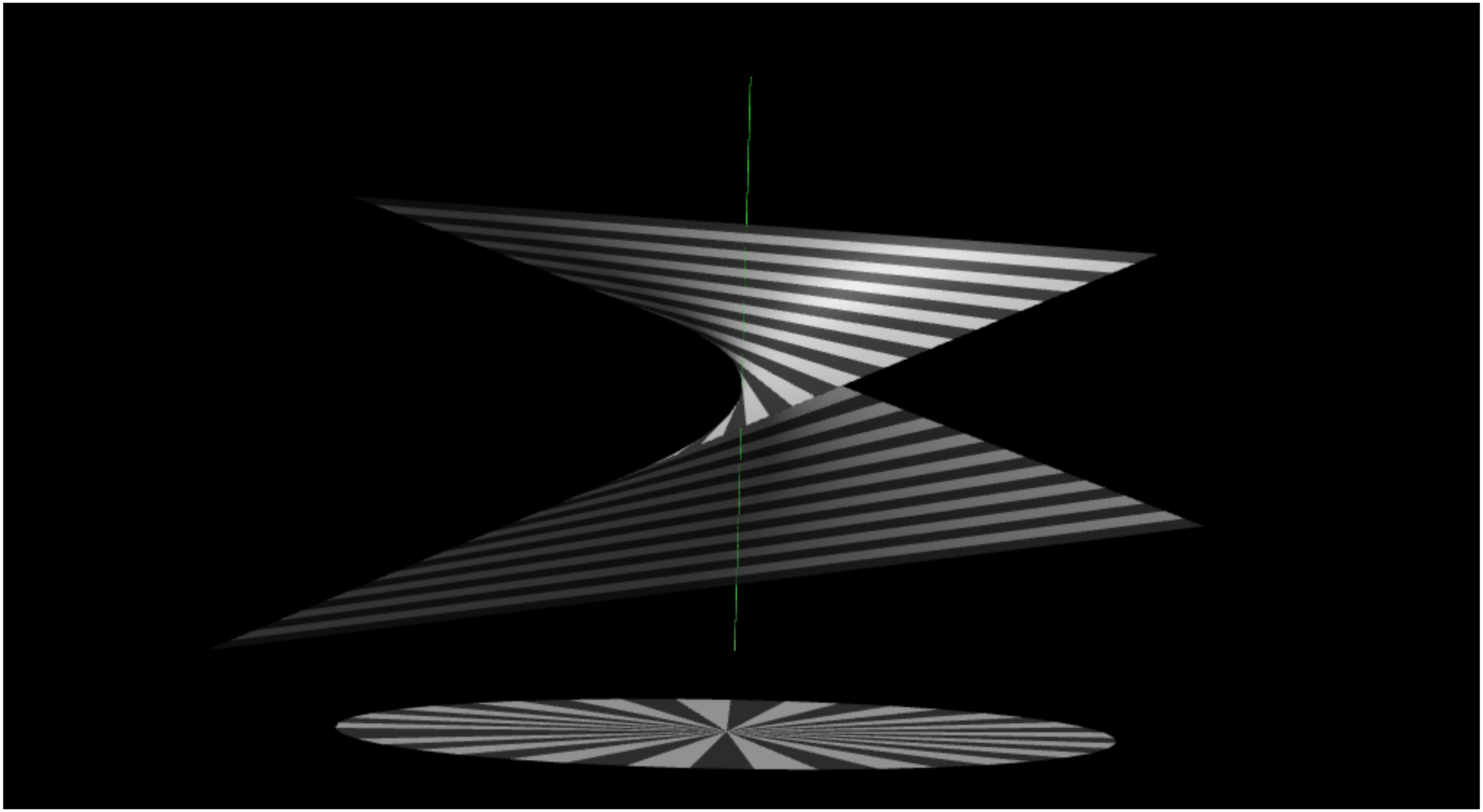
Blowing up:

If N is a complex surface, may replace $p \in N$ with $\mathbb{CP}_1$ to obtain blow-up

$$M \approx N \# \overline{\mathbb{CP}}_2$$

in which added $\mathbb{CP}_1$ has normal bundle $\mathcal{O}(-1)$.



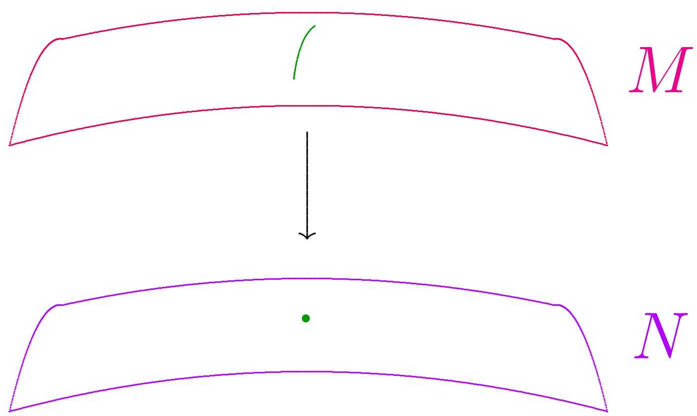


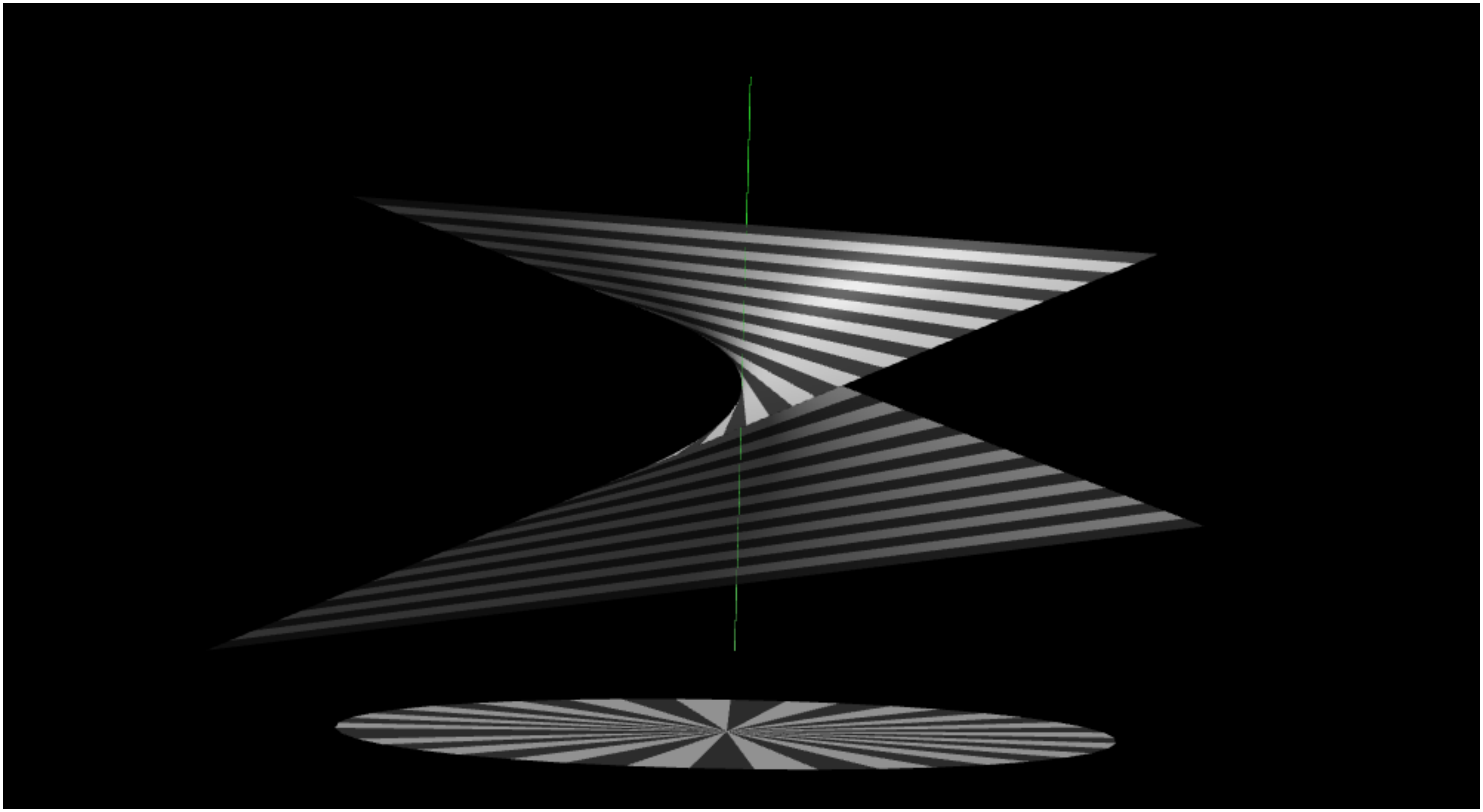
Blowing up:

If N is a complex surface, may replace $p \in N$ with $\mathbb{CP}_1$ to obtain blow-up

$$M \approx N \# \overline{\mathbb{CP}}_2$$

in which added $\mathbb{CP}_1$ has normal bundle $\mathcal{O}(-1)$.



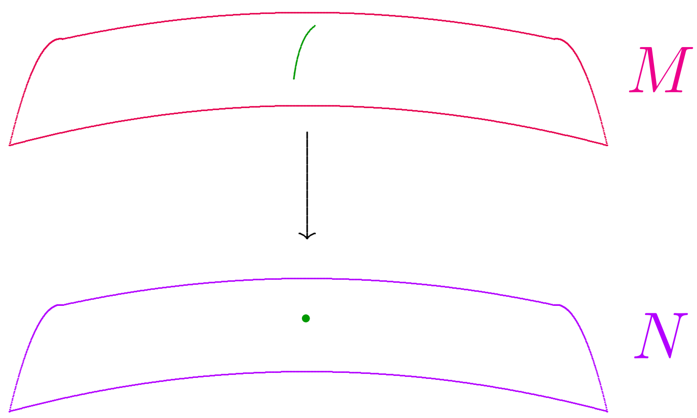


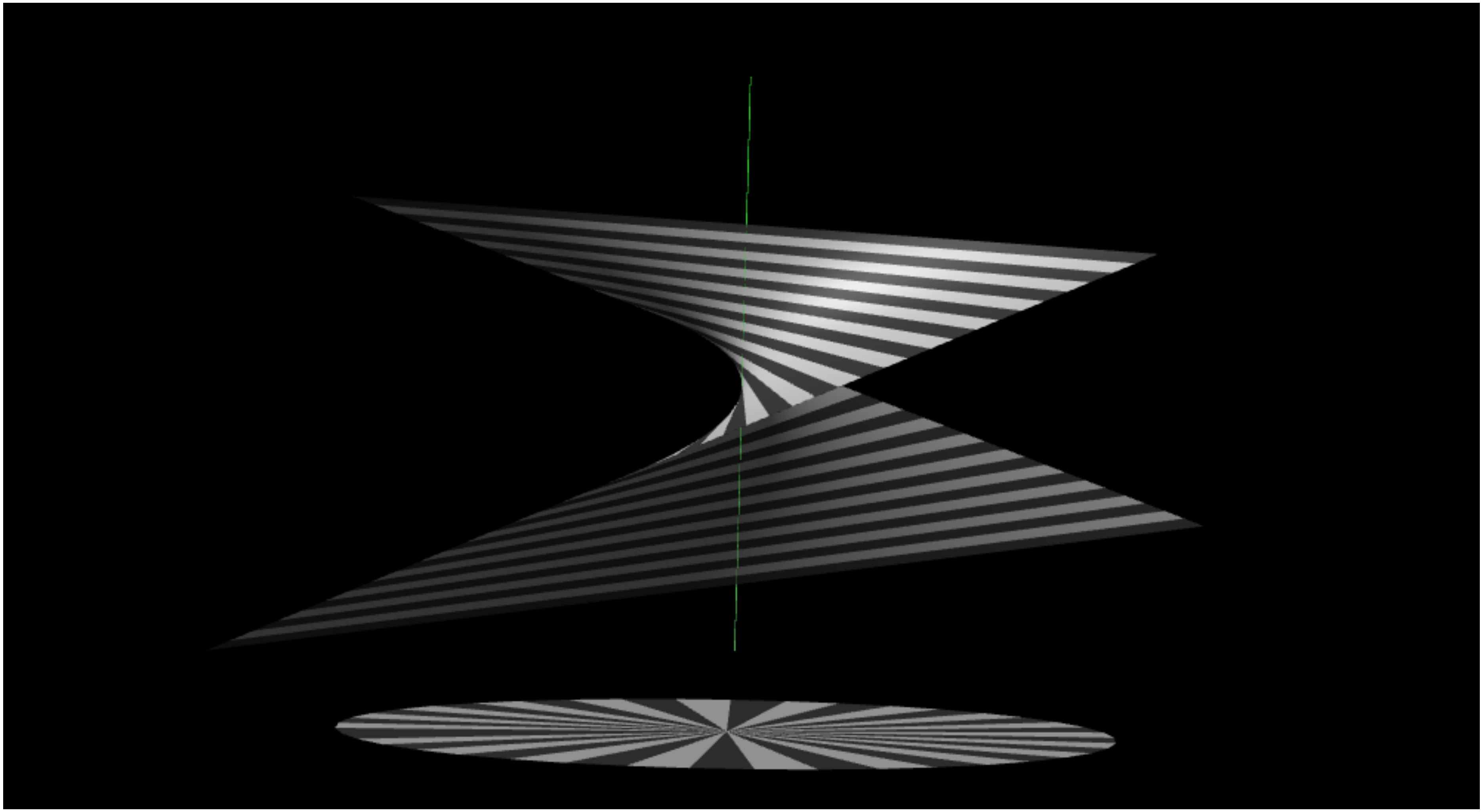
Blowing up:

If N is a complex surface, may replace $p \in N$ with $\mathbb{CP}_1$ to obtain blow-up

$$M \approx N \# \overline{\mathbb{CP}}_2$$

in which added $\mathbb{CP}_1$ has normal bundle $\mathcal{O}(-1)$.



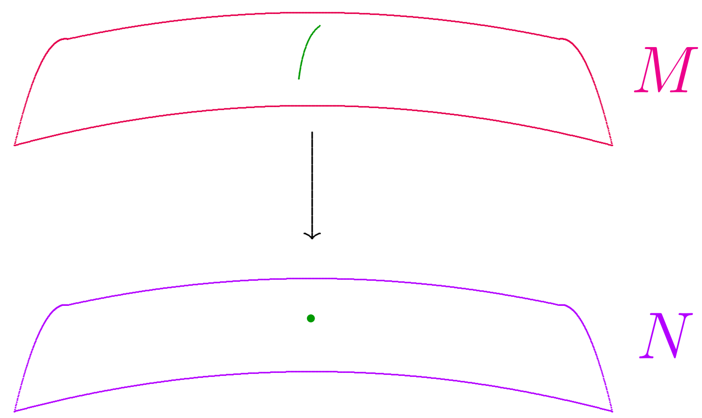


Blowing up:

If N is a complex surface, may replace $p \in N$ with $\mathbb{CP}_1$ to obtain blow-up

$$M \approx N \# \overline{\mathbb{CP}}_2$$

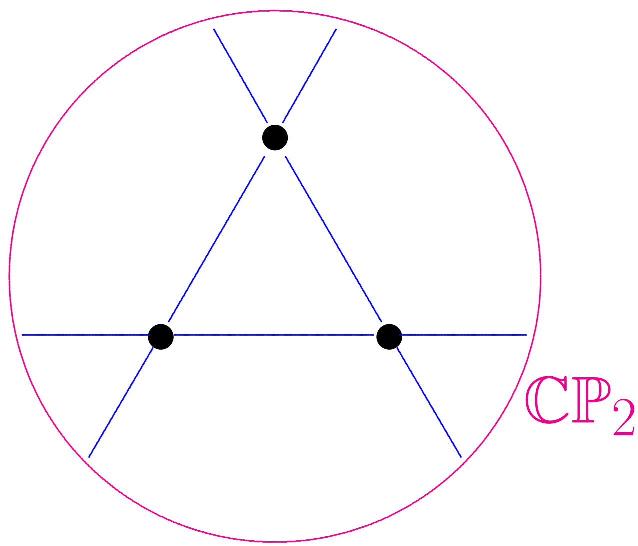
in which added $\mathbb{CP}_1$ has normal bundle $\mathcal{O}(-1)$.



Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

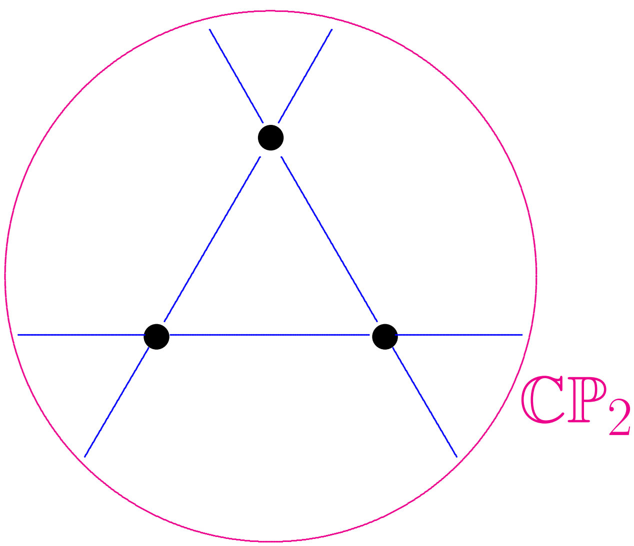
Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.



Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

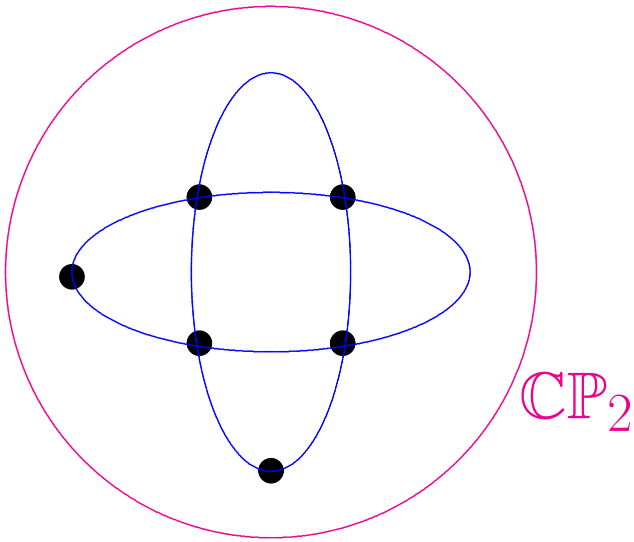


No 3 on a line,

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

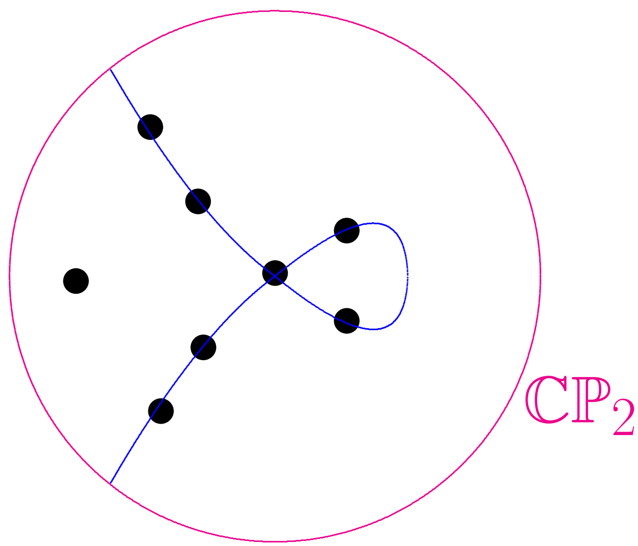


No 3 on a line, no 6 on conic,

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.



No 3 on a line, no 6 on conic, no 8 on nodal cubic.

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Theorem.

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Theorem. *Each del Pezzo* (M^4, J) *admits a* J*-compatible*

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Theorem. *Each del Pezzo (M^4, J) admits a J-compatible conformally Kähler,*

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Theorem. *Each del Pezzo* (M^4, J) *admits a* J*-compatible conformally Kähler, Einstein metric,*

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Theorem. *Each del Pezzo (M^4, J) admits a J-compatible conformally Kähler, Einstein metric, and this metric is unique*

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Theorem. *Each del Pezzo (M^4, J) admits a J-compatible conformally Kähler, Einstein metric, and this metric is unique up to complex automorphisms and constant rescalings.*

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Theorem. *Each del Pezzo (M^4, J) admits a J-compatible conformally Kähler, Einstein metric, and this metric is geometrically unique.*

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Theorem. *Each del Pezzo (M^4, J) admits a J-compatible conformally Kähler, Einstein metric, and this metric is geometrically unique.*

Existence: Page

1978

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Theorem. *Each del Pezzo (M^4, J) admits a J-compatible conformally Kähler, Einstein metric, and this metric is geometrically unique.*

Existence: Page-Derdziński,

1983

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Theorem. *Each del Pezzo (M^4, J) admits a J-compatible conformally Kähler, Einstein metric, and this metric is geometrically unique.*

Existence: Page-Derdziński, Siu,

1986

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Theorem. *Each del Pezzo (M^4, J) admits a J-compatible conformally Kähler, Einstein metric, and this metric is geometrically unique.*

Existence: Page-Derdziński, Siu, Tian-Yau,

1987

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Theorem. *Each del Pezzo (M^4, J) admits a J-compatible conformally Kähler, Einstein metric, and this metric is geometrically unique.*

Existence: Page-Derdziński, Siu, Tian-Yau, Tian,

1990

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Theorem. *Each del Pezzo (M^4, J) admits a J-compatible conformally Kähler, Einstein metric, and this metric is geometrically unique.*

Existence: Page-Derdziński, Siu, Tian-Yau, Tian, Odaka-Spotti-Sun,

2016

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Theorem. *Each del Pezzo (M^4, J) admits a J-compatible conformally Kähler, Einstein metric, and this metric is geometrically unique.*

Existence: Page-Derdziński, Siu, Tian-Yau, Tian, Odaka-Spotti-Sun, Chen-L-Weber.

2008

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Theorem. *Each del Pezzo (M^4, J) admits a J-compatible conformally Kähler, Einstein metric, and this metric is geometrically unique.*

Uniqueness: Bando-Mabuchi '87

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.

Theorem. *Each del Pezzo (M^4, J) admits a J-compatible conformally Kähler, Einstein metric, and this metric is geometrically unique.*

Uniqueness: Bando-Mabuchi '87, L '12.

One reason this seems satisfying. . .

Theorem (CLW '08). *Suppose that M is a smooth compact oriented 4-manifold which carries some symplectic form ω.*

Theorem (CLW '08). *Suppose that* M *is a smooth compact oriented* 4*-manifold which carries some symplectic form* ω. *Then* M *admits an (unrelated) Einstein metric* g *with* $\lambda > 0$

Theorem (CLW '08). *Suppose that M is a smooth compact oriented 4-manifold which carries some symplectic form ω. Then M admits an (unrelated) Einstein metric g with $\lambda > 0$*

$$\Longleftrightarrow \quad M \approx \begin{cases} \mathbb{CP}_2 \# k \overline{\mathbb{CP}}_2, & 0 \leq k \leq 8, \\ \quad or & \\ S^2 \times S^2 & \end{cases}$$

Theorem (CLW '08). *Suppose that M is a smooth compact oriented 4-manifold which carries some symplectic form ω. Then M admits an (unrelated) Einstein metric g with $\lambda > 0$*

$$\Longleftrightarrow \quad M \approx \begin{cases} \mathbb{CP}_2 \# k \overline{\mathbb{CP}}_2, & 0 \leq k \leq 8, \\ \quad or & \\ S^2 \times S^2 & \end{cases}$$

Diffeotypes: exactly the Del Pezzo surfaces.

Theorem (CLW '08). *Suppose that M is a smooth compact oriented 4-manifold which carries some symplectic form ω. Then M admits an (unrelated) Einstein metric g with $\lambda > 0$*

$$\Longleftrightarrow \quad M \approx \begin{cases} \mathbb{CP}_2 \# k \overline{\mathbb{CP}}_2, & 0 \leq k \leq 8, \\ \quad or & \\ S^2 \times S^2 & \end{cases}$$

Diffeotypes: exactly the Del Pezzo surfaces.

For known g, can take ω harmonic self-dual 2-form.

Theorem (CLW '08). *Suppose that M is a smooth compact oriented 4-manifold which carries some symplectic form ω. Then M admits an (unrelated) Einstein metric g with $\lambda > 0$*

$$\Longleftrightarrow \quad M \approx \begin{cases} \mathbb{CP}_2 \# k \overline{\mathbb{CP}}_2, & 0 \leq k \leq 8, \\ \quad or & \\ S^2 \times S^2 & \end{cases}$$

Diffeotypes: exactly the Del Pezzo surfaces.

For known g, can take ω harmonic self-dual 2-form.

But this is not needed in above result.

Osamu Kobayashi '86:

What about $S^2 \times S^2$?

Conjecture (Kobayashi)**.** *The Kähler-Einstein product metric on* $S^2 \times S^2$ *minimizes the Weyl functional* $\mathscr{W}$.

Natural Generalization:

Natural Generalization:

Conjecture. *On any del Pezzo surface* (M^4, J)*, the conformally Kähler, Einstein product metric minimizes the Weyl functional* $\mathscr{W}$ *.*

Natural Generalization:

Conjecture. *On any del Pezzo surface* (M^4, J)*, the conformally Kähler, Einstein product metric minimizes the Weyl functional* $\mathscr{W}$.

Persuasive partial results.

Natural Generalization:

Conjecture. *On any del Pezzo surface* (M^4, J), *the conformally Kähler, Einstein product metric minimizes the Weyl functional* $\mathscr{W}$.

Persuasive partial results.

But problem still not settled!

Theorem (Gursky '98). *Let M be a smooth compact 4-manifold with $b_+(M) \neq 0$.*

Theorem (Gursky '98). *Let M be a smooth compact 4-manifold with $b_+(M) \neq 0$. Then any conformal class* $[g]$

Theorem (Gursky '98). *Let* M *be a smooth compact* 4*-manifold with* $b_+(M) \neq 0$*. Then any conformal class* $[g]$ *with* $Y([g]) > 0$

Theorem (Gursky '98). *Let M be a smooth compact 4-manifold with $b_+(M) \neq 0$. Then any conformal class $[g]$ with $Y([g]) > 0$*

i.e. represented by metric with $s > 0$.

$$Y([g]) = \inf_{\widehat{g}=u^2 g} \frac{\int_M s_{\widehat{g}} \; d\mu_{\widehat{g}}}{\sqrt{\int_M d\mu_{\widehat{g}}}} \; ;$$

Theorem (Gursky '98). *Let* M *be a smooth compact* 4*-manifold with* $b_+(M) \neq 0$*. Then any conformal class* $[g]$ *with* $Y([g]) > 0$ *satisfies*

Theorem (Gursky '98). *Let* M *be a smooth compact* 4*-manifold with* $b_+(M) \neq 0$*. Then any conformal class* $[g]$ *with* $Y([g]) > 0$ *satisfies*

$$\int_M |W_+|^2 d\mu \geq \frac{4\pi^2}{3}(2\chi + 3\tau)(M),$$

Theorem (Gursky '98). *Let* M *be a smooth compact* 4*-manifold with* $b_+(M) \neq 0$. *Then any conformal class* $[g]$ *with* $Y([g]) > 0$ *satisfies*

$$\int_M |W_+|^2 d\mu \geq \frac{4\pi^2}{3}(2\chi + 3\tau)(M),$$

with equality $\Leftrightarrow$ $[g]$ *contains Kähler-Einstein* $\widehat{g}$ *with* $s > 0$.

Theorem (Gursky '98). *Let M be a smooth compact 4-manifold with $b_+(M) \neq 0$. Then any conformal class $[g]$ with $Y([g]) > 0$ satisfies*

$$\int_M |W_+|^2 d\mu \geq \frac{4\pi^2}{3}(2\chi + 3\tau)(M),$$

with equality $\Leftrightarrow$ $[g]$ contains Kähler-Einstein $\widehat{g}$ with $s > 0$.

In particular, any K-E g with $s > 0$ minimizes restriction of $\mathscr{W}$ to $s > 0$ metrics.

Theorem (Gursky '98). *Let M be a smooth compact 4-manifold with $b_+(M) \neq 0$. Then any conformal class $[g]$ with $Y([g]) > 0$ satisfies*

$$\int_M |W_+|^2 d\mu \geq \frac{4\pi^2}{3}(2\chi + 3\tau)(M),$$

with equality $\Leftrightarrow$ $[g]$ contains Kähler-Einstein $\widehat{g}$ with $s > 0$.

In particular, any K-E g with $s > 0$ minimizes restriction of $\mathscr{W}$ to $s > 0$ metrics.

Big step towards generalized Kobayashi conjecture!

Theorem (Gursky '98). *Let M be a smooth compact 4-manifold with $b_+(M) \neq 0$. Then any conformal class $[g]$ with $Y([g]) > 0$ satisfies*

$$\int_M |W_+|^2 d\mu \geq \frac{4\pi^2}{3}(2\chi + 3\tau)(M),$$

with equality $\Leftrightarrow$ $[g]$ contains Kähler-Einstein $\widehat{g}$ with $s > 0$.

In particular, any K-E g with $s > 0$ minimizes restriction of $\mathscr{W}$ to $s > 0$ metrics.

Big step towards generalized Kobayashi conjecture!

But applies in much greater generality.

Theorem (Gursky '98). *Let* M *be a smooth compact 4-manifold with* $b_+(M) \neq 0$. *Then any conformal class* $[g]$ *with* $Y([g]) > 0$ *satisfies*

$$\int_M |W_+|^2 d\mu \geq \frac{4\pi^2}{3}(2\chi + 3\tau)(M),$$

with equality $\Leftrightarrow$ $[g]$ *contains Kähler-Einstein* $\widehat{g}$ *with* $s > 0$.

In particular, any K-E g with $s > 0$ minimizes restriction of $\mathscr{W}$ to $s > 0$ metrics.

Big step towards generalized Kobayashi conjecture.

But says nothing about $Y([g]) < 0$ realm.

Theorem (Gursky '98). *Let M be a smooth compact 4-manifold with $b_+(M) \neq 0$. Then any conformal class $[g]$ with $Y([g]) > 0$ satisfies*

$$\int_M |W_+|^2 d\mu \geq \frac{4\pi^2}{3}(2\chi + 3\tau)(M),$$

with equality $\Leftrightarrow$ $[g]$ contains Kähler-Einstein $\widehat{g}$ with $s > 0$.

In particular, any K-E g with $s > 0$ minimizes restriction of $\mathscr{W}$ to $s > 0$ metrics.

Big step towards generalized Kobayashi conjecture.

Hence says nothing about "most" conformal classes.

Theorem (Gursky '98). *Let M be a smooth compact 4-manifold with $b_+(M) \neq 0$. Then any conformal class $[g]$ with $Y([g]) > 0$ satisfies*

$$\int_M |W_+|^2 d\mu \geq \frac{4\pi^2}{3}(2\chi + 3\tau)(M),$$

with equality $\Leftrightarrow$ $[g]$ contains Kähler-Einstein $\hat{g}$ with $s > 0$.

Method: Weitzenböck formula

Theorem (Gursky '98). *Let M be a smooth compact 4-manifold with $b_+(M) \neq 0$. Then any conformal class $[g]$ with $Y([g]) > 0$ satisfies*

$$\int_M |W_+|^2 d\mu \geq \frac{4\pi^2}{3}(2\chi + 3\tau)(M),$$

with equality $\Leftrightarrow$ $[g]$ contains Kähler-Einstein $\widehat{g}$ with $s > 0$.

Method: Weitzenböck formula

$$0 = \frac{1}{2}\Delta|\omega|^2 + |\nabla\omega|^2 - 2W_+(\omega,\omega) + \frac{s}{3}|\omega|^2$$

for self-dual harmonic 2-form ω.

A different use of self-dual harmonic forms yields a complementary result.

A different use of self-dual harmonic forms yields a complementary result.

Definition. *A conformal class* $[g]$ *on a compact oriented* 4*-manifold* M *is said to be of symplectic type*

A different use of self-dual harmonic forms yields a complementary result.

Definition. *A conformal class* $[g]$ *on a compact oriented* 4*-manifold* M *is said to be of symplectic type* $\Leftrightarrow$

A different use of self-dual harmonic forms yields a complementary result.

Definition. *A conformal class* $[g]$ *on a compact oriented* 4*-manifold* M *is said to be of symplectic type* $\Leftrightarrow \exists$ *self-dual harmonic* 2*-form* ω *on* (M, g)

A different use of self-dual harmonic forms yields a complementary result.

Definition. *A conformal class* $[g]$ *on a compact oriented* 4*-manifold* M *is said to be of symplectic type* $\Leftrightarrow \exists$ *self-dual harmonic* 2*-form* ω *on* (M, g) *such that* $\omega \neq 0$ *everywhere.*

A different use of self-dual harmonic forms yields a complementary result.

Definition. *A conformal class* $[g]$ *on a compact oriented* 4*-manifold* M *is said to be of symplectic type* $\Leftrightarrow \exists$ *self-dual harmonic* 2*-form* ω *on* (M, g) *such that* $\omega \neq 0$ *everywhere.*

Implies ω is orientation-compatible symplectic form.

A different use of self-dual harmonic forms yields a complementary result.

Definition. *A conformal class $[g]$ on a compact oriented 4-manifold M is said to be of symplectic type* $\Leftrightarrow \exists$ *self-dual harmonic 2-form ω on (M, g) such that $\omega \neq 0$ everywhere.*

Implies ω is orientation-compatible symplectic form.

Every symplectic 4-manifold arises this way.

A different use of self-dual harmonic forms yields a complementary result.

Definition. *A conformal class* $[g]$ *on a compact oriented* 4*-manifold* M *is said to be of symplectic type* $\Leftrightarrow \exists$ *self-dual harmonic* 2*-form* ω *on* (M, g) *such that* $\omega \neq 0$ *everywhere.*

Implies ω is orientation-compatible symplectic form.

Every symplectic 4-manifold arises this way.

Choose $g \in [g]$ so that $|\omega| \equiv \sqrt{2}$.

A different use of self-dual harmonic forms yields a complementary result.

Definition. *A conformal class* $[g]$ *on a compact oriented* 4*-manifold* M *is said to be of symplectic type* $\Leftrightarrow \exists$ *self-dual harmonic* 2*-form* ω *on* (M, g) *such that* $\omega \neq 0$ *everywhere.*

Implies ω is orientation-compatible symplectic form.

Every symplectic 4-manifold arises this way.

Choose $g \in [g]$ so that $|\omega| \equiv \sqrt{2}$.

Then (M, g, ω) is almost-Kähler manifold:

A different use of self-dual harmonic forms yields a complementary result.

Definition. *A conformal class* $[g]$ *on a compact oriented* 4*-manifold* M *is said to be of symplectic type* $\Leftrightarrow \exists$ *self-dual harmonic* 2*-form* ω *on* (M, g) *such that* $\omega \neq 0$ *everywhere.*

Implies ω is orientation-compatible symplectic form.

Every symplectic 4-manifold arises this way.

Choose $g \in [g]$ so that $|\omega| \equiv \sqrt{2}$.

Then (M, g, ω) is almost-Kähler manifold:

$$\exists J \quad s.t. \quad \omega = g(J\cdot, \cdot)$$

A different use of self-dual harmonic forms yields a complementary result.

Definition. *A conformal class* $[g]$ *on a compact oriented* 4*-manifold* M *is said to be of symplectic type* $\Leftrightarrow \exists$ *self-dual harmonic* 2*-form* ω *on* (M, g) *such that* $\omega \neq 0$ *everywhere.*

Open condition in C^2 topology on metrics.

A different use of self-dual harmonic forms yields a complementary result.

Definition. *A conformal class* $[g]$ *on a compact oriented* 4*-manifold* M *is said to be of symplectic type* $\Leftrightarrow \exists$ *self-dual harmonic* 2*-form* ω *on* (M, g) *such that* $\omega \neq 0$ *everywhere.*

Open condition in C^2 topology on metrics.

(Harmonic forms depend continuously on metric.)

Theorem (L '15). *Let* M *be the underlying smooth oriented* 4*-manifold of a del Pezzo surface.*

Theorem (L '15). *Let M be the underlying smooth oriented* 4*-manifold of a del Pezzo surface. Then any conformal class $[g]$ of symplectic type on M satisfies*

Theorem (L '15). *Let M be the underlying smooth oriented 4-manifold of a del Pezzo surface. Then any conformal class $[g]$ of symplectic type on M satisfies*

$$\int_M |W_+|^2 d\mu \geq \frac{4\pi^2}{3}(2\chi + 3\tau)(M),$$

Theorem (L '15). *Let M be the underlying smooth oriented 4-manifold of a del Pezzo surface. Then any conformal class $[g]$ of symplectic type on M satisfies*

$$\int_M |W_+|^2 d\mu \geq \frac{4\pi^2}{3}(2\chi + 3\tau)(M),$$

with equality $\Leftrightarrow [g]$ contains a Kähler-Einstein metric g.

Theorem (L '15). *Let M be the underlying smooth oriented* 4*-manifold of a del Pezzo surface. Then any conformal class* $[g]$ *of symplectic type on* M *satisfies*

$$\int_M |W_+|^2 d\mu \geq \frac{4\pi^2}{3}(2\chi + 3\tau)(M),$$

with equality $\Leftrightarrow [g]$ *contains a Kähler-Einstein metric* g.

This recovers Gursky's inequality

Theorem (L '15). *Let M be the underlying smooth oriented* 4*-manifold of a del Pezzo surface. Then any conformal class $[g]$ of symplectic type on M satisfies*

$$\int_M |W_+|^2 d\mu \geq \frac{4\pi^2}{3}(2\chi + 3\tau)(M),$$

with equality $\Leftrightarrow [g]$ contains a Kähler-Einstein metric g.

This recovers Gursky's inequality — but for a different open set of conformal classes!

Theorem (L '15). *Let M be the underlying smooth oriented* 4*-manifold of a del Pezzo surface. Then any conformal class $[g]$ of symplectic type on M satisfies*

$$\int_M |W_+|^2 d\mu \geq \frac{4\pi^2}{3}(2\chi + 3\tau)(M),$$

with equality $\Leftrightarrow [g]$ contains a Kähler-Einstein metric g.

$\exists$ conformal classes of symplectic type with

$$Y([g_j]) \to -\infty.$$

Theorem (L '15). *Let M be the underlying smooth oriented* 4*-manifold of a* del Pezzo surface. *Then any conformal class $[g]$ of symplectic type on M satisfies*

$$\int_M |W_+|^2 d\mu \geq \frac{4\pi^2}{3}(2\chi + 3\tau)(M),$$

with equality $\Leftrightarrow [g]$ contains a Kähler-Einstein metric g.

$\exists$ conformal classes of symplectic type with

$$Y([g_j]) \to -\infty.$$

Inequality not limited to the positive Yamabe realm!

Theorem (L '15). *Let M be the underlying smooth oriented* 4*-manifold of a del Pezzo surface. Then any conformal class $[g]$ of symplectic type on M satisfies*

$$\int_M |W_+|^2 d\mu \geq \frac{4\pi^2}{3}(2\chi + 3\tau)(M),$$

with equality $\Leftrightarrow [g]$ *contains a Kähler-Einstein metric* g.

Method: Almost-Kähler geometry:

$$3\int_M W_+(\omega, \omega)\, d\mu \geq 4\pi c_1 \bullet [\omega]$$

Theorem (L '15). *Let* M *be the underlying smooth oriented* 4*-manifold of a del Pezzo surface. Then any conformal class* $[g]$ *of symplectic type on* M *satisfies*

$$\int_M |W_+|^2 d\mu \geq \frac{4\pi^2}{3}(2\chi + 3\tau)(M),$$

with equality $\Leftrightarrow [g]$ *contains a Kähler-Einstein metric* g.

However, only works for M del Pezzo.

Theorem (L '15)**.** *Let M be the underlying smooth oriented* 4*-manifold of a del Pezzo surface. Then any conformal class $[g]$ of symplectic type on M satisfies*

$$\int_M |W_+|^2 d\mu \geq \frac{4\pi^2}{3}(2\chi + 3\tau)(M),$$

with equality $\Leftrightarrow [g]$ contains a Kähler-Einstein metric g.

However, only works for M del Pezzo.

This is apparently not an accident!

Kobayashi's conjecture concerned $S^2 \times S^2$.

Kobayashi's conjecture concerned $S^2 \times S^2$.

But Gursky's theorem also works for $(S^2 \times S^2)\#(S^2 \times S^2)$.

Kobayashi's conjecture concerned $S^2 \times S^2$.

But Gursky's theorem also works for $(S^2 \times S^2)\#(S^2 \times S^2)$.

And indeed for all iterated connect-sums $m(S^2 \times S^2)$.

Kobayashi's conjecture concerned $S^2 \times S^2$.

But Gursky's theorem also works for $(S^2 \times S^2)\#(S^2 \times S^2)$.

And indeed for all iterated connect-sums $m(S^2 \times S^2)$.

What happens there in the Yamabe-negative realm?

Theorem (L '22). *For any sufficiently large integer* m*,*

Theorem (L '22). *For any sufficiently large integer m, the smooth compact simply-connected spin manifold*

$$M = m(S^2 \times S^2) := \underbrace{(S^2 \times S^2) \# \cdots \# (S^2 \times S^2)}_{m}$$

Theorem (L '22). *For any sufficiently large integer* m*, the smooth compact simply-connected spin manifold*

$$M = m(S^2 \times S^2) := \underbrace{(S^2 \times S^2)\# \cdots \#(S^2 \times S^2)}_{m}$$

admits Riemannian conformal classes $[g]$ *such that*

Theorem (L '22). *For any sufficiently large integer m, the smooth compact simply-connected spin manifold*

$$M = m(S^2 \times S^2) := \underbrace{(S^2 \times S^2) \# \cdots \# (S^2 \times S^2)}_{m}$$

admits Riemannian conformal classes $[g]$ such that

$$\int_M |W_+|^2 d\mu < \frac{4\pi^2}{3}(2\chi + 3\tau)(M).$$

Theorem (L '22). *For any sufficiently large integer* m*, the smooth compact simply-connected spin manifold*

$$M = m(S^2 \times S^2) := \underbrace{(S^2 \times S^2) \# \cdots \# (S^2 \times S^2)}_{m}$$

admits Riemannian conformal classes $[g]$ *such that*

$$\int_M |W_+|^2 d\mu < \frac{4\pi^2}{3}(2\chi + 3\tau)(M).$$

Similarly, for any any sufficiently large integer m *and any integer* n *such that* $\frac{n}{m}$ *is sufficiently close to* 1*,*

Theorem (L '22). *For any sufficiently large integer m, the smooth compact simply-connected spin manifold*

$$M = m(S^2\times S^2) := \underbrace{(S^2\times S^2)\#\cdots\#(S^2\times S^2)}_{m}$$

admits Riemannian conformal classes $[g]$ such that

$$\int_M |W_+|^2 d\mu < \frac{4\pi^2}{3}(2\chi + 3\tau)(M).$$

Similarly, for any any sufficiently large integer m and any integer n such that $\frac{n}{m}$ is sufficiently close to 1, the smooth compact simply-connected non-spin manifold

$$M = m\mathbb{CP}_2\# n\overline{\mathbb{CP}}_2 := \underbrace{\mathbb{CP}_2\#\cdots\#\mathbb{CP}_2}_{m}\#\underbrace{\overline{\mathbb{CP}}_2\#\cdots\#\overline{\mathbb{CP}}_2}_{n}$$

Theorem (L '22). *For any sufficiently large integer* m*, the smooth compact simply-connected spin manifold*

$$M = m(S^2 \times S^2) := \underbrace{(S^2 \times S^2) \# \cdots \# (S^2 \times S^2)}_{m}$$

admits Riemannian conformal classes $[g]$ *such that*

$$\int_M |W_+|^2 d\mu < \frac{4\pi^2}{3}(2\chi + 3\tau)(M).$$

Similarly, for any any sufficiently large integer m *and any integer* n *such that* $\frac{n}{m}$ *is sufficiently close to* 1*, the smooth compact simply-connected non-spin manifold*

$$M = m\mathbb{CP}_2 \# n\overline{\mathbb{CP}}_2 := \underbrace{\mathbb{CP}_2 \# \cdots \# \mathbb{CP}_2}_{m} \# \underbrace{\overline{\mathbb{CP}}_2 \# \cdots \# \overline{\mathbb{CP}}_2}_{n}$$

admits conformal classes $[g]$ *where the above inequality holds.*

Key facts used in proof:

Key facts used in proof:

Wall '66: If Y and Z are homotopy-equivalent simply-connected smooth compact 4-manifolds,

Key facts used in proof:

Wall '66: If Y and Z are homotopy-equivalent simply-connected smooth compact 4-manifolds,

then $Y\#\ell(S^2\times S^2) \approx Z\#\ell(S^2\times S^2)$ for all $\ell \gg 0$.

Key facts used in proof:

Wall '66: If Y and Z are homotopy-equivalent simply-connected smooth compact 4-manifolds,

then $Y\#\ell(S^2\times S^2) \approx Z\#\ell(S^2\times S^2)$ for all $\ell \gg 0$.

In proof, we apply this to

$$M = (k+\ell)(X\#\overline{X})\#(k+2\ell)(S^2\times S^2)$$

where X simply-connected minimal complex surface of general type with $\tau(X) > 0$.

Key facts used in proof:

Wall '66: If Y and Z are homotopy-equivalent simply-connected smooth compact 4-manifolds,

then $Y \# \ell(S^2 \times S^2) \approx Z \# \ell(S^2 \times S^2)$ for all $\ell \gg 0$.

In proof, we apply this to

$$M = (k+\ell)(X \# \overline{X}) \# (k+2\ell)(S^2 \times S^2)$$

where X simply-connected minimal complex surface of general type with $\tau(X) > 0$.

Such X now known to exist in profusion!

Key facts used in proof:

Wall '66: If Y and Z are homotopy-equivalent simply-connected smooth compact 4-manifolds,

then $Y\#\ell(S^2\times S^2) \approx Z\#\ell(S^2\times S^2)$ for all $\ell \gg 0$.

In proof, we apply this to

$$M = (k+\ell)(X\#\overline{X})\#(k+2\ell)(S^2\times S^2)$$

where X simply-connected minimal complex surface of general type with $\tau(X) > 0$.

Such X now known to exist in profusion!

Roulleau-Urzúa '15: $\exists$ sequences with $\tau/\chi \to 1/3$.

$\to$ Miyaoka-Yau line! Can choose spin or non-spin!

Theorem (L '22). *For any sufficiently large integer m, the smooth compact simply-connected spin manifold*

$$M = m(S^2 \times S^2) := \underbrace{(S^2 \times S^2) \# \cdots \# (S^2 \times S^2)}_{m}$$

admits Riemannian conformal classes $[g]$ such that

$$\int_M |W_+|^2 d\mu < \frac{4\pi^2}{3}(2\chi + 3\tau)(M).$$

Theorem (L '22). *For any sufficiently large integer m, the smooth compact simply-connected spin manifold*

$$M = m(S^2\times S^2) := \underbrace{(S^2\times S^2)\#\cdots\#(S^2\times S^2)}_{m}$$

admits Riemannian conformal classes $[g]$ such that

$$\int_M |W_+|^2 d\mu < \frac{4\pi^2}{3}(2\chi + 3\tau)(M).$$

Similarly, for any any sufficiently large integer m and any integer n such that $\frac{n}{m}$ is sufficiently close to 1, the smooth compact simply-connected non-spin manifold

$$M = m\mathbb{CP}_2\#n\overline{\mathbb{CP}}_2 := \underbrace{\mathbb{CP}_2\#\cdots\#\mathbb{CP}_2}_{m}\#\underbrace{\overline{\mathbb{CP}}_2\#\cdots\#\overline{\mathbb{CP}}_2}_{n}$$

admits conformal classes $[g]$ where the above inequality holds.

It’s a real pleasure to be here!

It's a real pleasure to be here!

Thanks for the invitation!