Mass, Scalar Curvature, Kähler Geometry, and All That

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Stony Brook University

Mathematics Colloquium
University of Florida
Gainesville, FL, April 15, 2019

Core results joint with

Core results joint with

Hans-Joachim Hein
Fordham University

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Mass in Kähler Geometry
Comm. Math. Phys. 347 (2016) 621–653.

Recent technical improvements:

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Mass, Kähler Manifolds, and Symplectic Geometry

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arXiv: 1810.11417 [math.DG]

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Ann. Glob. An. Geom. *to appear*
`doi: 10.1007/s10455-019-09658-9`

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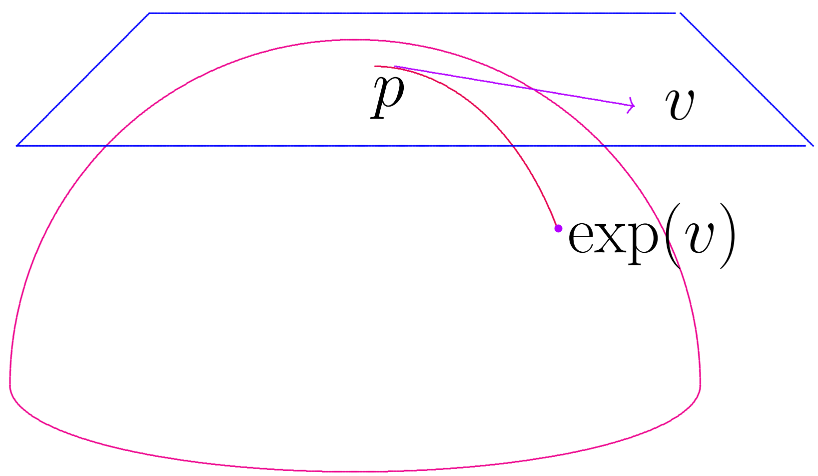

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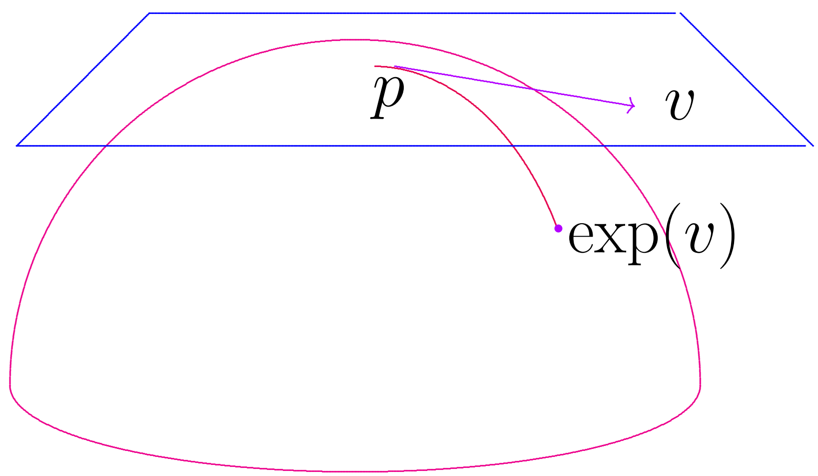

Now choosing $T_pM \xrightarrow{\cong} \mathbb{R}^n$ via some orthonormal basis gives us special coordinates on M.

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Components like $\mathcal{R}_{1212}$ are "sectional curvatures"...

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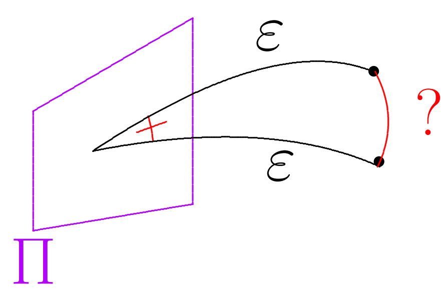

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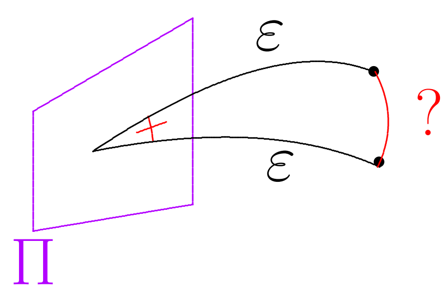

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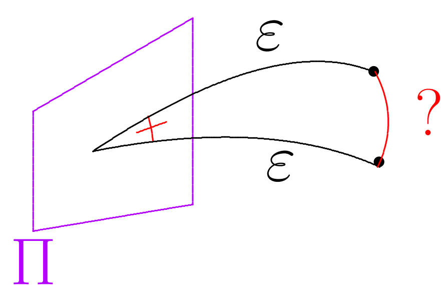

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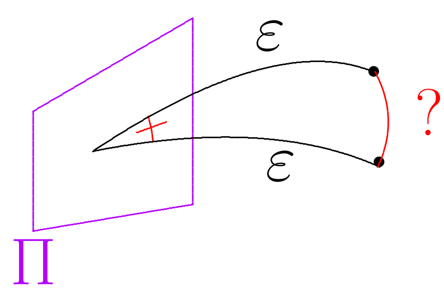

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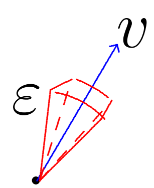

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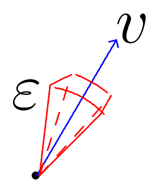

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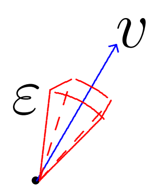

The metric g is called Ricci-flat if it satisfies $r \equiv 0$.

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But this has a simple geometric interpretation. . .

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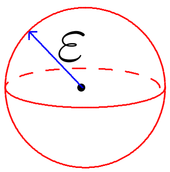

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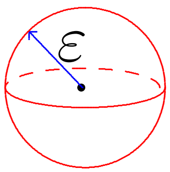

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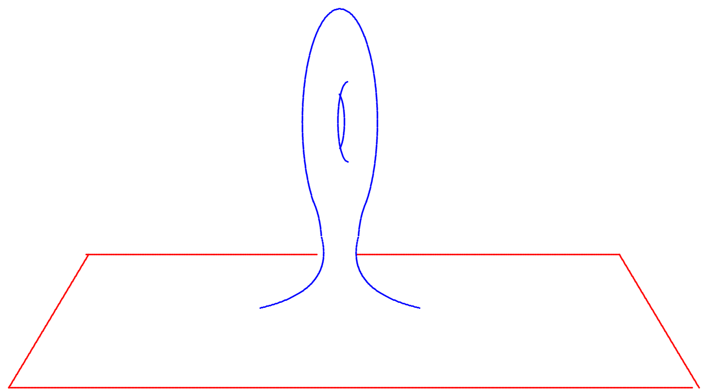

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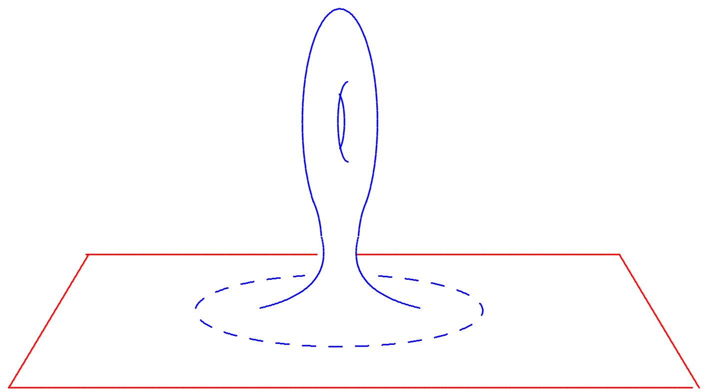

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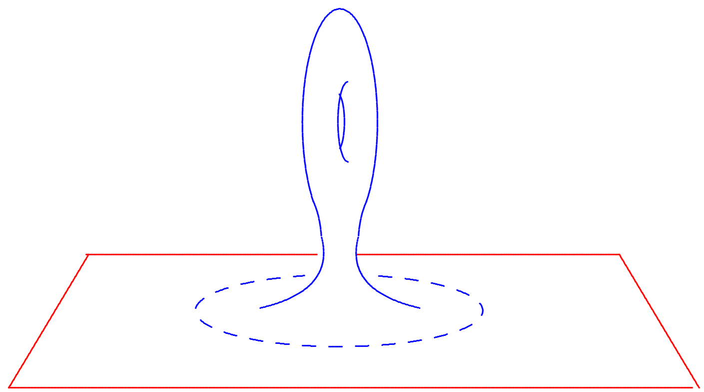

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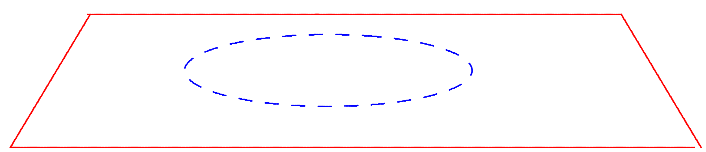

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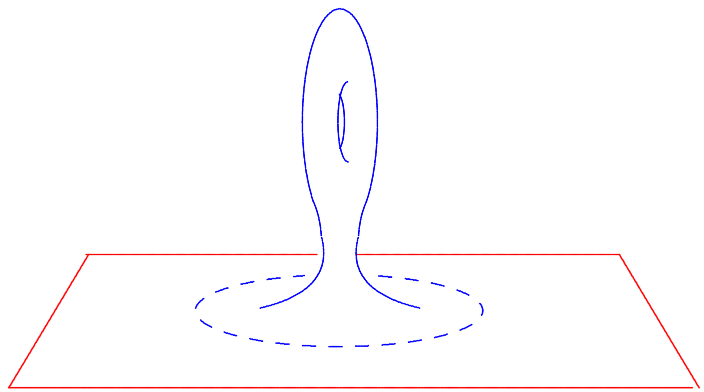

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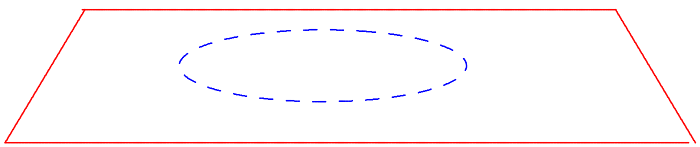

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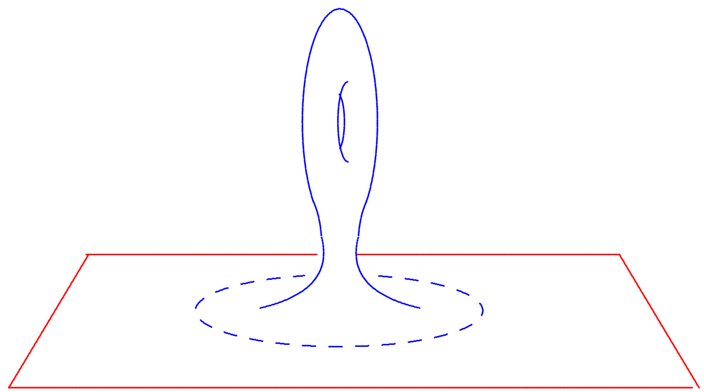

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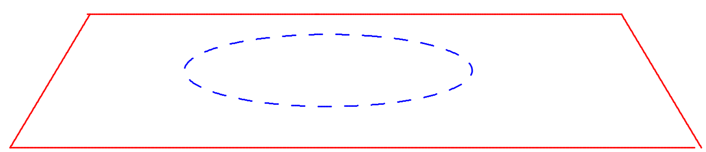

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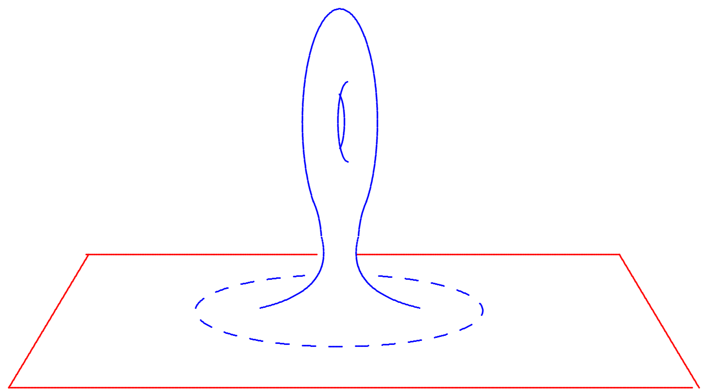

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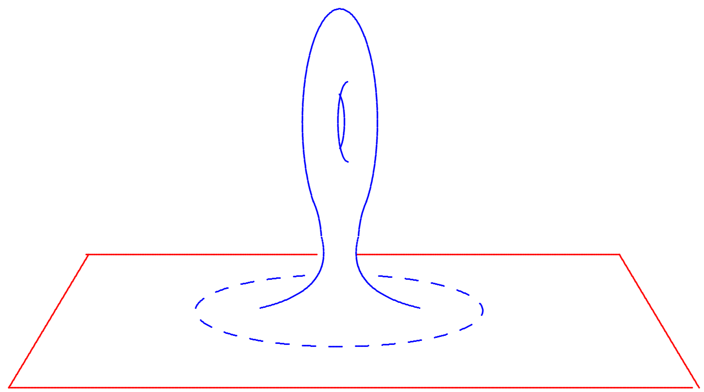

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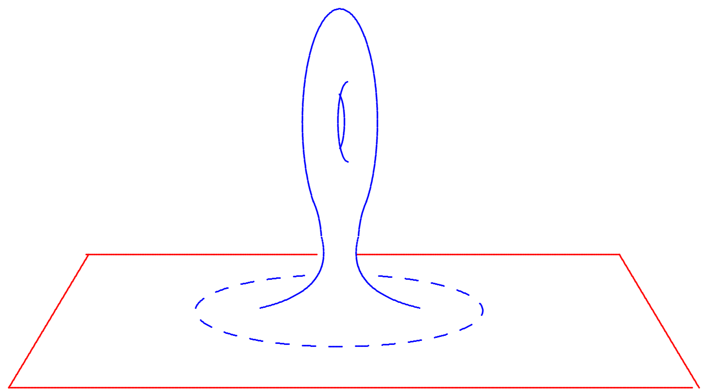

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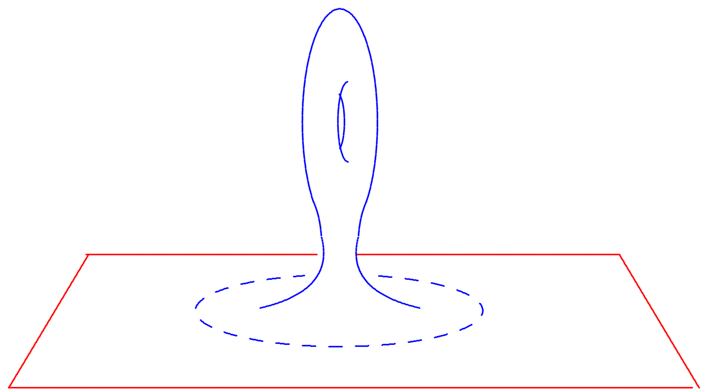

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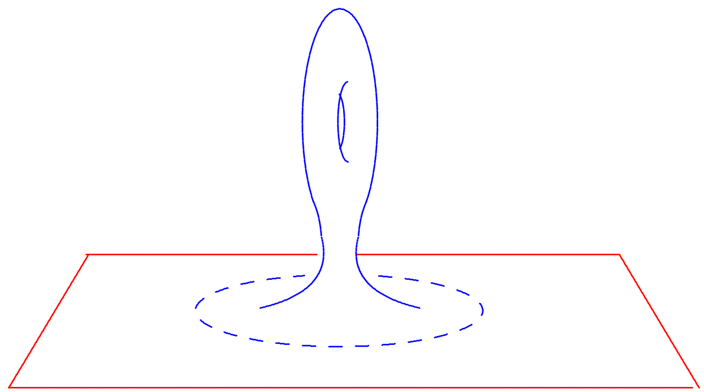

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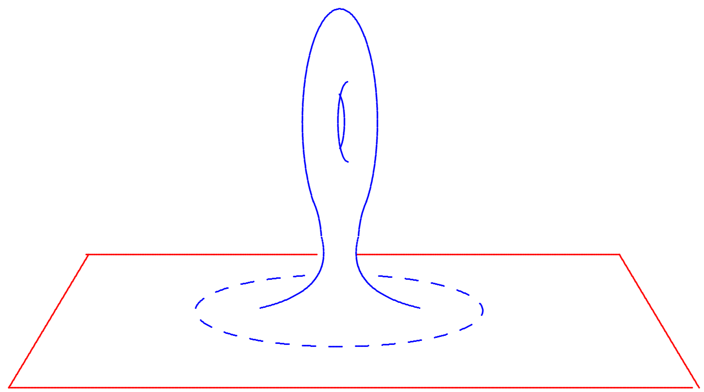

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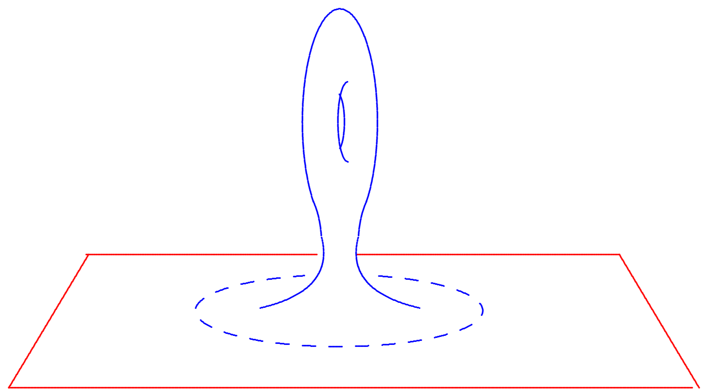

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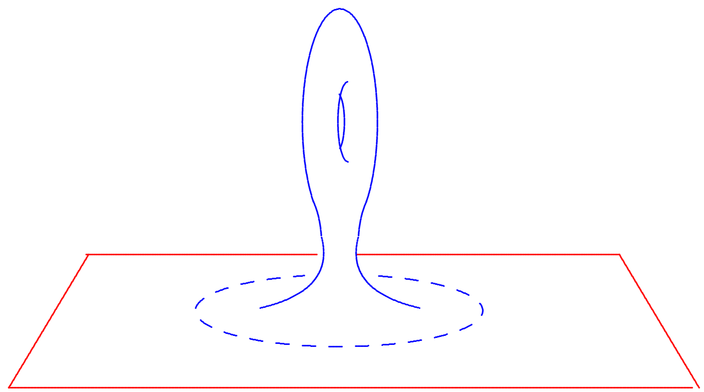

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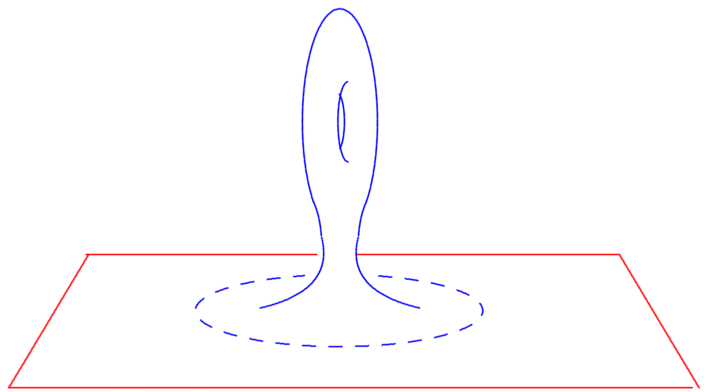

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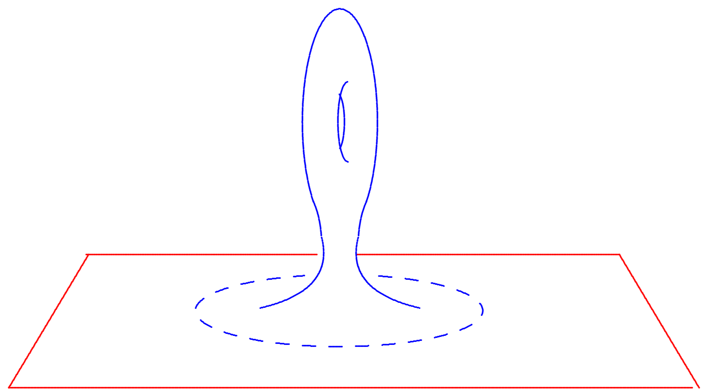

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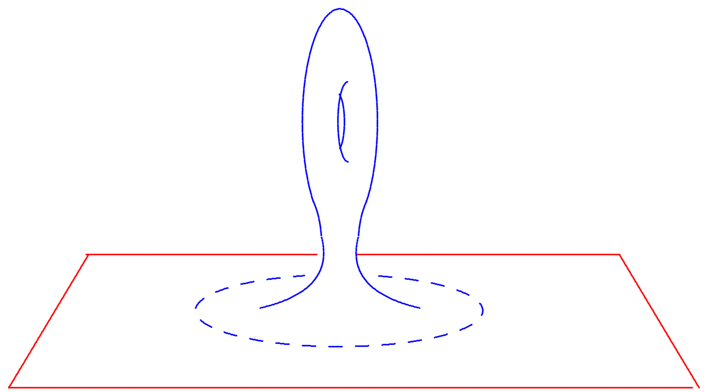

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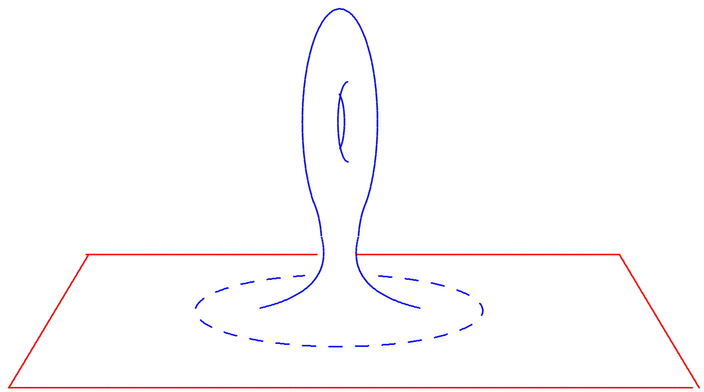

and an isometry $M - K \to \mathbb{R}^n - D^n$. (Euclidean)

If M has scalar curvature ≥ 0, is it flat?

Much harder!

Simple, natural problem:

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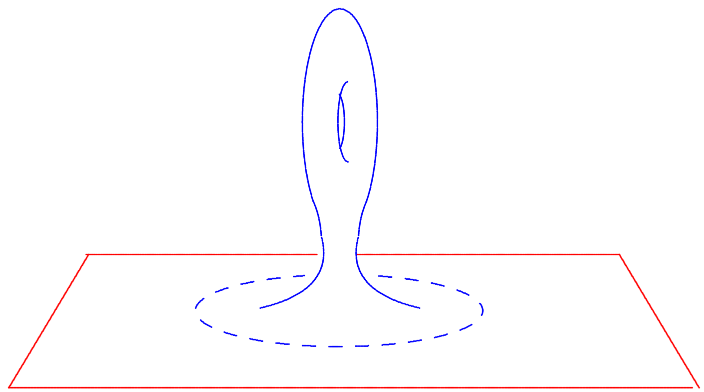

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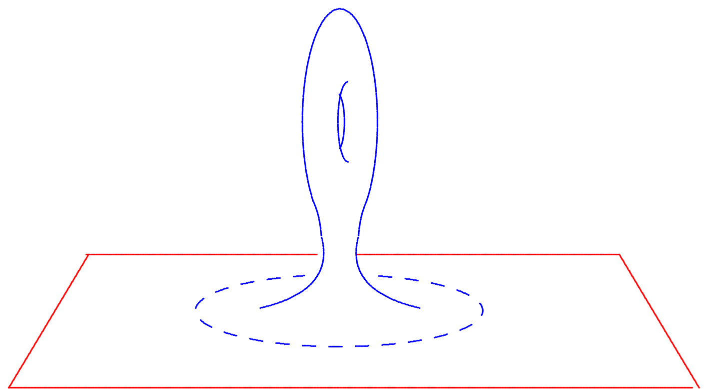

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This time, the inspiration comes from physics!

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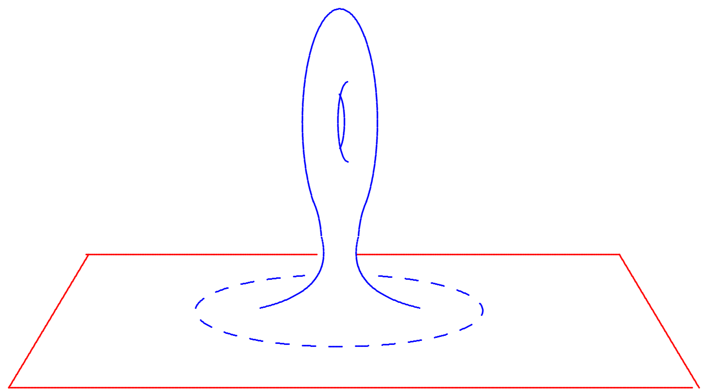

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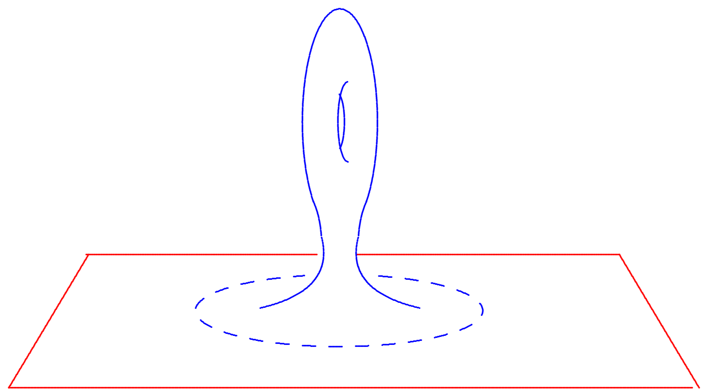

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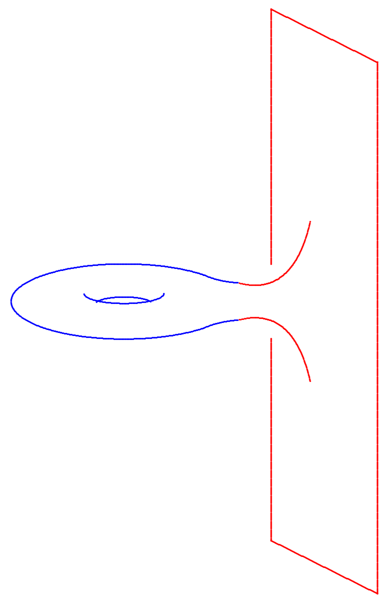
"Positive Mass Theorem"

Get result even with appropriate fall-off to Euclidean...

Definition. *A complete, non-compact Riemannian* n*-manifold* (M^n, g)



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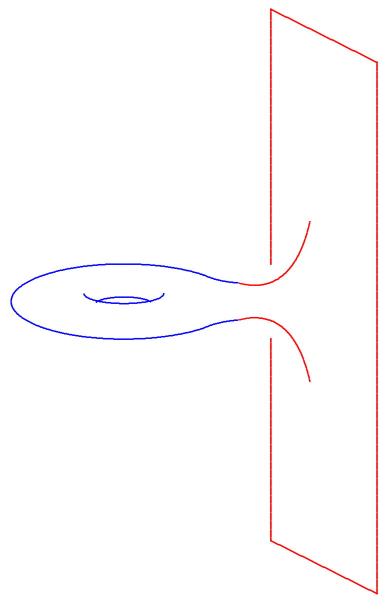

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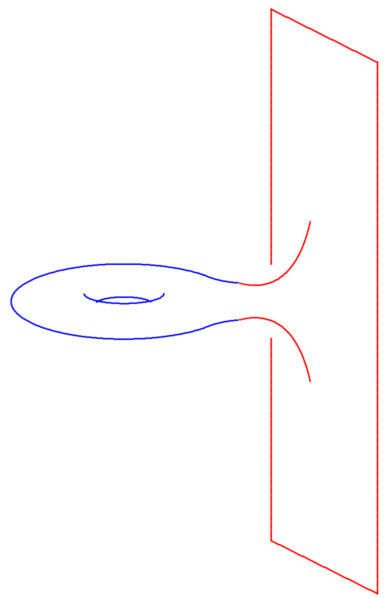

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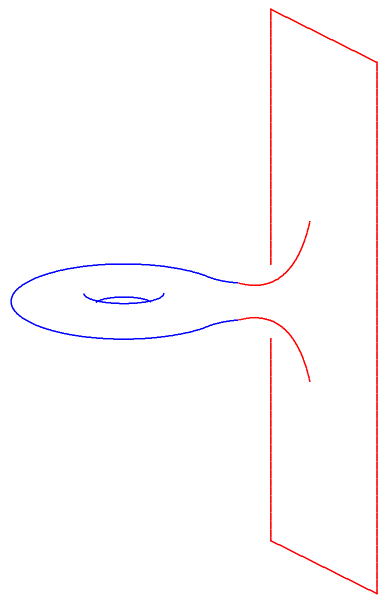

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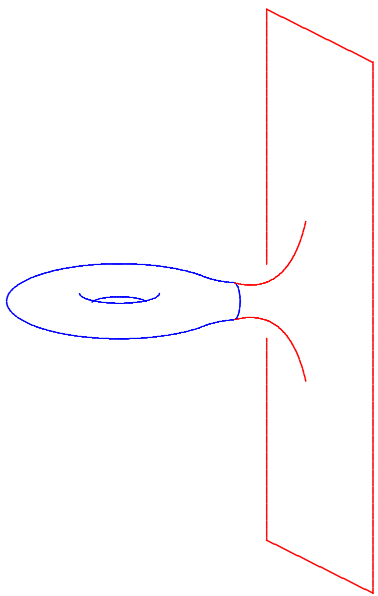

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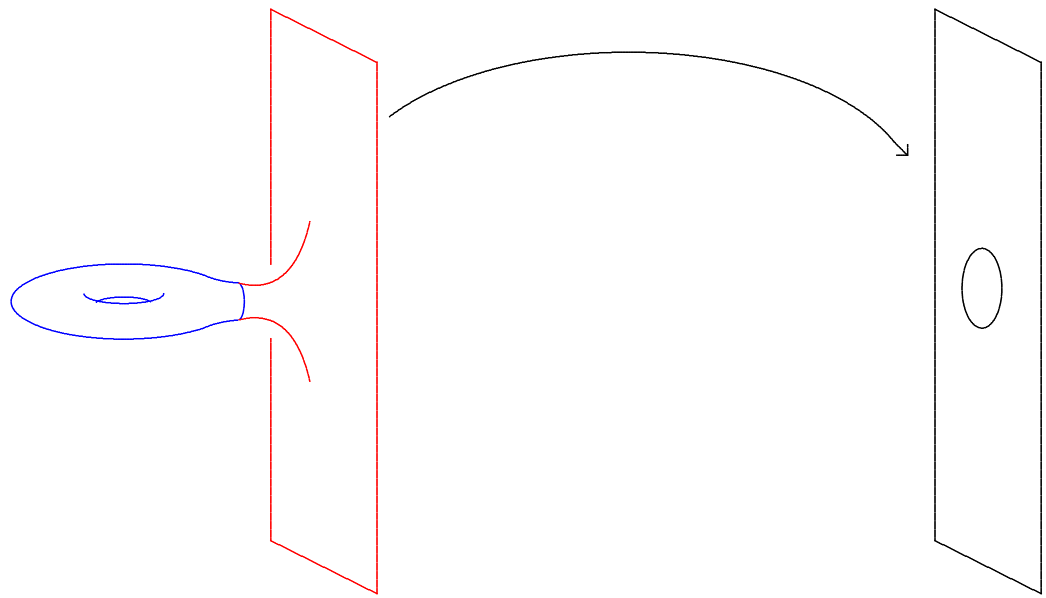

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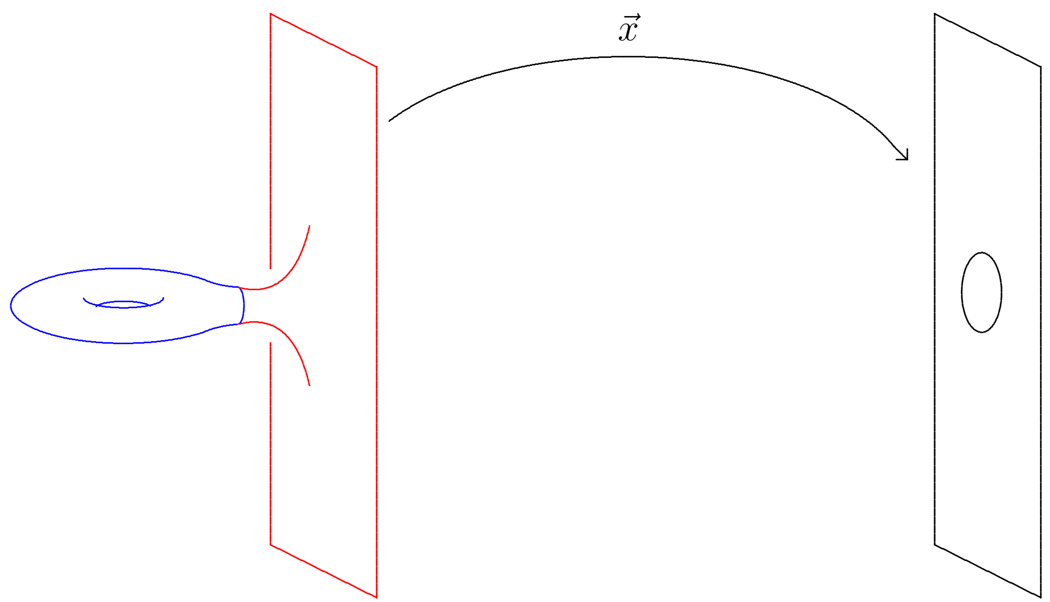

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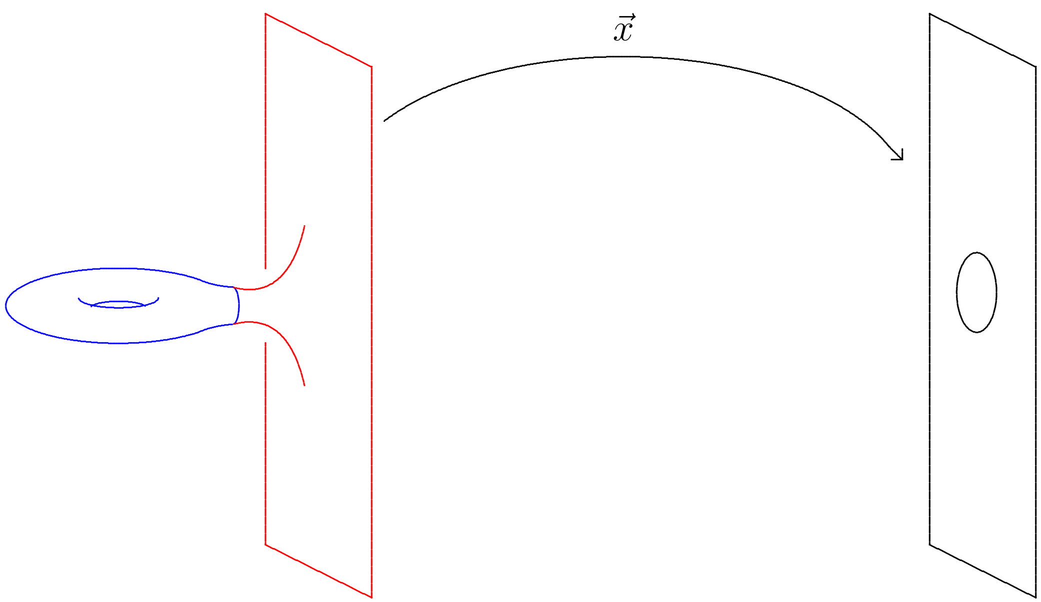

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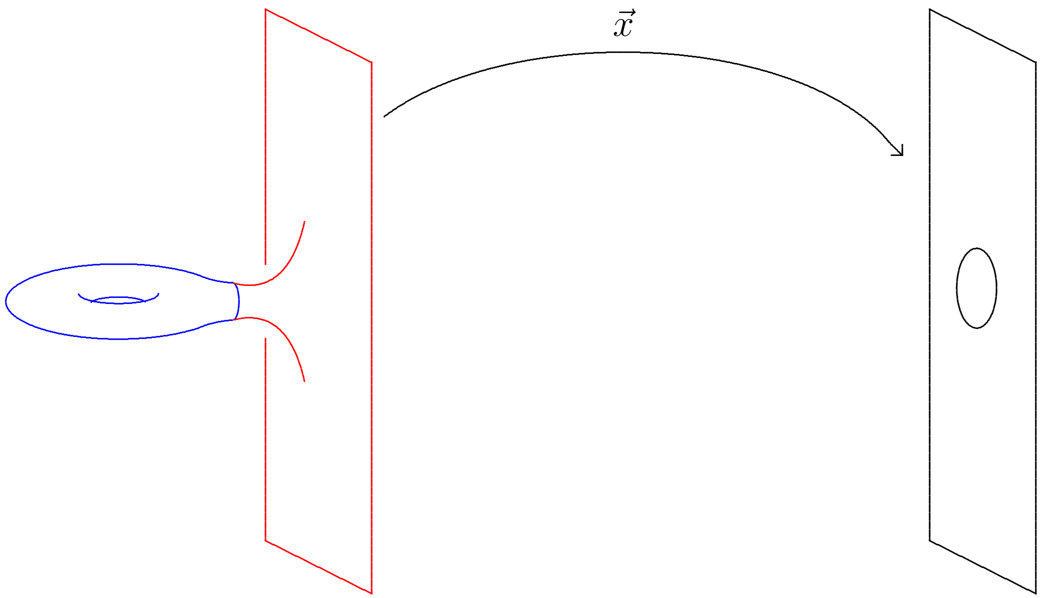

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Chruściel-type fall-off:

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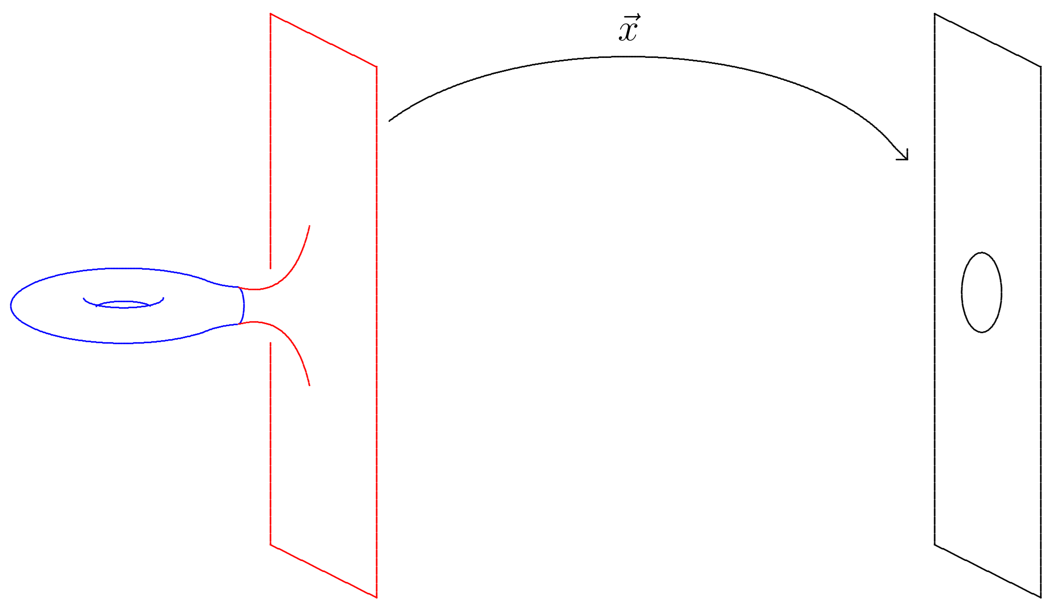

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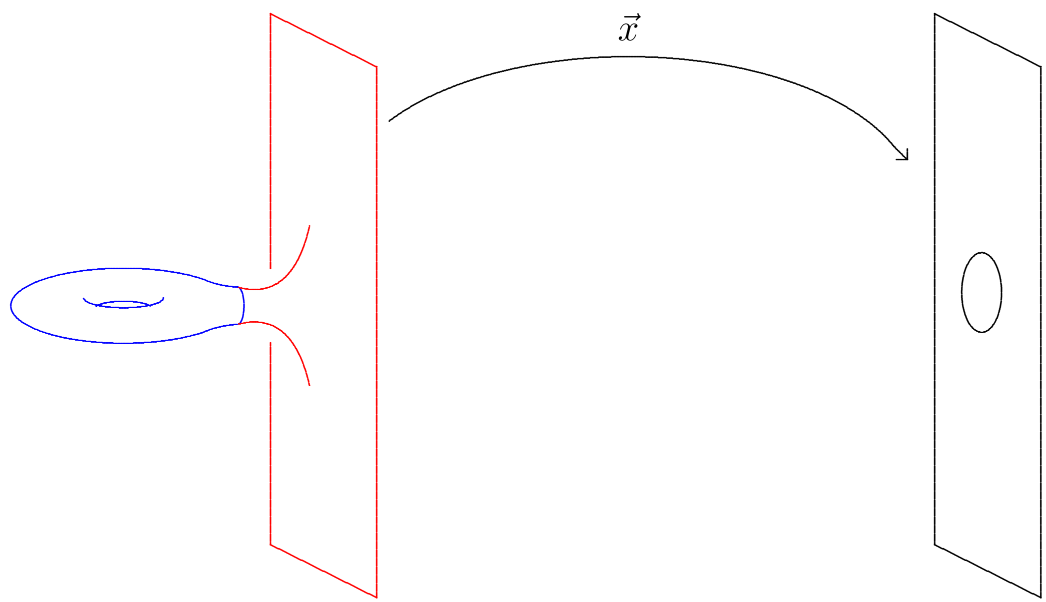

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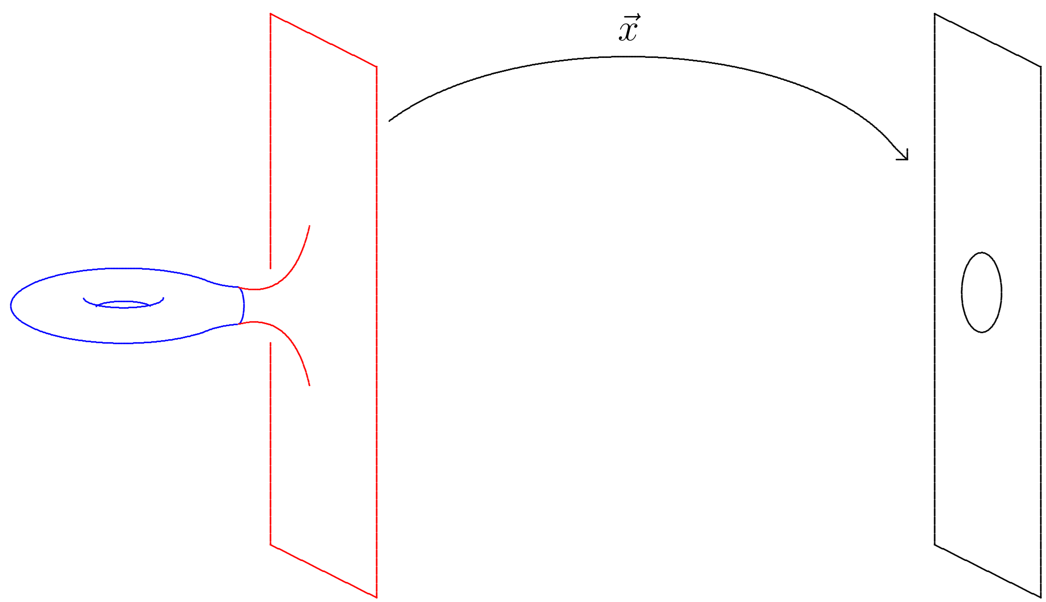

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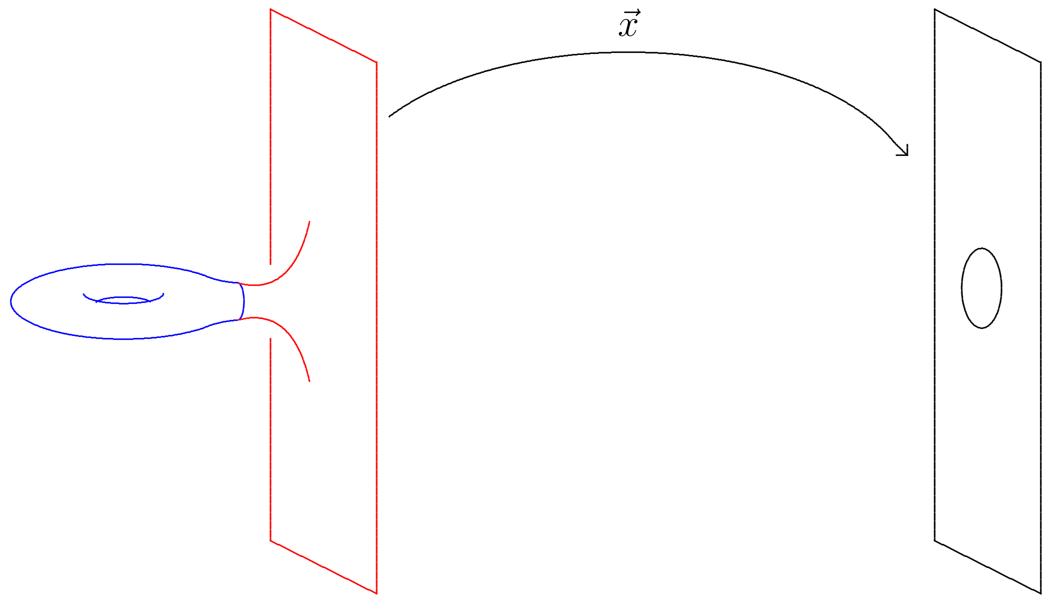

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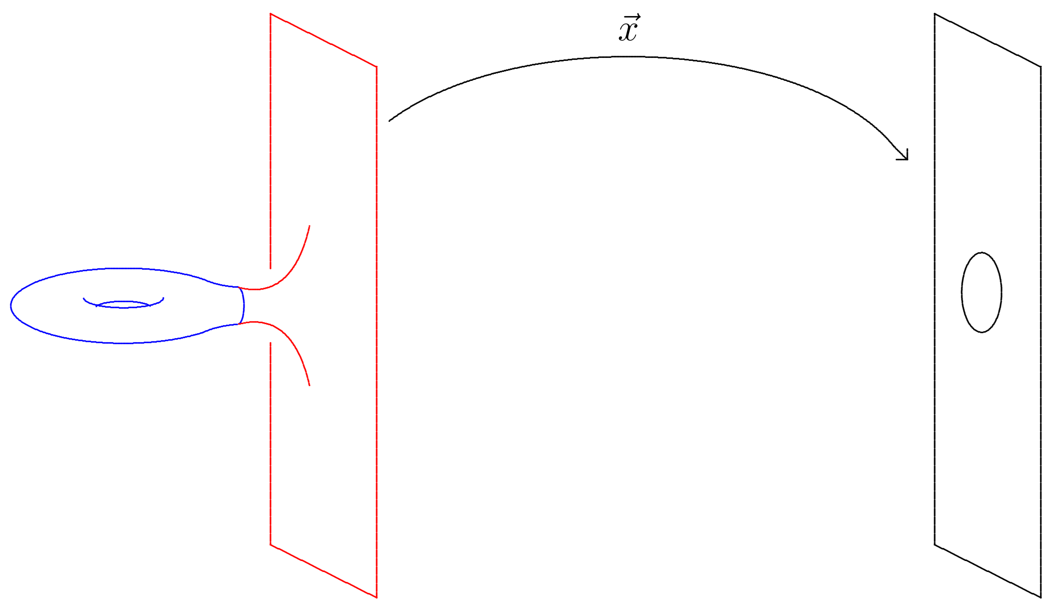
Definition. *A complete, non-compact Riemannian n-manifold (M^n, g) is called* asymptotically Euclidean *(AE) if there is a compact set $K \subset M$ such that*  *$M - K$ is diffeomorphic to $\mathbb{R}^n - D^n$ in such a manner that*

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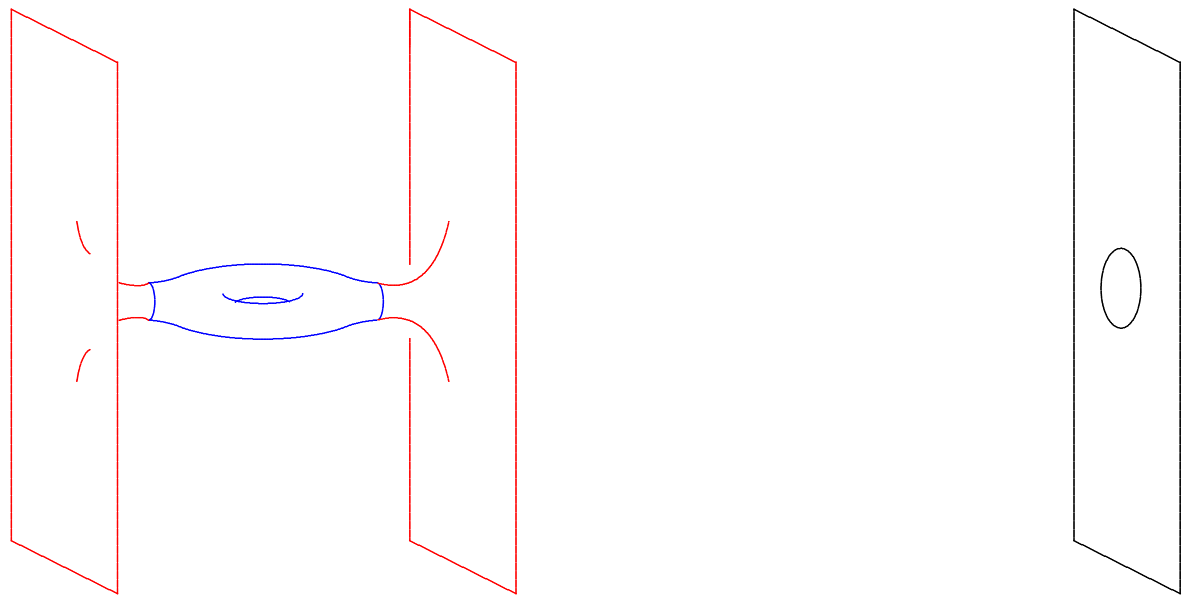

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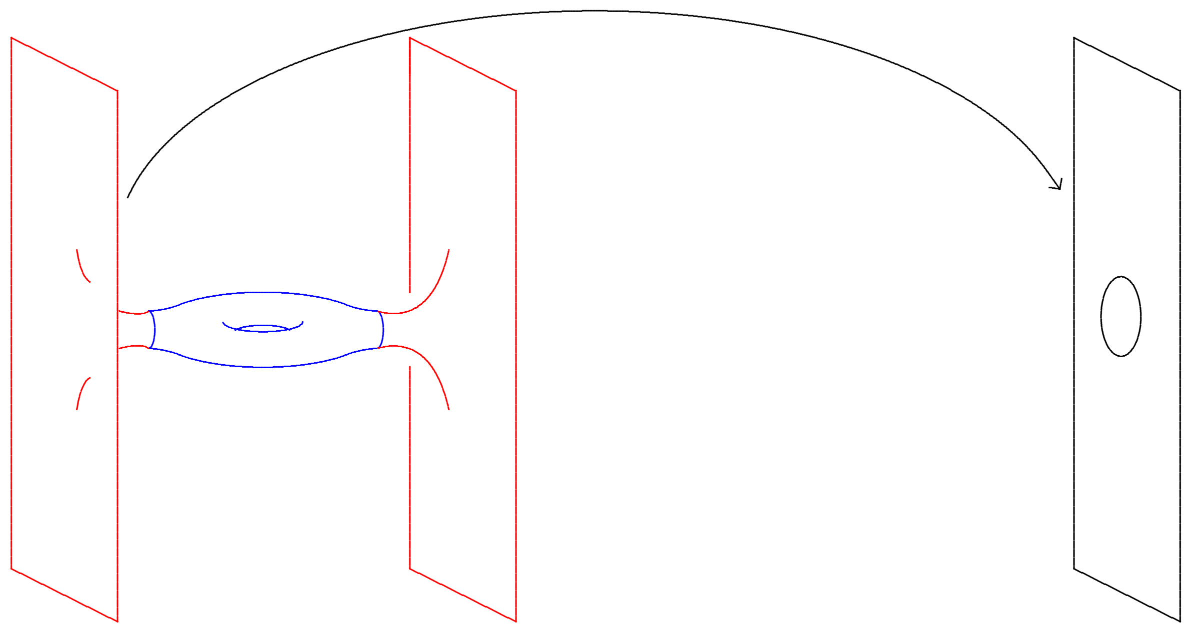

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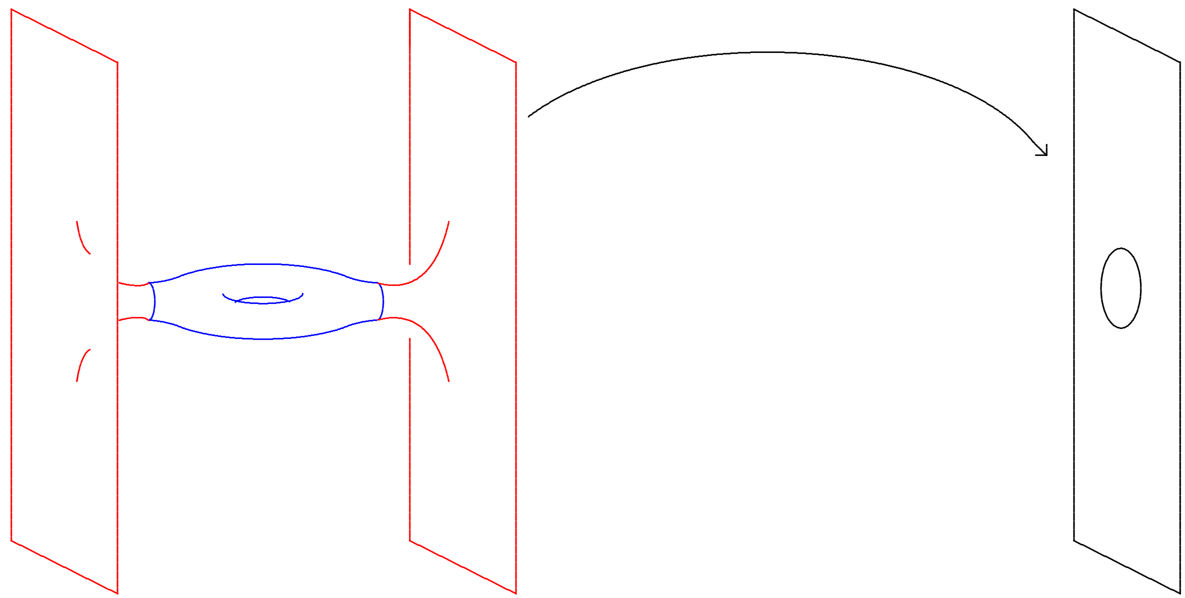

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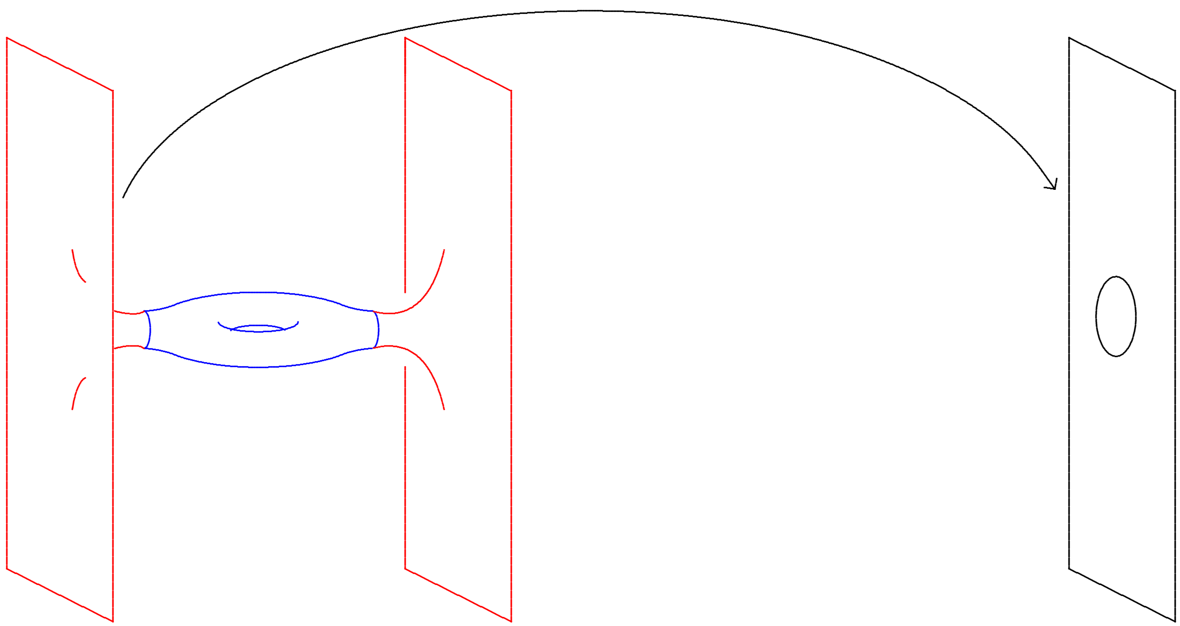

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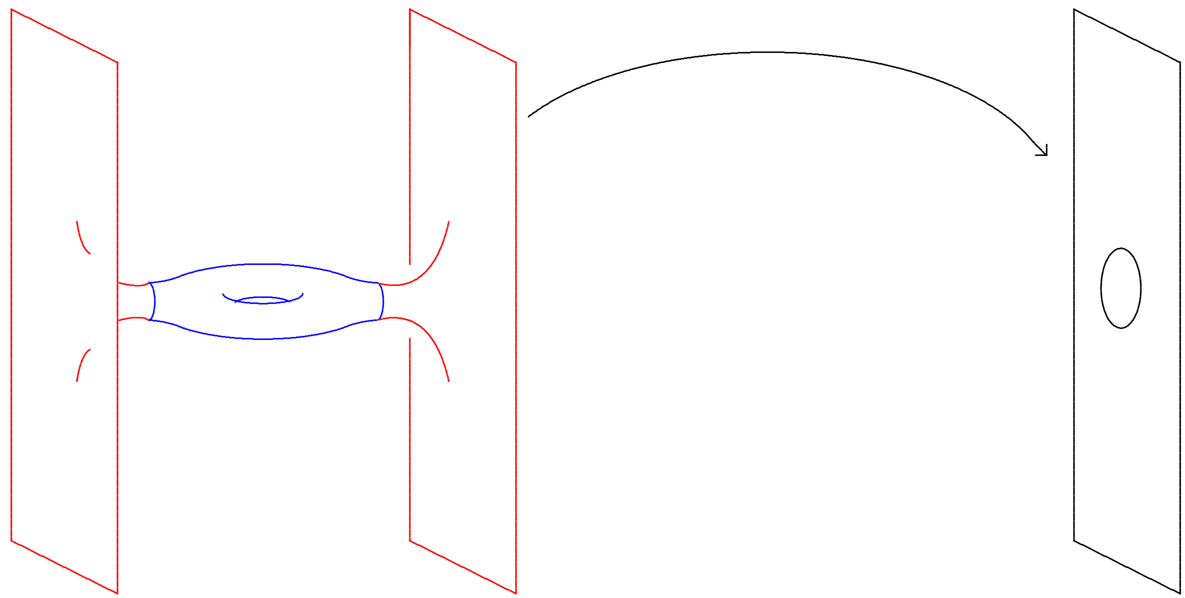

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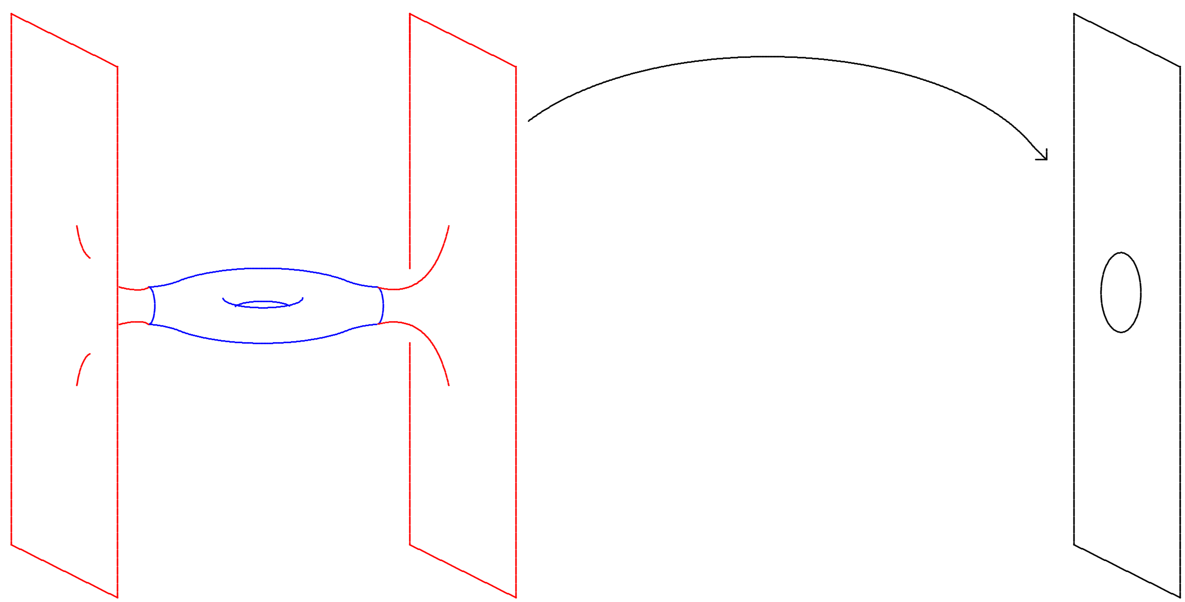

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Seems to depend on choice of coordinates!

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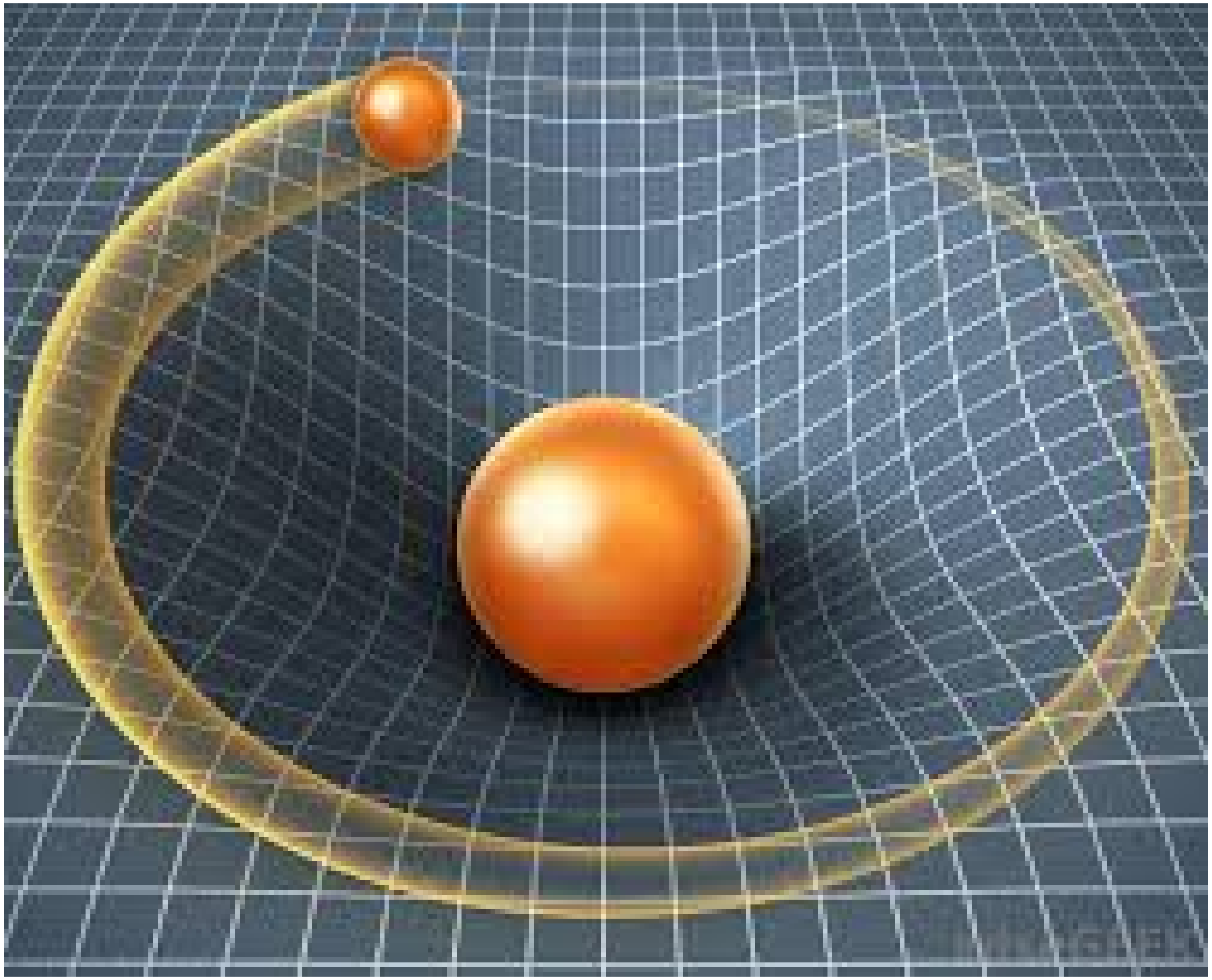

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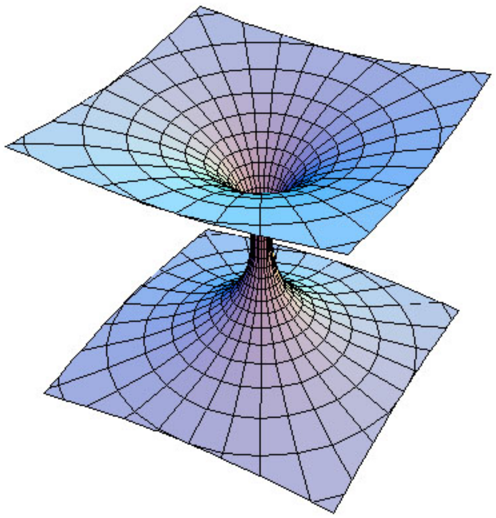
Reads off "apparent mass" from strength of the gravitational field far from an isolated source.

In any dimension, reproduces "mass" of $t = 0$ hypersurface in $(n+1)$-dimensional Schwarzschild

$$g = \left(1 - \frac{2m}{\varrho^{n-2}}\right)^{-1} d\varrho^2 + \varrho^2 h_{S^{n-1}}$$

Two such regions fit together to form the wormhole metric.



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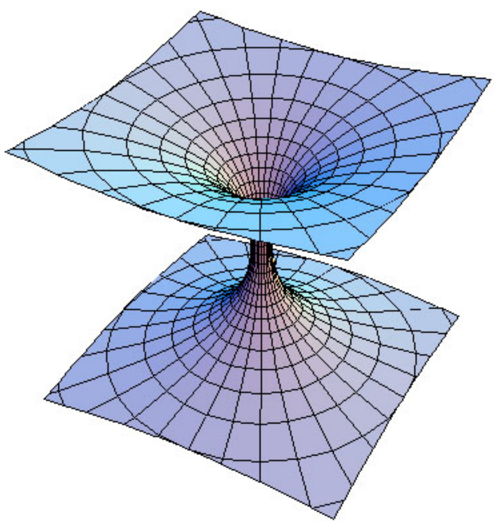

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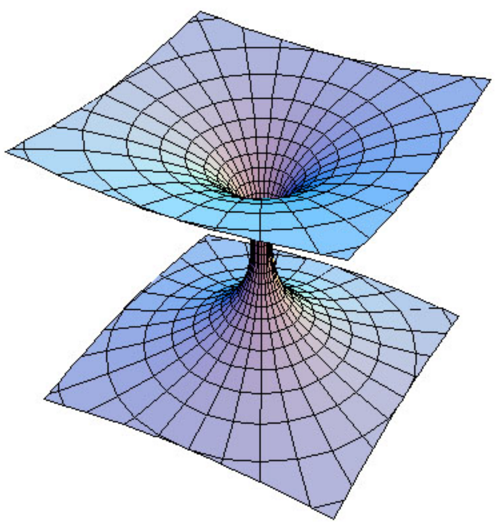

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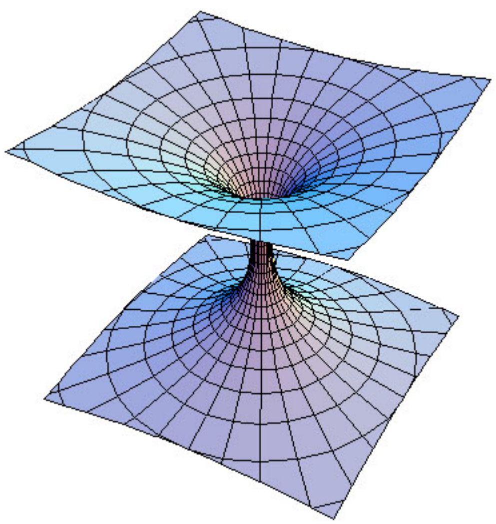

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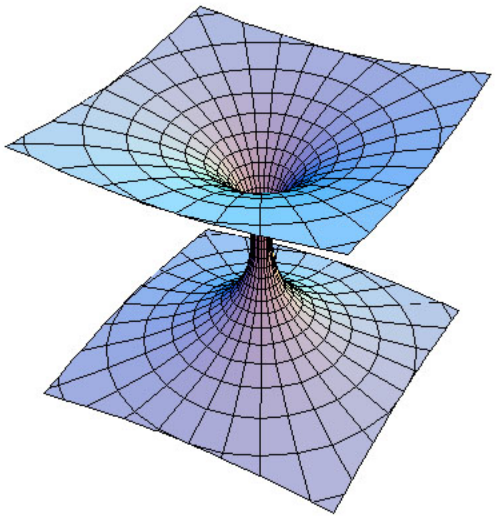

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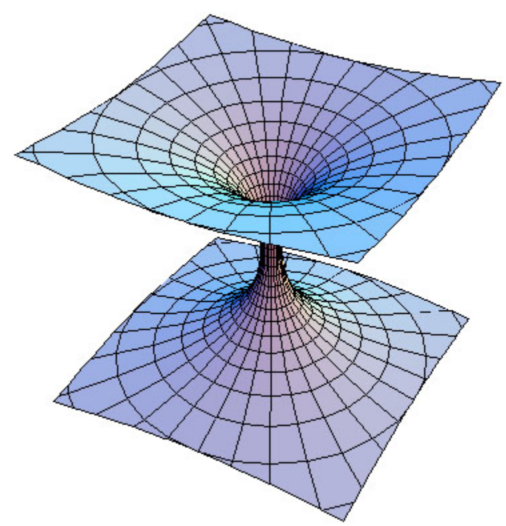

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$$g_{jk} = \left(1 + \frac{2m}{(n-2)r^{n-2}}\right)\delta_{jk} + \cdots$$

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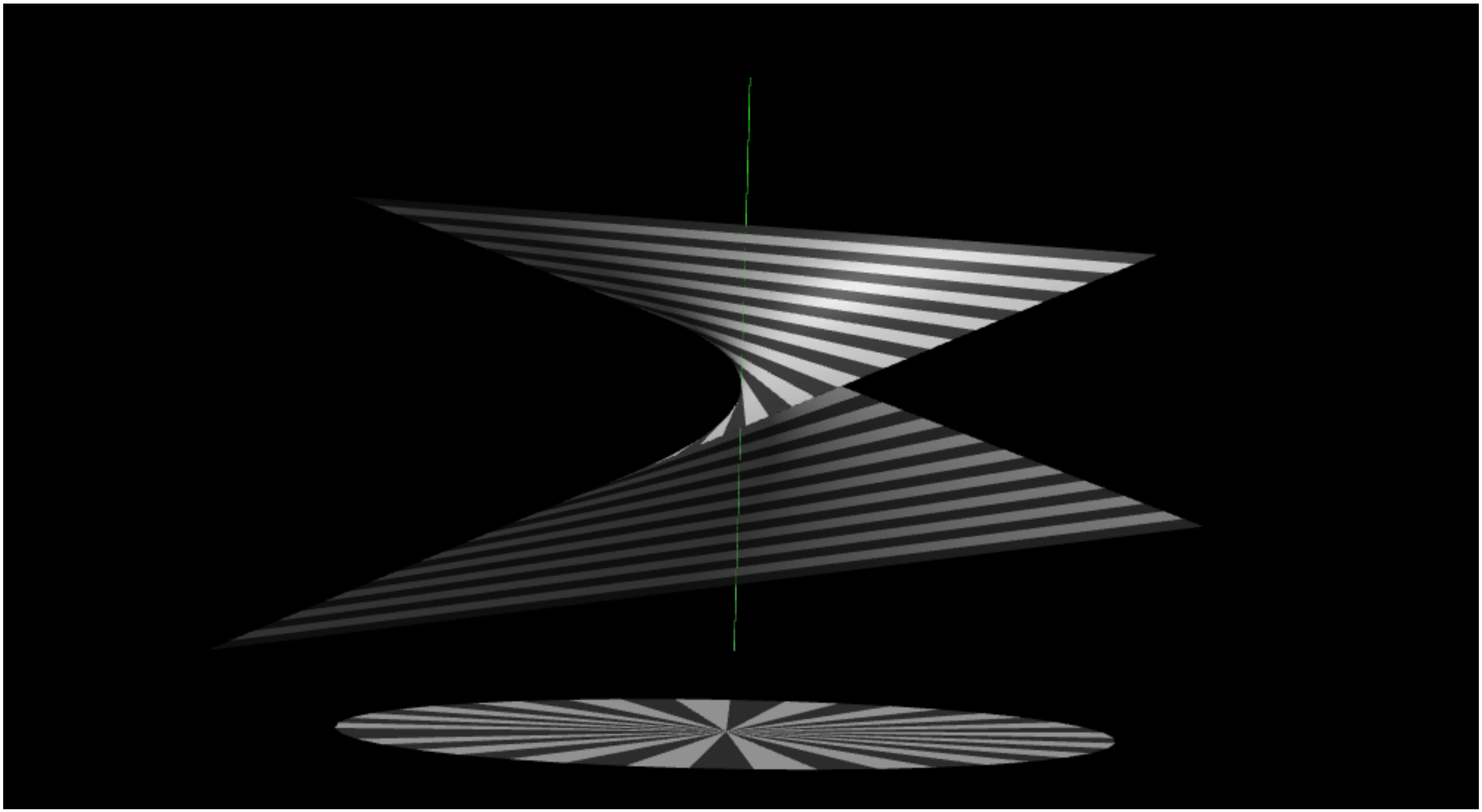

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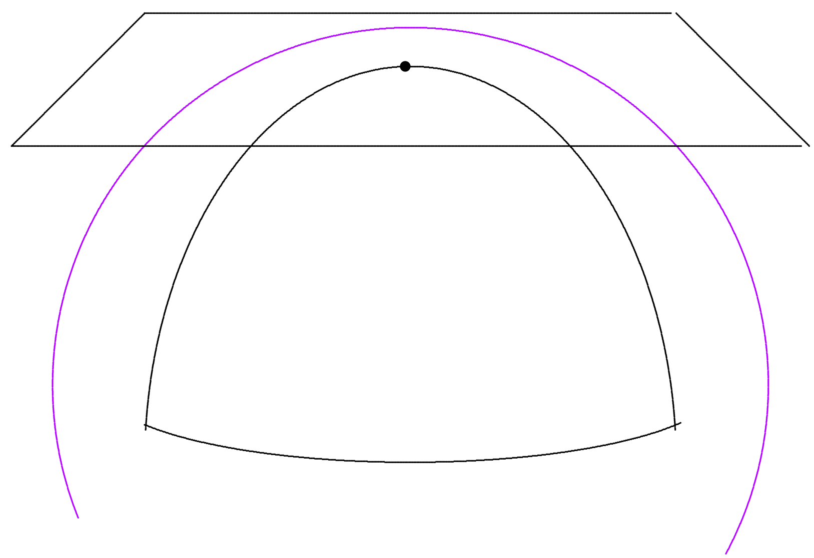

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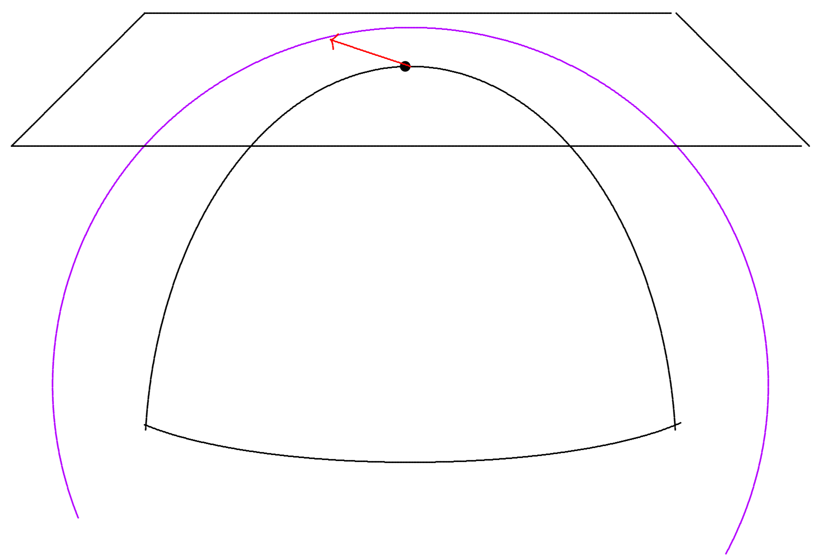

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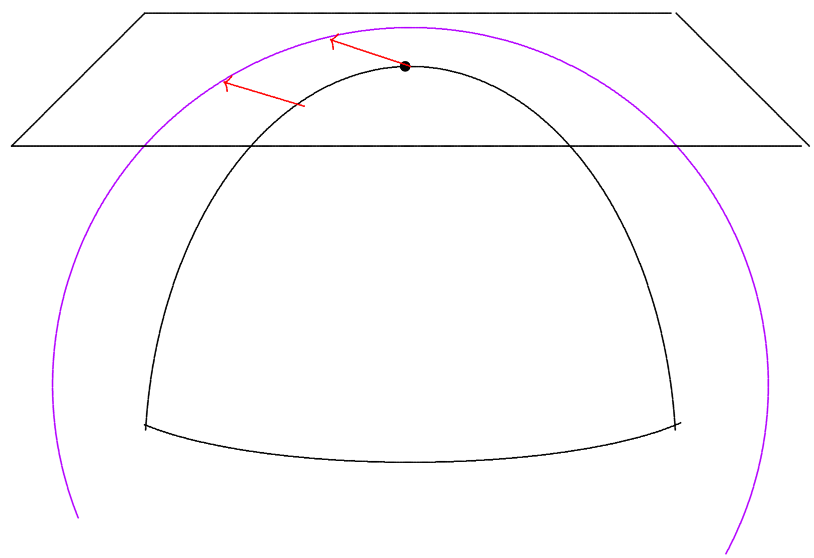

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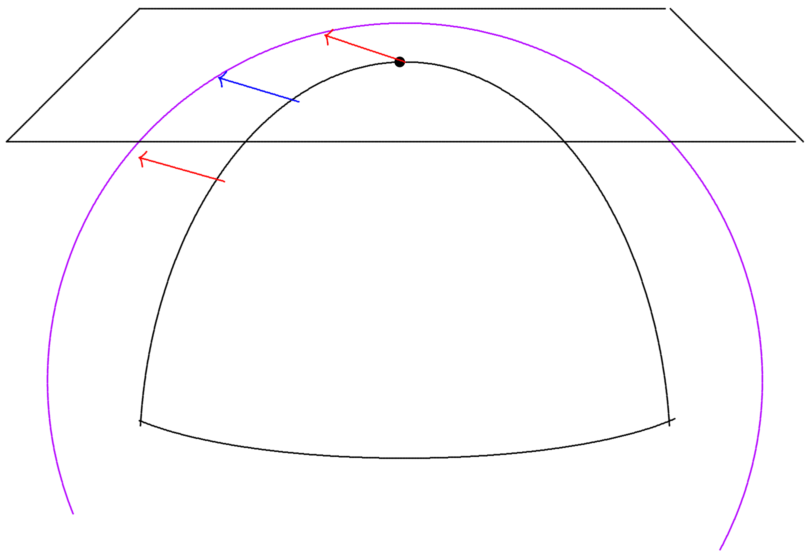

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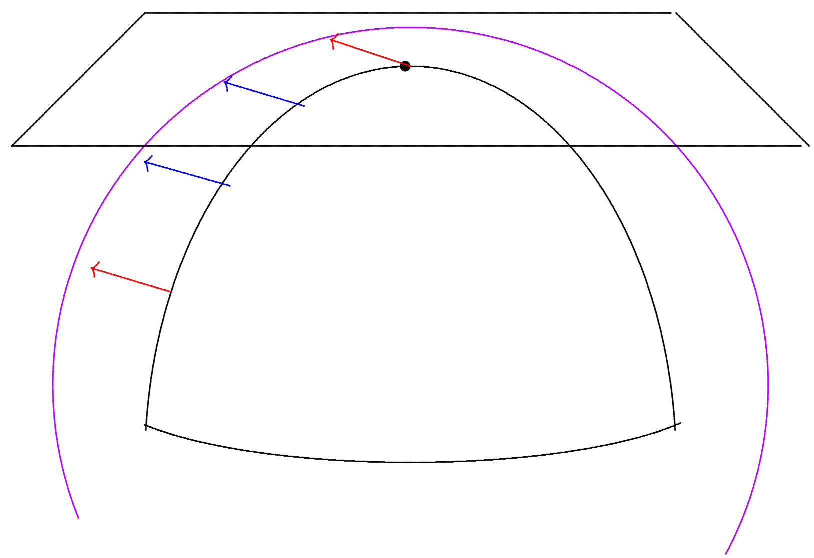

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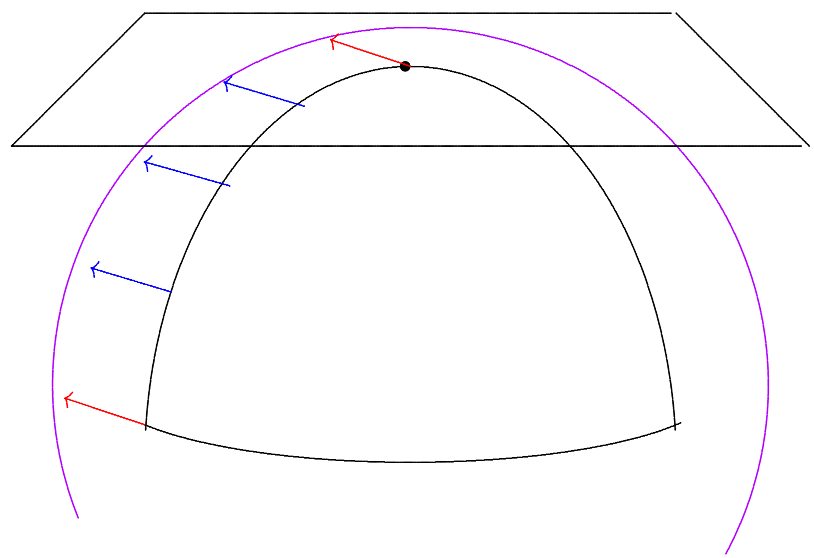

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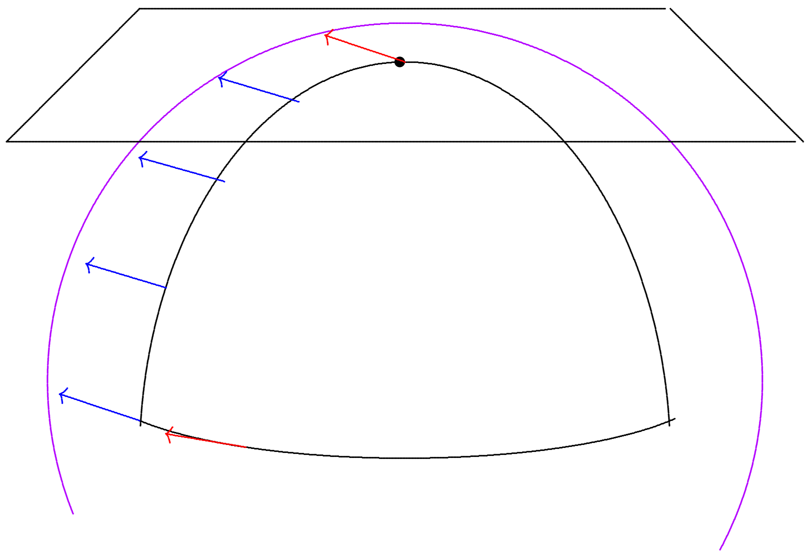

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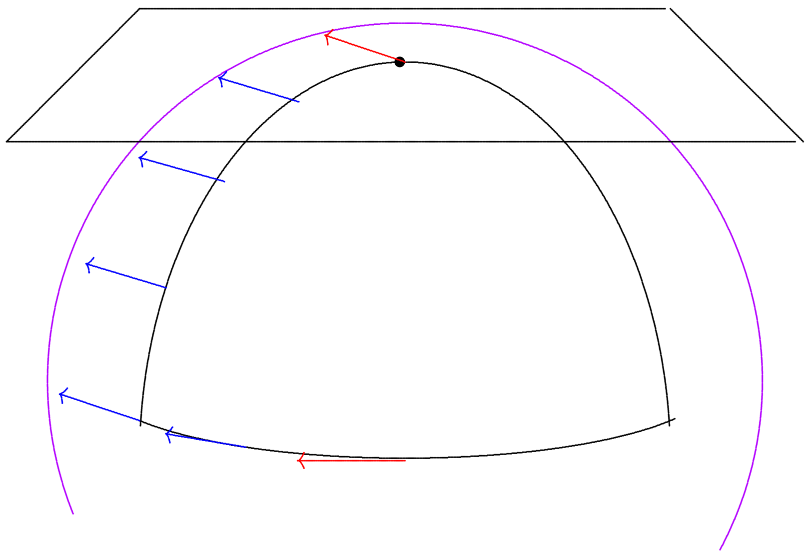

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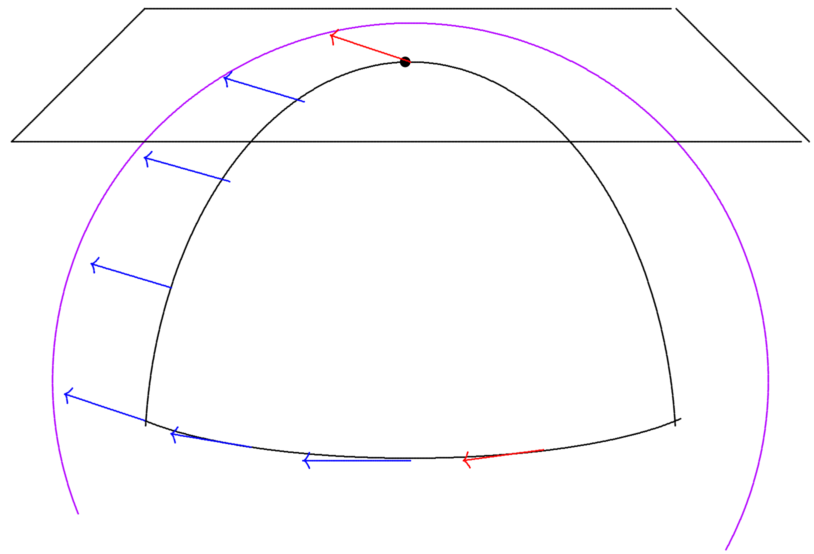

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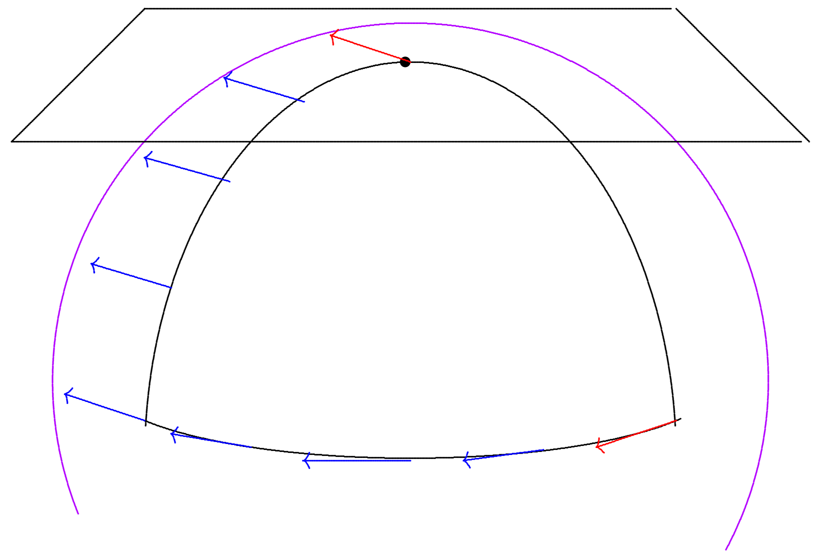

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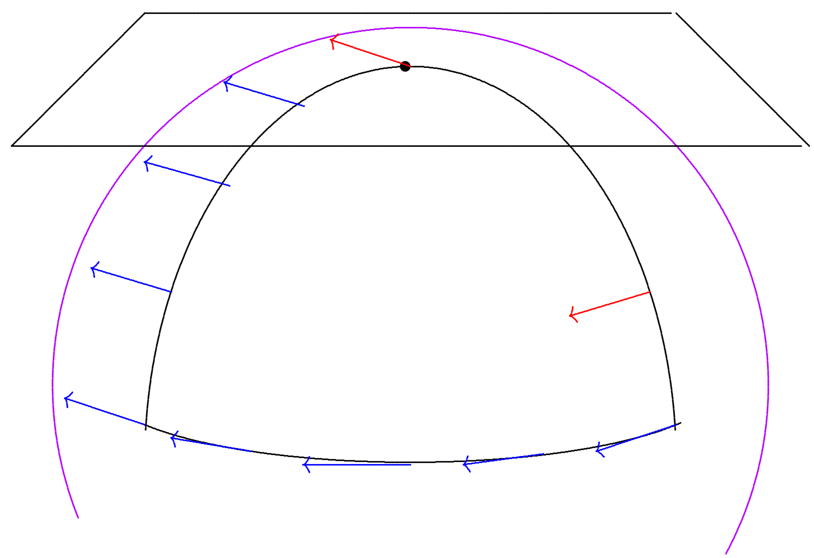

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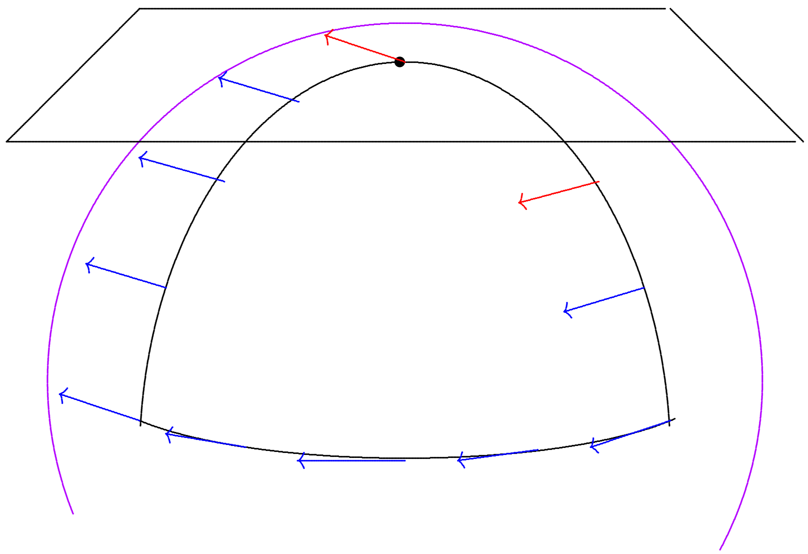

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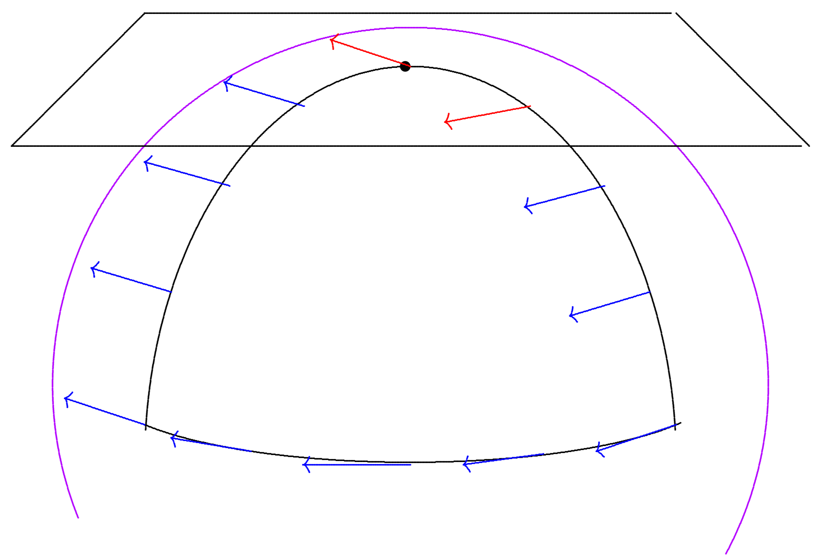

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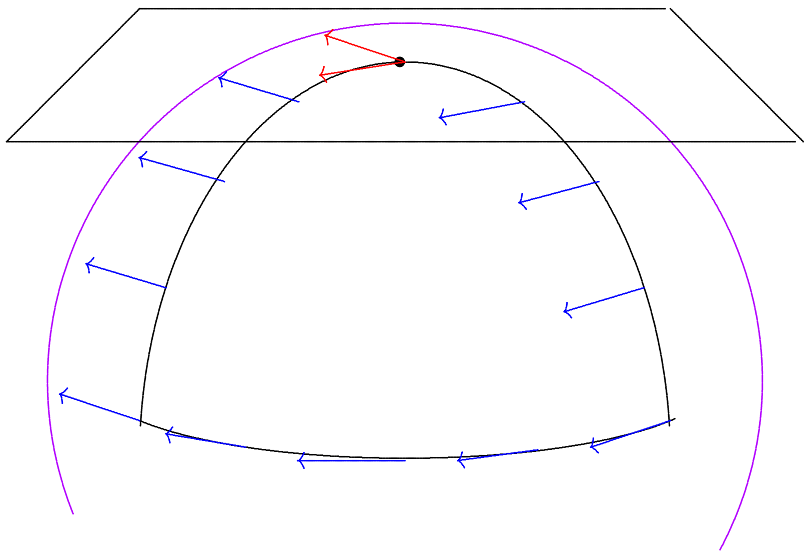

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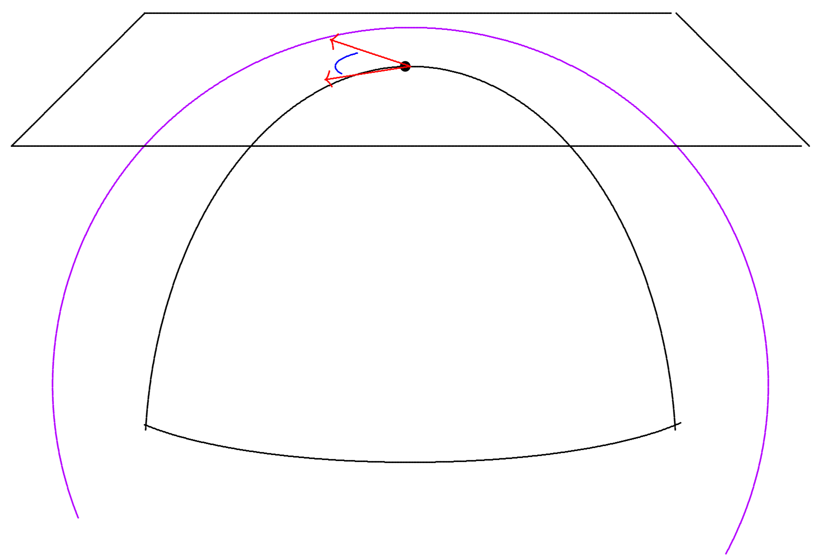
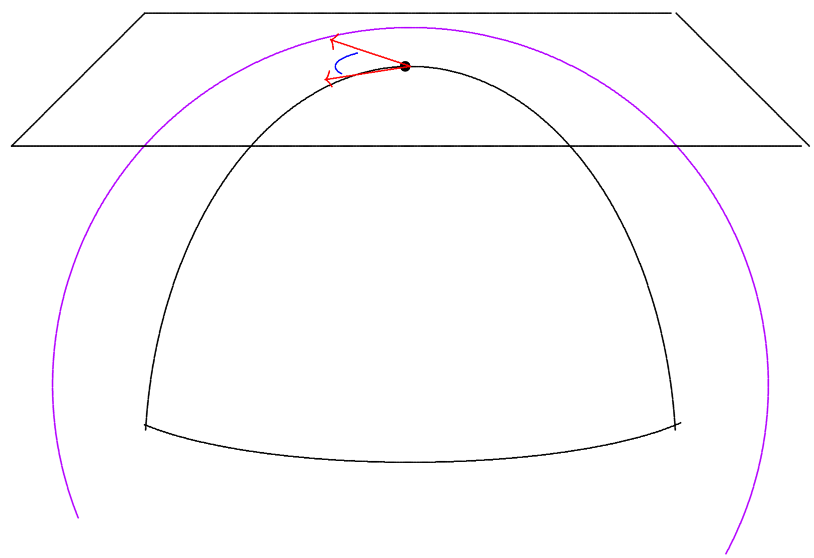
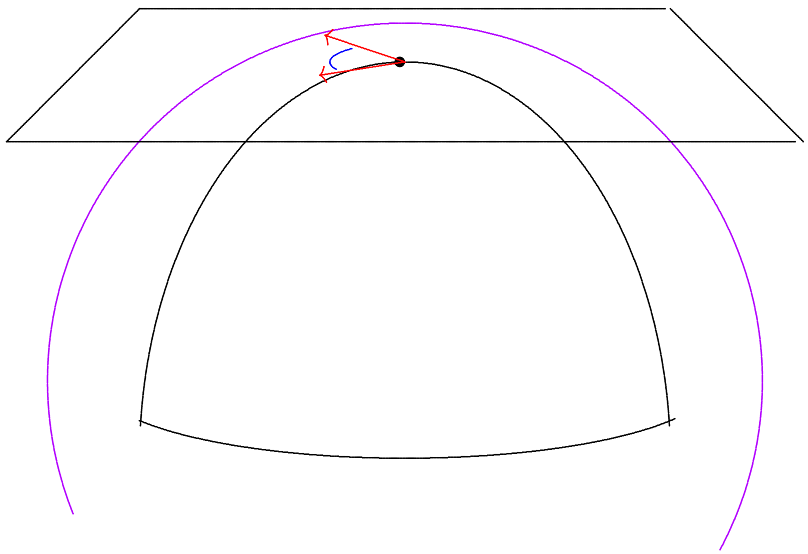


(M^n, g): holonomy $\subset \mathbf{O}(n)$



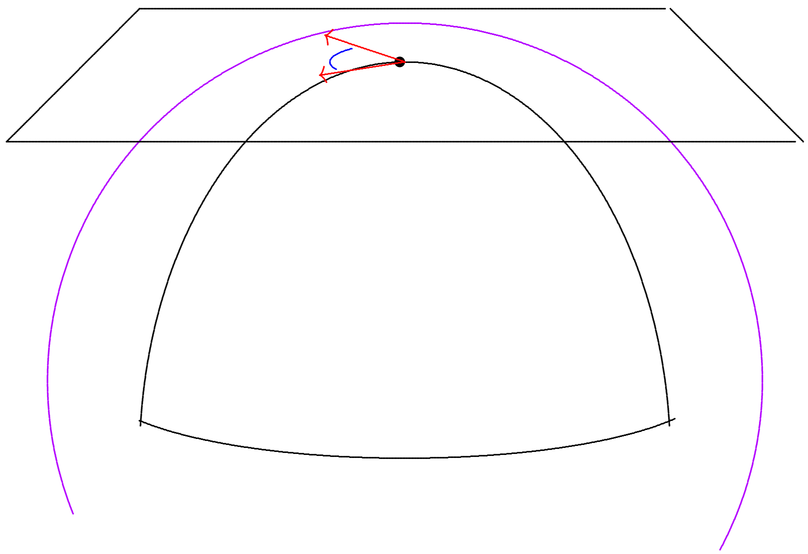
Kähler metrics:

(M^{2m}, g): holonomy



Kähler metrics:

(M^{2m}, g) Kähler $\Longleftrightarrow$ holonomy $\subset \mathbf{U}(m)$



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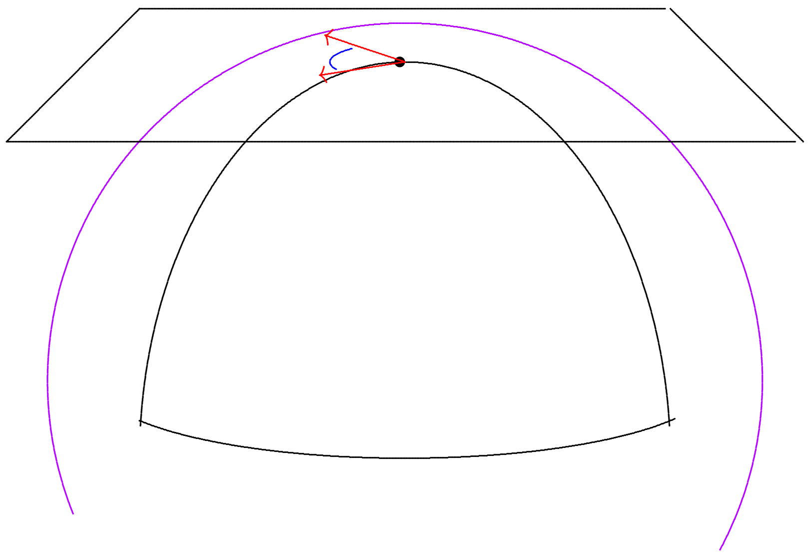

$$\mathbf{U}(m) := \mathbf{O}(2m) \cap \mathbf{GL}(m, \mathbb{C})$$

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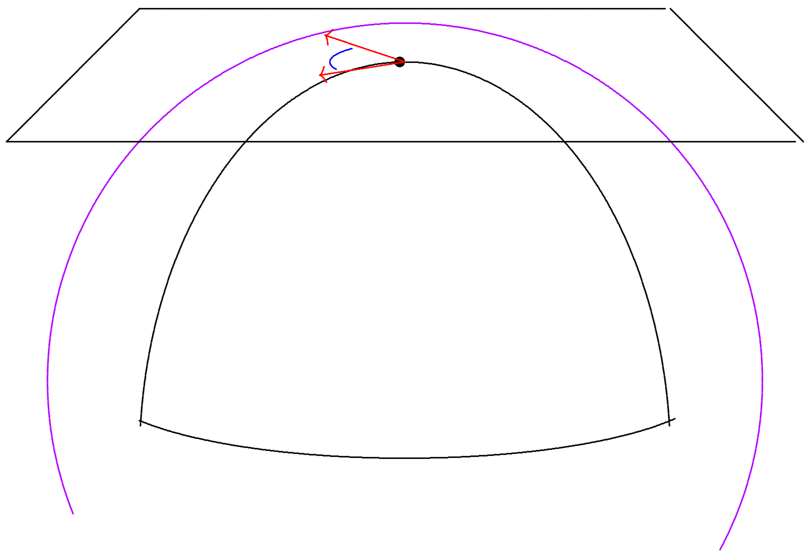

Makes tangent space a complex vector space!

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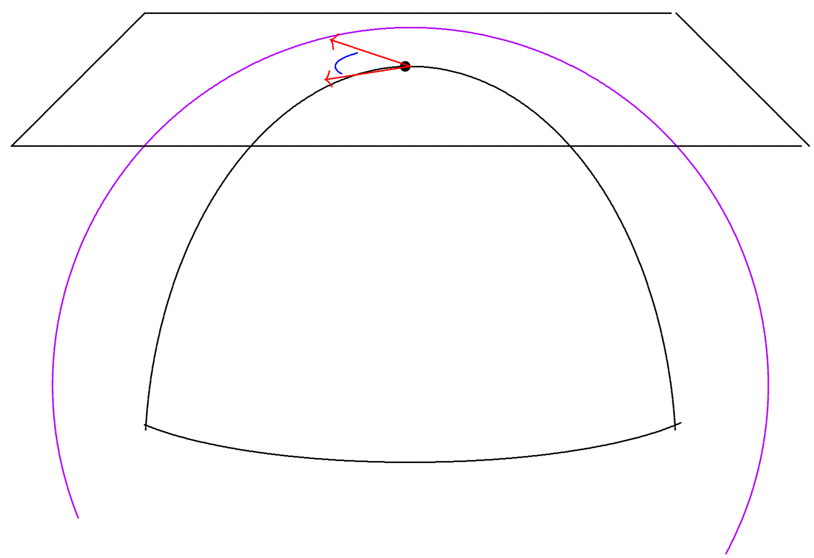

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$$J : TM \to TM, \qquad J^2 = -\text{identity}$$

"almost-complex structure"

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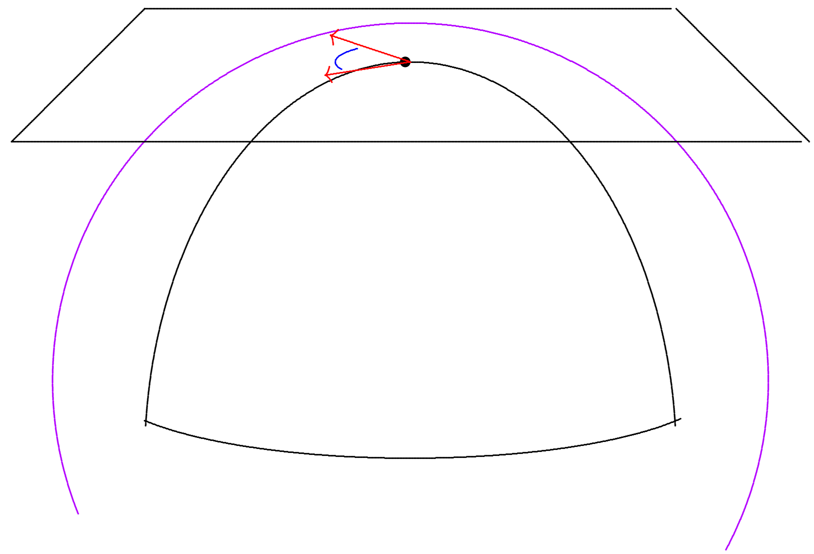

Makes tangent space a complex vector space!

Invariant under parallel transport!

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$$\omega = i \sum_{j,k=1}^{m} \frac{\partial^2 f}{\partial z^j \partial \bar{z}^k} \, dz^j \wedge d\bar{z}^k$$

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Kähler magic:

$$r = -\sum_{j,k=1}^{m} \frac{\partial^2}{\partial z^j \partial \bar{z}^k} \log \det[g_{p\bar{q}}] \left[dz^j \otimes d\bar{z}^k + d\bar{z}^k \otimes dz^j\right]$$

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Kähler magic:

If we define the Ricci form by

$$\rho = r(J\cdot, \cdot)$$

then $i\rho$ is curvature of canonical line bundle $\Lambda^{m,0}$.

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$$[\omega] \in H^2(M)$$

"Kähler class"

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ω non-degenerate closed 2-form: symplectic form

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$$g = \left(1 - \frac{2m}{\varrho^{n-2}}\right)^{-1} d\varrho^2 + \varrho^2 h_{S^{n-1}}$$

Scalar-flat-Kähler Burns metric on $\widetilde{\mathbb{C}^2} \subset \mathbb{C}^2 \times \mathbb{CP}_1$:

Restrict Euclidean $\times$ round metric to $\begin{vmatrix} z_1 & z_2 \\ \zeta_1 & \zeta_2 \end{vmatrix} = 0.$

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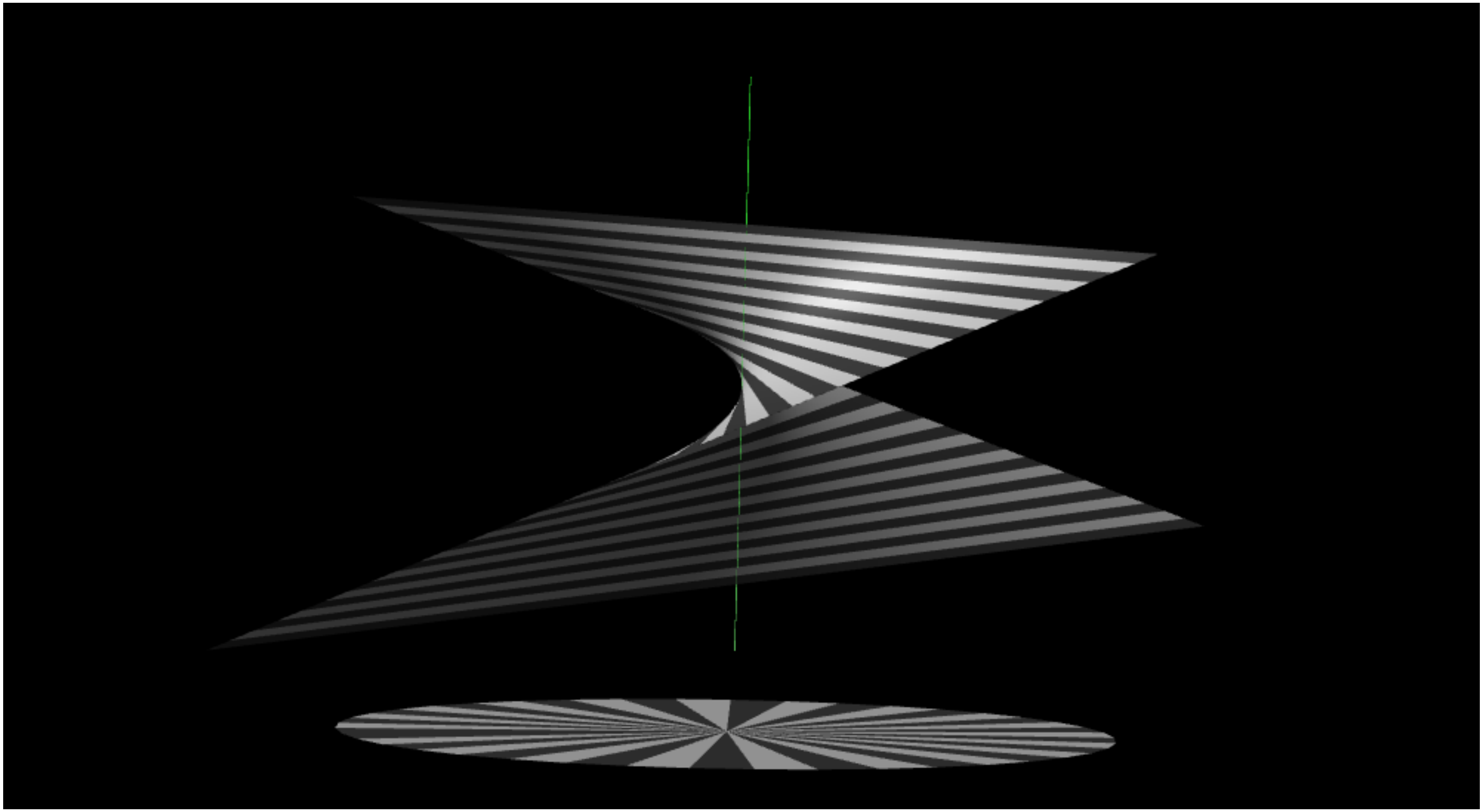

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Scalar-flat-Kähler Burns metric on $\widetilde{\mathbb{C}^2} \subset \mathbb{C}^2 \times \mathbb{CP}_1$:

Restrict Euclidean $\times$ round metric to $\begin{vmatrix} z_1 & z_2 \\ \zeta_1 & \zeta_2 \end{vmatrix} = 0.$

Motivation:

When $n = 3$, ADM mass in general relativity.

Reads off "apparent mass" from strength of the gravitational field far from an isolated source.

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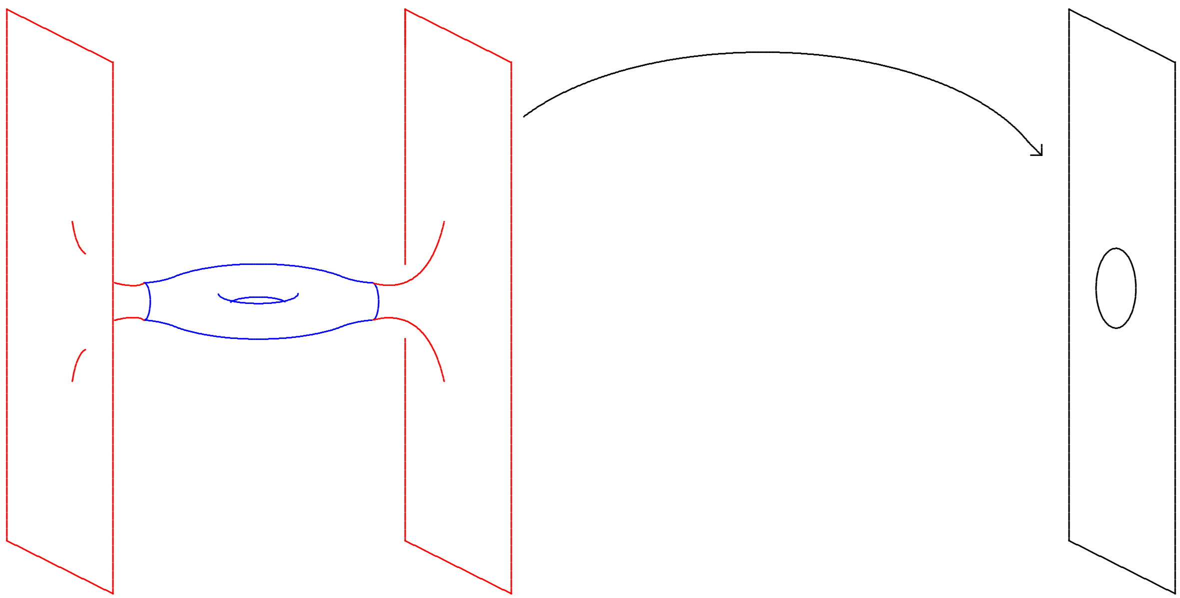
Definition. *A complete, non-compact Riemannian* n*-manifold* (M^n, g) *is called* asymptotically Euclidean *(AE) if there is a compact set* $K \subset M$ *such that each* "end" *is diffeomorphic to* $\mathbb{R}^n - D^n$ *in such a manner that*

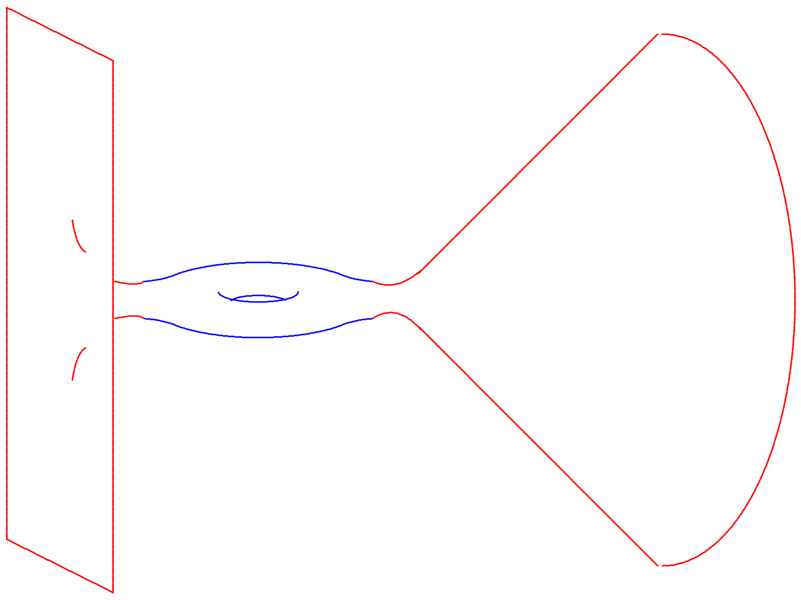


$$g_{jk} = \delta_{jk} + O(|x|^{1-\frac{n}{2}-\varepsilon})$$

$$g_{jk,\ell} = O(|x|^{-\frac{n}{2}-\varepsilon}), \quad s \in L^1$$

Interesting generalization. . .

Definition. *Complete, non-compact* n*-manifold* (M^n, g) *is* asymptotically locally Euclidean



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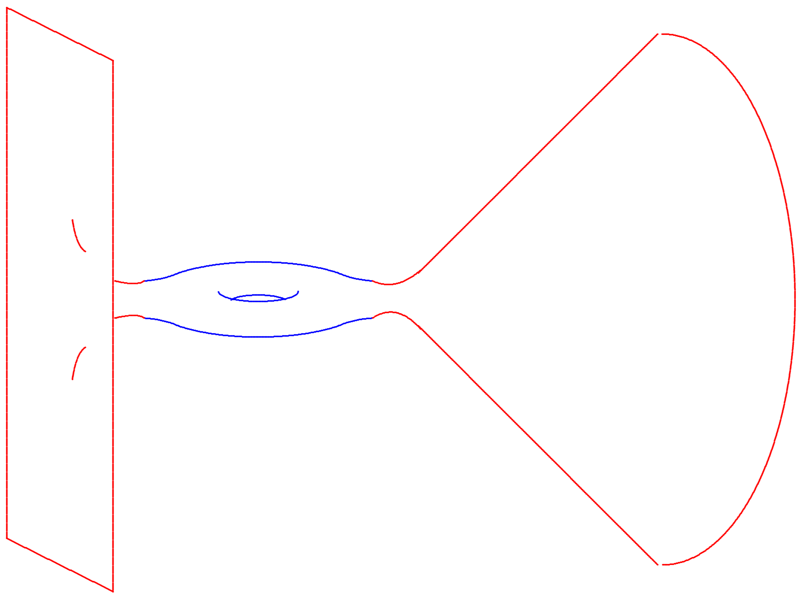

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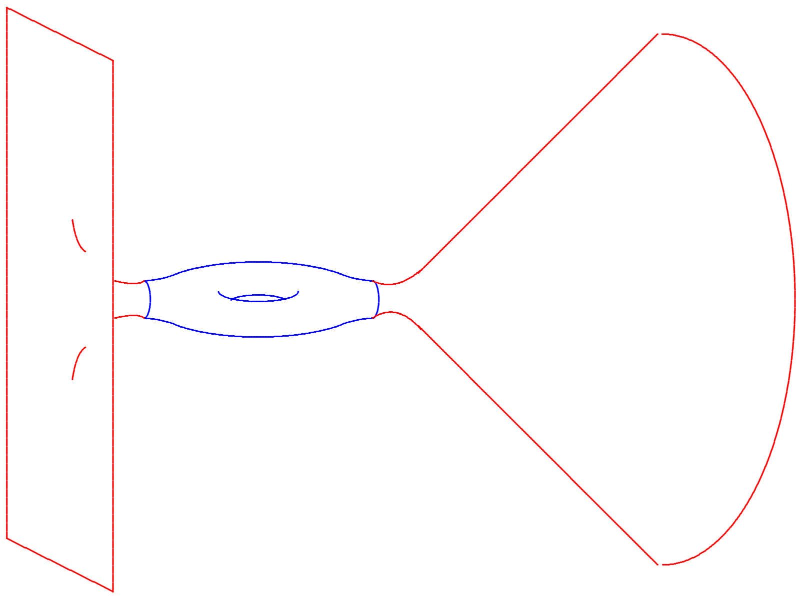

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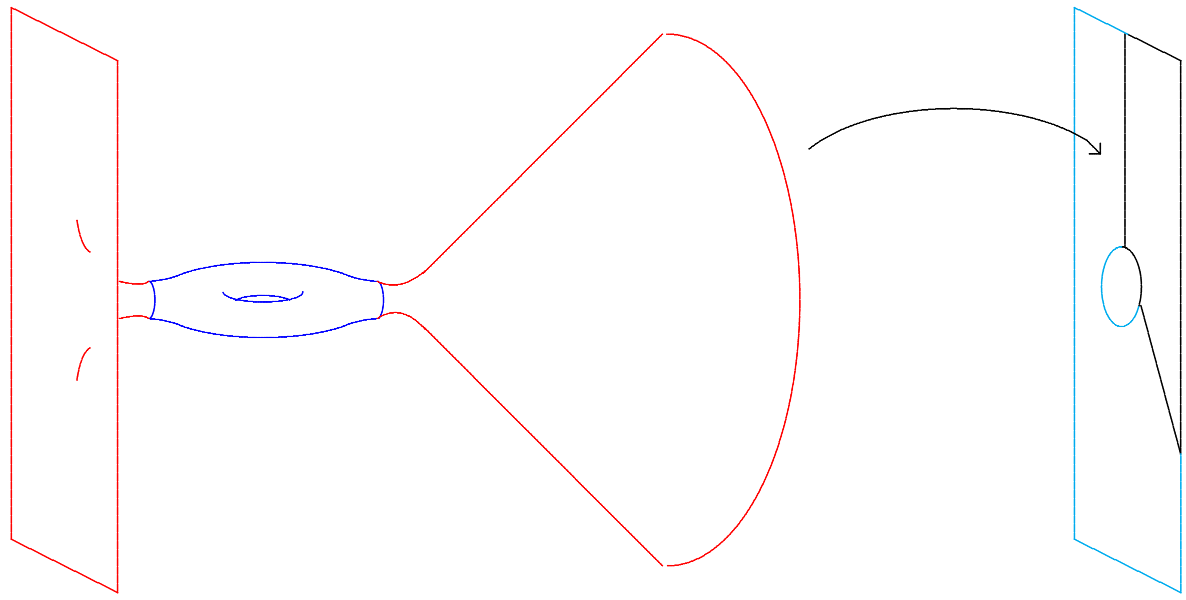

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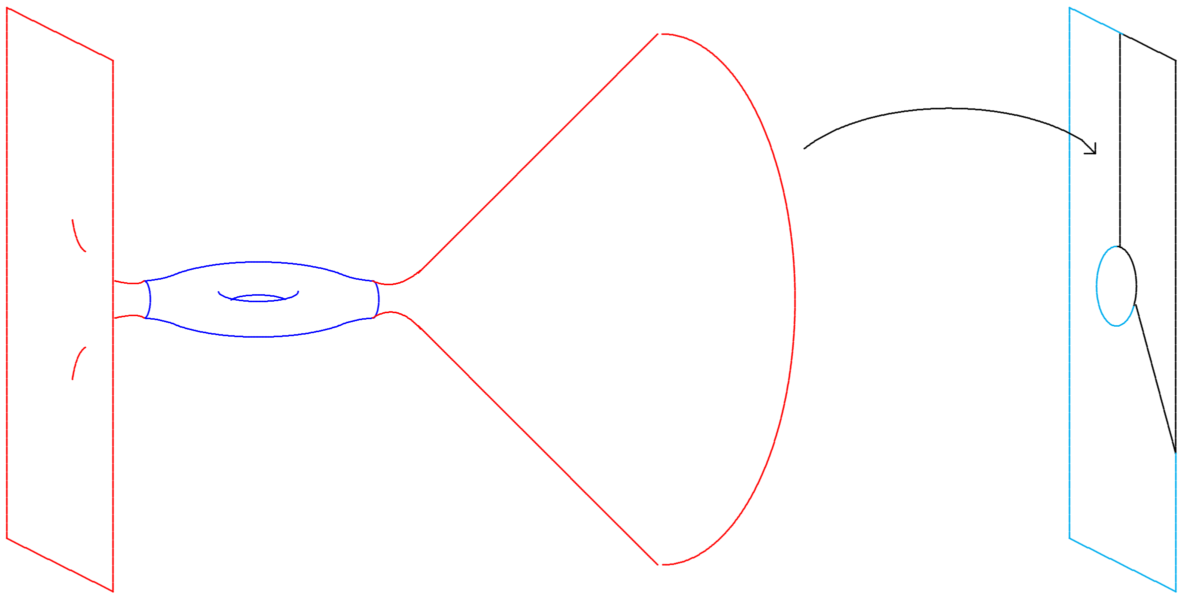

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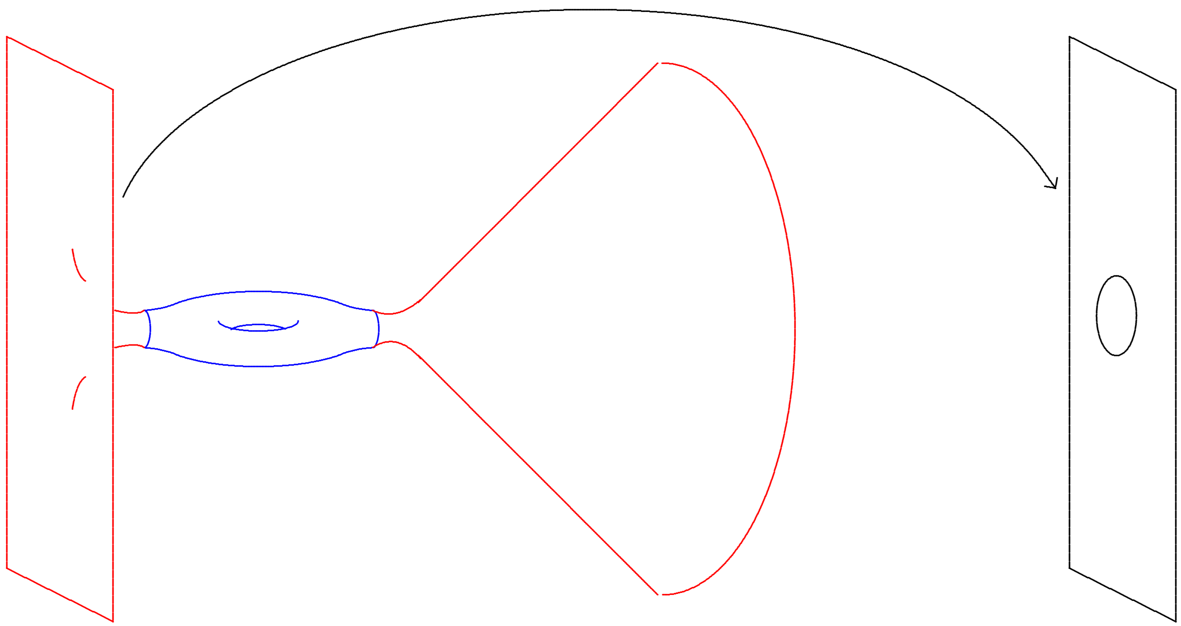

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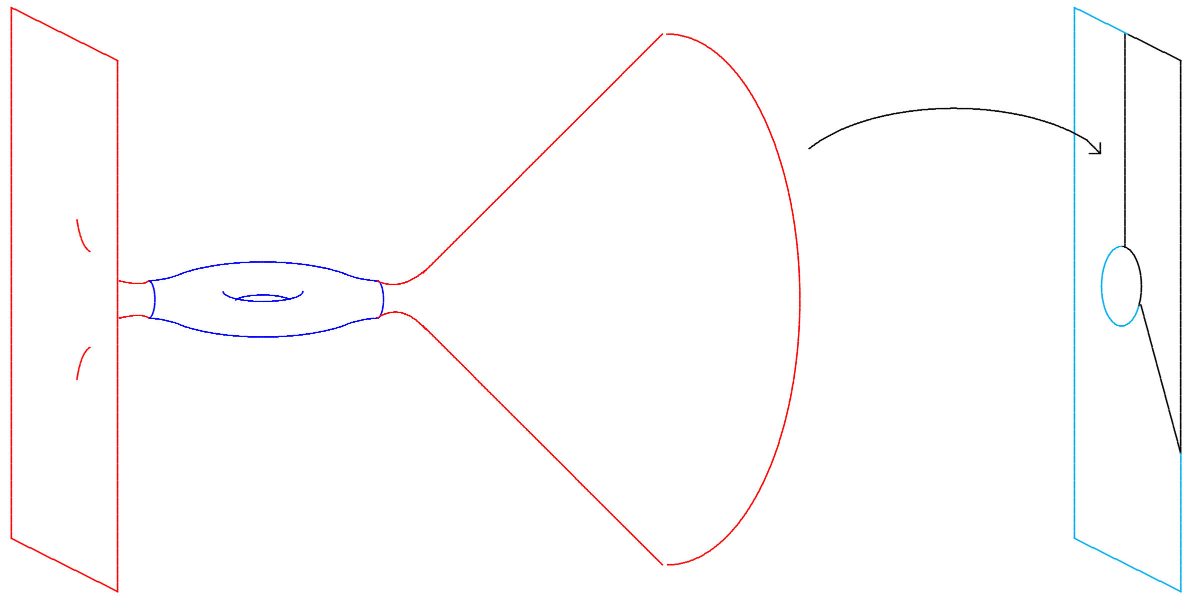

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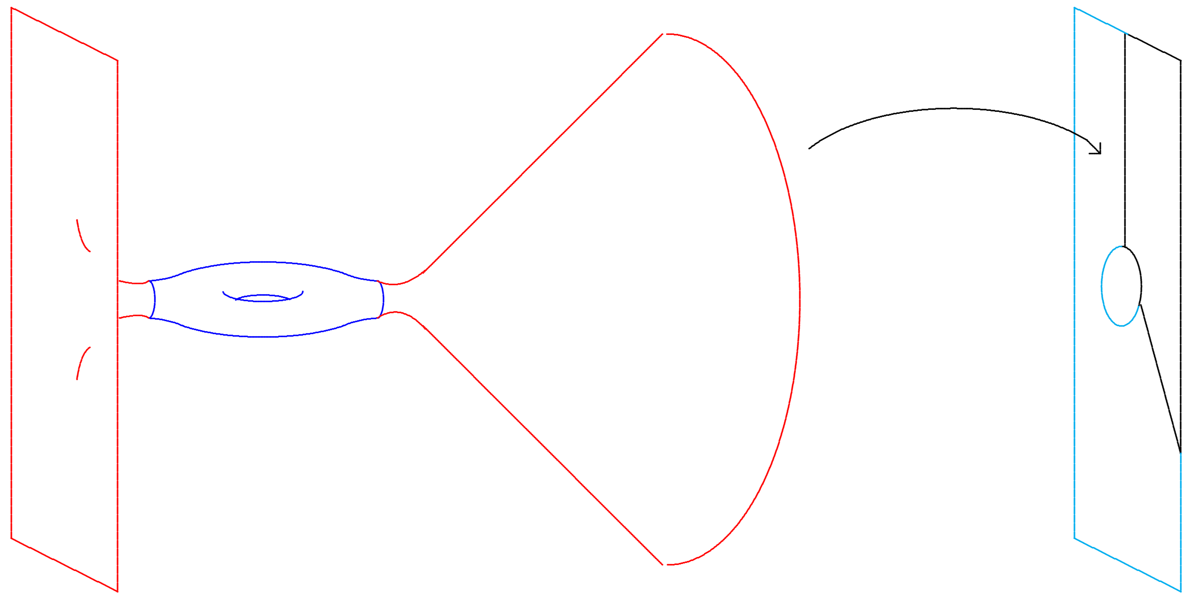

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Why consider ALE spaces?

Key examples:

Key examples:

Gravitational Instantons:

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ALE Ricci-flat Kähler manifolds.

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Bubbling modes for sequences of Einstein metrics.

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Chen-L-Weber '08.

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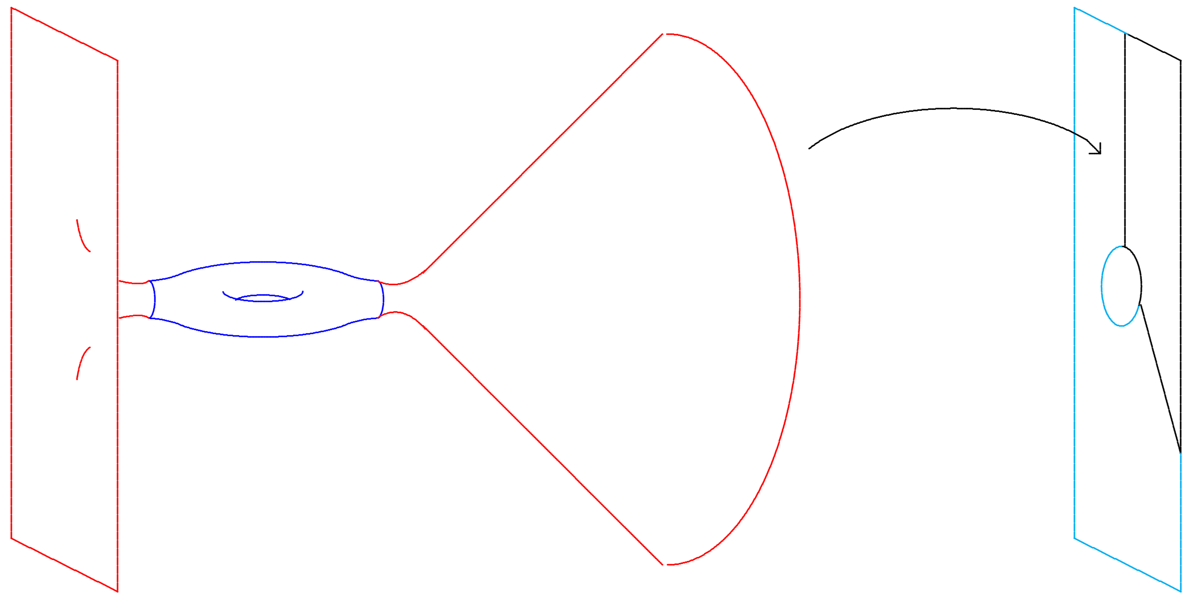

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Mass still meaningful in this context. . .

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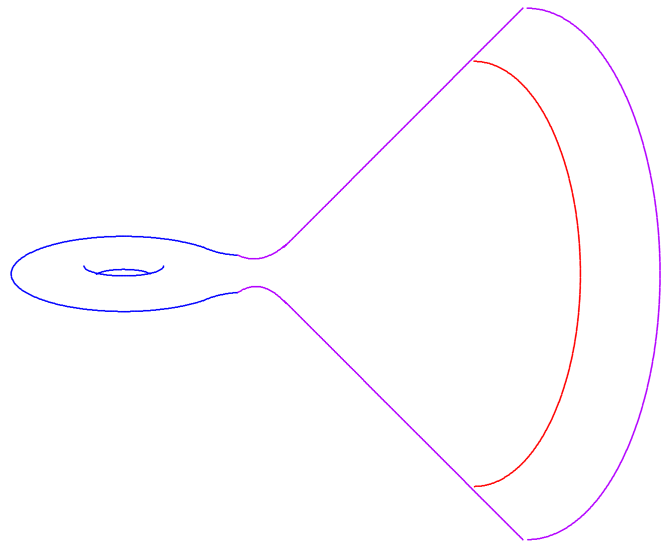

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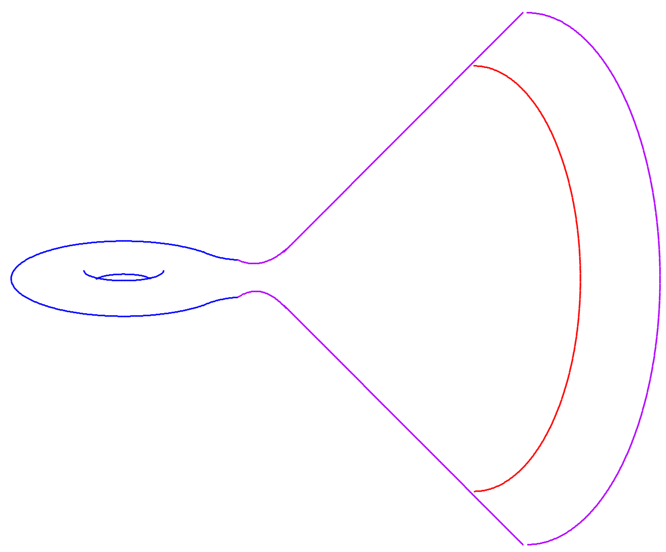

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Bartnik/Chruściel (1986): With weak fall-off conditions, the mass is well-defined & coordinate independent.

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General case in arbitrary dimension n.

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Lemma. *Any ALE Kähler manifold*

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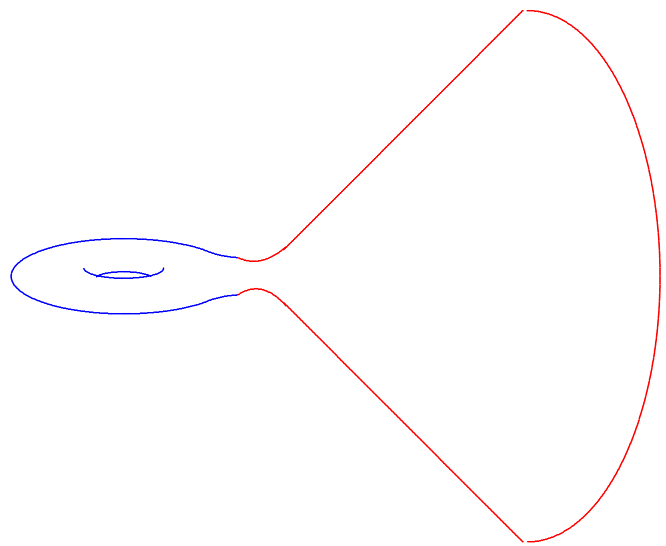

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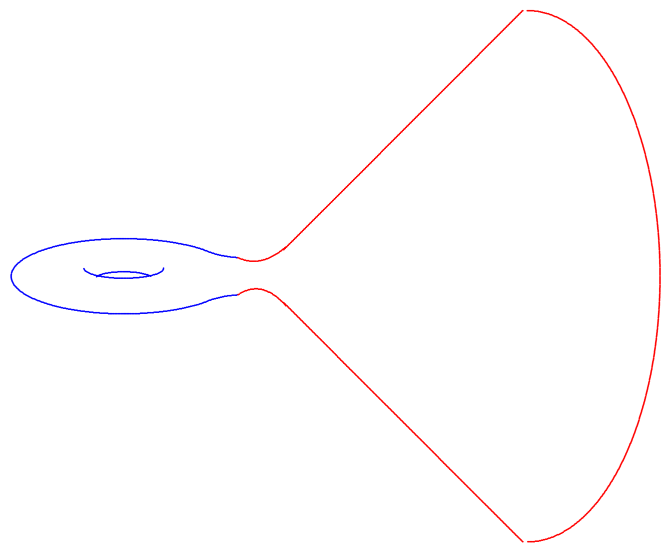

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Uses methods of symplectic topology.

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Mass of an ALE Kähler manifold is unambiguous.

Does not depend on the choice of an end!

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Theorem A.

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In fact, we will see that there is an explicit formula for the mass in terms of these data!

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minimal: containing no (-1)-curves

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minimal: not the blow-up of another resolution

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Non-minimal resolutions typically admit families of such metrics for which the mass can be continuously deformed from negative to positive.

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So the mass is a "boundary correction" to the topological formula for the total scalar curvature.

Theorem C. *Any ALE Kähler manifold* (M, g, J) *of complex dimension* m *has mass given by*

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So **Theorem A** is an immediate consequence!

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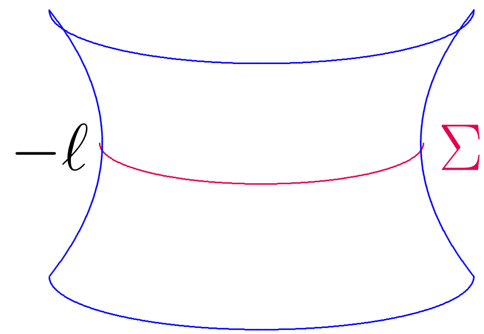
Example. Blow up Chern class $-\ell$ line bundle over $\mathbb{CP}_1$ at k points on zero section Σ.



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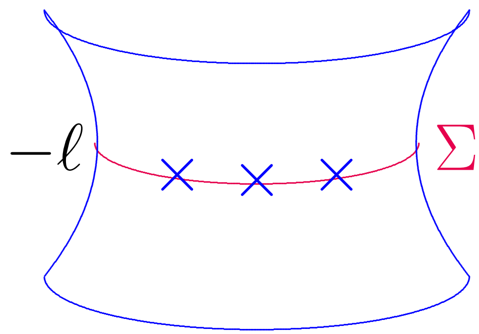

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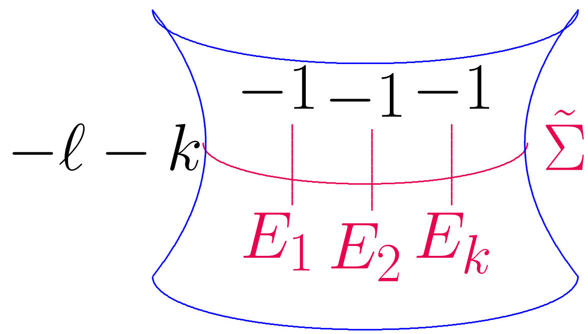

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Proof actually shows something stronger!

Theorem E (Penrose Inequality).

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canonical $\Longleftrightarrow$ Poincaré dual to $-c_1$.

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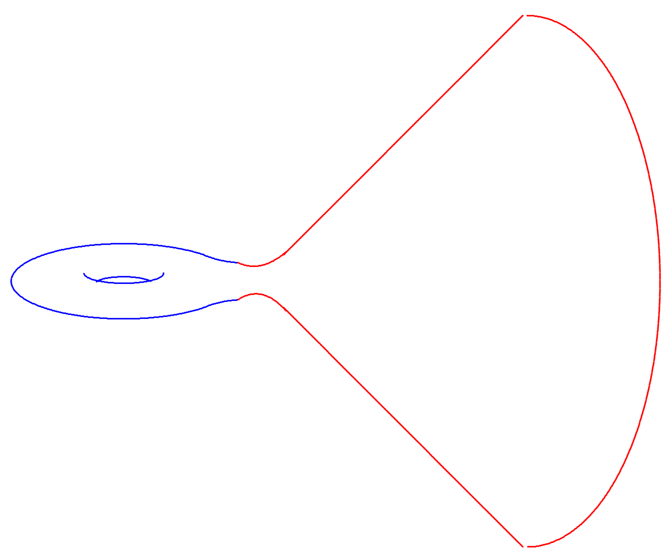

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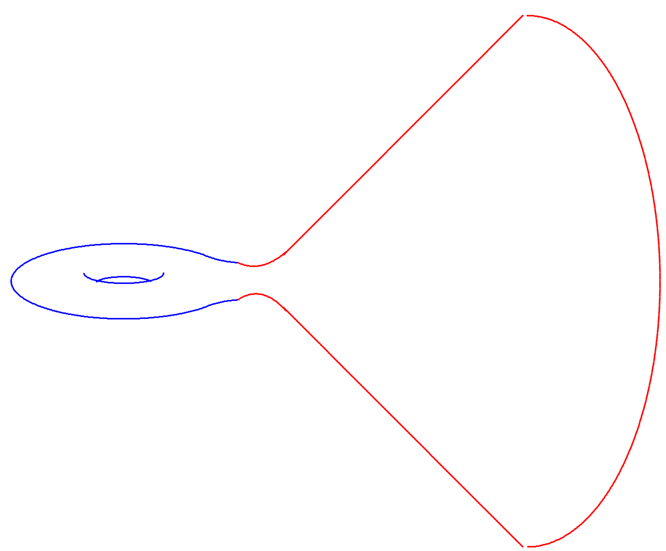

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