

*Mass in*

*Kähler Geometry*

Claude LeBrun

Stony Brook University

Geometric Analysis in Geometry and Topology  
Tokyo University of Science, November 10, 2015

Joint work with

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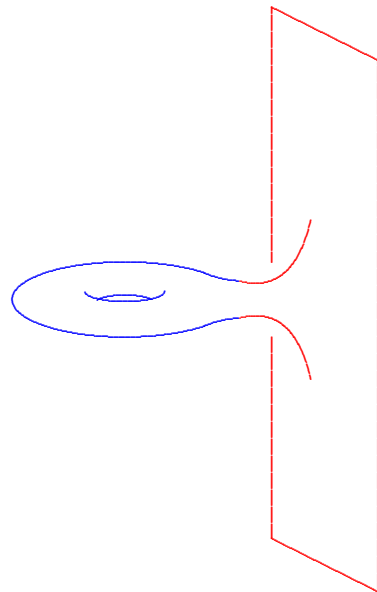
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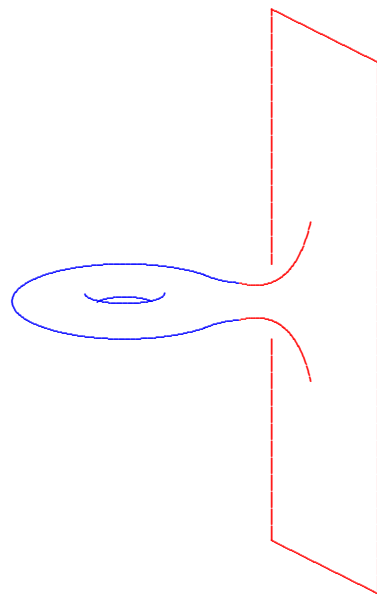
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e-print: [arXiv:1507.08885](https://arxiv.org/abs/1507.08885) [math.DG]

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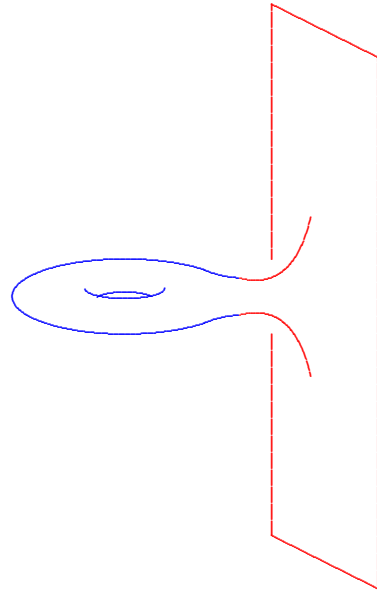


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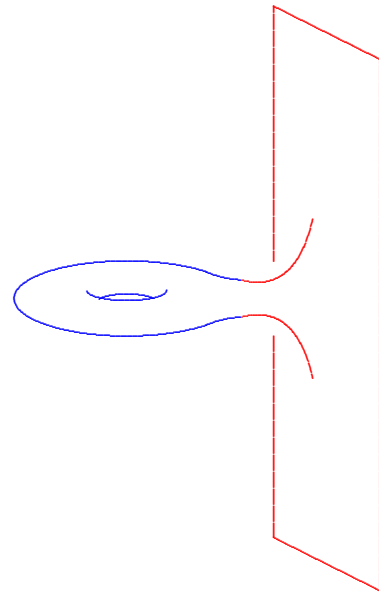
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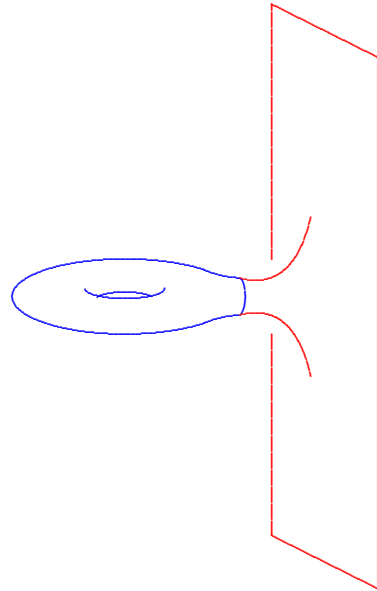
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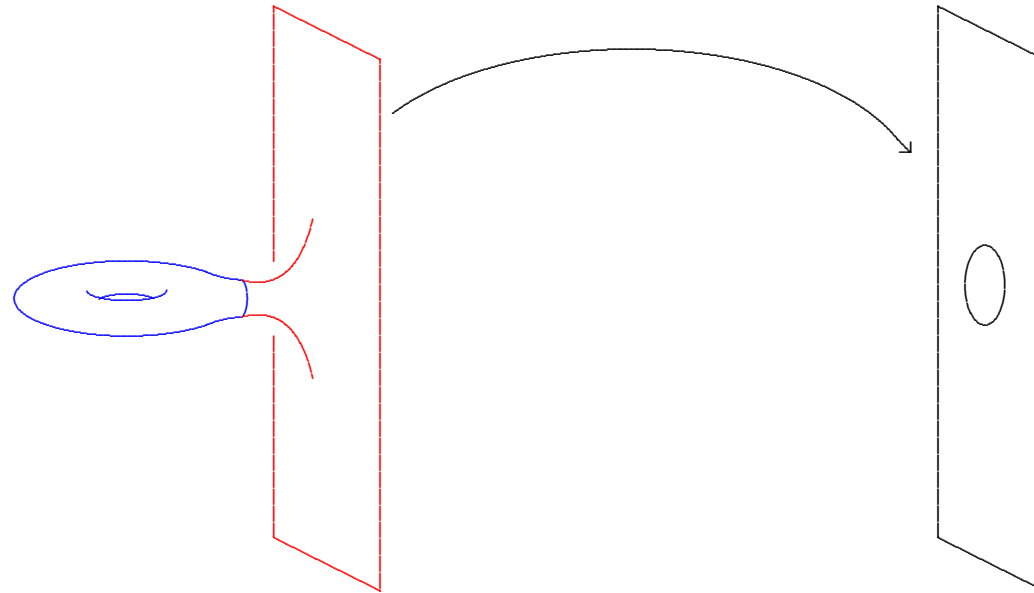
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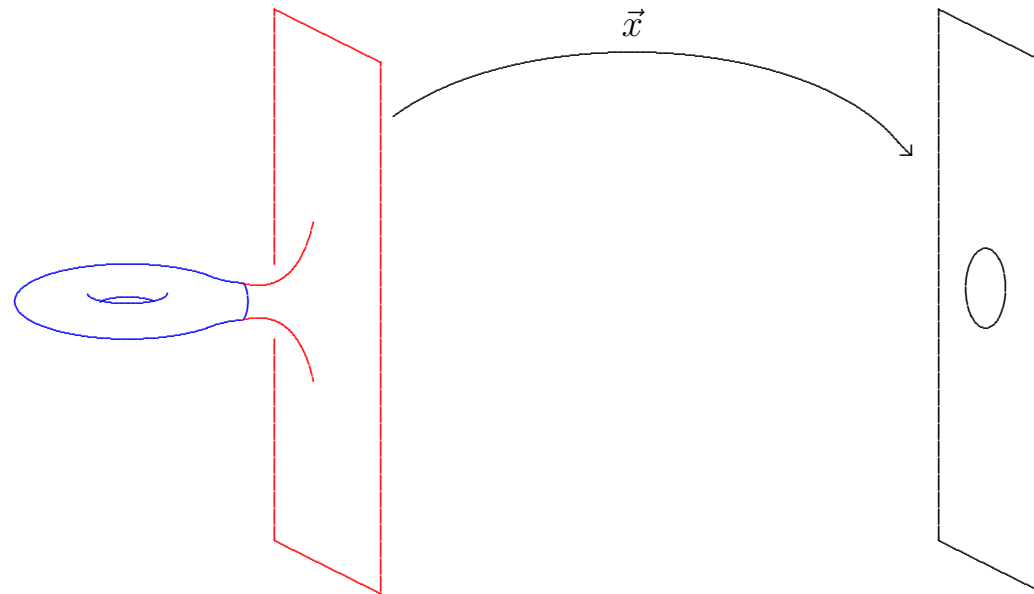
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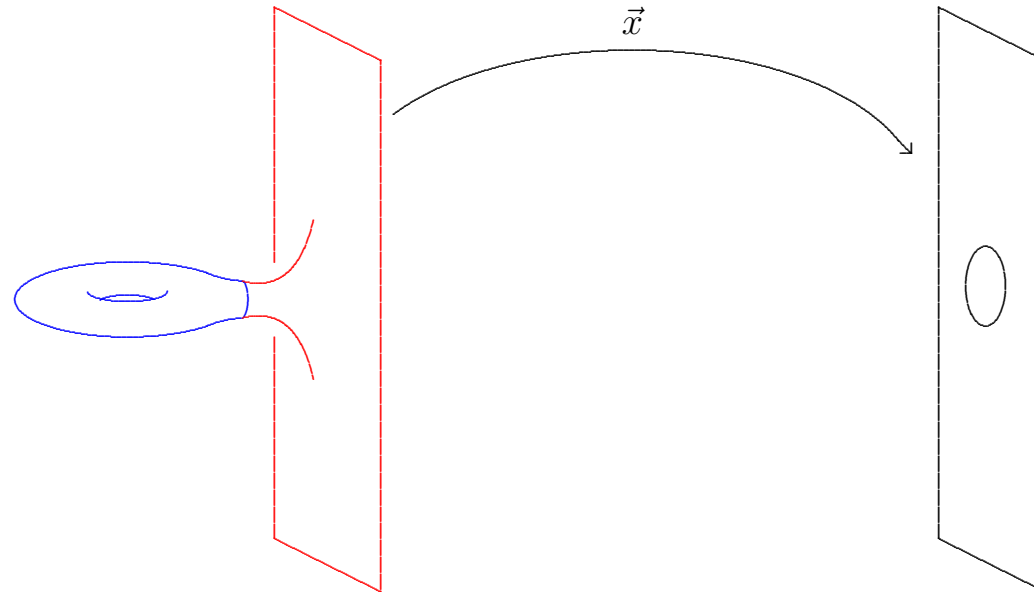


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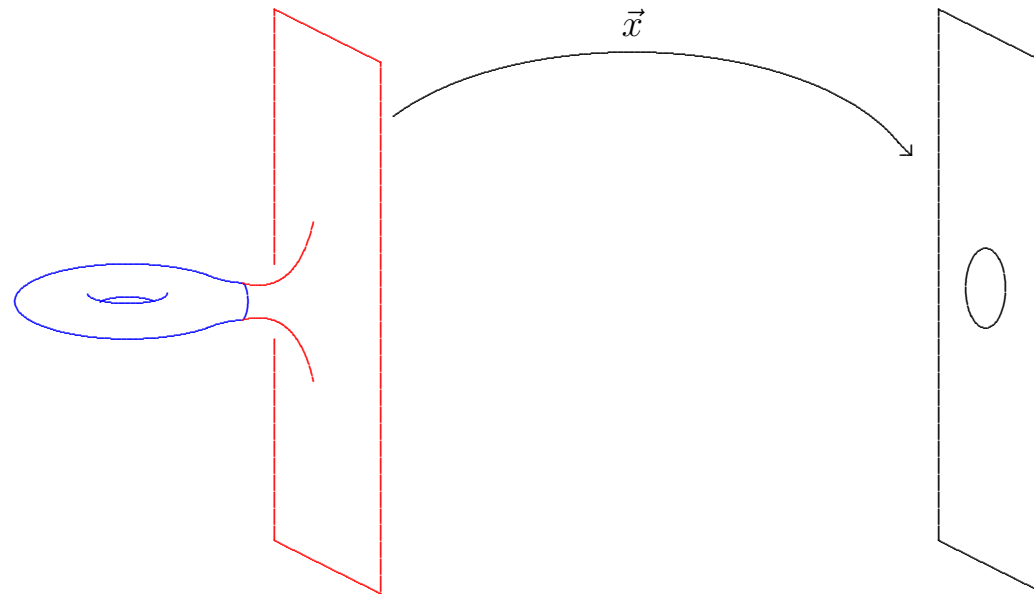
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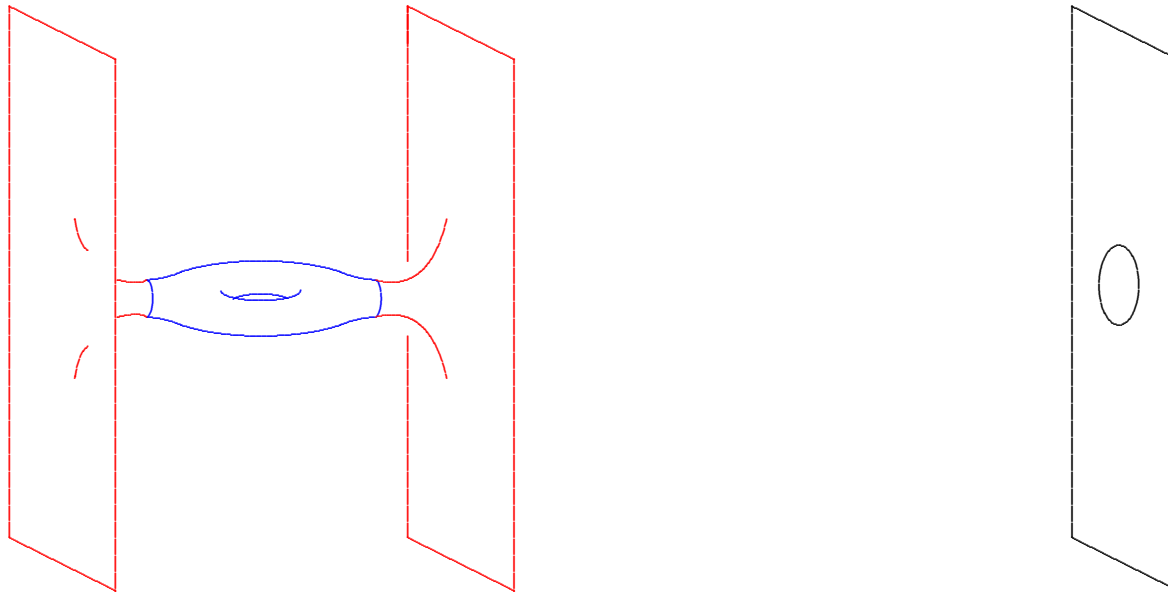
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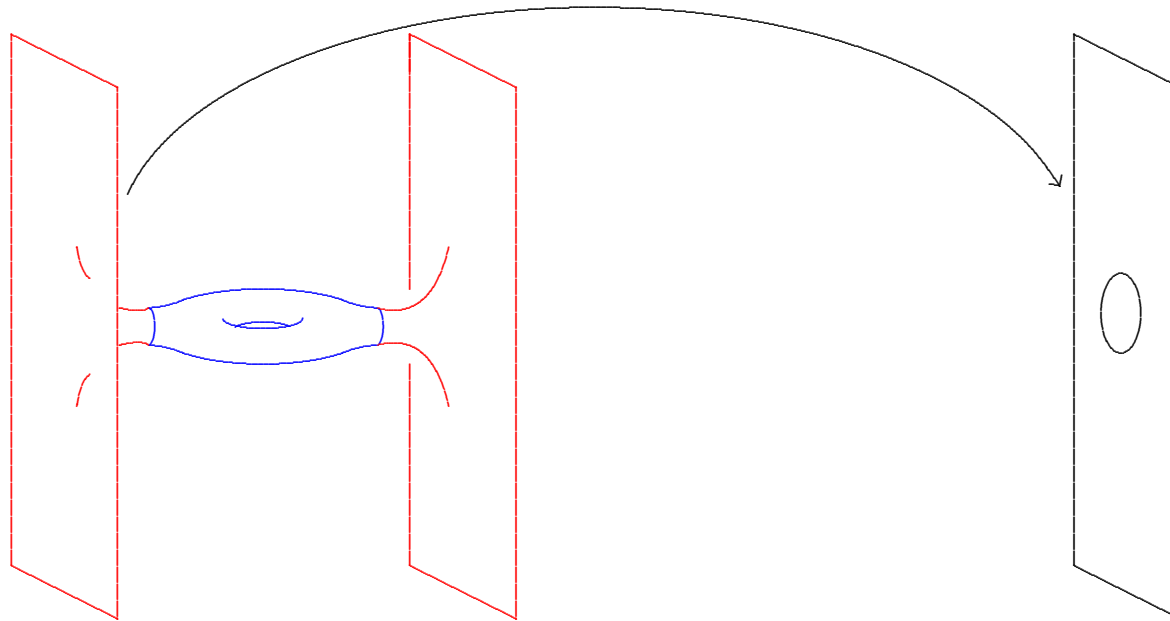
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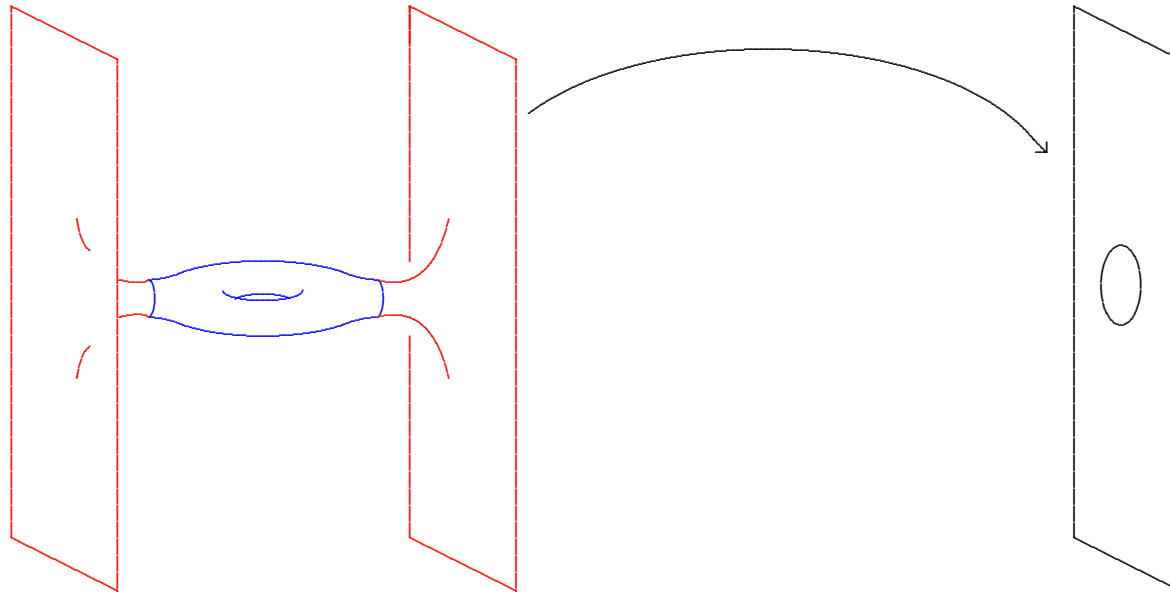
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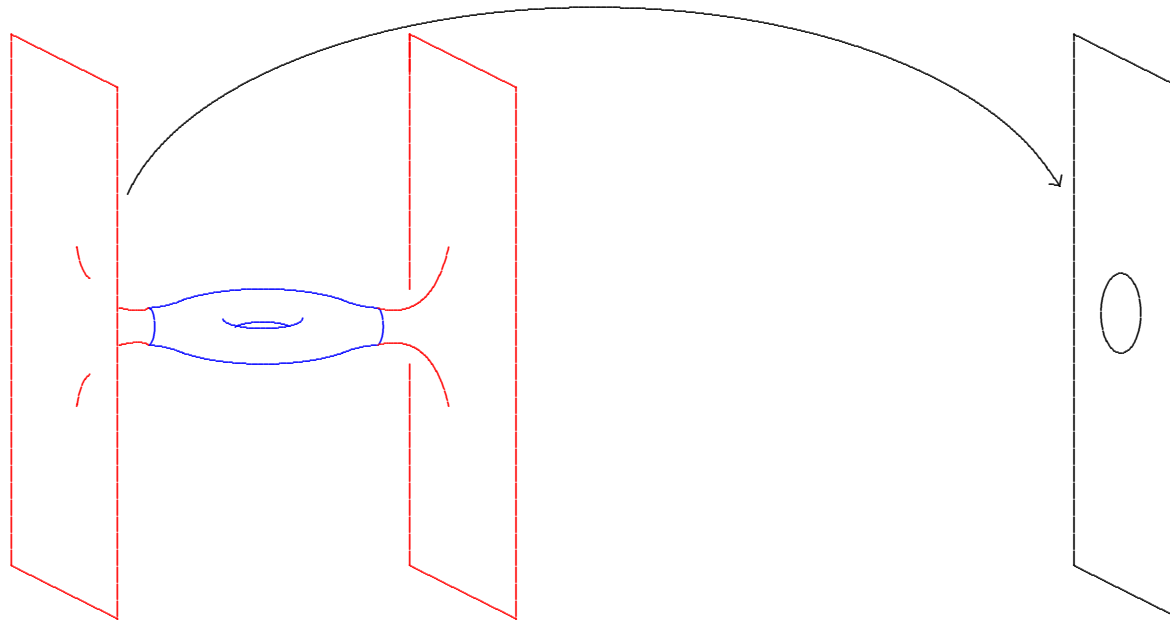


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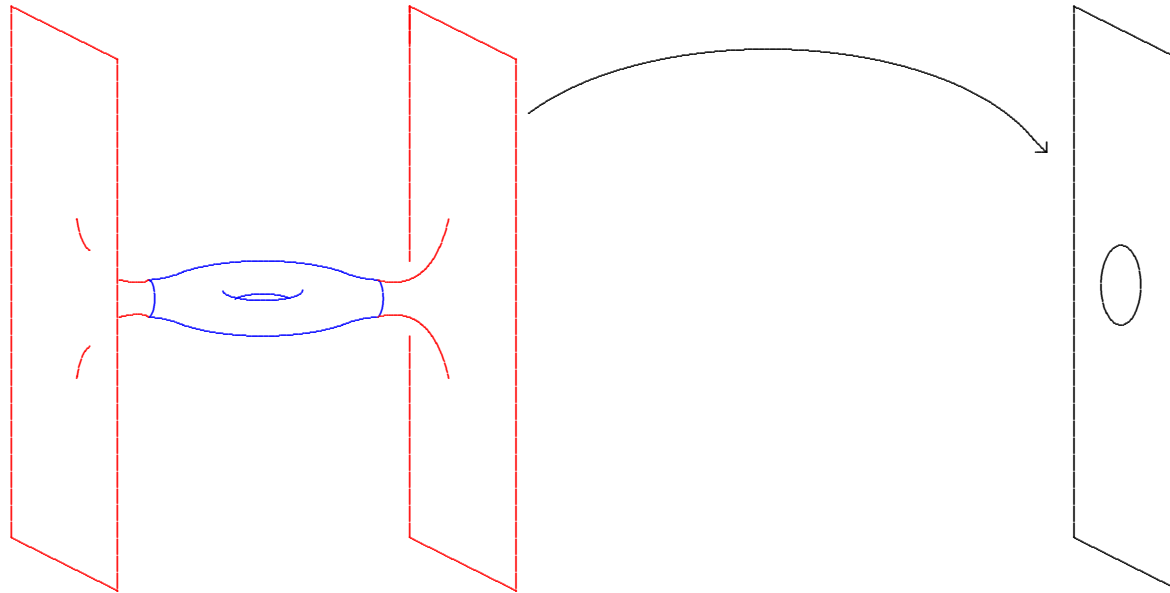
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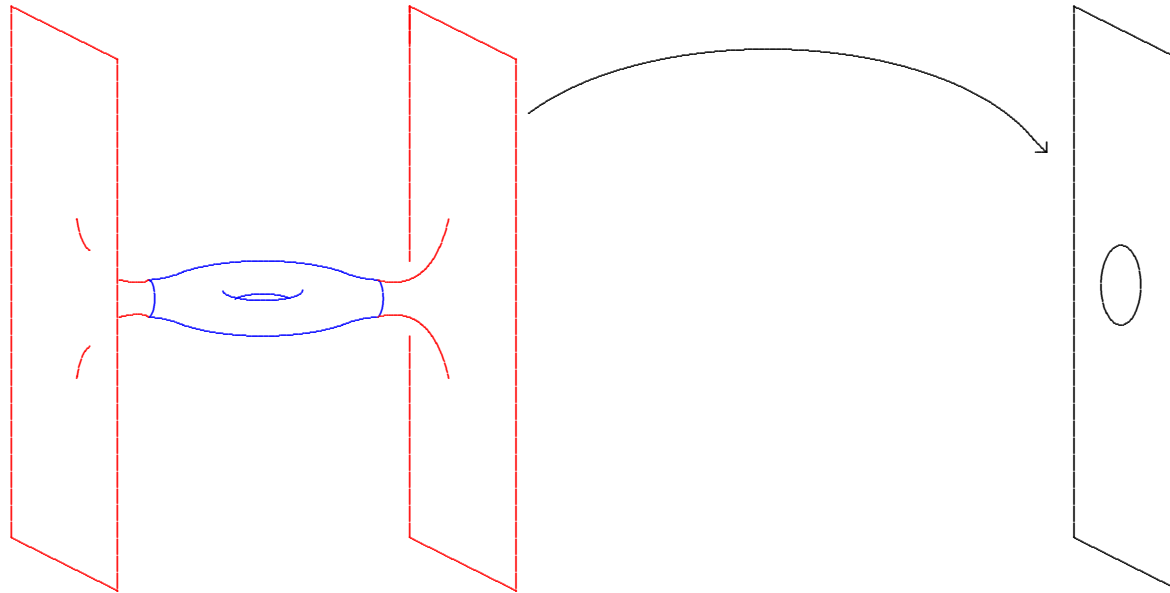
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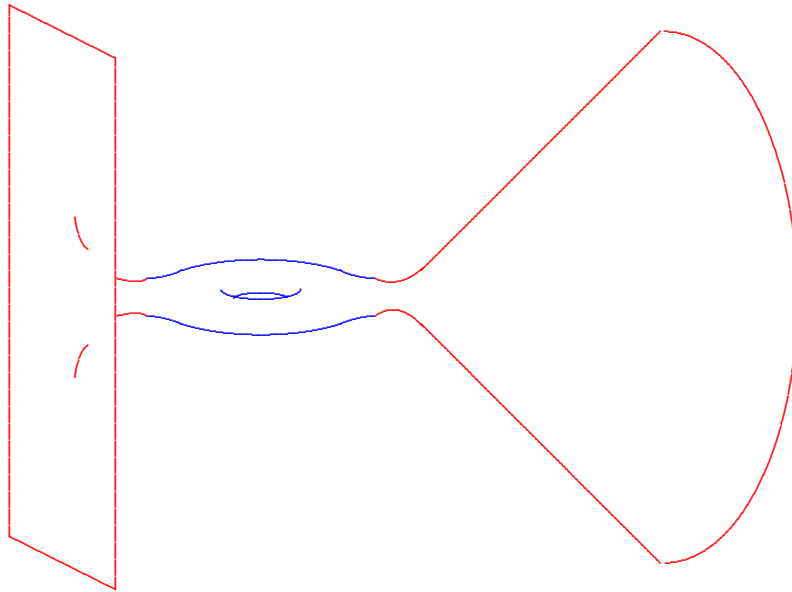
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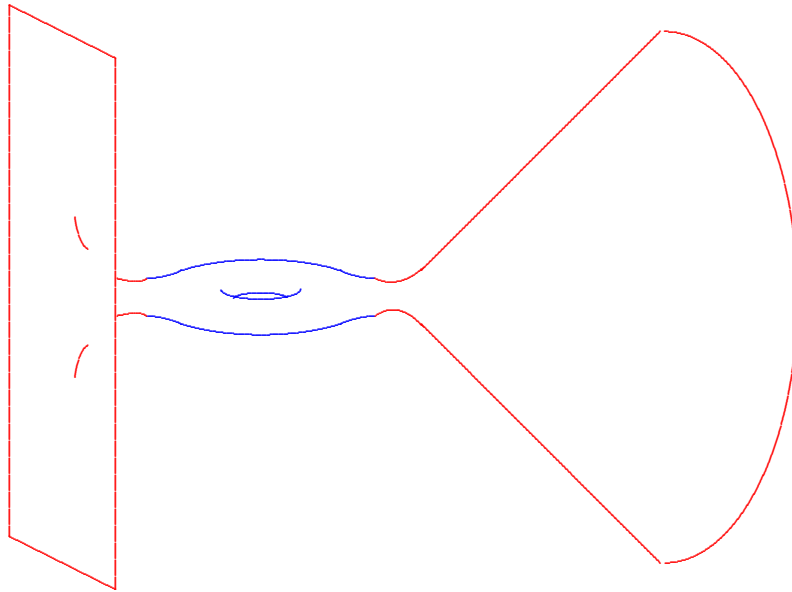
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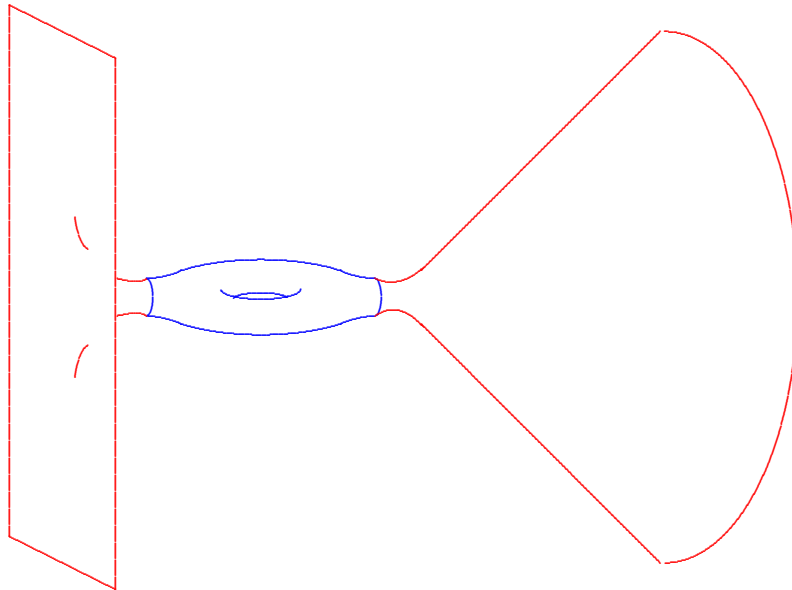
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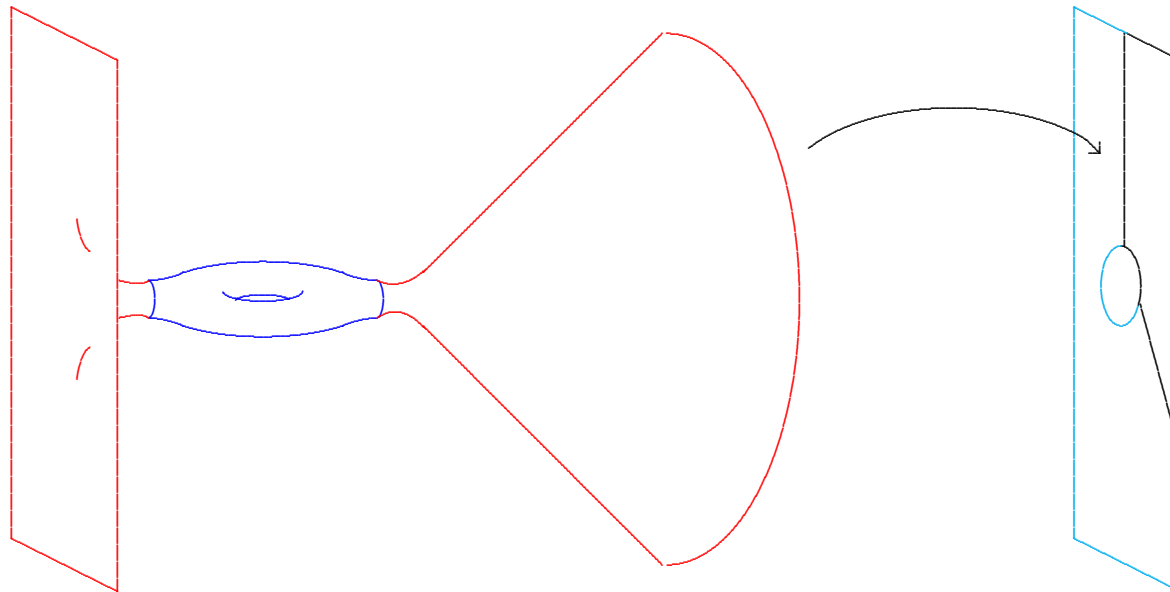
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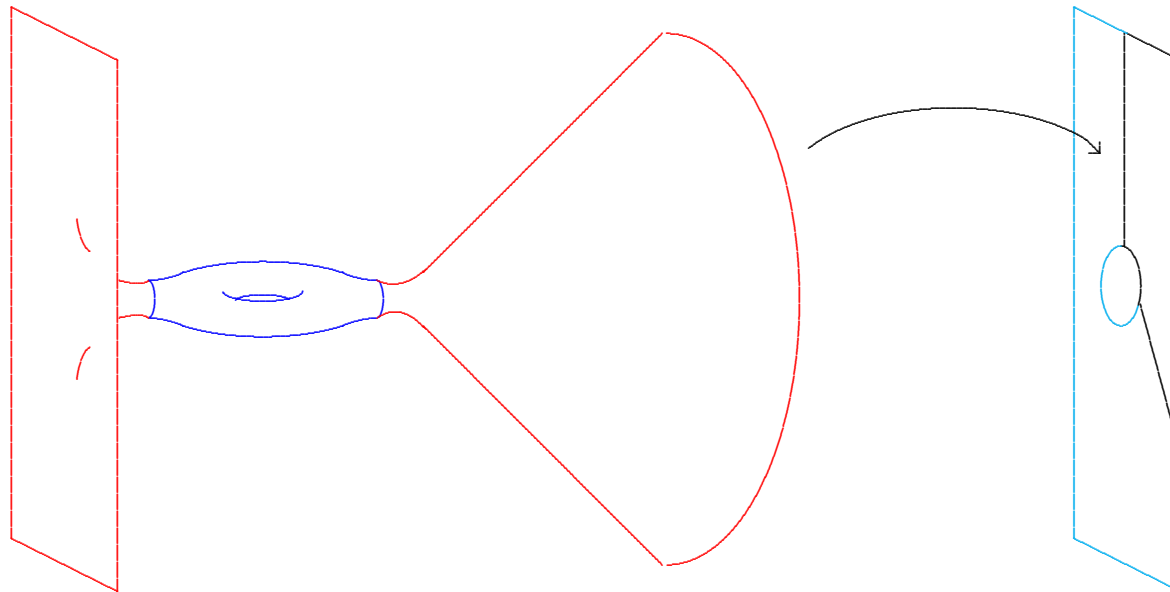
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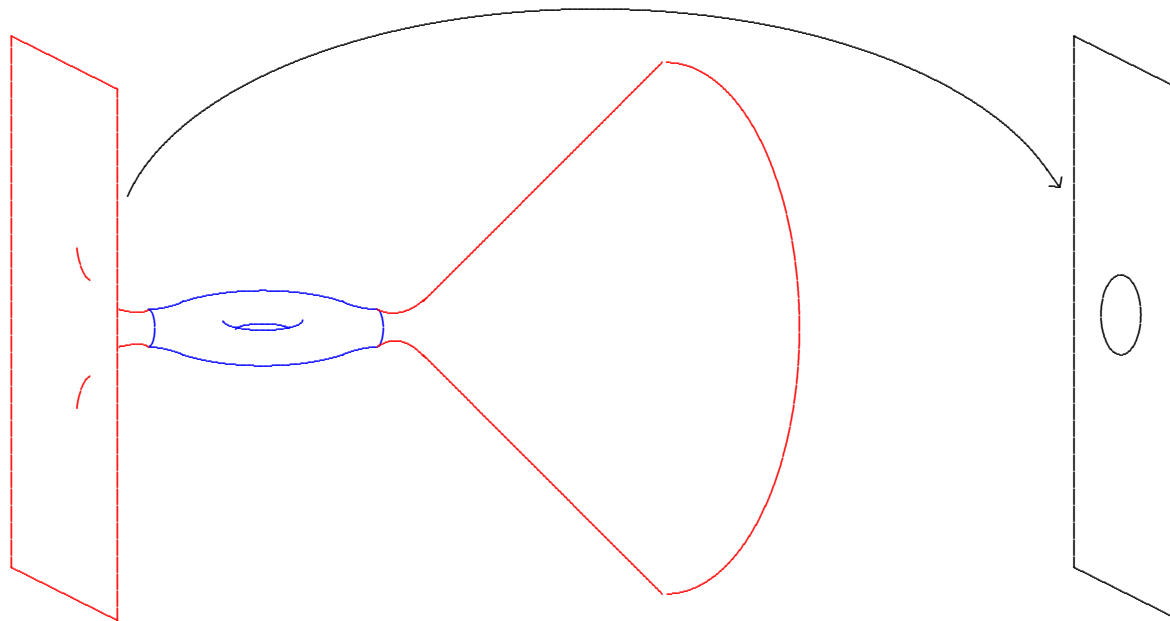


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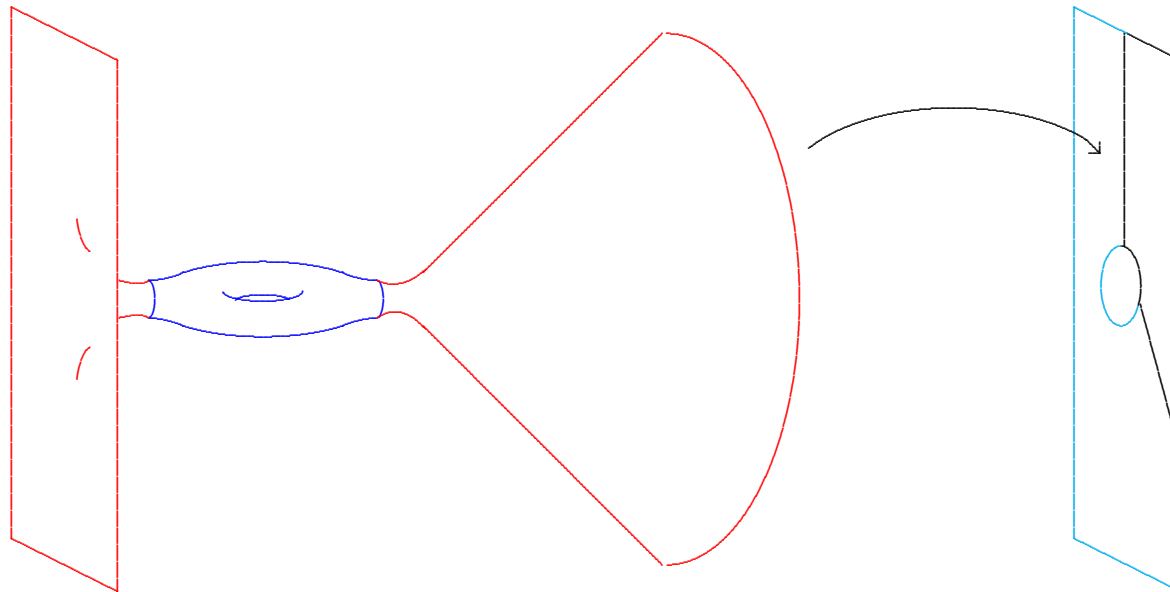




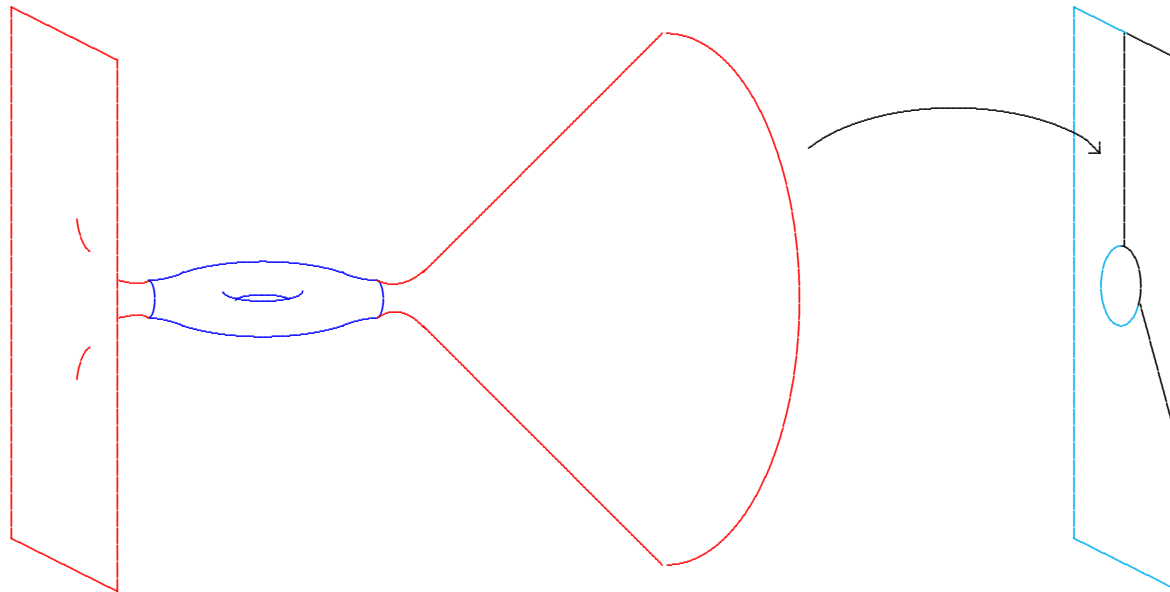
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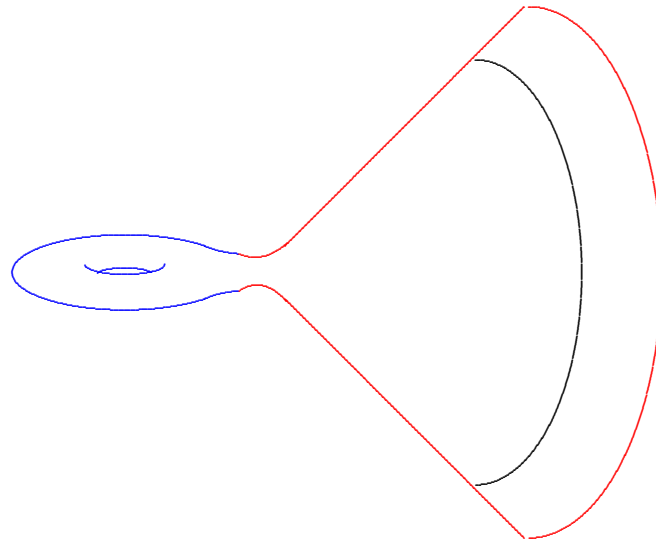
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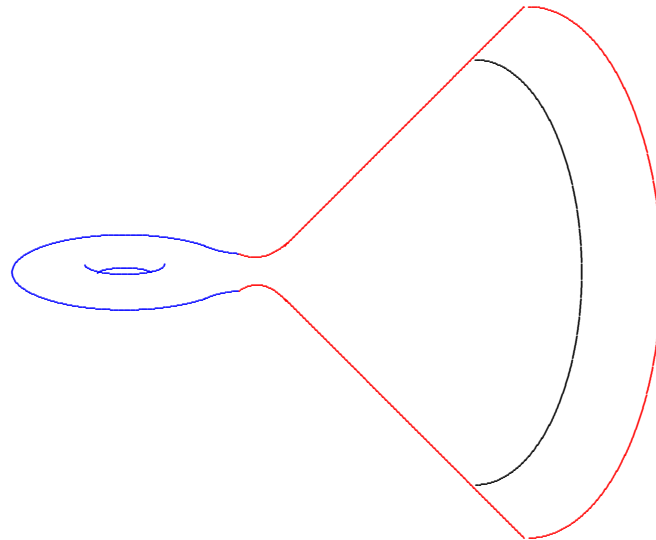


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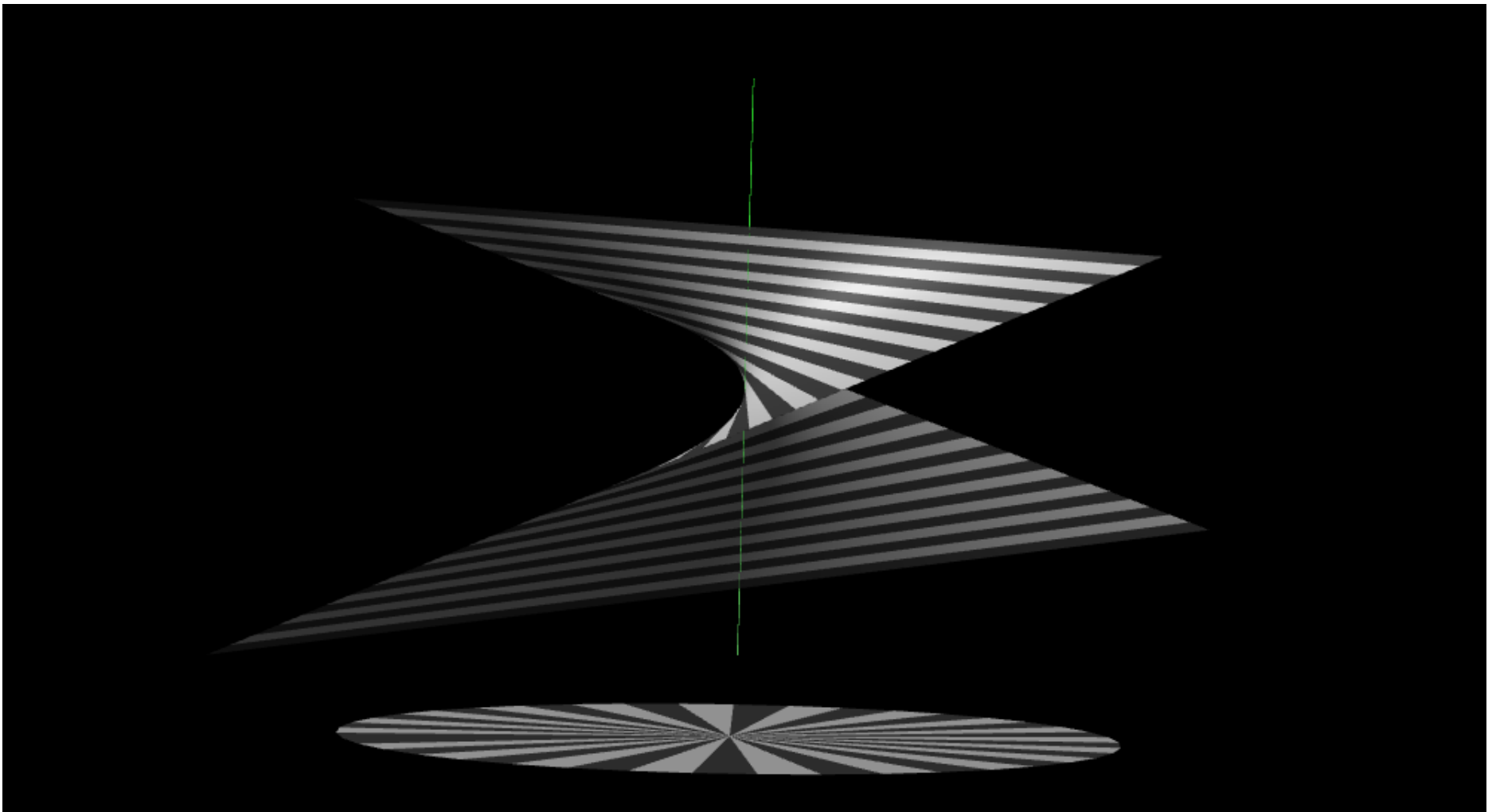
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Scalar-flat-Kähler Burns metric on  $\widetilde{\mathbb{C}^2} \subset \mathbb{C}^2 \times \mathbb{CP}_1$ :

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also has mass  $m$ .

# Positive Mass Conjecture:

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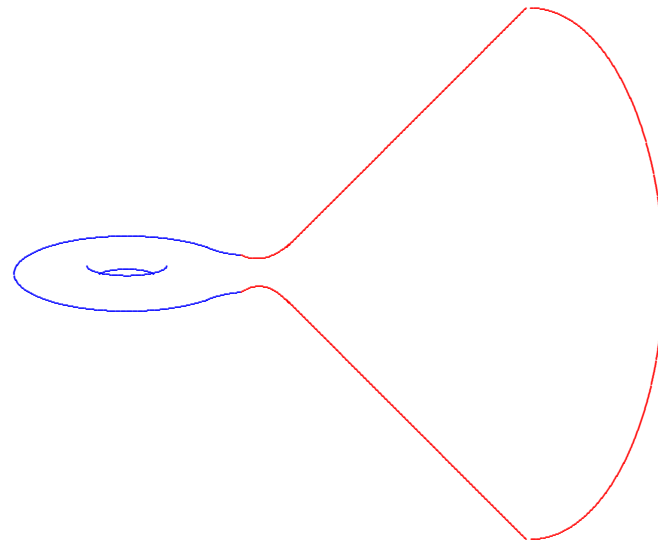
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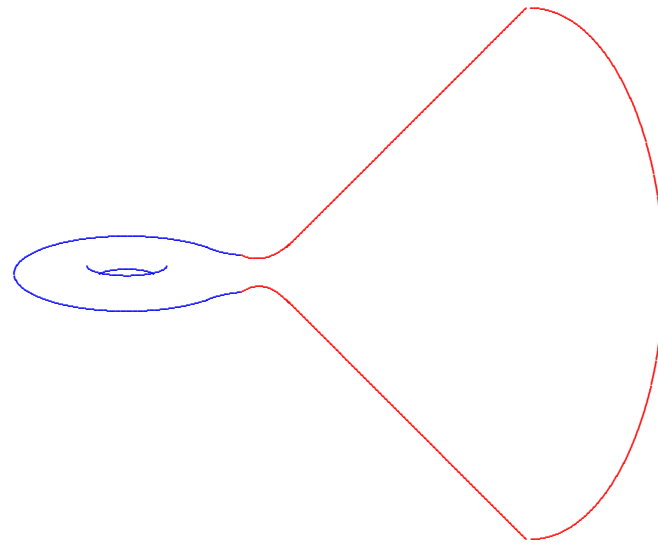
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**Upshot:**

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Mass of an **ALE** Kähler manifold is unambiguous.

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**Upshot:**

Mass of an **ALE Kähler** manifold is unambiguous.

Does not depend on the choice of an end!

We begin with the scalar-flat Kähler case.

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**Theorem A.**

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In fact, we will see that there is an explicit formula for the mass in terms of these data!

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(Discovered independently by Rollin, Singer, & Şuvaina, using different methods.)

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So the mass is a “**boundary correction**” to the topological formula for the total scalar curvature.

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**Corollary.** Any ALE scalar-flat Kähler manifold  $(M, g, J)$  of complex dimension  $m$  has mass given by

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So **Theorem A** is an immediate consequence!

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$$g^{jk} (g_{j\ell,k} - g_{jk,\ell}) \nu^\ell \alpha_E = -\star d \log \left( \sqrt{\det g} \right) + O(\varrho^{-3-\varepsilon}).$$

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However, since  $s = 0$ ,

$$d(\theta \wedge \omega) = \rho \wedge \omega = \frac{s}{4} \omega^2 = 0.$$

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So

$$m(M, g) = -\frac{1}{3\pi} \langle \clubsuit(c_1), \omega \rangle$$

as claimed.

## Theorem D.

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Proof actually shows something stronger!

# Theorem E.

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