Einstein Metrics,

Weyl Curvature, &

Symplectic 4-Manifolds

Claude LeBrun
Stony Brook University

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"…the greatest blunder of my life!"

— A. Einstein, to G. Gamow

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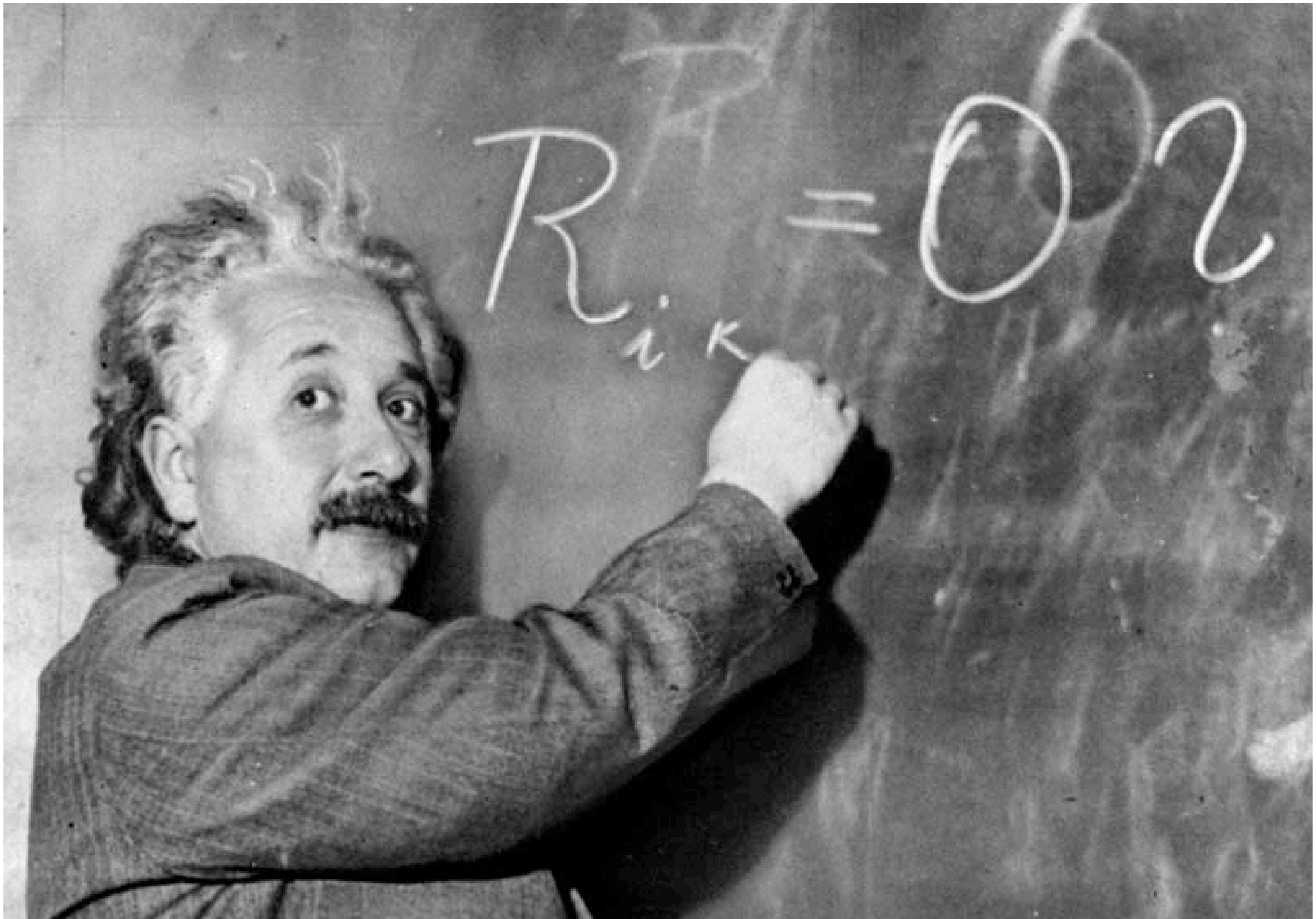

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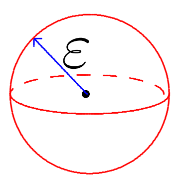
$$s = r_j^{\ j} = \mathcal{R}^{ij}{}_{ij}.$$

$$\frac{\mathrm{vol}_g(B_\varepsilon(p))}{c_n \varepsilon^n} = 1 - s\,\frac{\varepsilon^2}{6(n+2)} + O(\varepsilon^4)$$



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Perhaps reasonable in other dimensions?

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When $n = 4$, situation is more encouraging...

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$K3$ = underlying M^4 of a generic quartic in $\mathbb{CP}_3$.

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Key question:

For which M^4 is $\mathscr{E}(M)$ connected?

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$$\star^2 = 1.$$

Λ^+ self-dual 2-forms.
Λ^- anti-self-dual 2-forms.

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Measures deviation $[g]$ from conformal flatness.

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Ricci-flat product $K3 \times T^m$ never critical!

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4-dimensional Gauss-Bonnet formula

$$\chi(M) = \frac{1}{8\pi^2} \int_M \left(\frac{s^2}{24} + |W_+|^2 + |W_-|^2 - \frac{|\mathring{r}|^2}{2} \right) d\mu$$

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for Euler-characteristic $\chi(M) = \sum_j (-1)^j b_j(M)$.

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For signature $\tau(M) = b_+ - b_-$ of intersection form.

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With equality iff $W_+ \equiv 0$ or $W_- \equiv 0$.

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Connectedness of $\mathscr{E}(M)$: must also control $\int_M s^2 d\mu$.

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Connectedness of $\mathscr{E}(M)$: more difficult when $\lambda > 0$.

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Conjectured: Global minimizer.

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Proposed systematic study of invariant $\inf \mathscr{W}(M)$.

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We will see later that $Y > 0$ does not seem essential.

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Fortunately, a complete answer is available!

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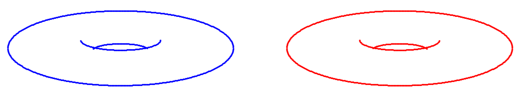

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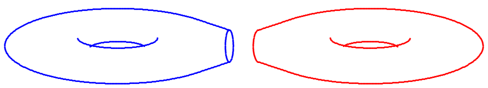

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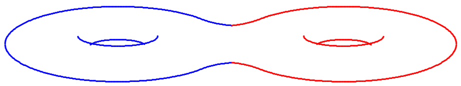

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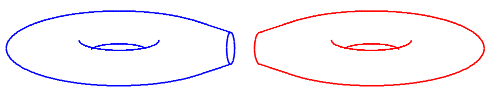

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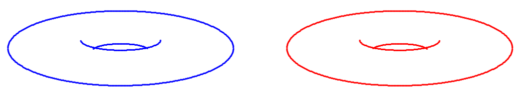

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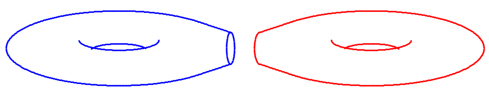

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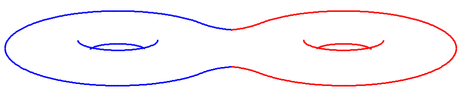

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Del Pezzo surfaces,
K3 surface, Enriques surface,
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No others: Hitchin-Thorpe, Seiberg-Witten, . . .

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Definitive list . . .

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$$

But we understand some cases better than others!

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Every Einstein metric is Ricci-flat Kähler.

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Moduli space $\mathscr{E}(M)$

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Moduli space $\mathscr{E}(M) = \{\text{Einstein } g\}/(\text{Diffeos} \times \mathbb{R}^+)$

But we understand some cases better than others!

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Below the line:

Every Einstein metric is Ricci-flat Kähler.

Moduli space $\mathscr{E}(M)$ completely understood.

But we understand some cases better than others!

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Moduli space $\mathscr{E}(M)$ connected!

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Below the line:

Every Einstein metric is Ricci-flat Kähler.

Moduli space $\mathscr{E}(M)$ connected!

Above the line:

Know an Einstein metric on each manifold.

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Moduli space $\mathscr{E}(M)$ connected!

Above the line:

Moduli space $\mathscr{E}(M) \neq \varnothing$.

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Every Einstein metric is Ricci-flat Kähler.

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Above the line:

Moduli space $\mathscr{E}(M) \neq \varnothing$. But is it connected?

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Below the line:

Every Einstein metric is Ricci-flat Kähler.

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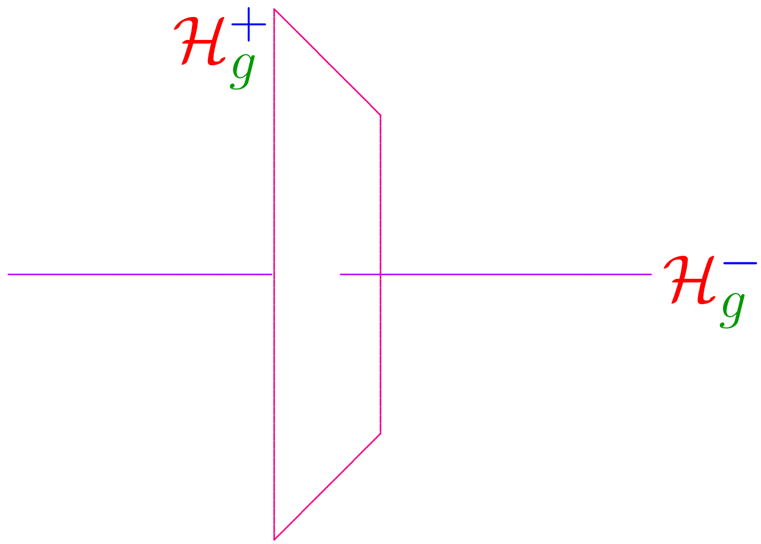
This decomposition is conformally invariant,

but does vary as we change $[g]$.



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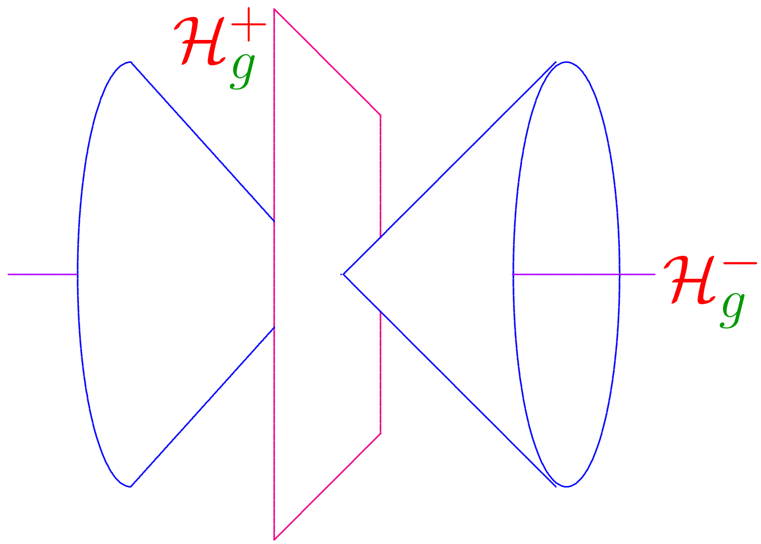

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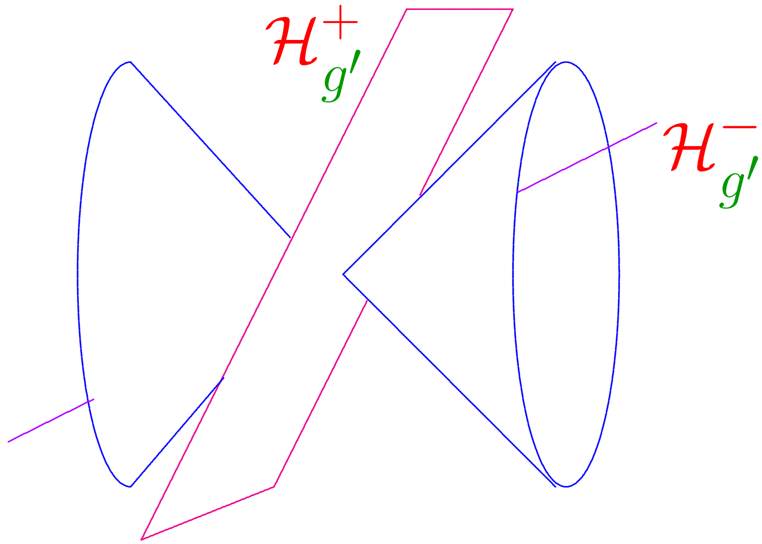

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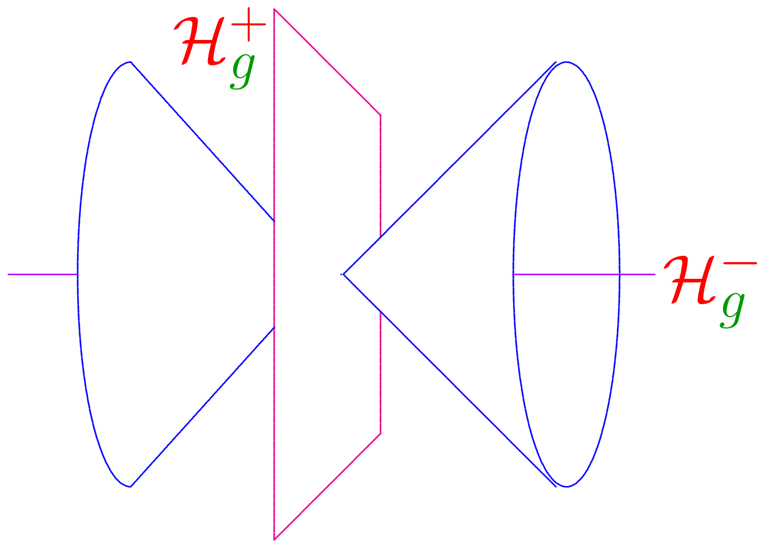

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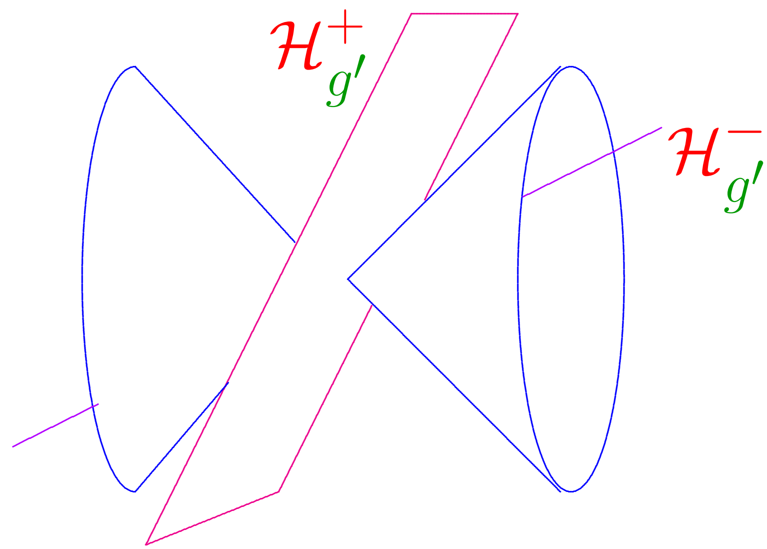

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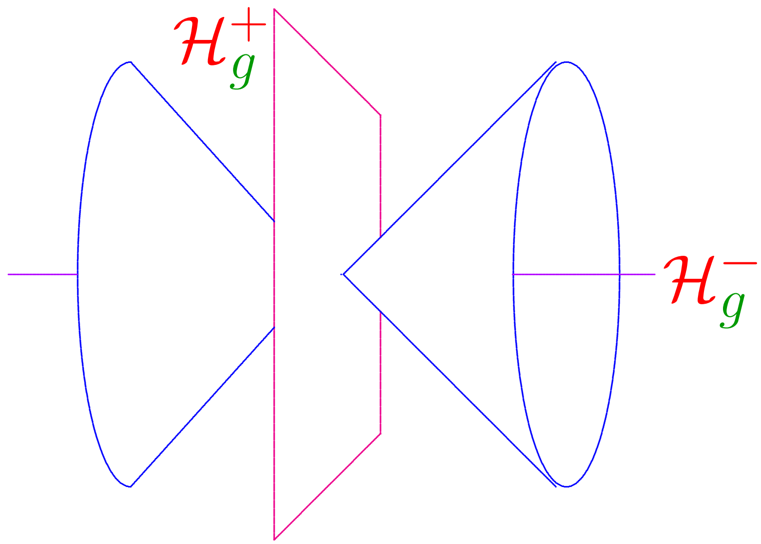

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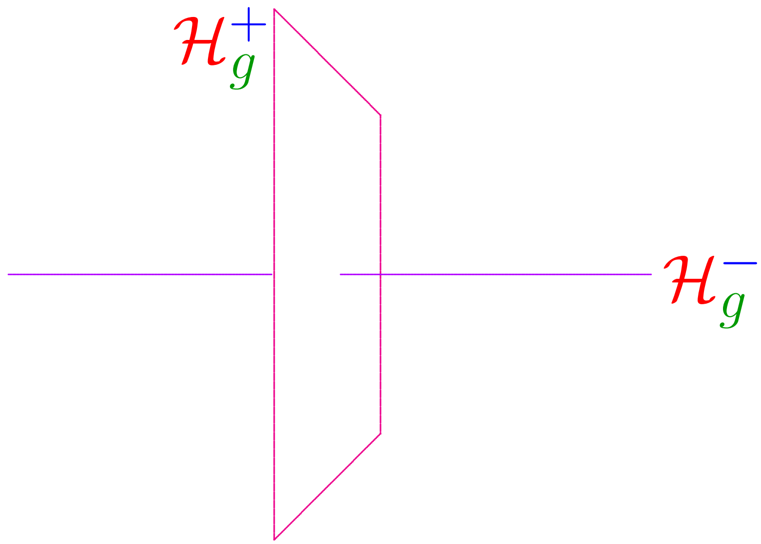

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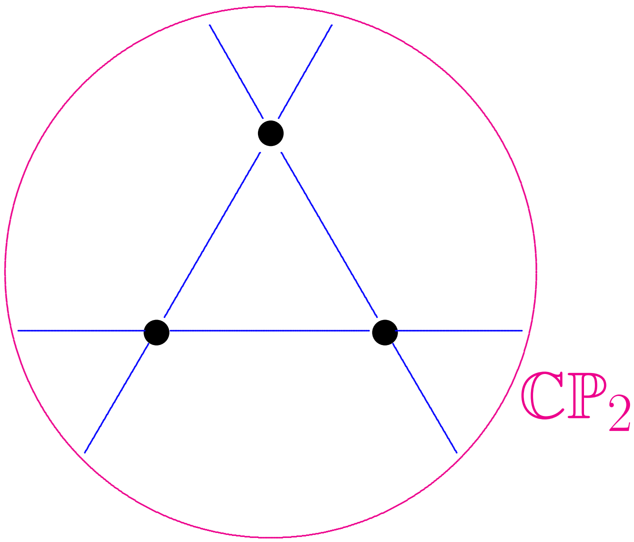

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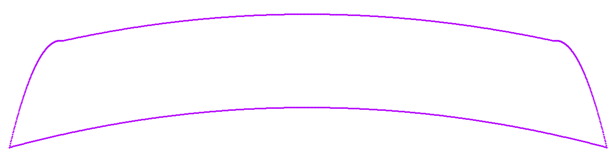

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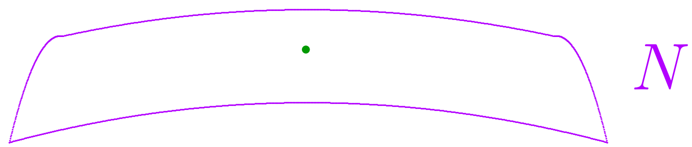

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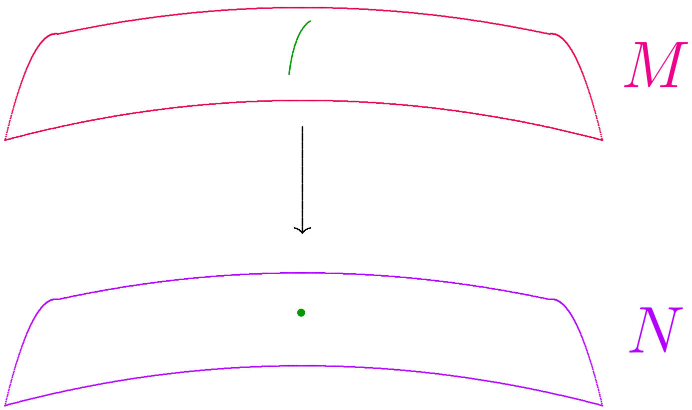

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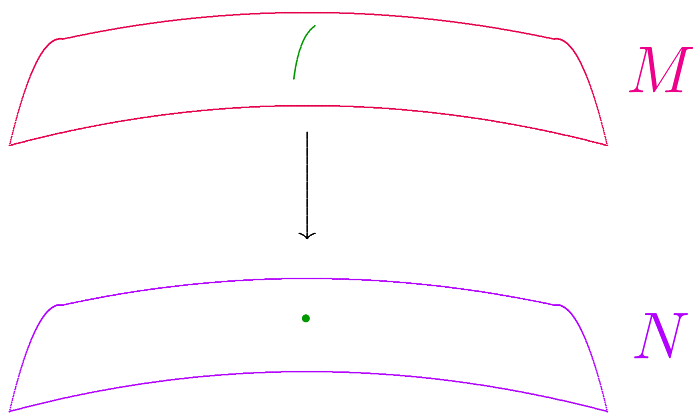

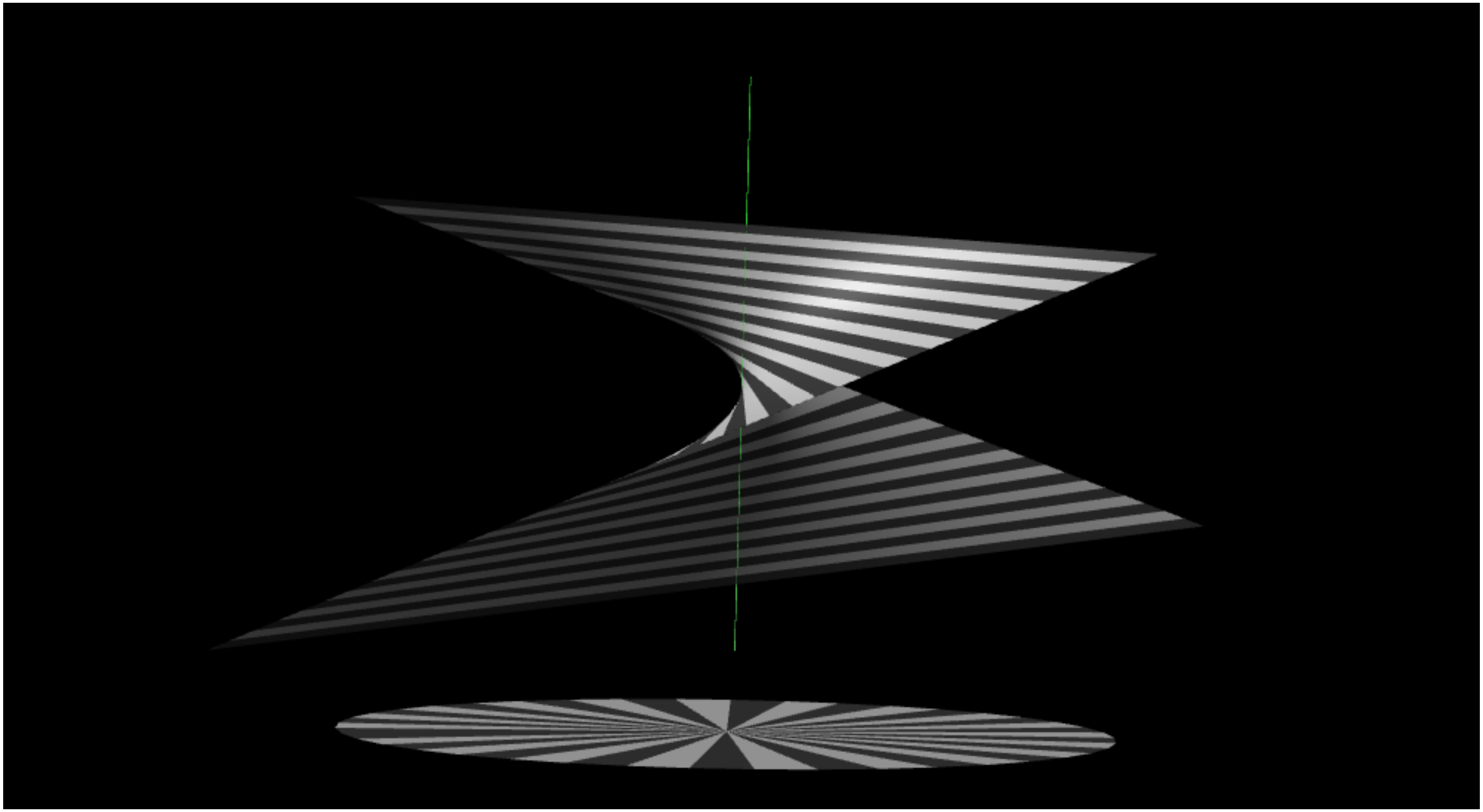

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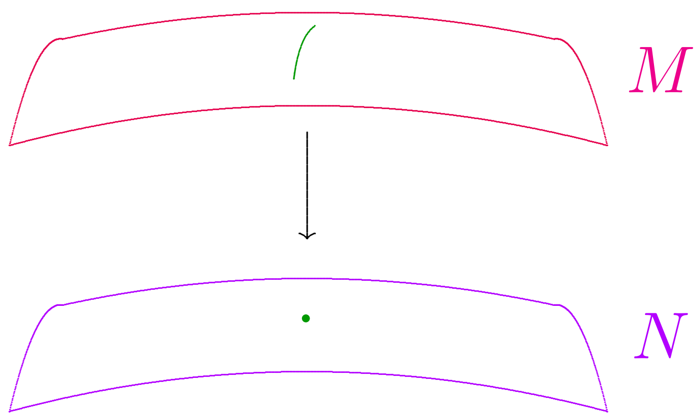

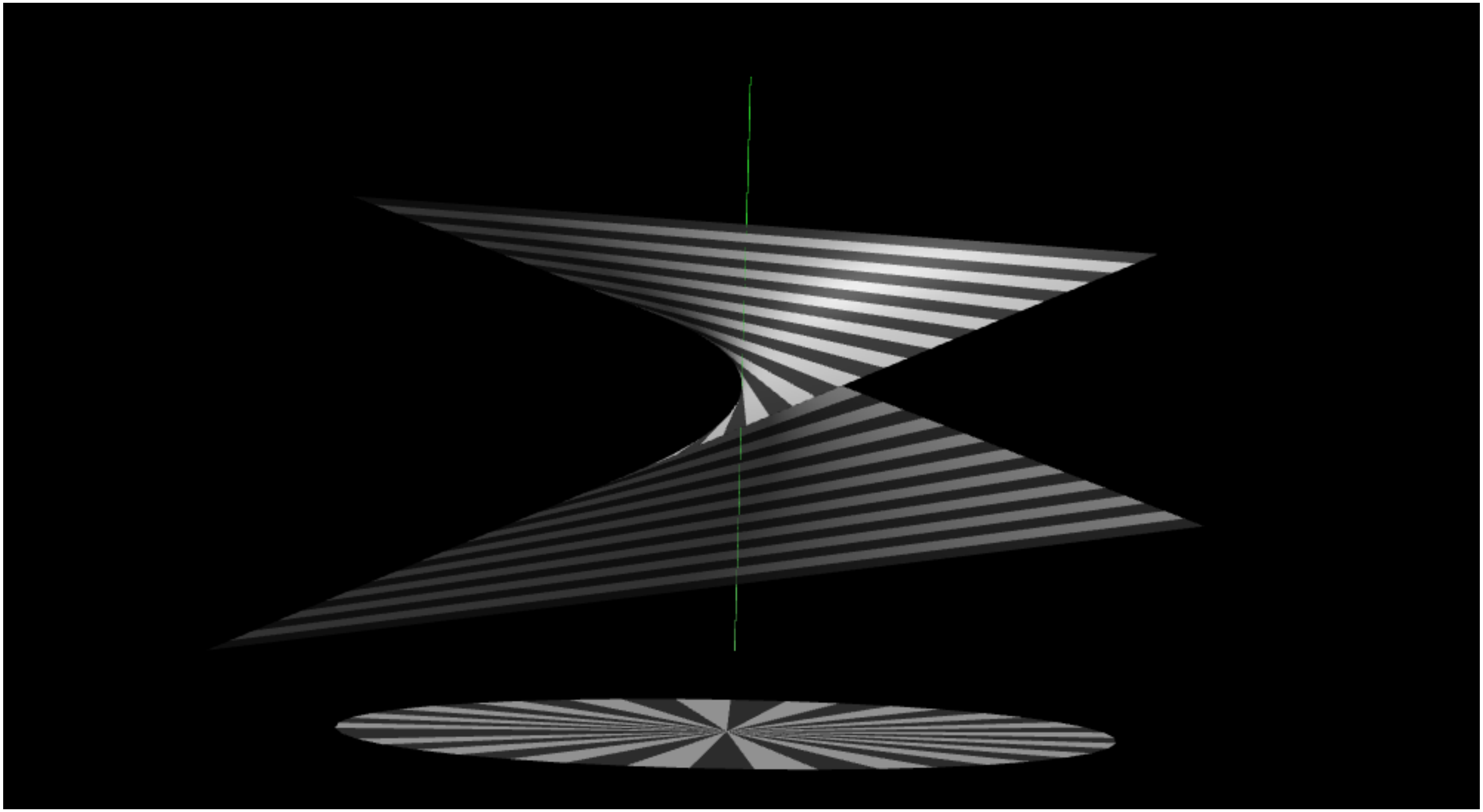

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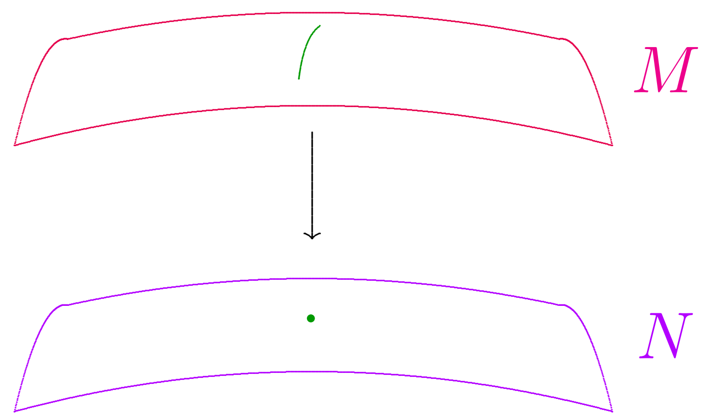

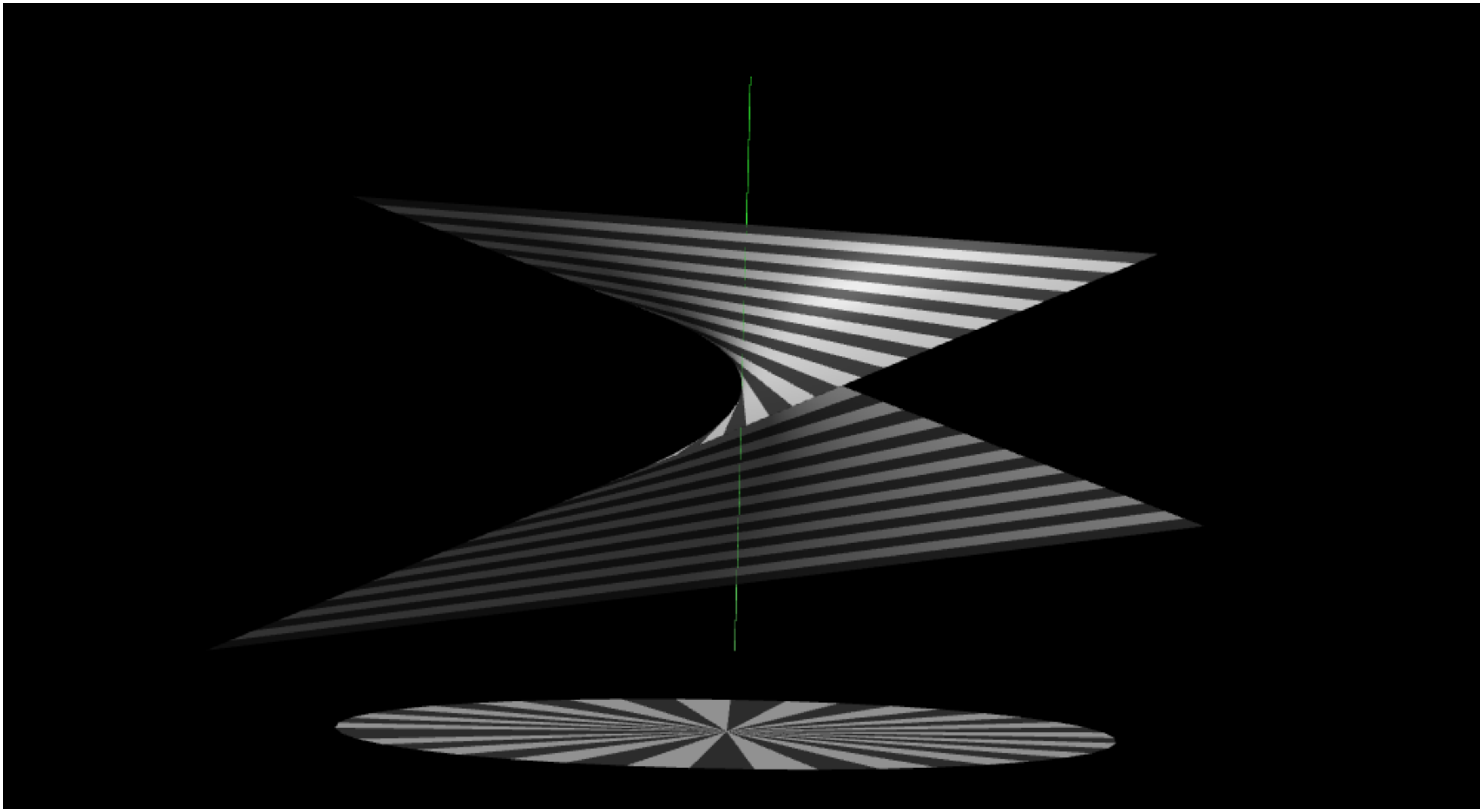

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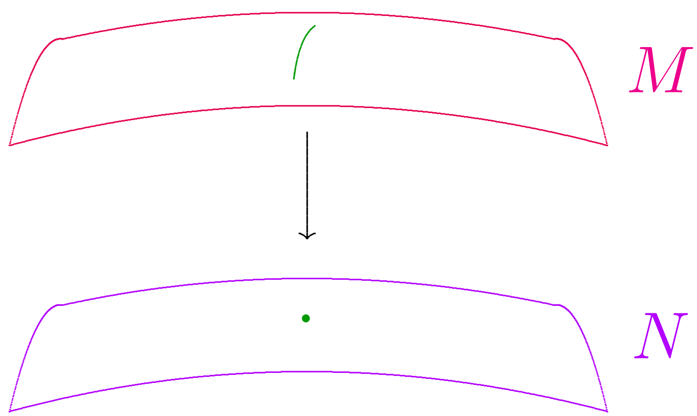

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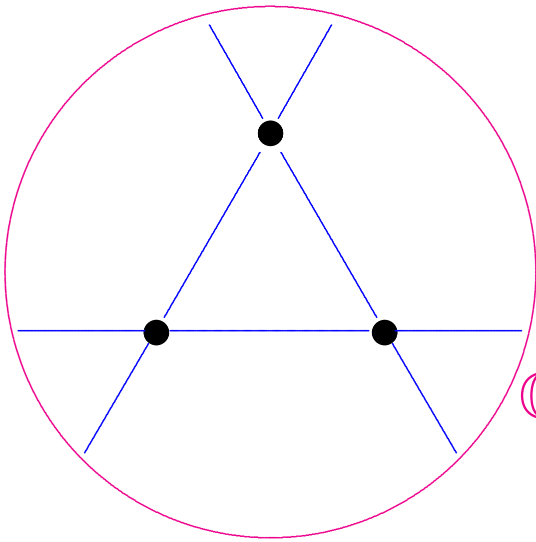

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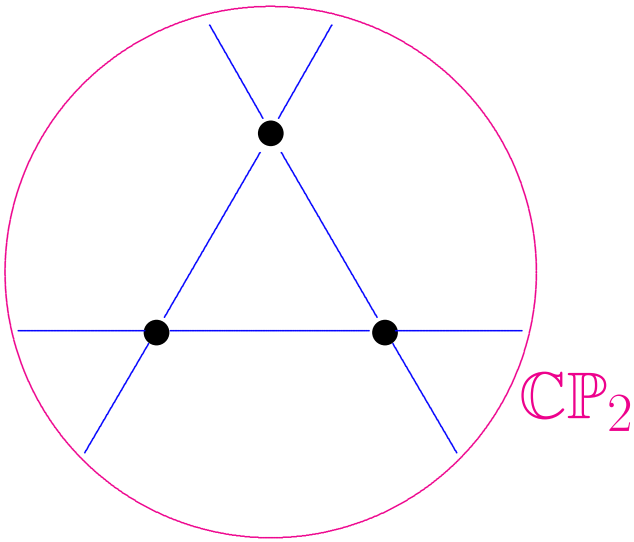

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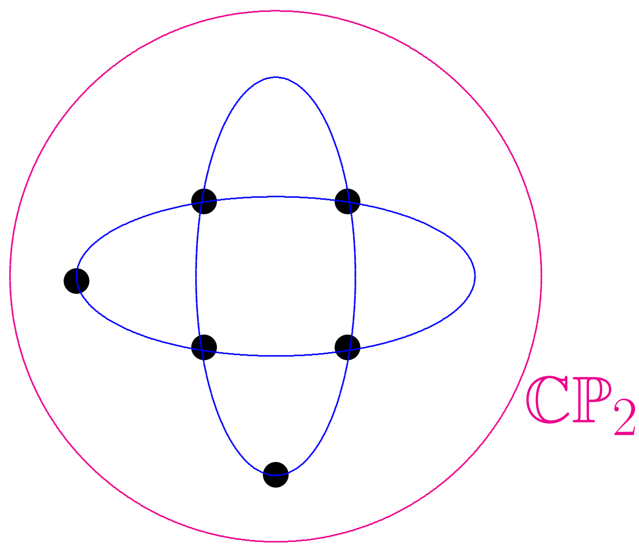

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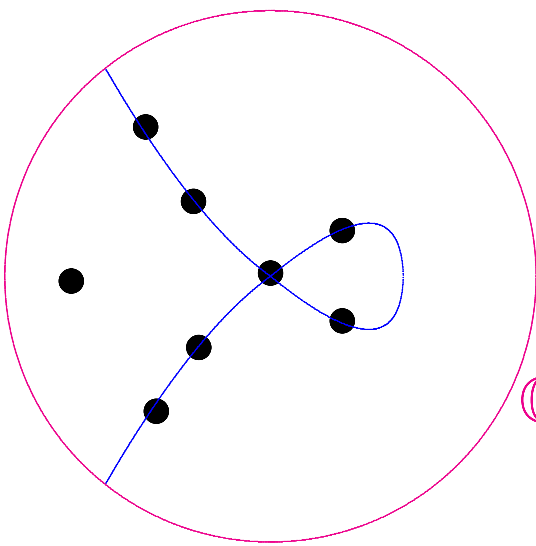

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Existence: Tian, Odaka-Spotti-Sun, Chen-L-Weber...

Uniqueness: Bando-Mabuchi, L 2012...

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Now works in a setting where $Y \to -\infty$ allowed.

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Strong evidence for O. Kobayashi's conjecture.

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Tempting to conjecture that these minimize, too!

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以上です。