

*Anti-Self-Dual 4-Manifolds,*

*Quasi-Fuchsian Groups, &*

*Almost-Kähler Geometry*

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Séminaire d'Analyse et Géométrie

Jussieu, September 25, 2018



Discussion will mention results from

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To appear in **Comm. An. Geom.**

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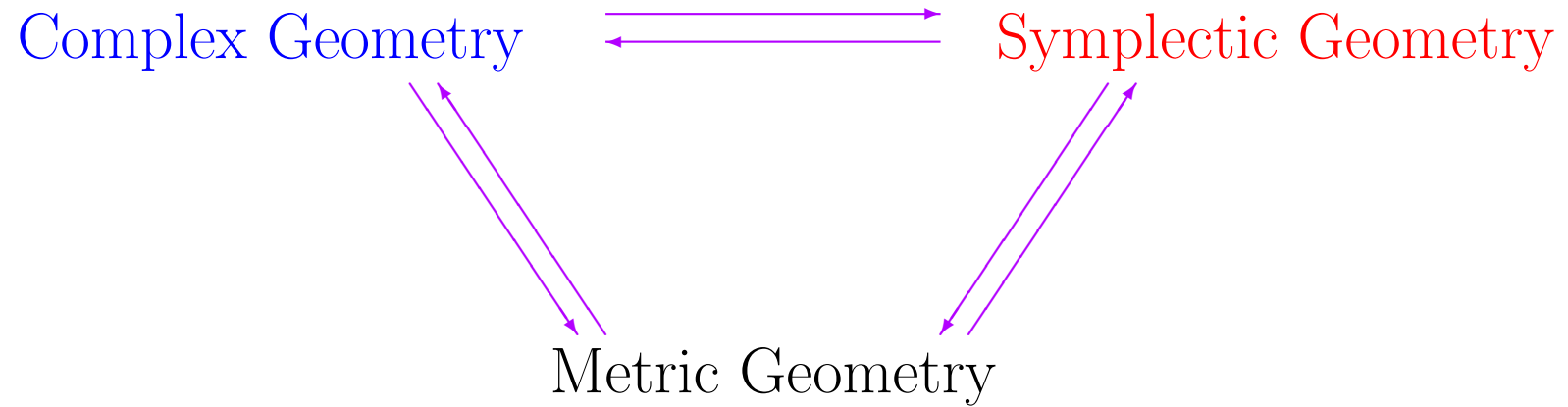
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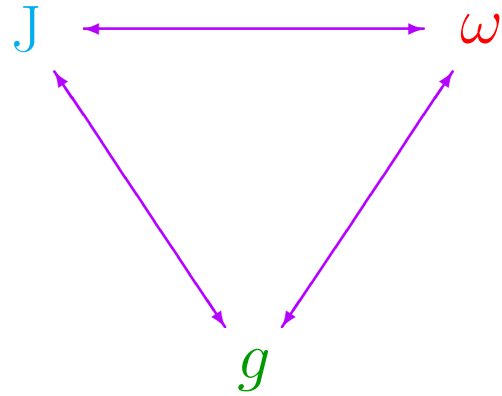
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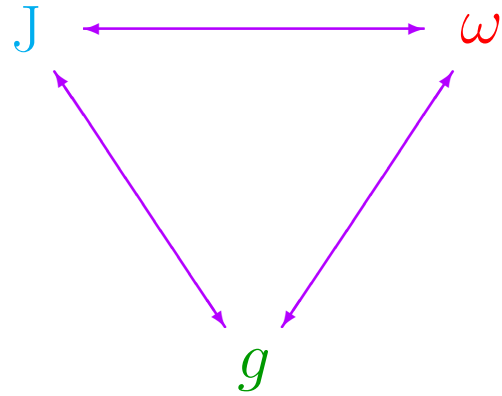
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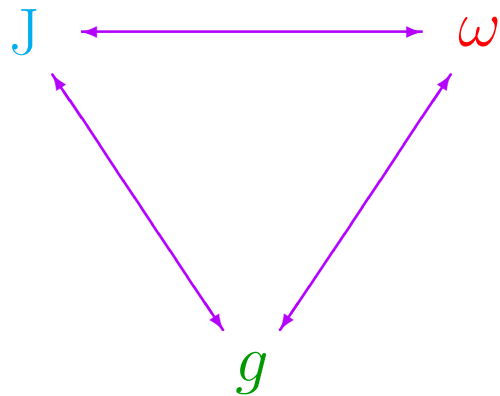


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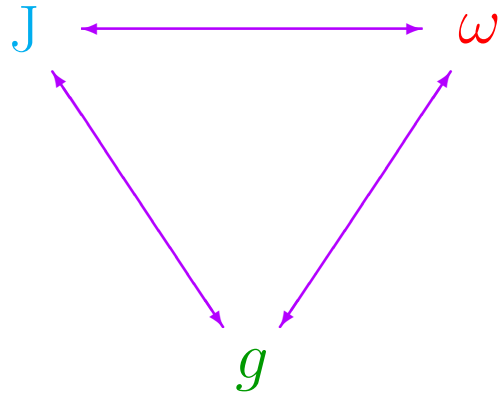


## Almost-Kähler Geometry:



Drop demand that  $J$  be integrable.

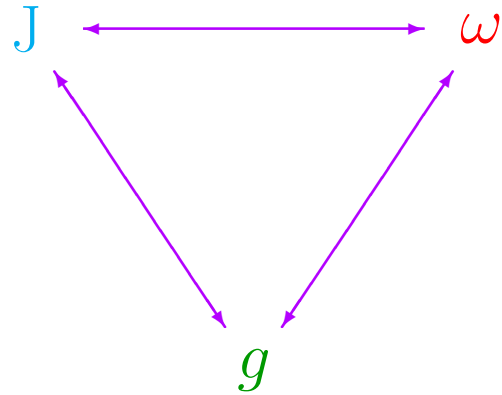
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Higher dimensions are demonstrably different.

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Imitates Kähler geometry in a non-Kähler setting.

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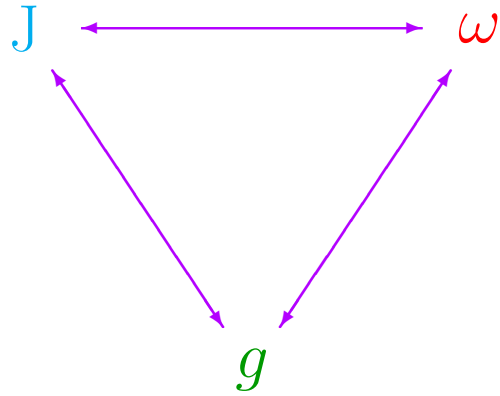
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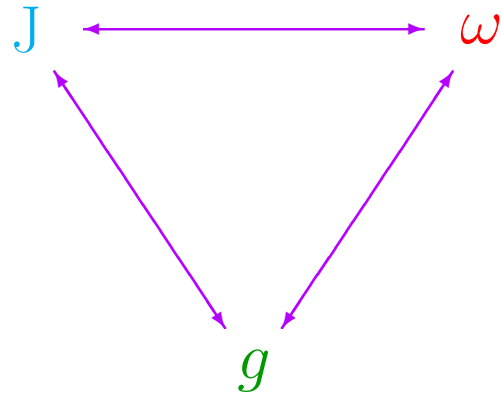
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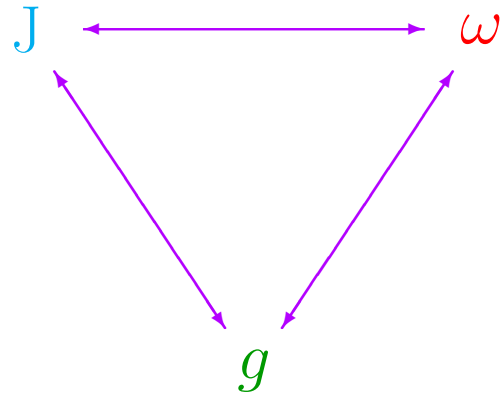


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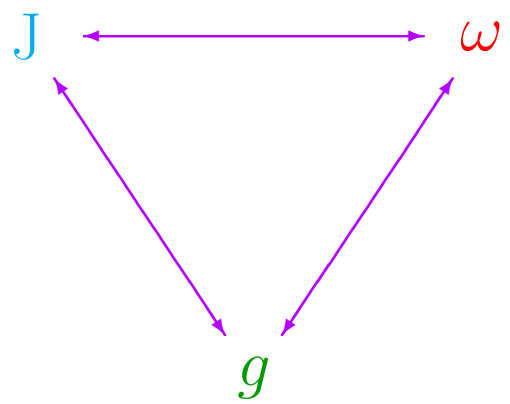
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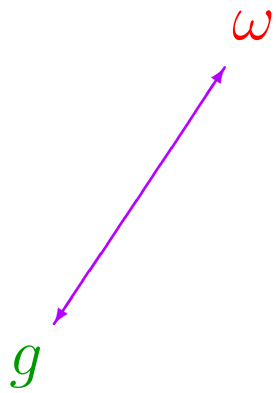


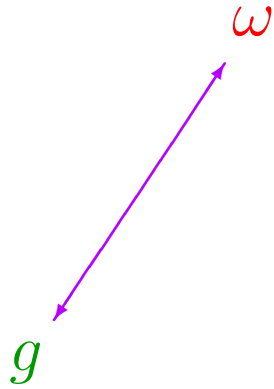
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For example, can avoid explicitly mentioning  $J$ .

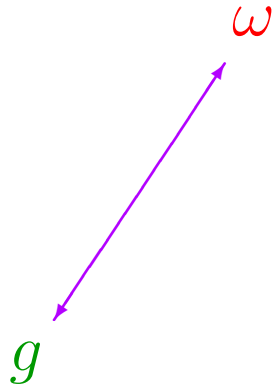




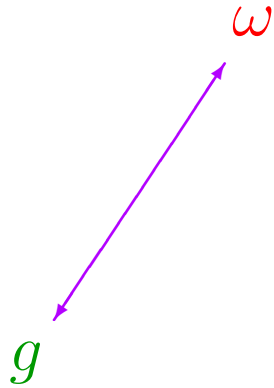




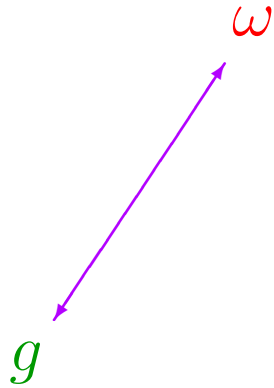
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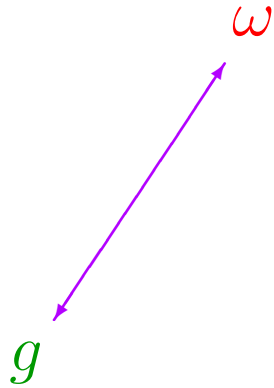
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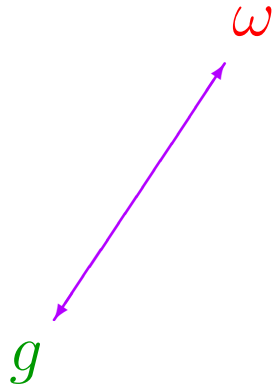


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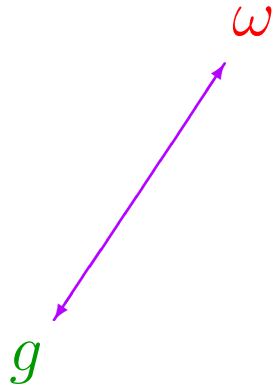
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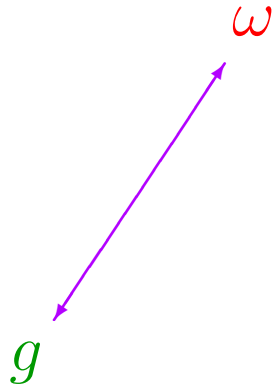
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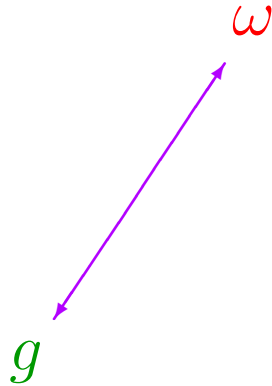




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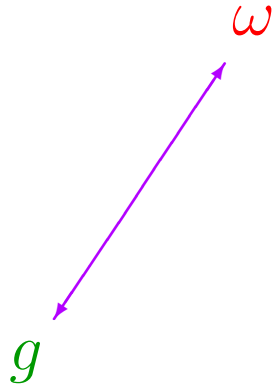
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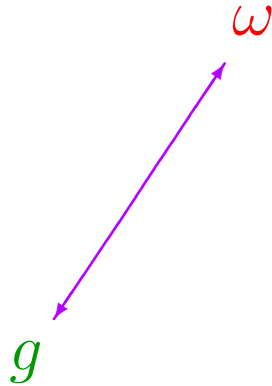
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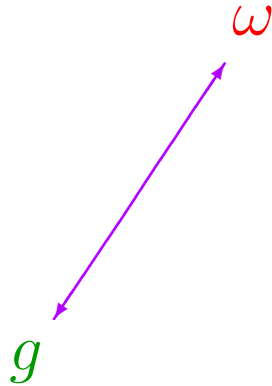
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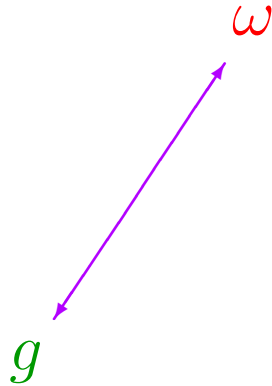
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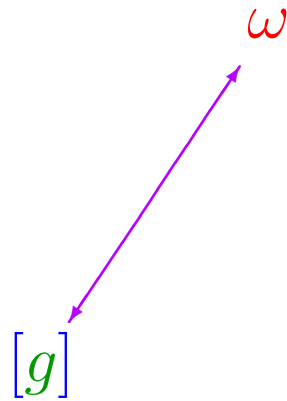
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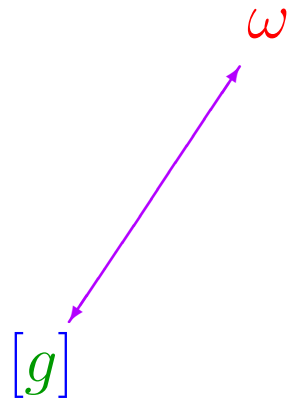


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**Proposition.** *A conformal class  $[g]$  on a smooth compact oriented 4-manifold  $M$  is represented by an almost-Kähler metric  $g$  iff it carries a self-dual harmonic 2-form  $\omega$  that is  $\neq 0$  everywhere.*



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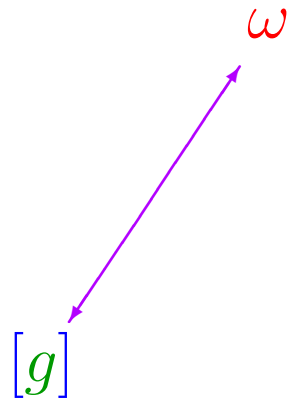
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## “Conformal classes of symplectic type”

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In particular, the numbers

$$b_\pm(M) = \dim \mathcal{H}_g^\pm$$

are independent of  $g$ , and so are invariants of  $M$ .

$b_{\pm}(M)?$



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Diagonalize:

$$\begin{bmatrix} +1 & & & & & & & \\ & \dots & & & & & & \\ & & +1 & & & & & \\ & & & & & & & \\ & & & & -1 & & & \\ & & & & & \dots & & \\ & & & & & & & -1 \\ & & & & & & & \end{bmatrix} \cdot$$

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 H^2(M, \mathbb{R}) \times H^2(M, \mathbb{R}) &\longrightarrow \mathbb{R} \\
 ([\varphi], [\psi]) &\longmapsto \int_M \varphi \wedge \psi
 \end{aligned}$$

Diagonalize:

$$\left[ \begin{array}{ccc}
 +1 & & \\
 & \dots & \\
 & & +1 \\
 \underbrace{\hspace{10em}}_{b_+(M)} & & \\
 & & -1 \\
 & \left. \begin{array}{c} \\ \\ \\ \end{array} \right\} b_-(M) & \dots \\
 & & -1
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$$b_2(M) = b_+(M) + b_-(M)$$

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“Signature” of  $M$ .

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but also expressible as a curvature integral:

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(Thom-Hirzebruch Signature Formula)

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Has major consequences in conformal geometry.

On oriented  $(M^4, g)$ ,

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^-$$

Riemann curvature of  $g$

$$\mathcal{R} : \Lambda^2 \rightarrow \Lambda^2$$

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Basic problems: For given smooth compact  $M^4$ ,

- What is  $\inf \mathcal{W}$ ?
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So Weyl functional is essentially equivalent to

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(ASD)



Twistor picture of anti-self-duality condition:

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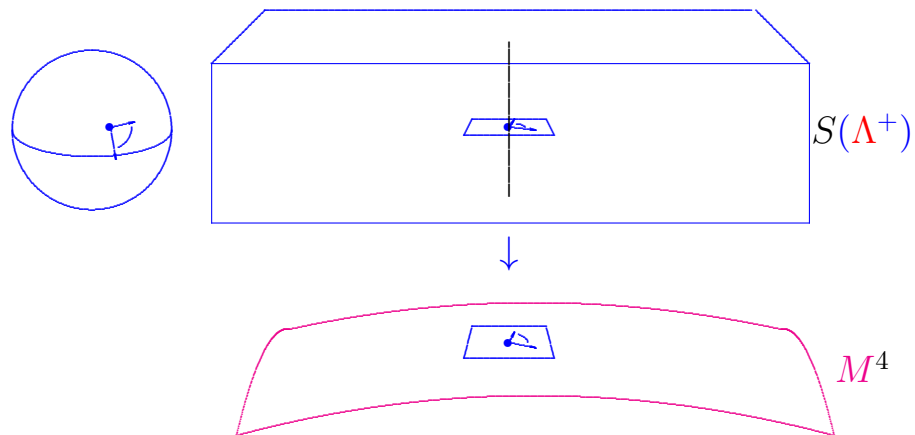
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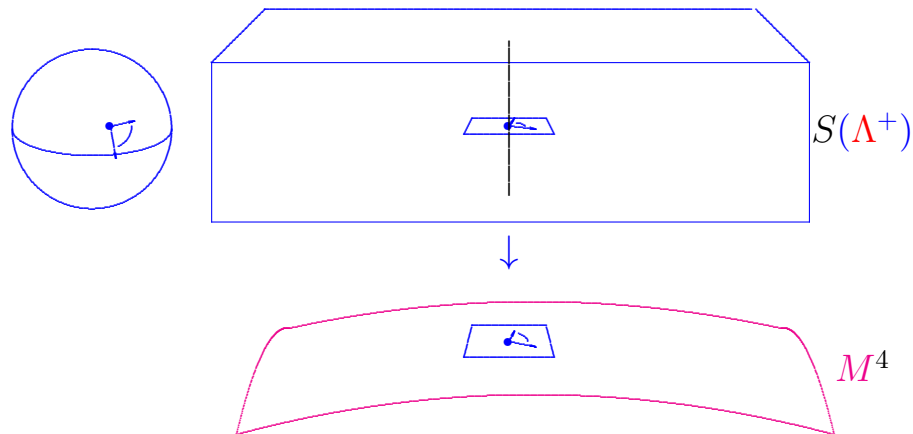
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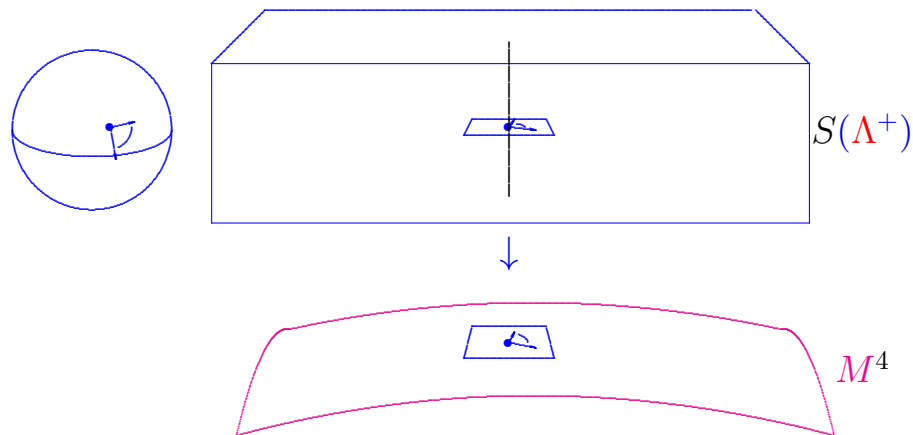


**Theorem** (Atiyah-Hitchin-Singer).  $(Z, J)$  is a complex 3-manifold iff  $W_+ = 0$ .

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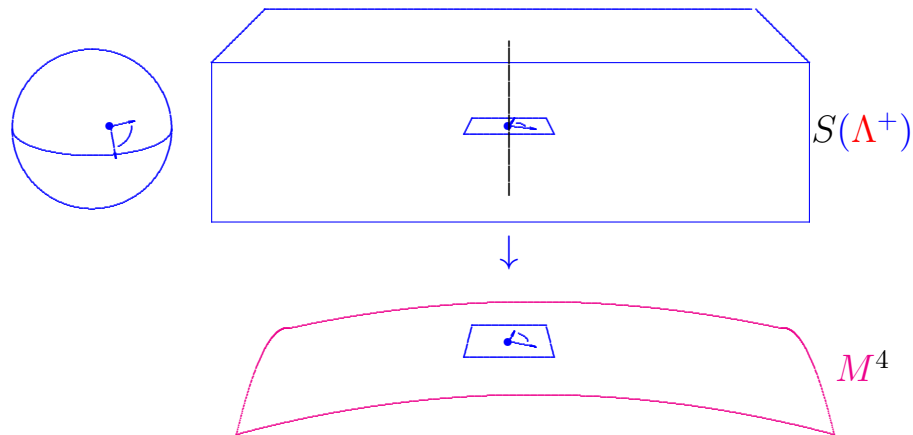
**Theorem** (Atiyah-Hitchin-Singer).  $(Z, J)$  is a complex 3-manifold iff  $W_+ = 0$ .

Reconceptualizes earlier work by Penrose.

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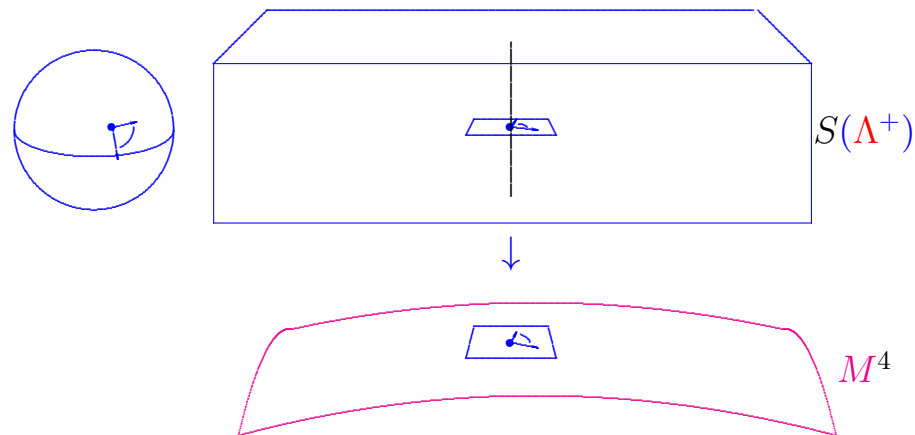


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**Theorem** (Atiyah-Hitchin-Singer).  $(Z, J)$  is a complex 3-manifold iff  $W_+ = 0$ .

Motivates study of ASD metrics,  
and yields methods for constructing them.



So ASD metrics are linked to complex geometry. . .

A different link with complex geometry:

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If  $(M^4, g, J)$  is a Kähler surface,

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special case of cscK manifolds,

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special case of cscK manifolds,  
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Results proved about SFK in '90s foreshadowed  
many more recent results about general case.



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Classification up to diffeomorphism:

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Classification up to diffeomorphism: (compact case)

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Scalar-flat Kähler surfaces:

Classification up to diffeomorphism:

- Ricci-flat case

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Scalar-flat Kähler surfaces:

Classification up to diffeomorphism:

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  - 
  - 
  -
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Scalar-flat Kähler surfaces:

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  - eight specific finite quotients of these
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Scalar-flat Kähler surfaces:

Classification up to diffeomorphism:

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Scalar-flat Kähler surfaces:

Classification up to diffeomorphism:

- Ricci-flat case
- Non-Ricci-flat case
  - $\mathbb{C}P_2 \# k \overline{\mathbb{C}P}_2, k \geq 10$
  - 
  - 
  -

Convention:

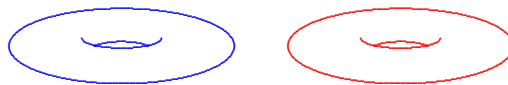
$\overline{\mathbb{C}P}_2$  = reverse oriented  $\mathbb{C}P_2$ .

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Connected sum #:



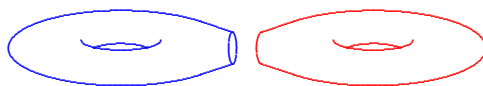


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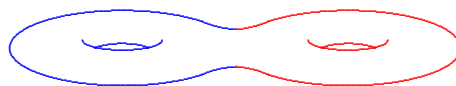


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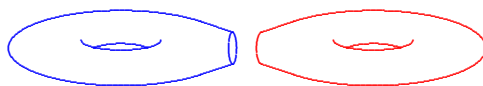


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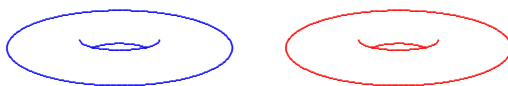


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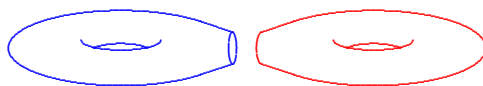


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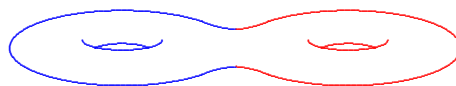


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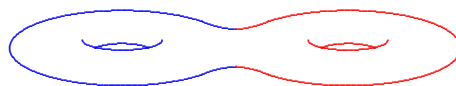


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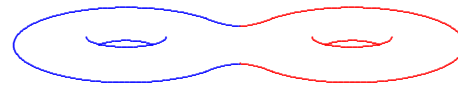


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Connected sum #:



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Blowing Up:  $M \rightsquigarrow M \# \overline{\mathbb{C}P}_2$

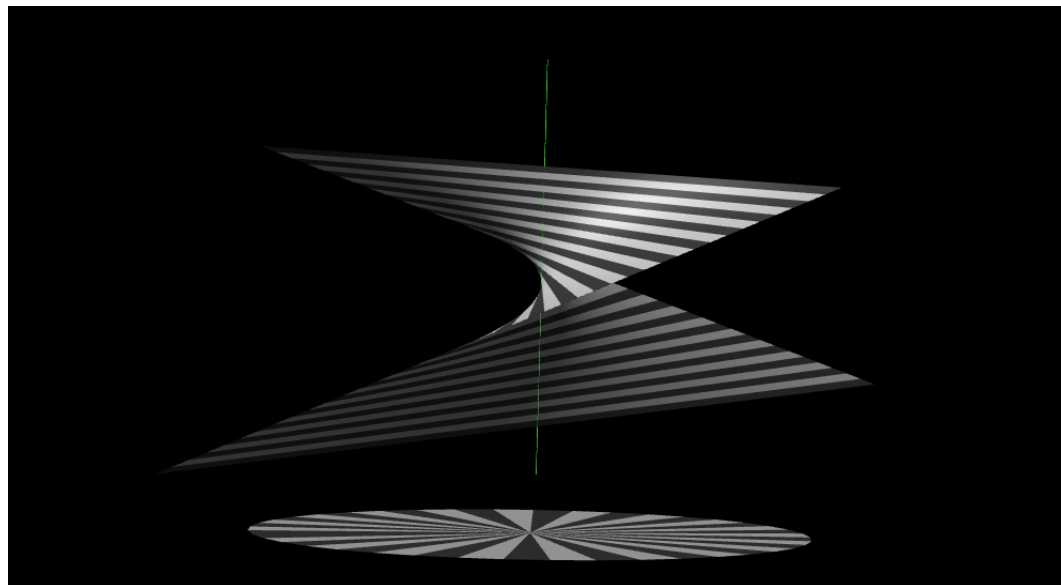
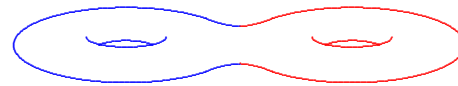


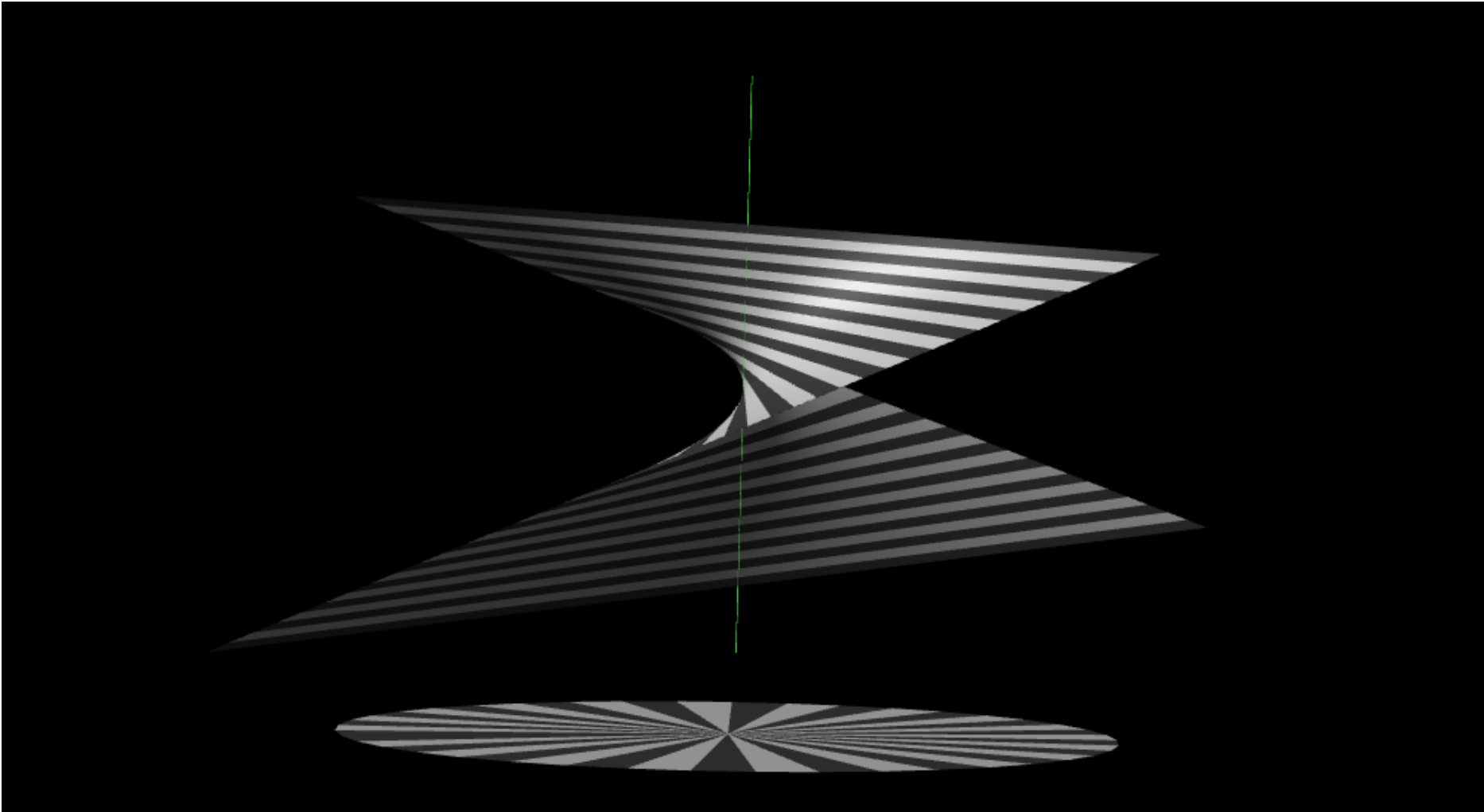
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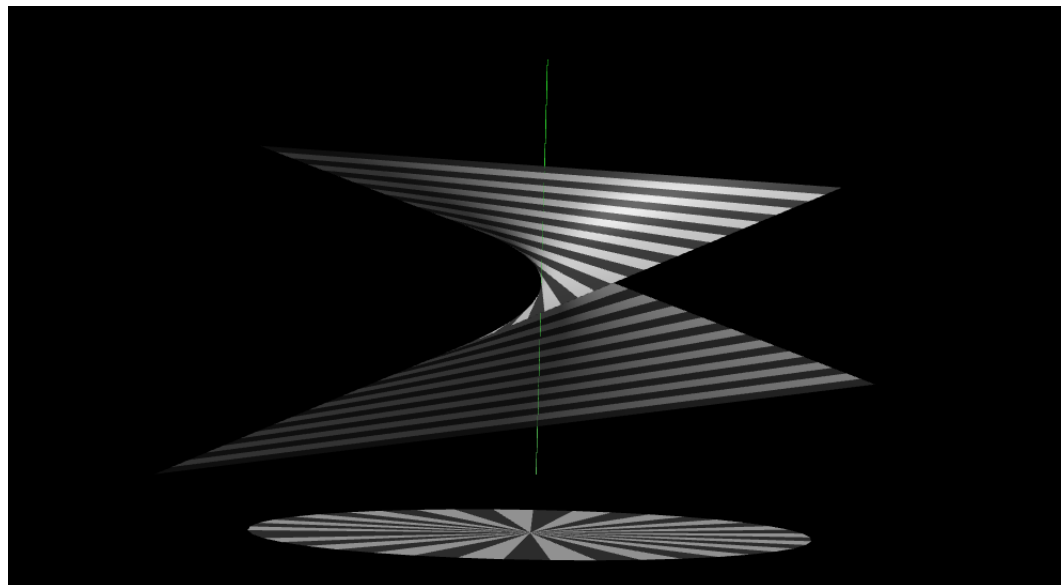
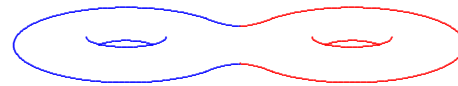


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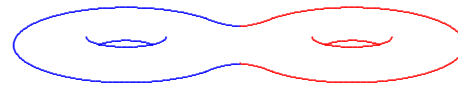


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---

Blowing Up:  $M \rightsquigarrow M \# \overline{\mathbb{C}P}_2$

A different link with complex geometry:

If  $(M^4, g, J)$  is a Kähler surface, then  $[g]$  is ASD  
 $\iff$  the scalar curvature  $s$  of  $g$  is identically zero.

Scalar-flat Kähler surfaces:

Classification up to diffeomorphism:

- Ricci-flat case
- Non-Ricci-flat case
  - $\mathbb{C}P_2 \# k \overline{\mathbb{C}P}_2, k \geq 10$
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with equality iff  $g$  conformal to Kähler-Einstein.

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$$\left( \int |W_+|^2 d\mu \right)^{1/2} \geq \frac{2\pi c_1 \cdot [\omega]}{\sqrt{3}[\omega]^2}$$

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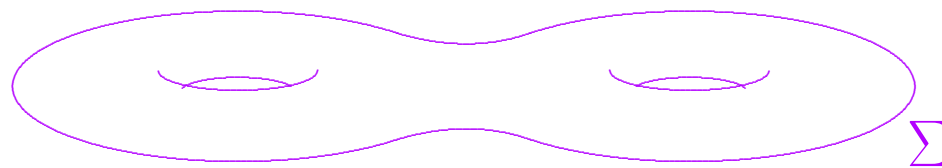
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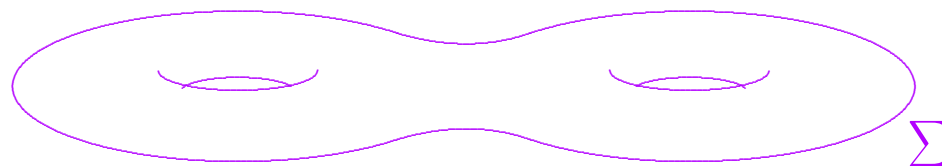
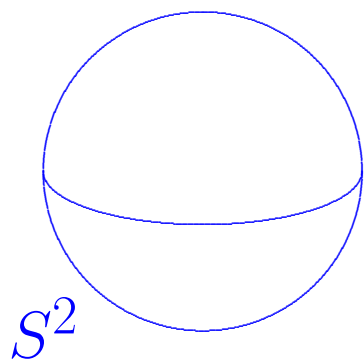
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Almost-Kähler ASD metrics sweep out an open set in the ASD moduli space.

Example.

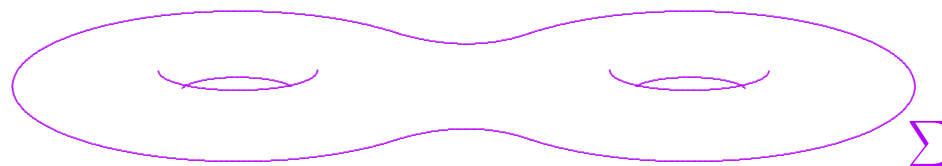
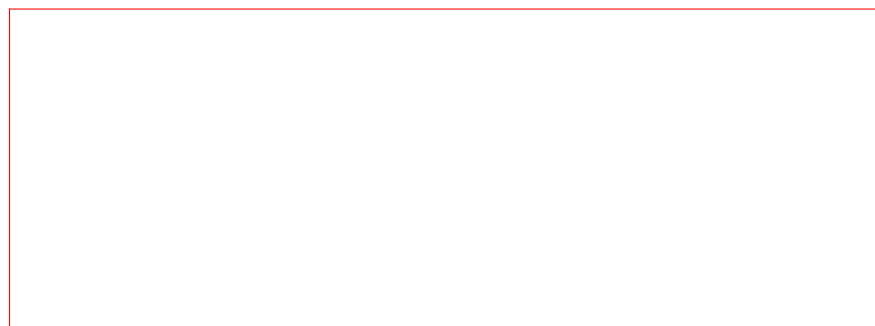
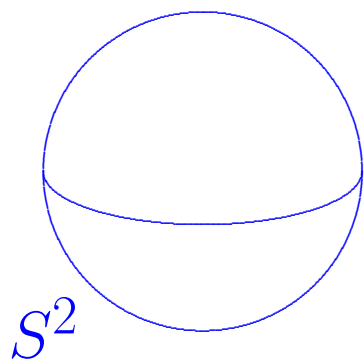


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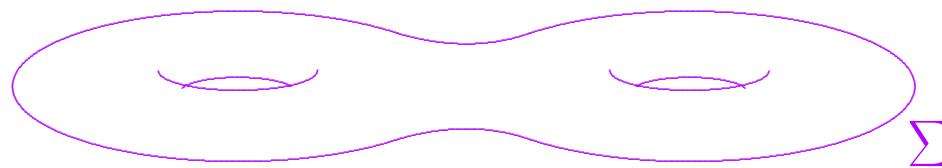
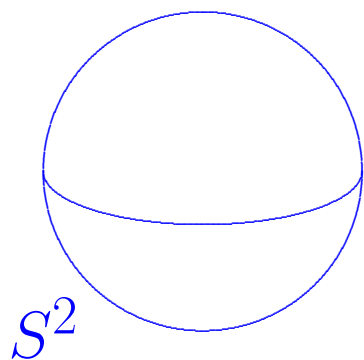
Example.

$$M = \Sigma \times S^2$$



# Example.

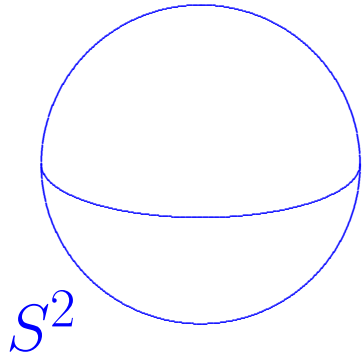
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$$K = -1$$

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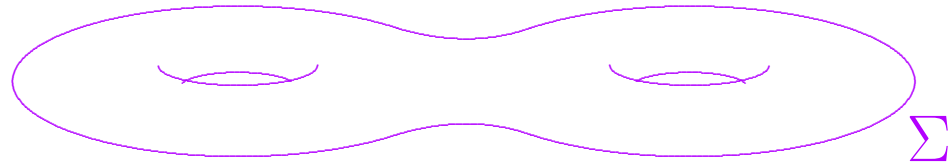
$$K = +1$$



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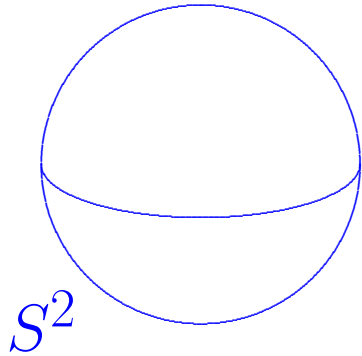


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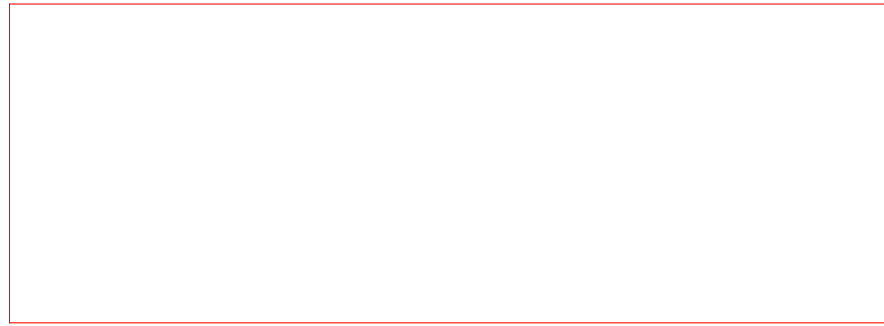


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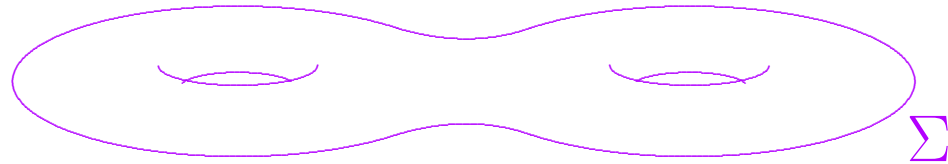
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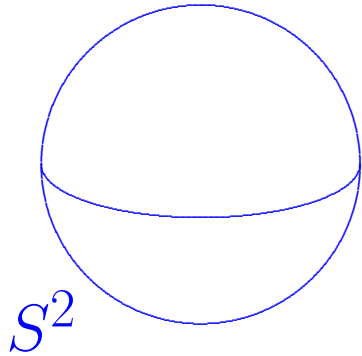


Product is scalar-flat

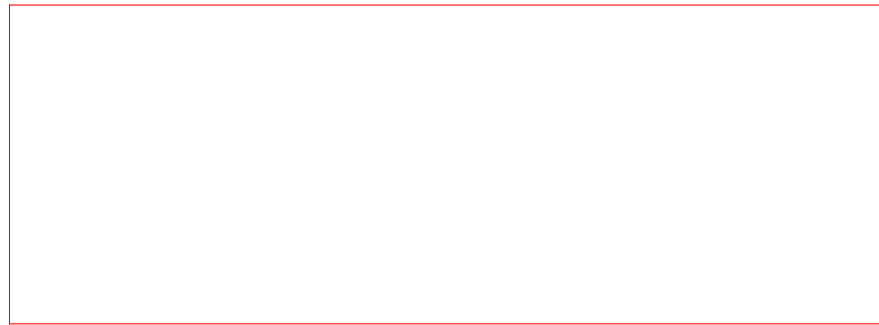


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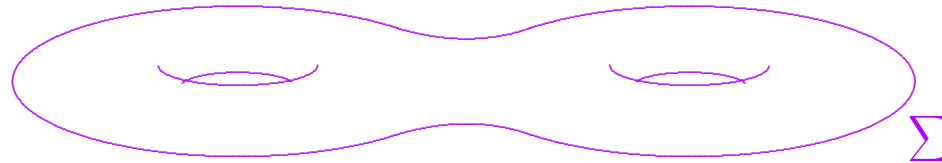
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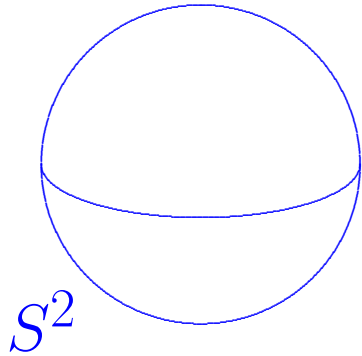
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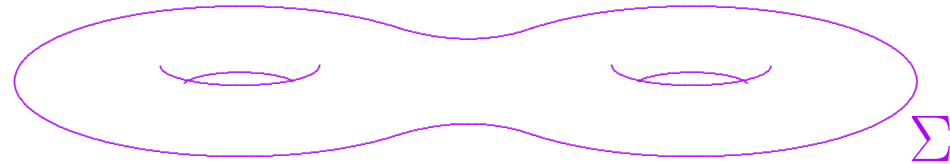
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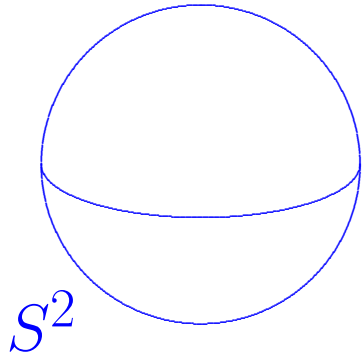
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For both orientations!

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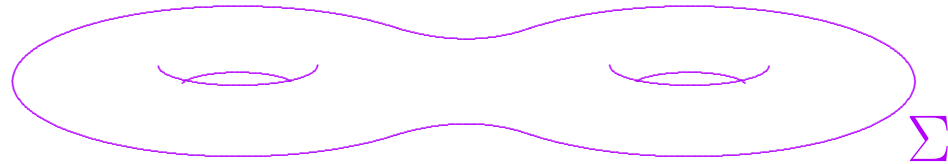
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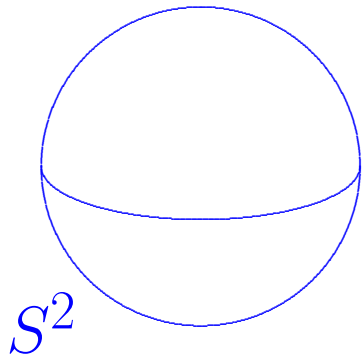
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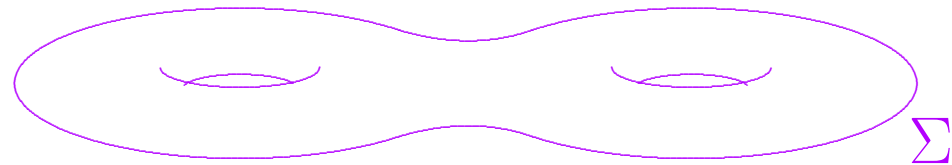
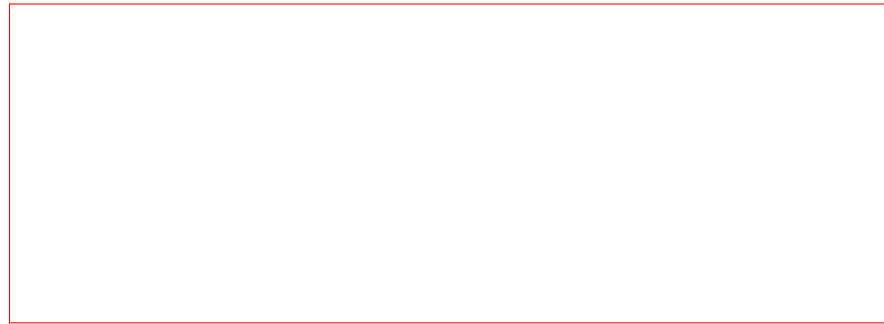
$$W_+ = 0.$$

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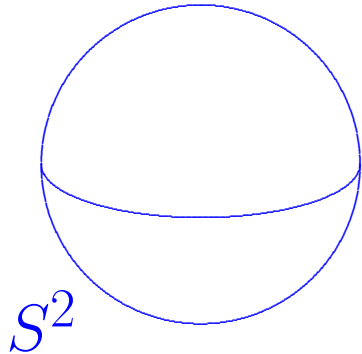
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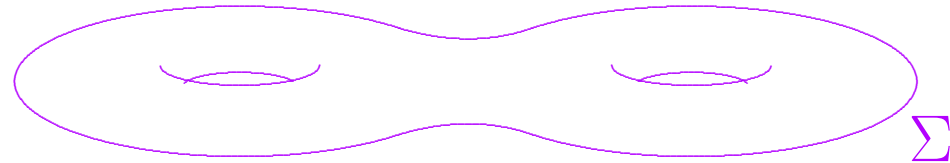
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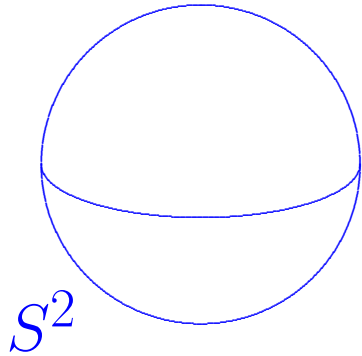
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Locally conformally flat!

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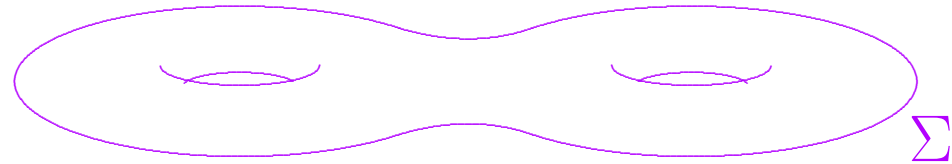
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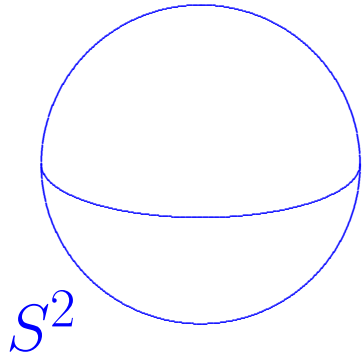
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$$\widetilde{M} = \mathcal{H}^2 \times S^2$$

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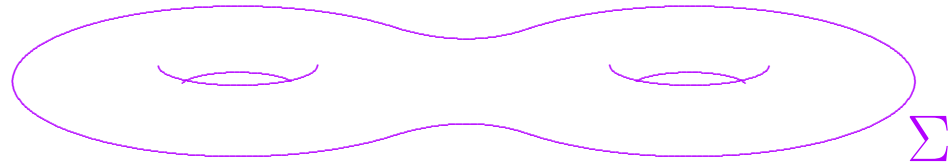
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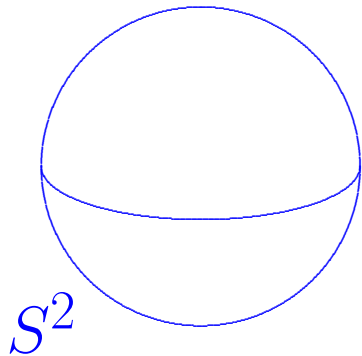
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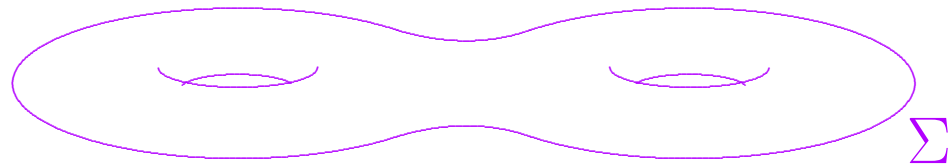
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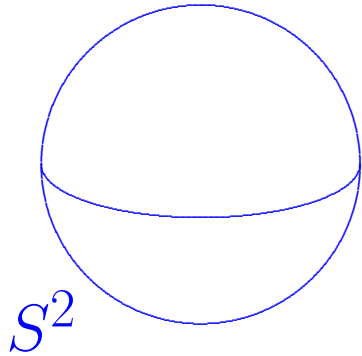
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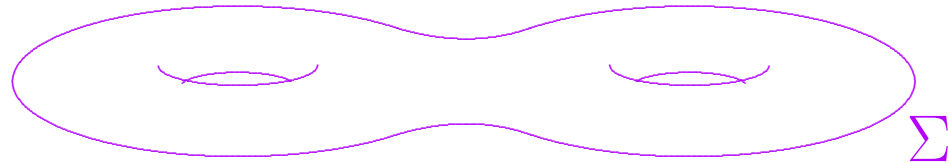
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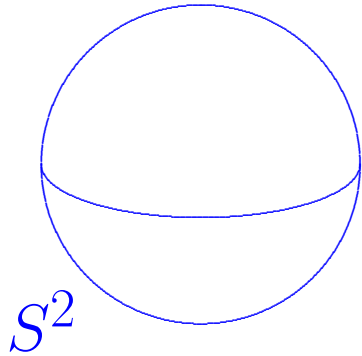


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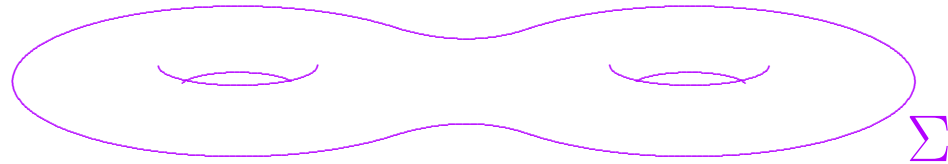
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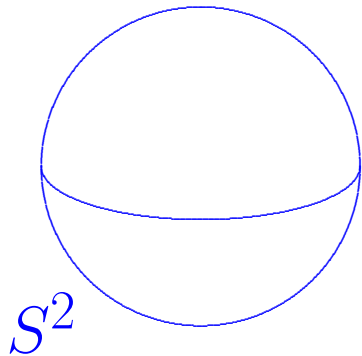


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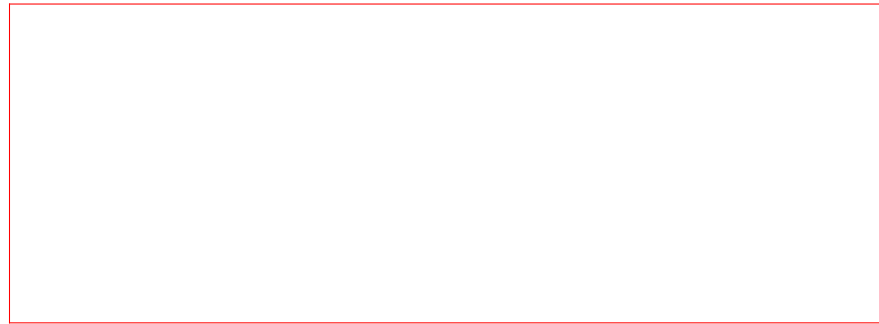
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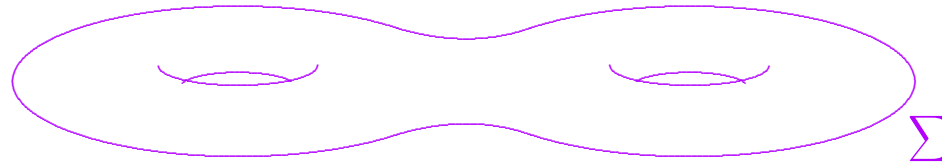
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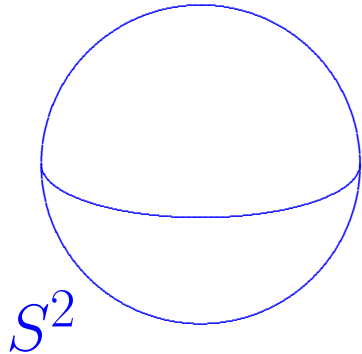
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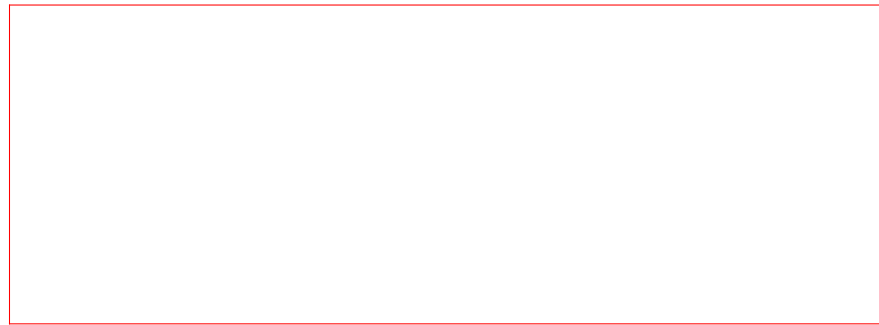
Scalar-flat Kähler deformations:  $12(g - 1)$  moduli

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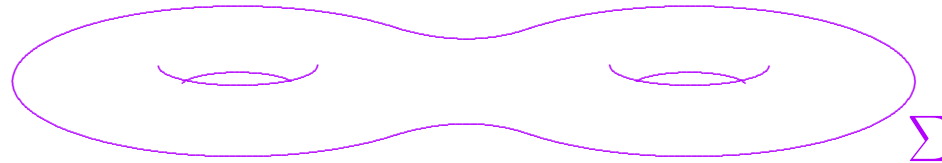
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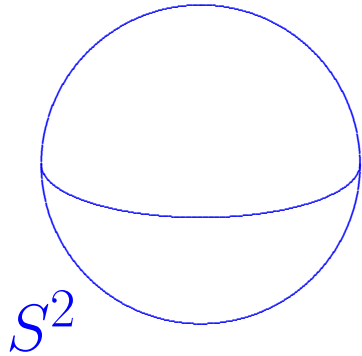
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Scalar-flat Kähler deformations:  $12(g - 1)$  moduli  
Locally conformally flat def'ms:  $30(g - 1)$  moduli

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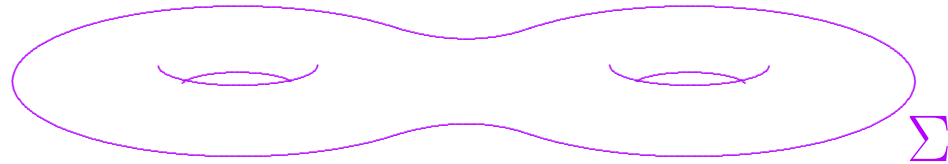
$$K = +1$$



$$M = \Sigma \times S^2$$



$$K = -1$$



Scalar-flat Kähler deformations:  $12(g-1)$  moduli  
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Inyoung Kim '16: classification of almost-Kähler ASD roughly the same as in scalar-flat Kähler case.

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Alas, **No!**

# Theorem A.

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Same method simultaneously proves...

## Theorem B.

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Proof hinges on a construction of hyperbolic 3-manifolds.

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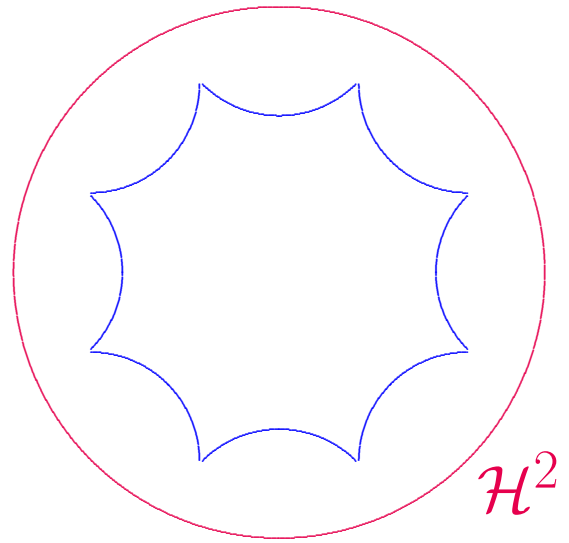
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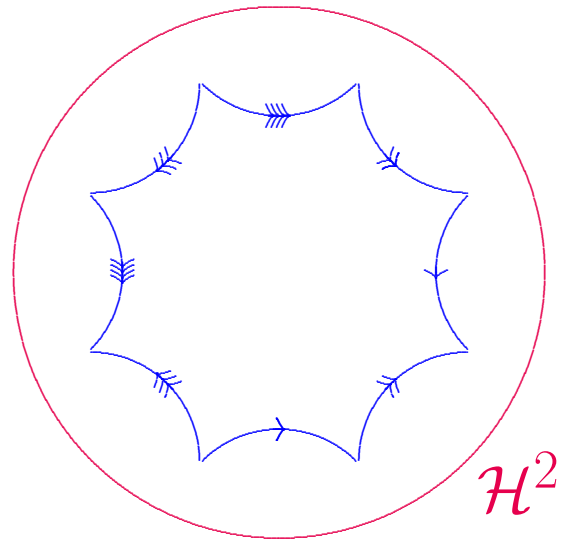
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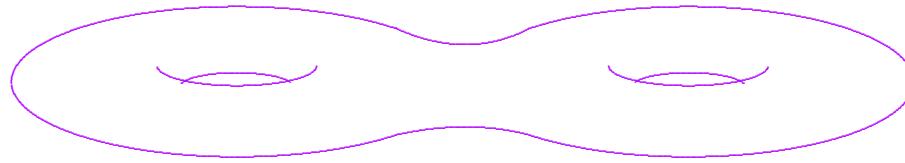
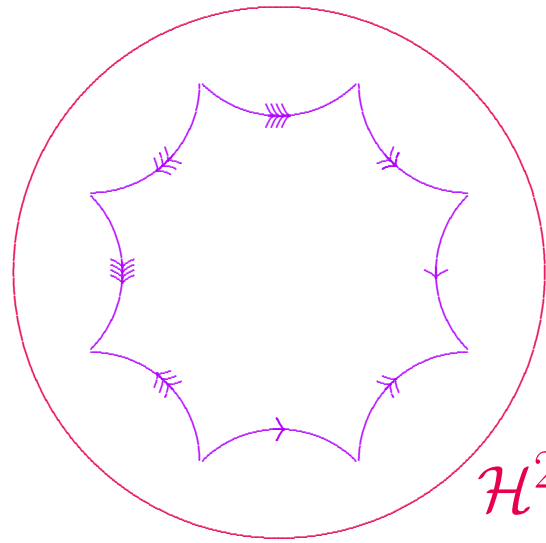
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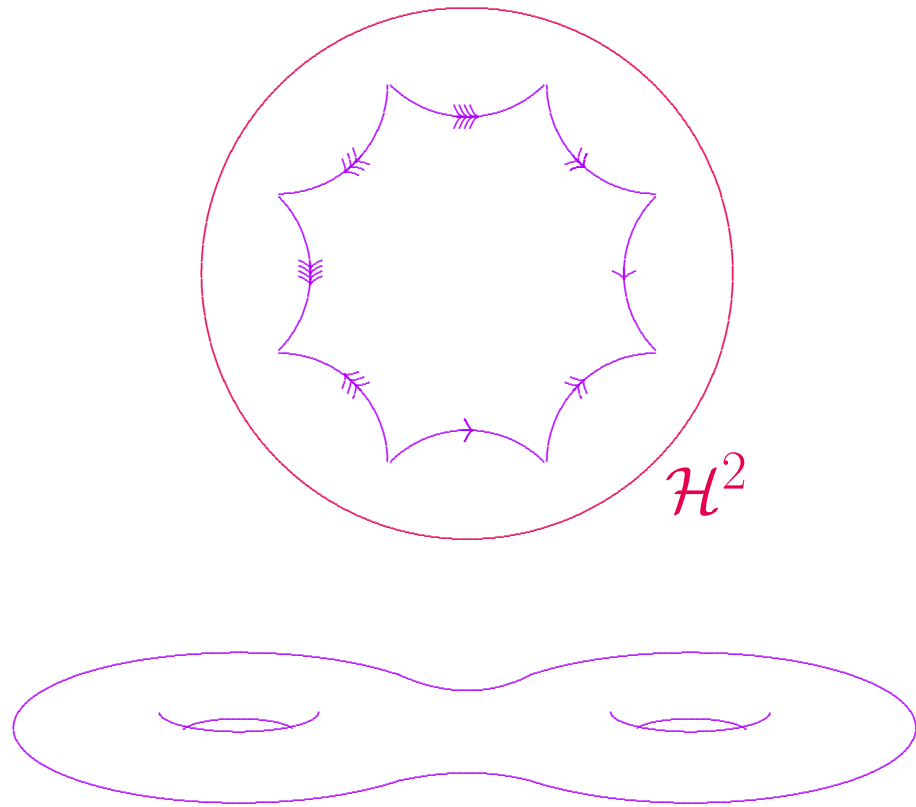
We begin by revisiting hyperbolic metrics on  $\Sigma$ .



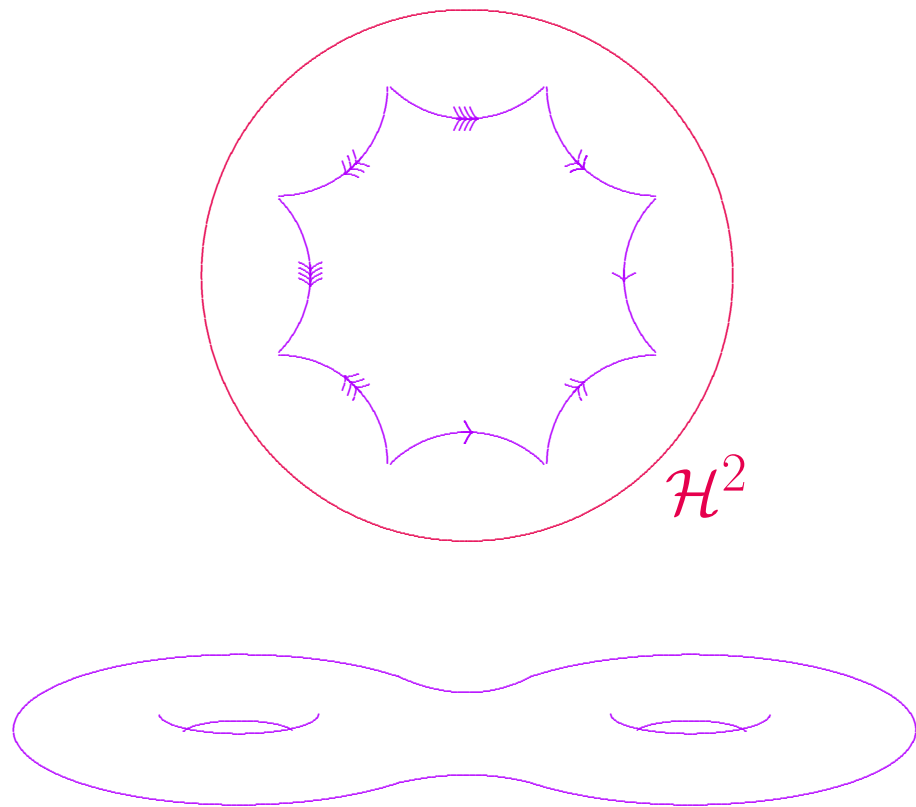




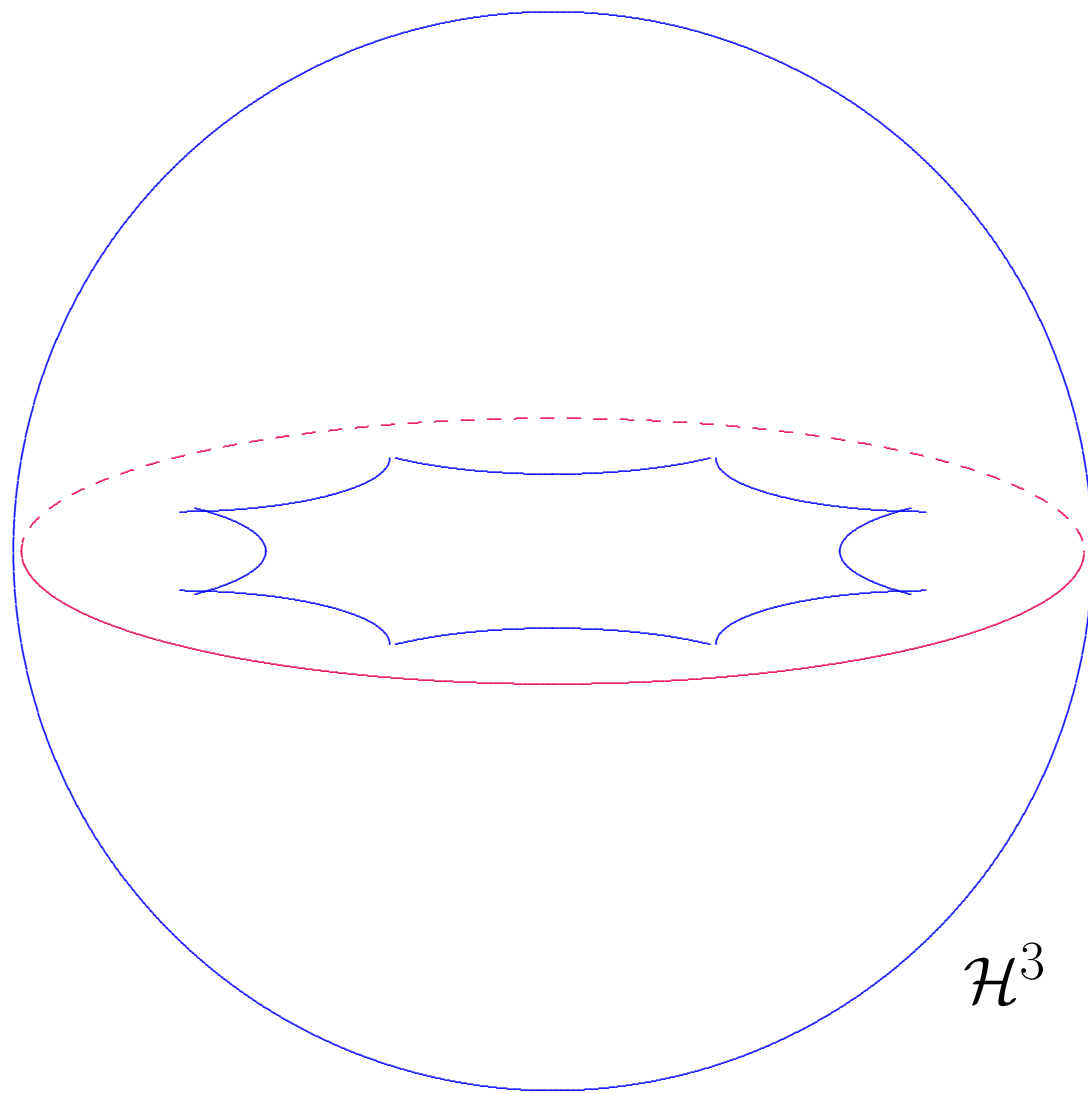




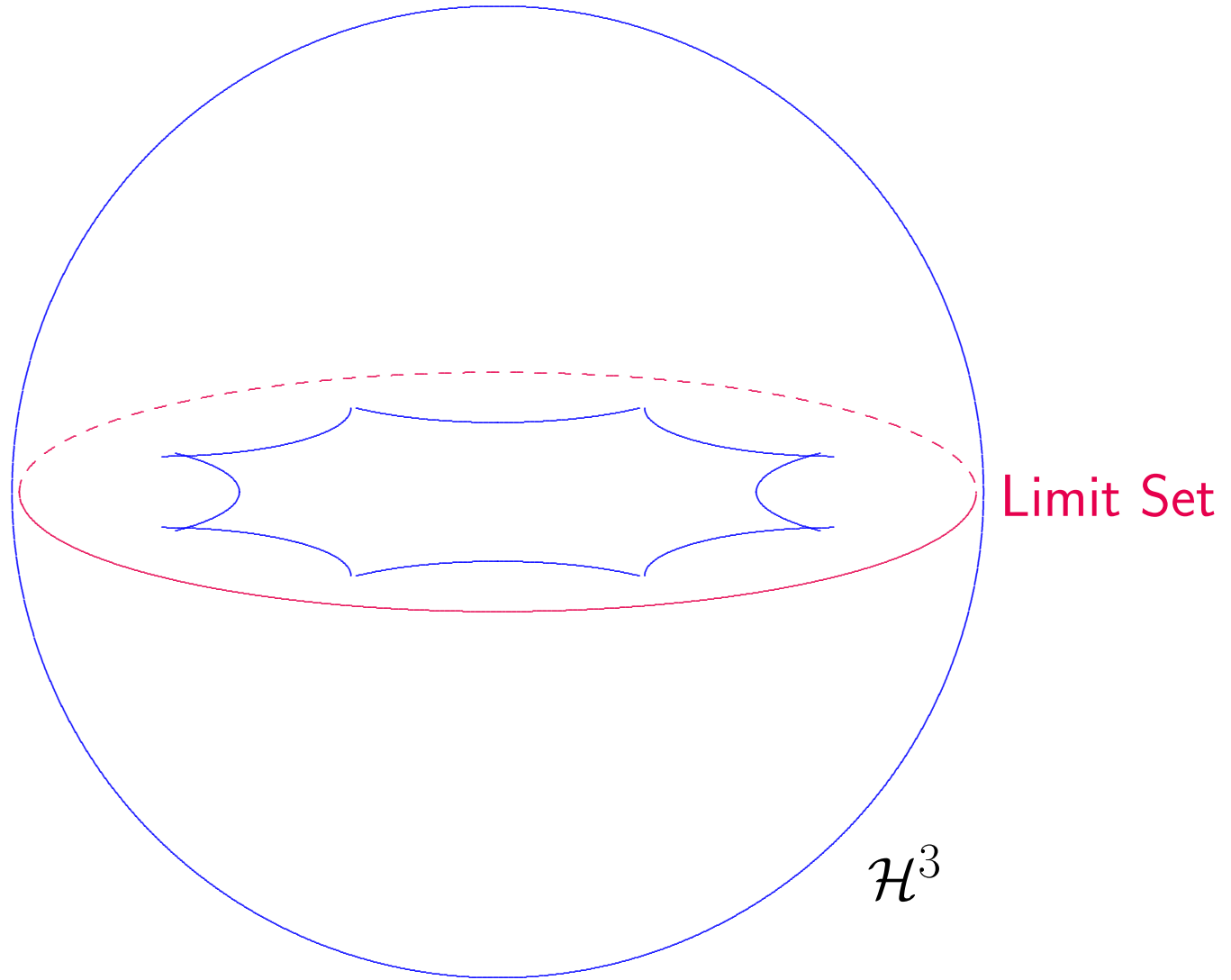
$$\pi_1(\Sigma) \hookrightarrow \mathbf{SO}_+(1, 2) = \mathbf{PSL}(2, \mathbb{R})$$

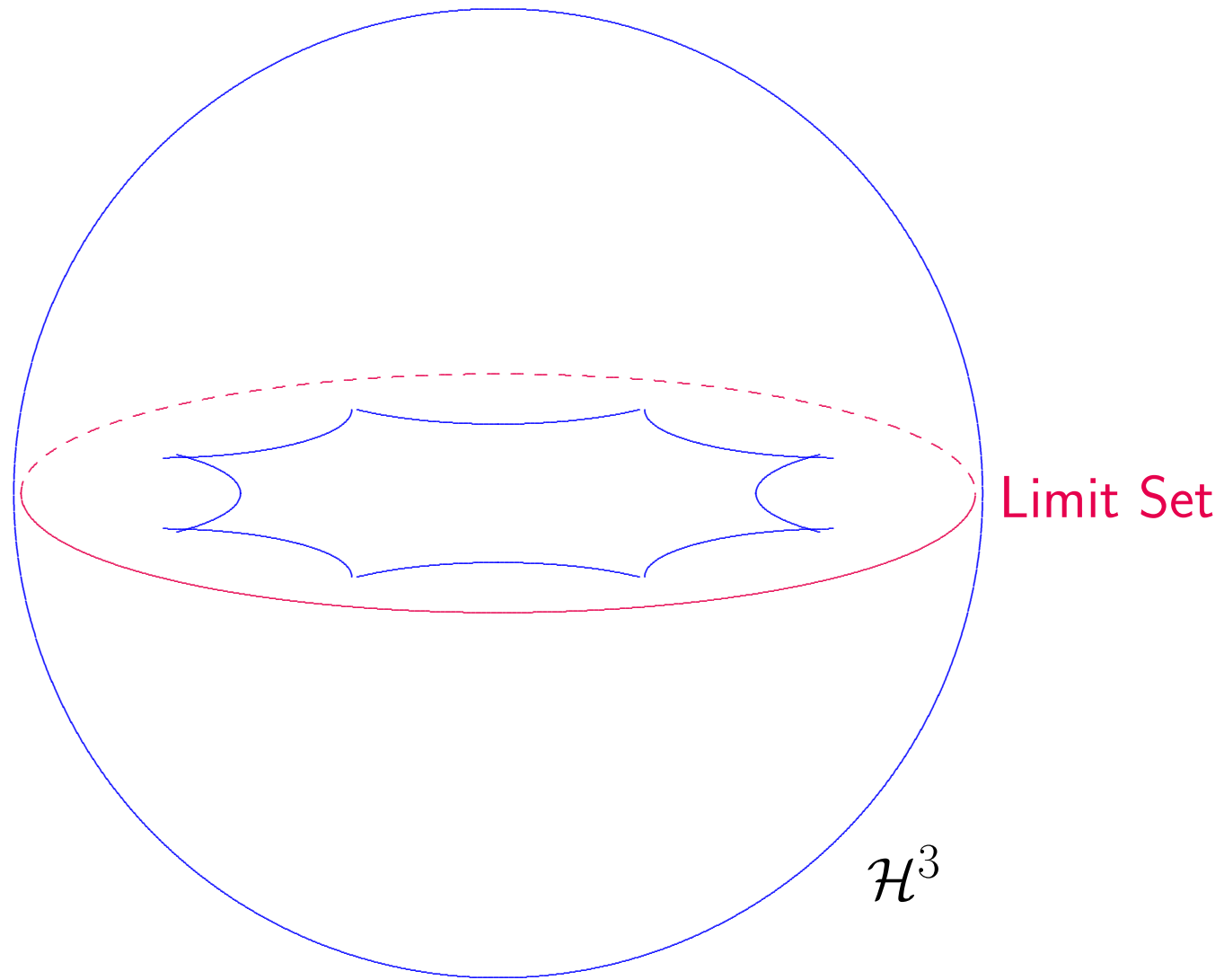


$$\begin{array}{ccc}
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 \end{array}$$



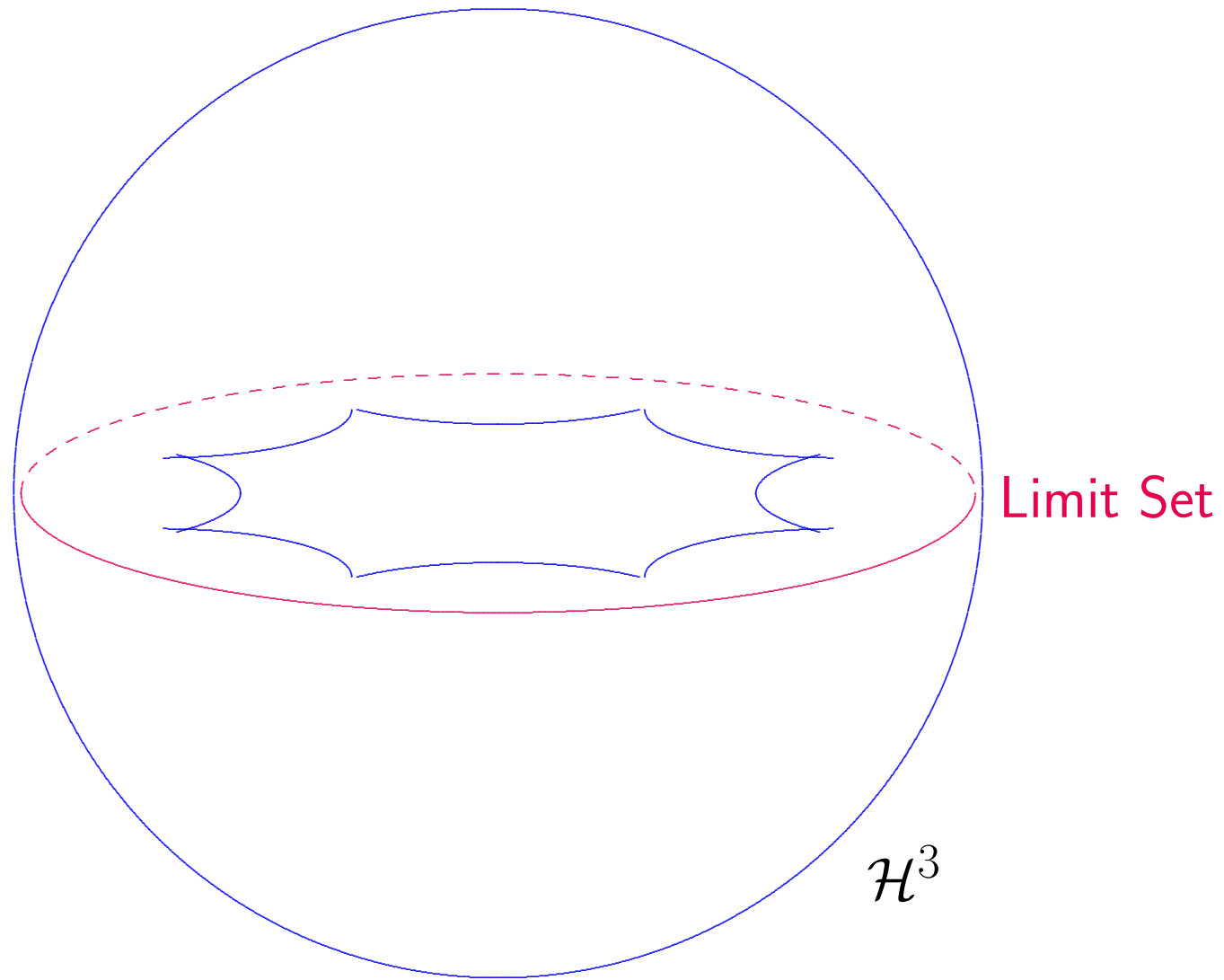
$\mathcal{H}^3$



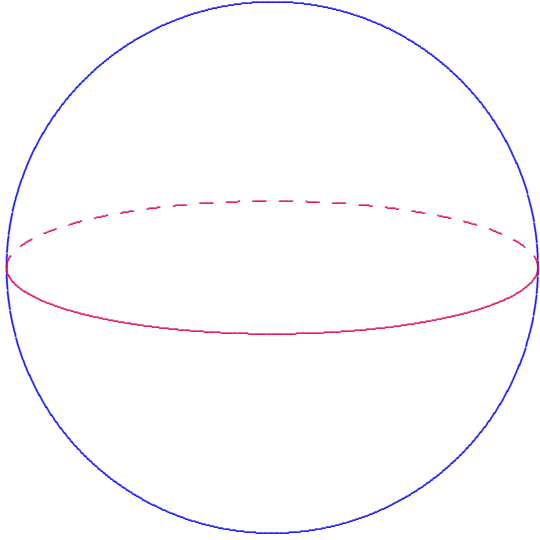


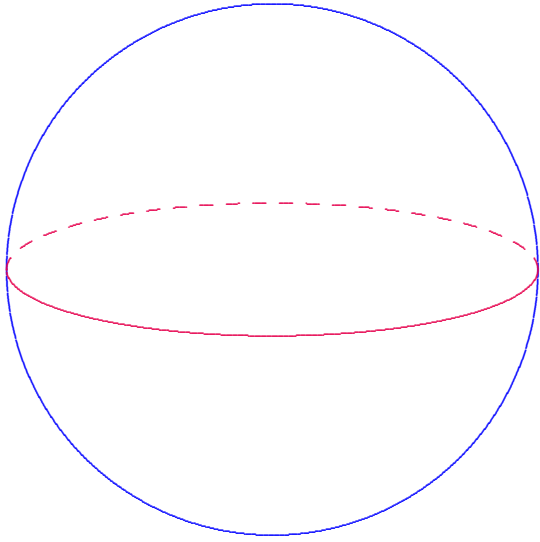
$$\pi_1(\Sigma) \xrightarrow{\cong} \Gamma \subset \mathbf{PSL}(2, \mathbb{R}) \text{ Fuchsian group}$$



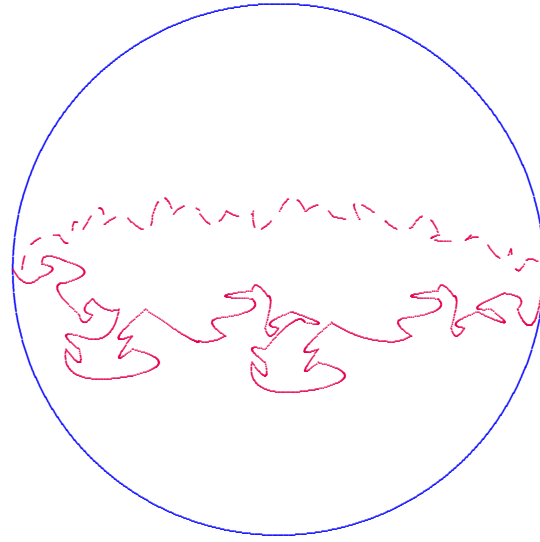
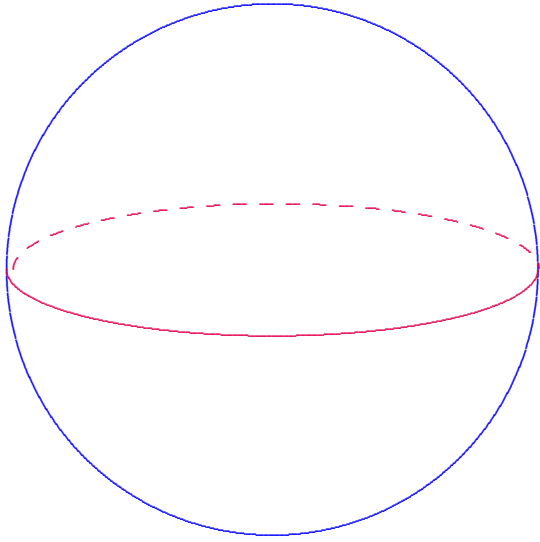


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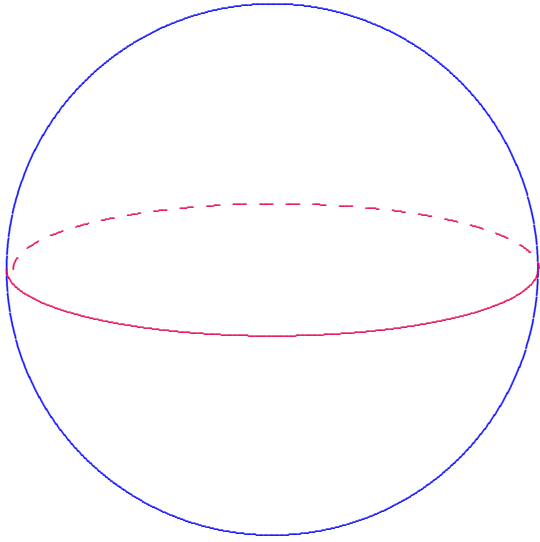




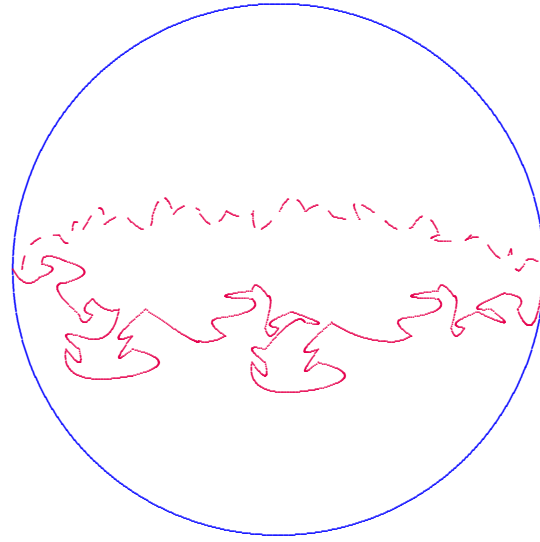
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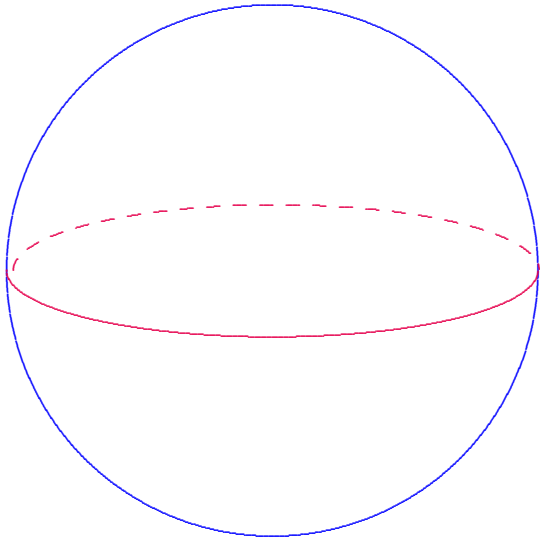
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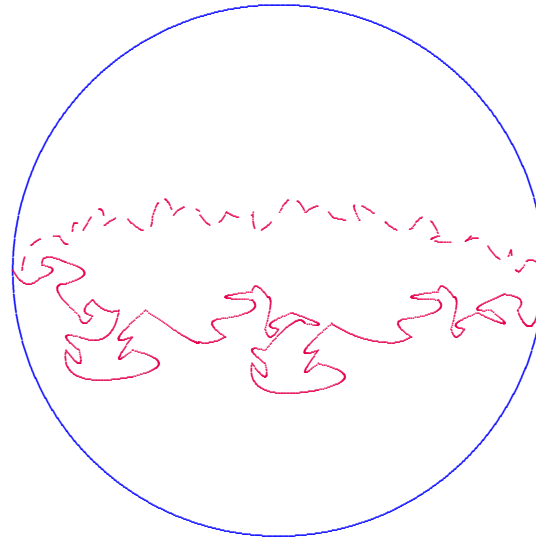
Fuchsian



quasi-Fuchsian

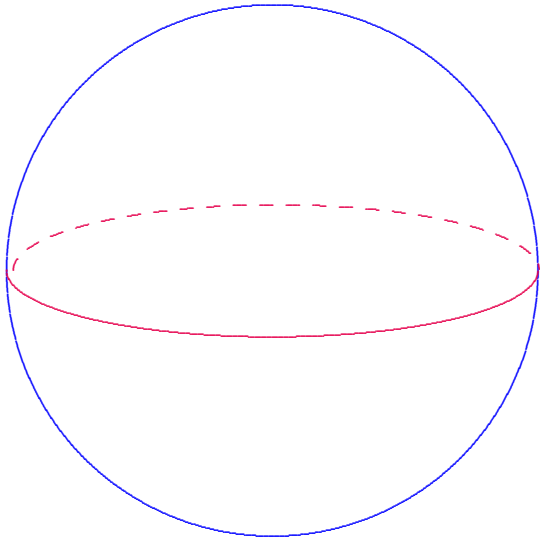


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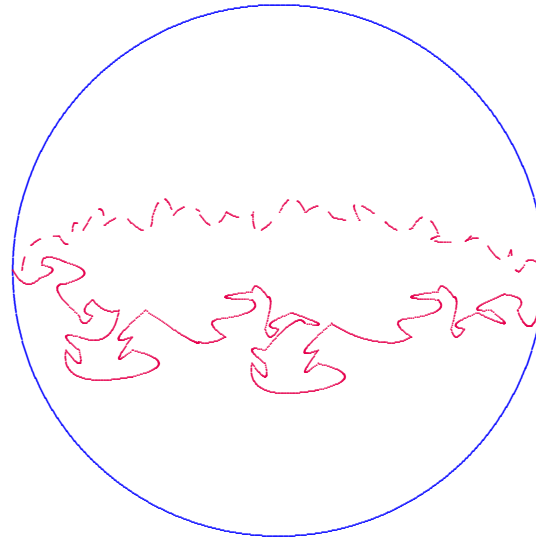


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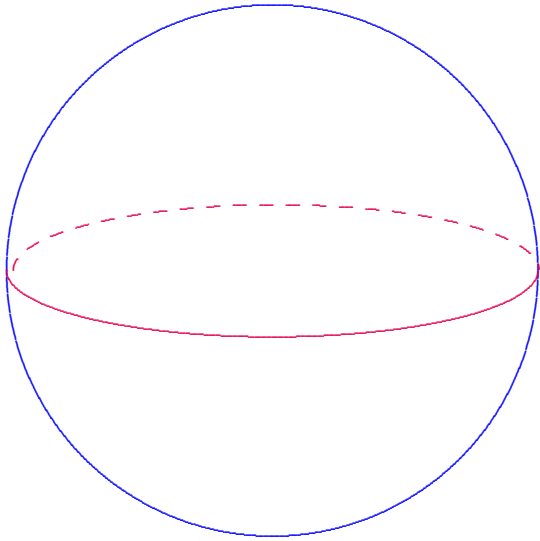


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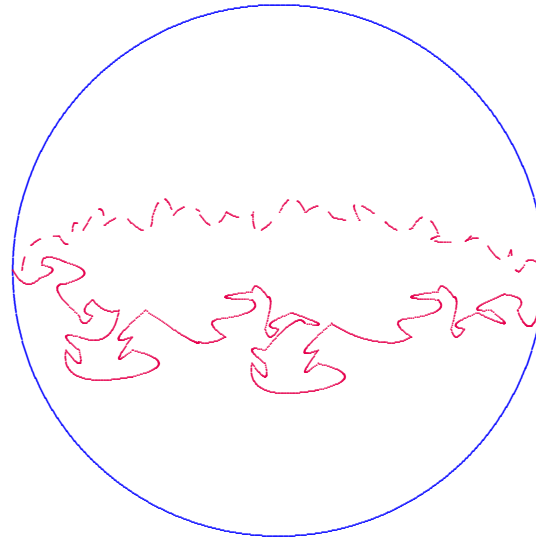


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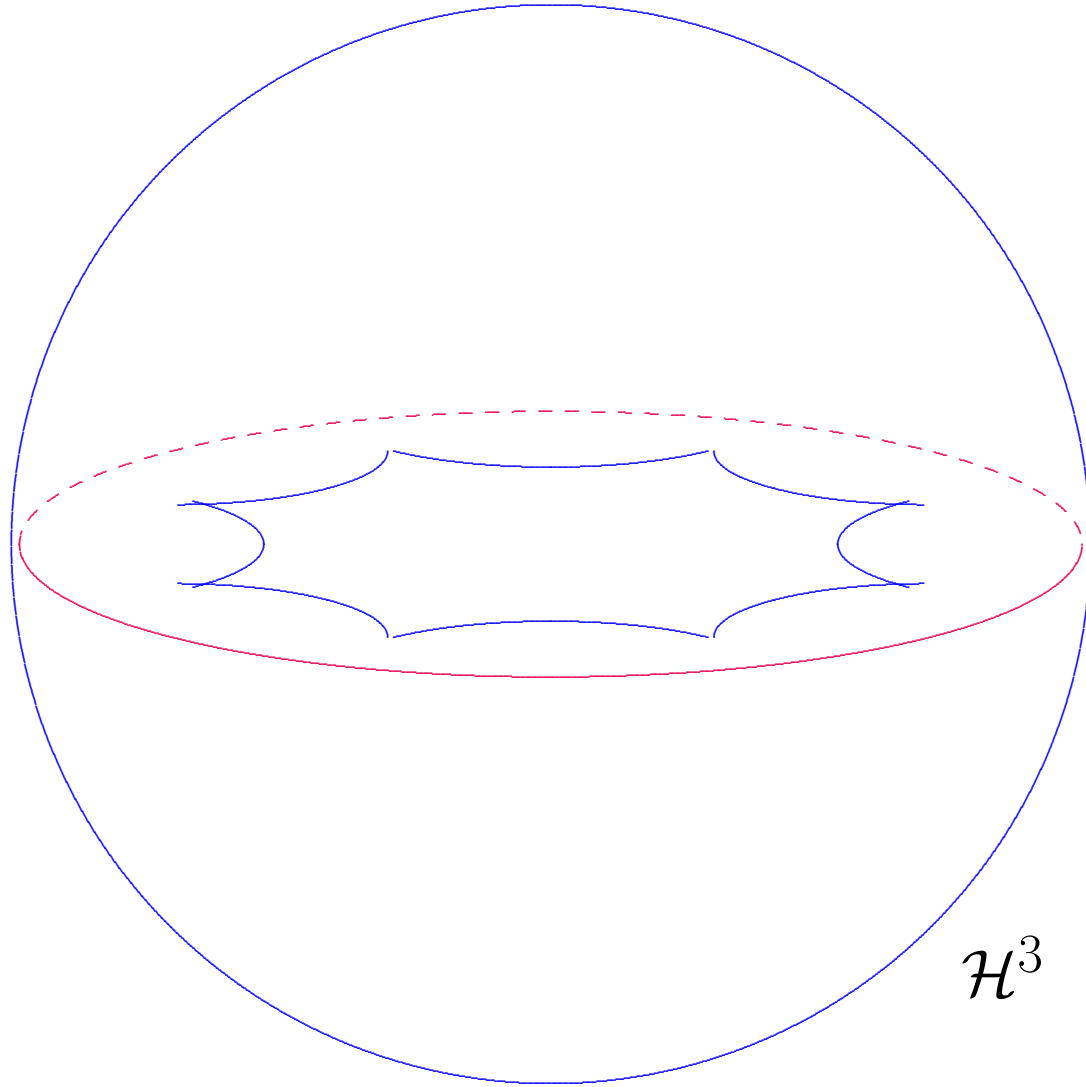


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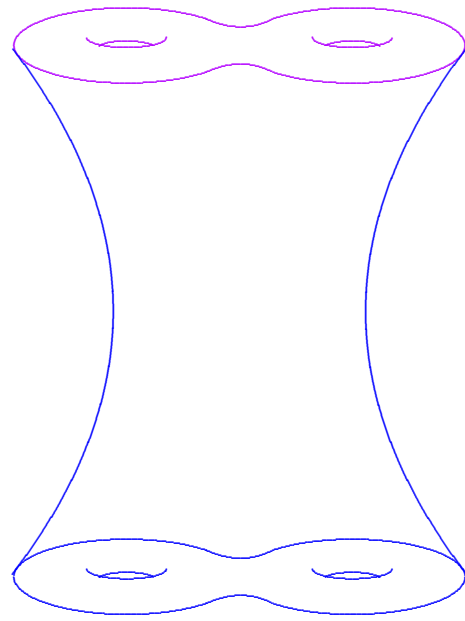
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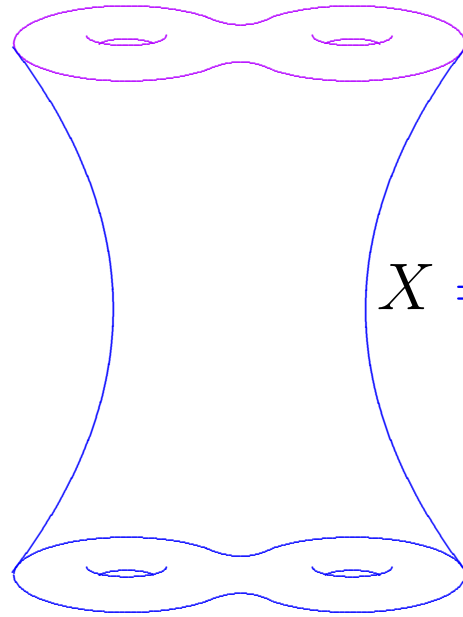
Quasi-conformally conjugate to Fuchsian.



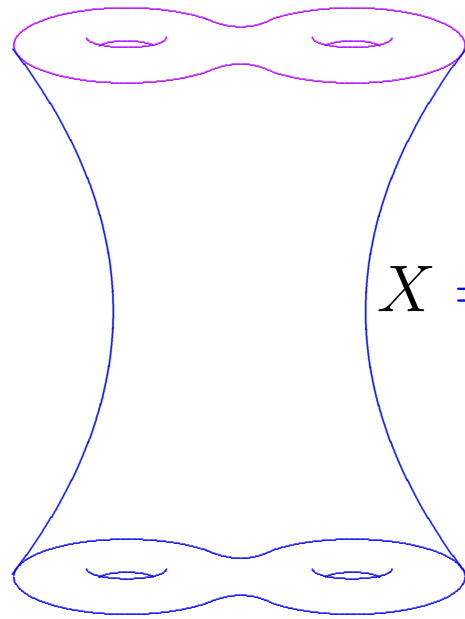


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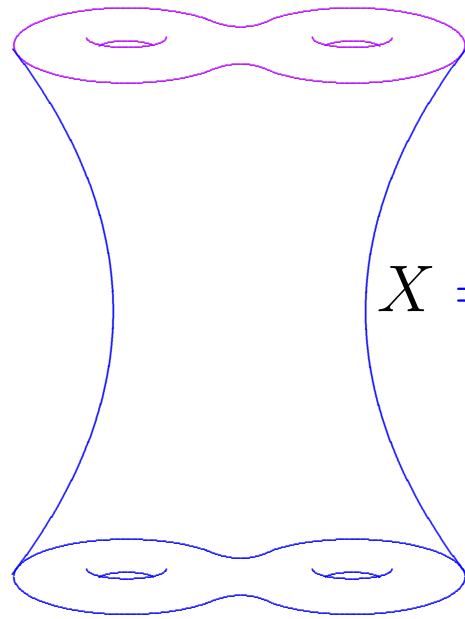


$$X = \mathcal{H}^3 / \Gamma$$



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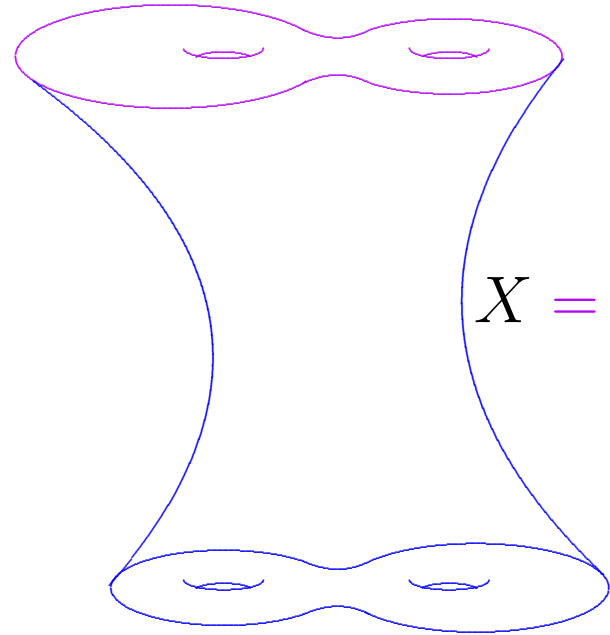
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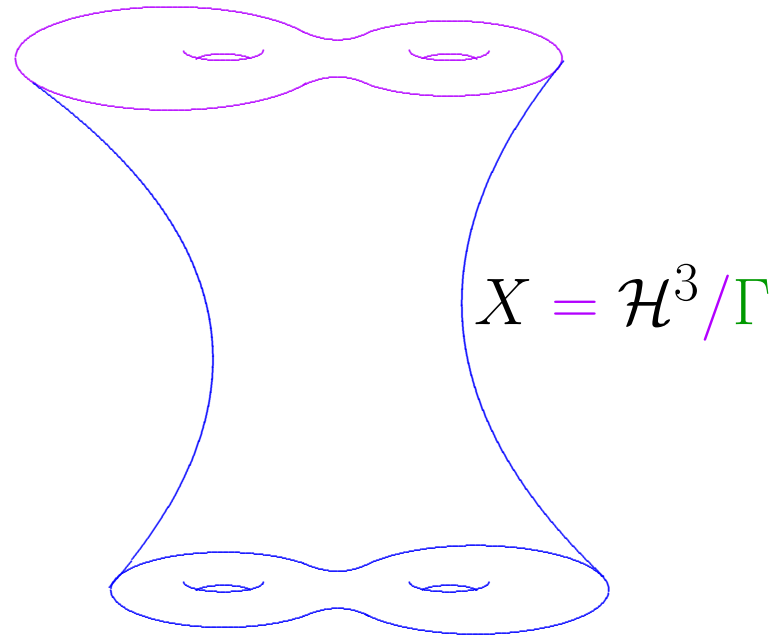
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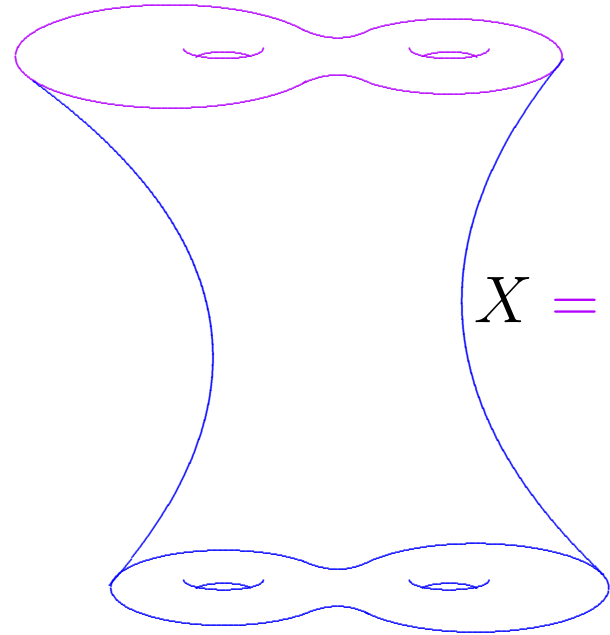
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Freedom: two points in Teichmüller space.

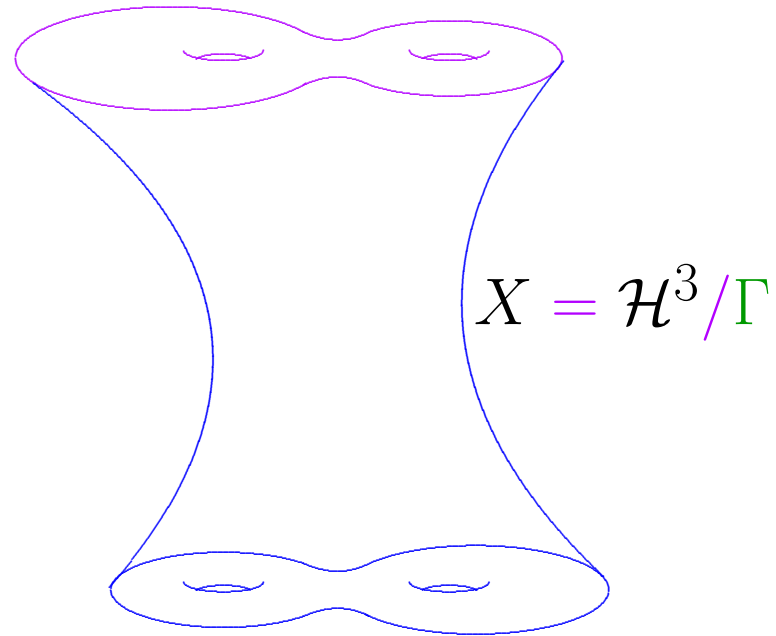


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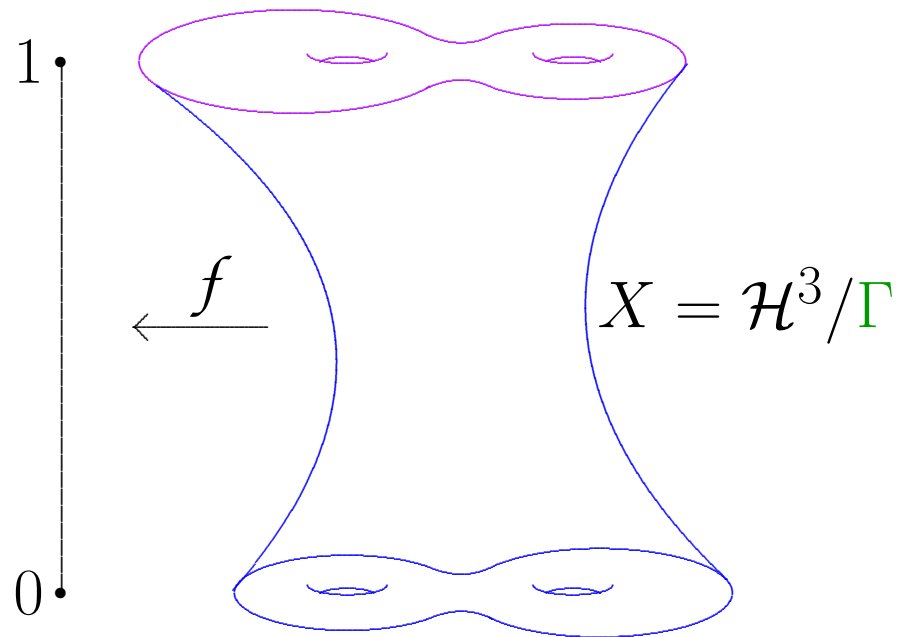
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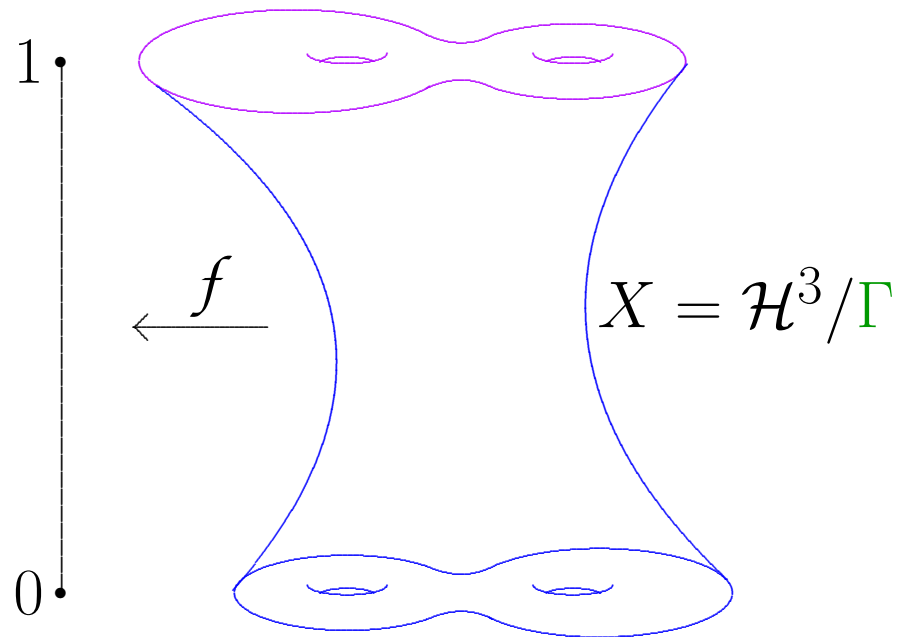


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Tunnel-Vision function:

$$f : \overline{X} \rightarrow [0, 1]$$



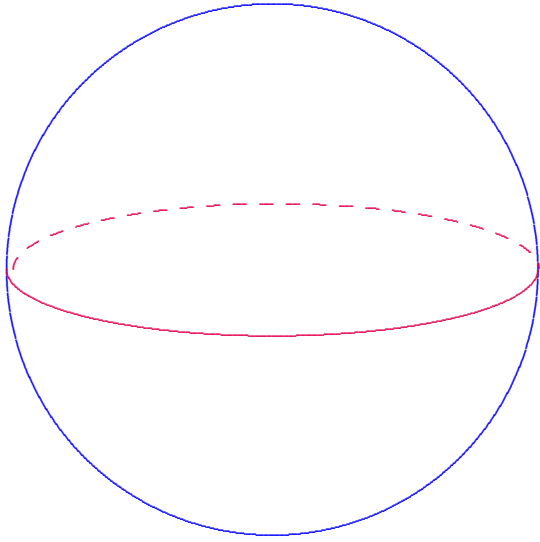
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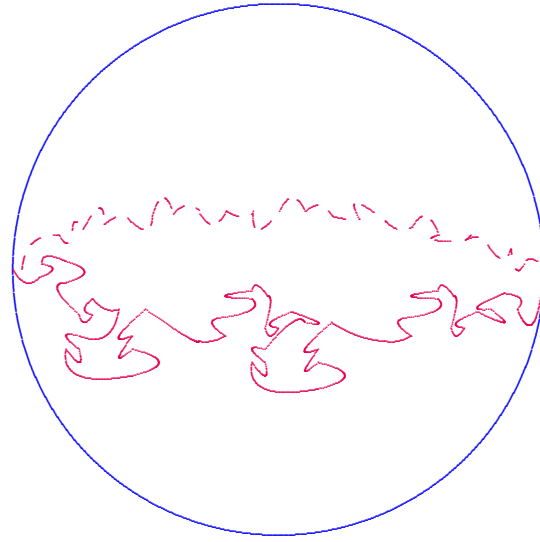
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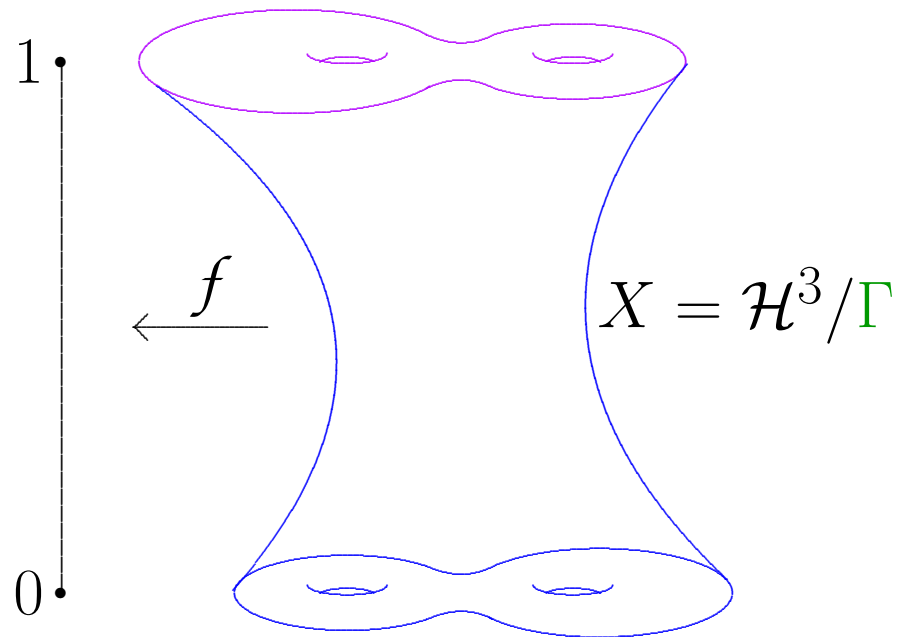
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Fuchsian



quasi-Fuchsian



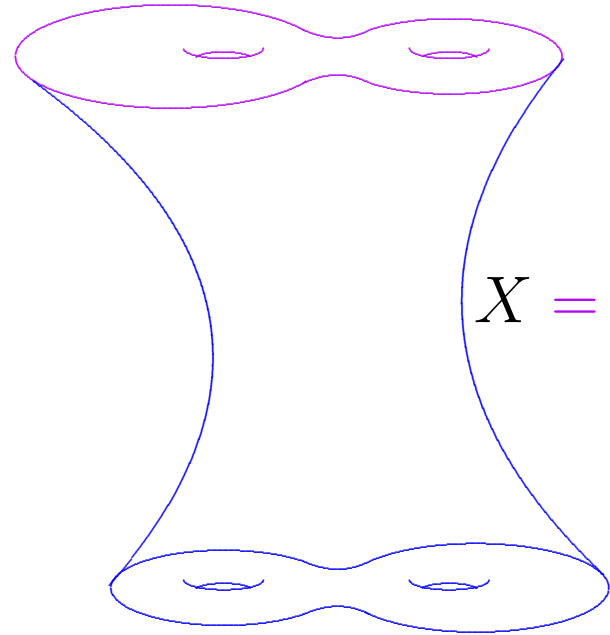
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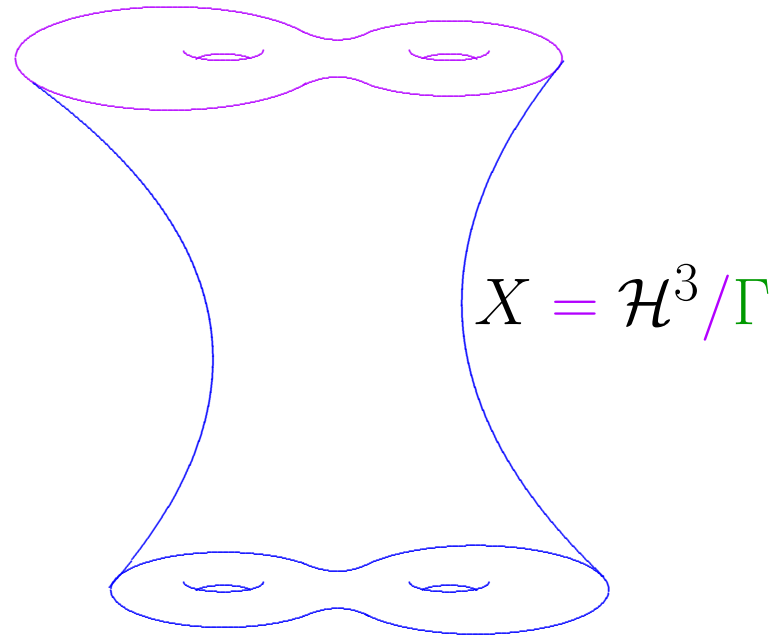
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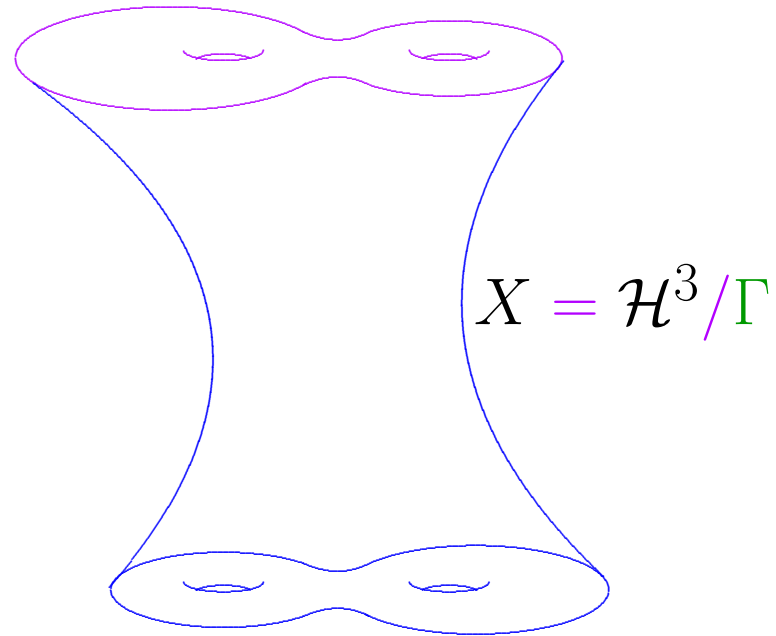
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$$X = \mathcal{H}^3 / \Gamma$$



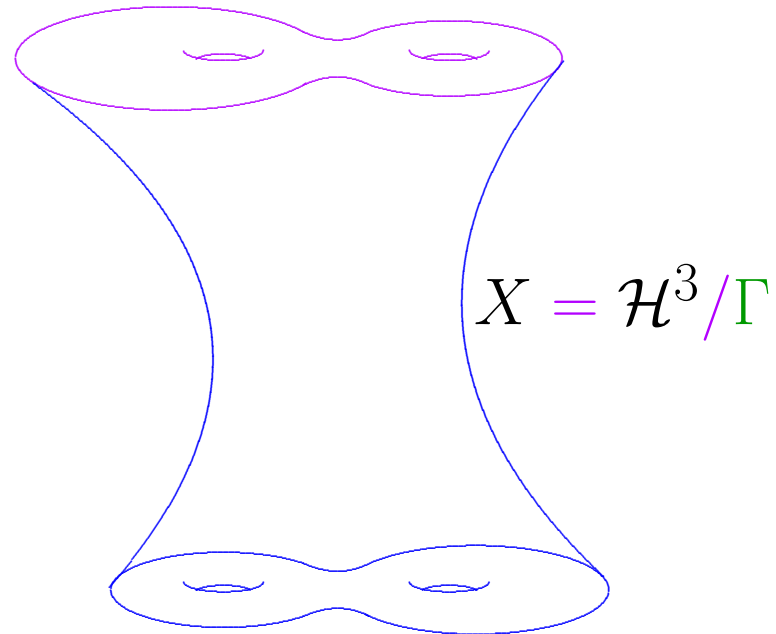
Construction of conformally flat 4-manifolds:



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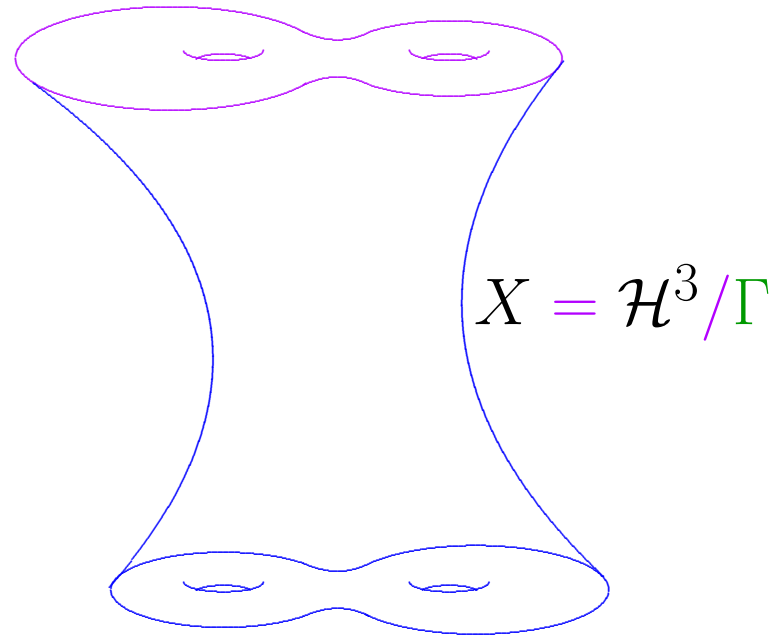




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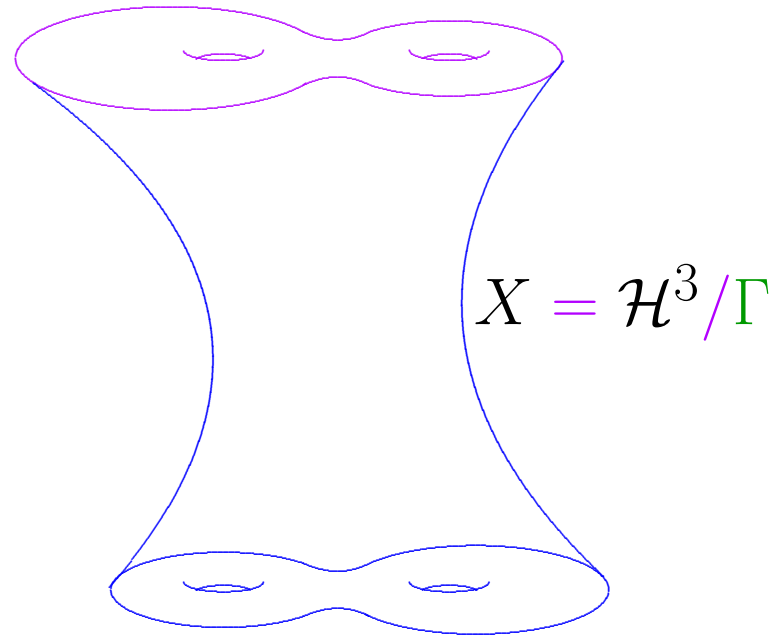
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$\sim$ : crush  $\partial\bar{X} \times S^1$  to  $\partial\bar{X}$ .



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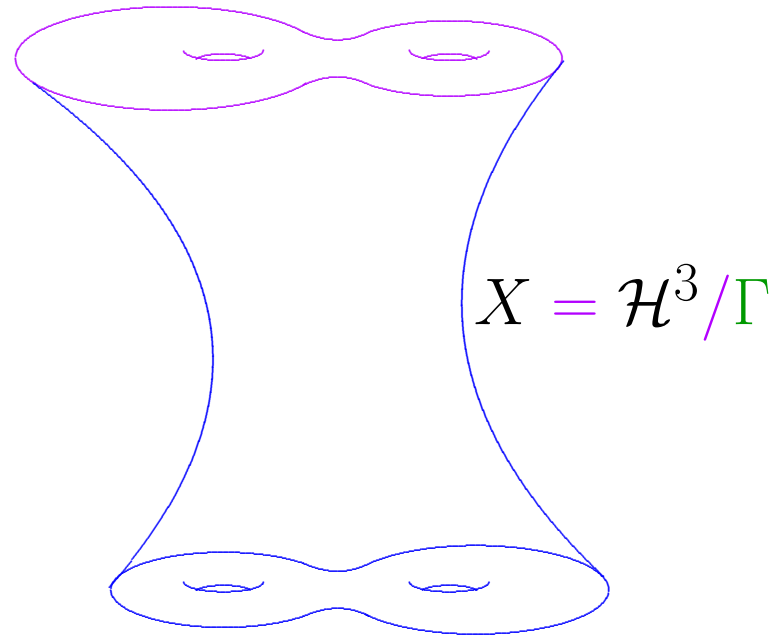
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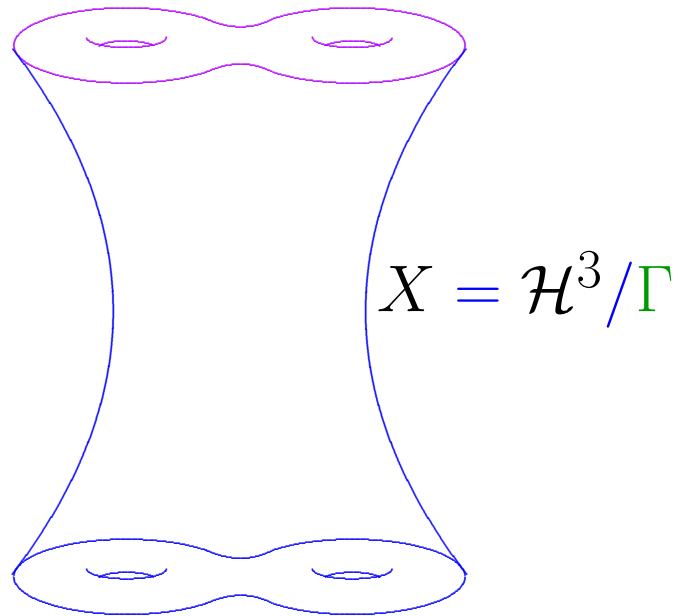
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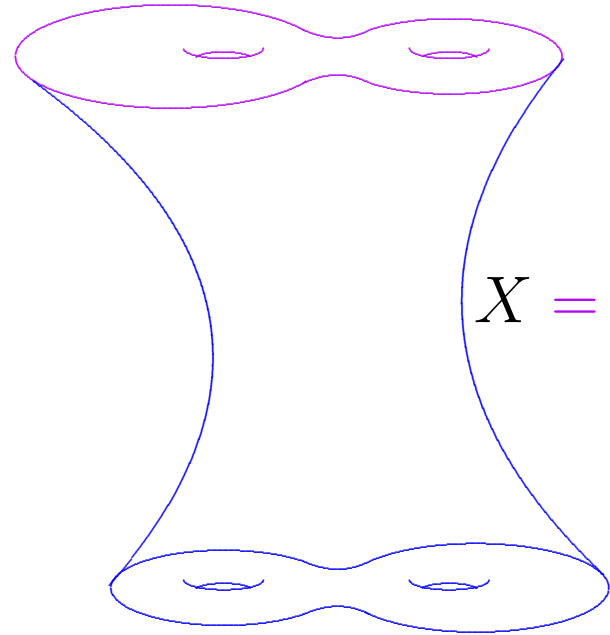


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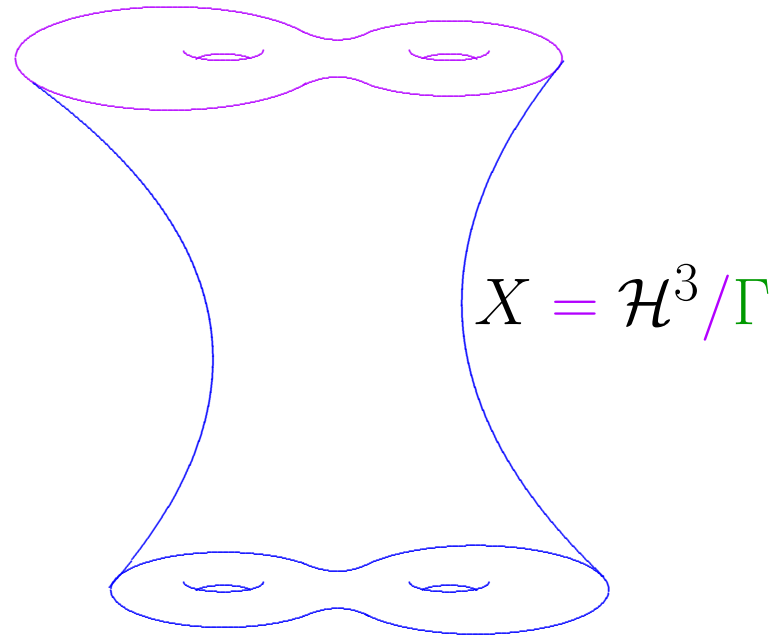
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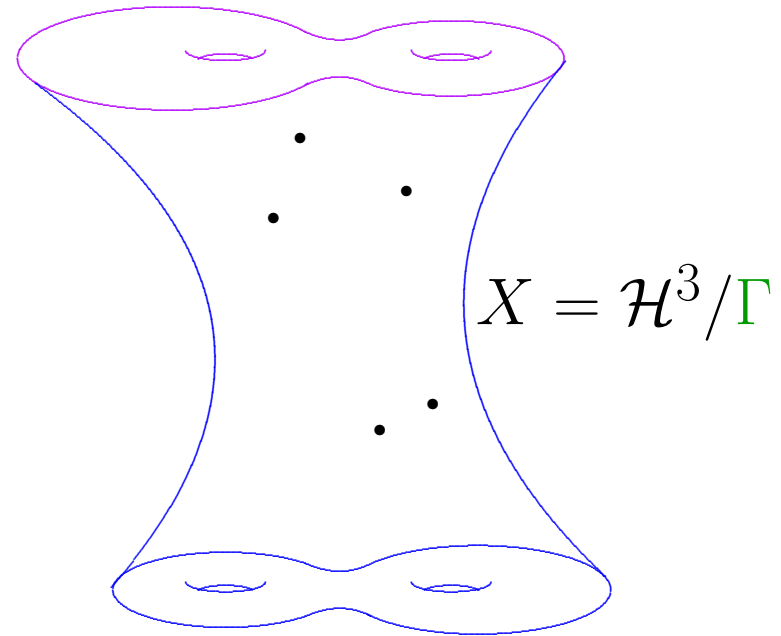
Fuchsian case:  $\Sigma \times S^2$  scalar-flat Kähler.



$$X = \mathcal{H}^3 / \Gamma$$



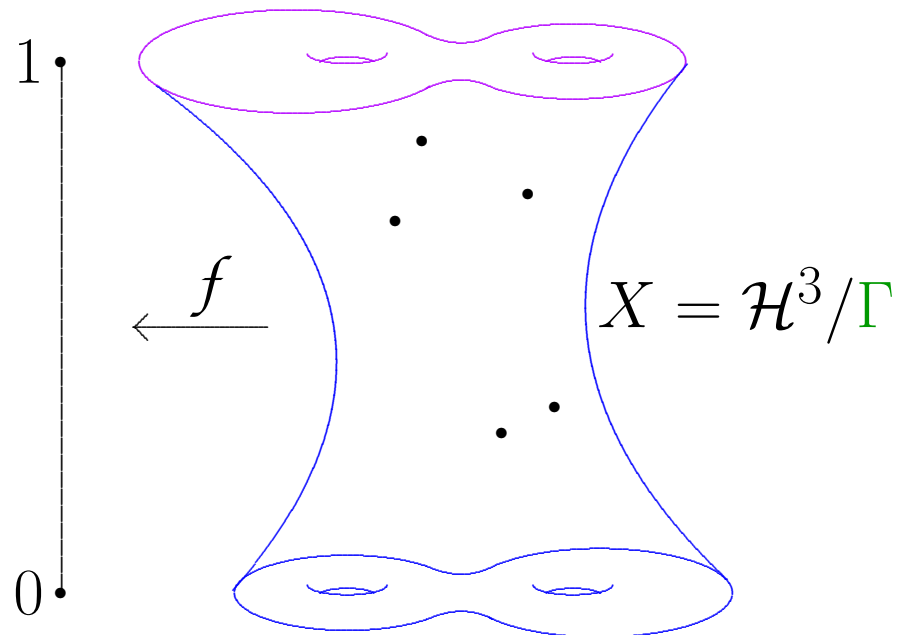
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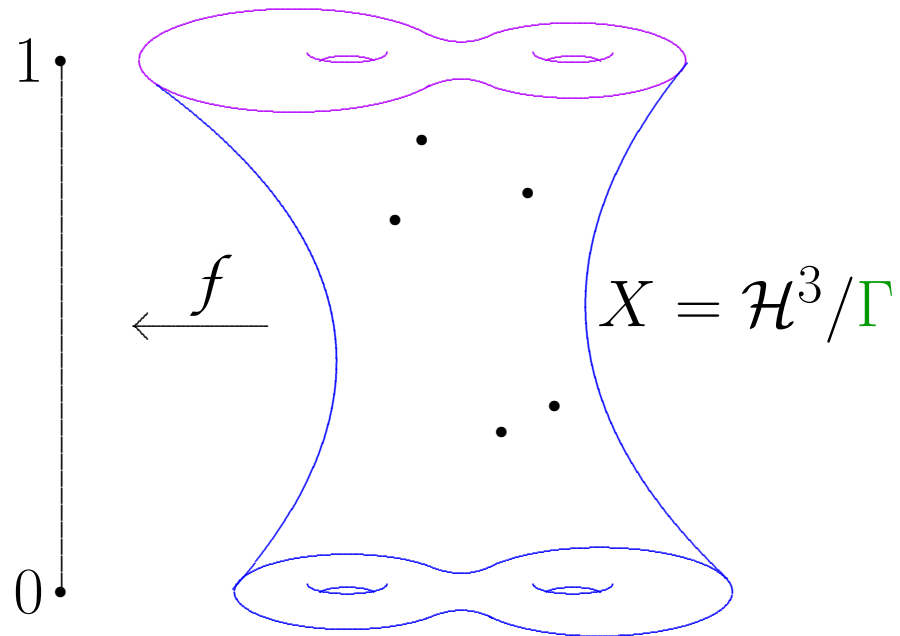




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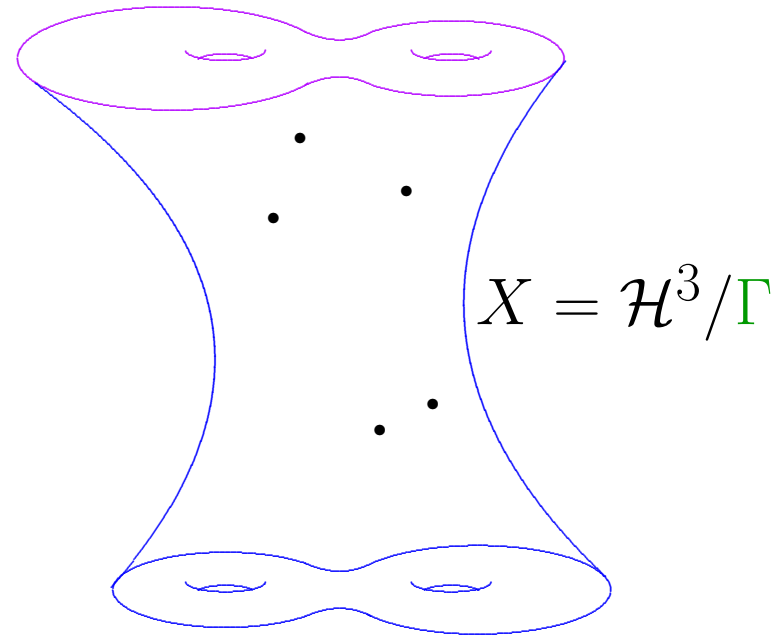


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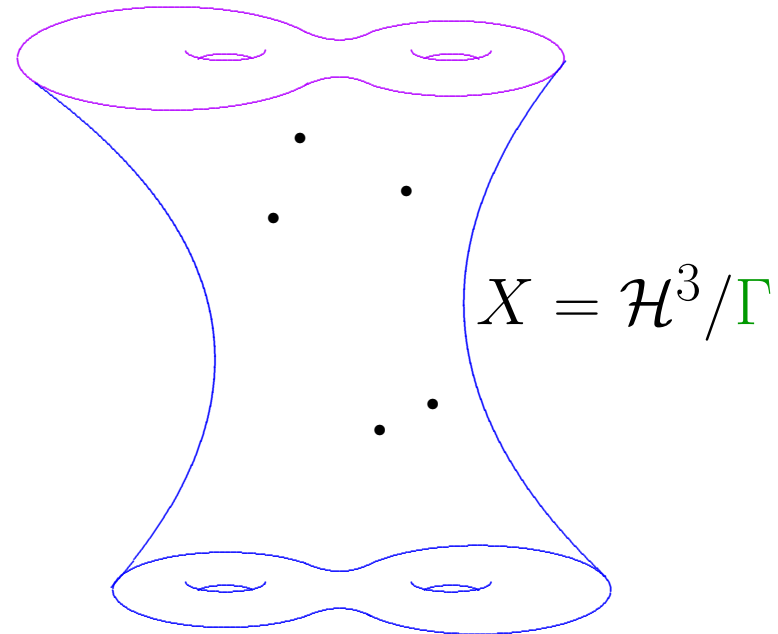
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Can do if  $k \neq 1$ .



Construction of ASD 4-manifolds:

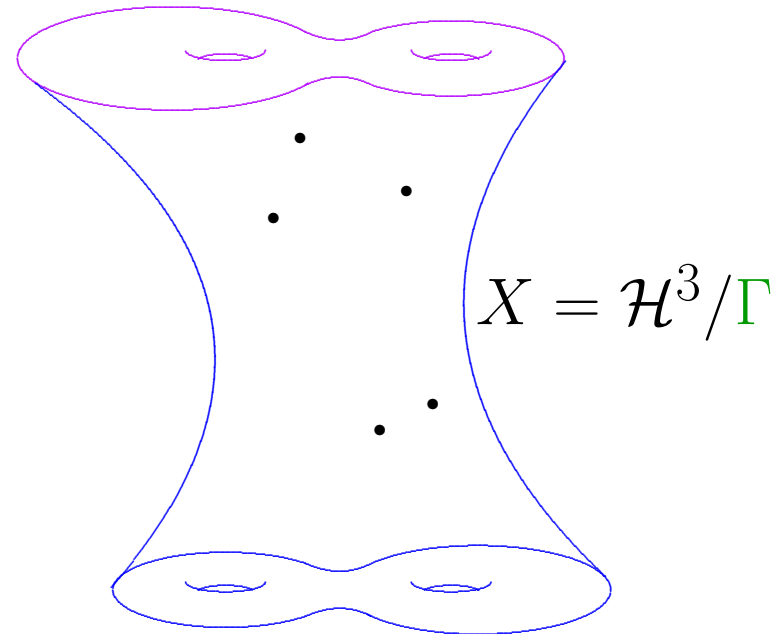
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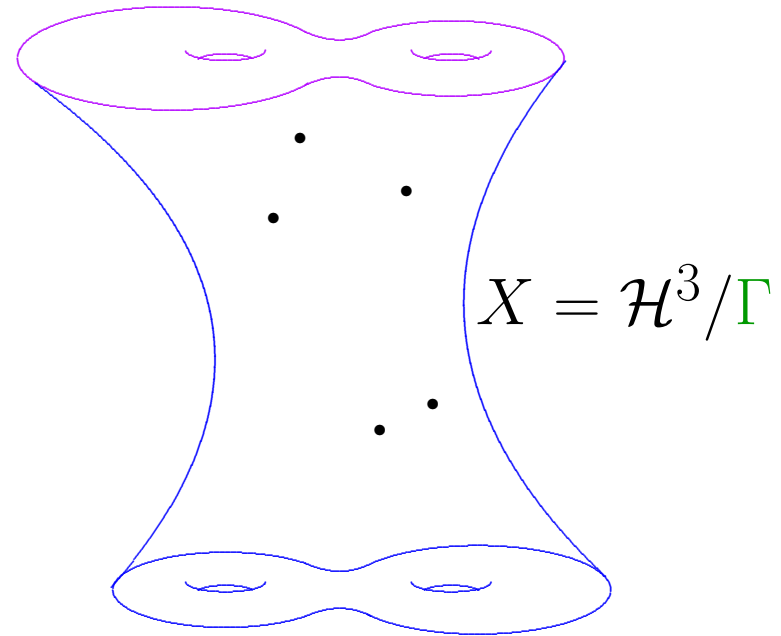
$$\Delta G_j = 2\pi\delta_{p_j}, \quad G_j \rightarrow 0 \text{ at } \partial\bar{X}$$



Construction of ASD 4-manifolds:

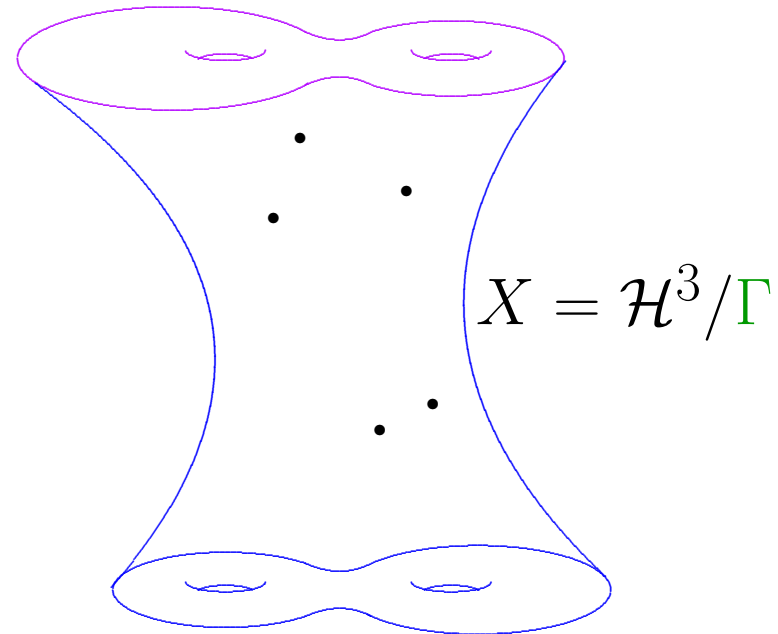
Let  $G_j$  be the Green's function of  $p_j$ , and set

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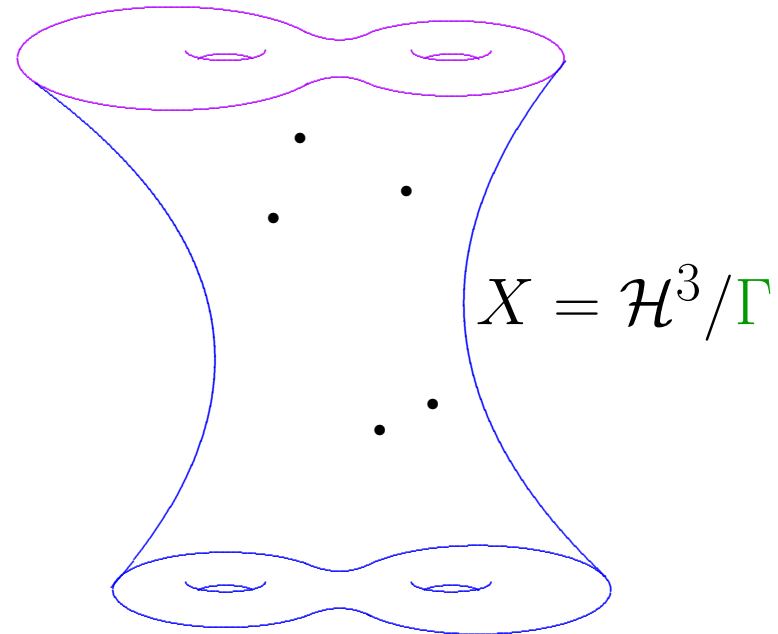


Construction of ASD 4-manifolds:

$$V = 1 + \sum_{j=1}^k G_j.$$

Choose  $P \rightarrow (X - \{p_1, \dots, p_k\})$  circle bundle with connection form  $\theta$  such that

$$d\theta = \star dV.$$



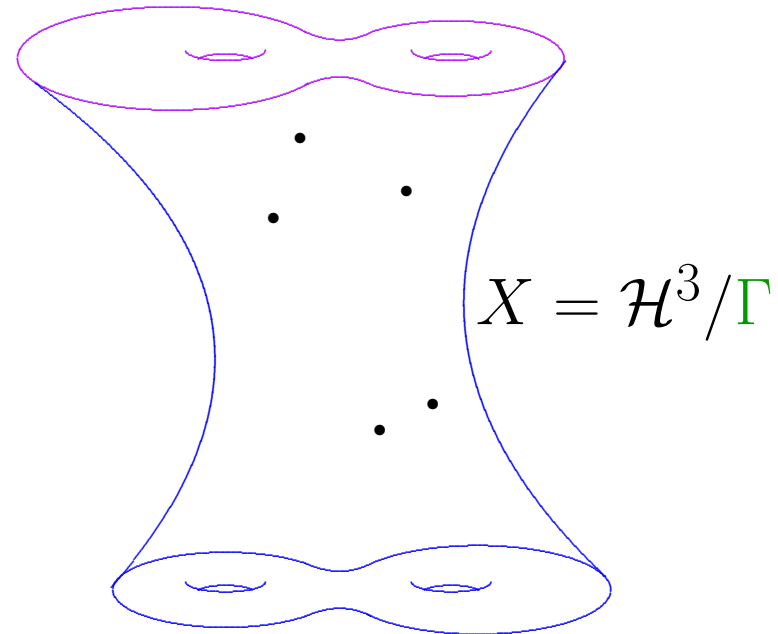
Construction of ASD 4-manifolds:

$$g = Vh + V^{-1}\theta^2$$

$$V = 1 + \sum_{j=1}^k G_j$$

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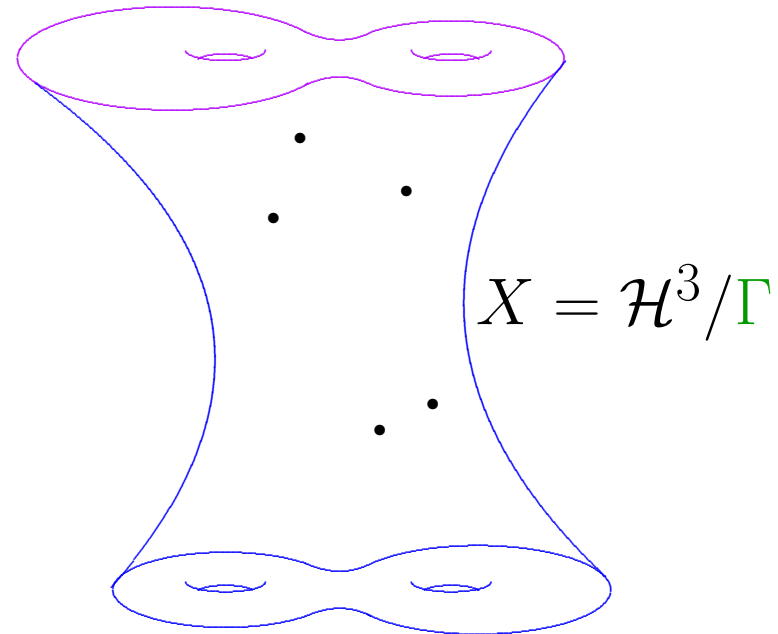


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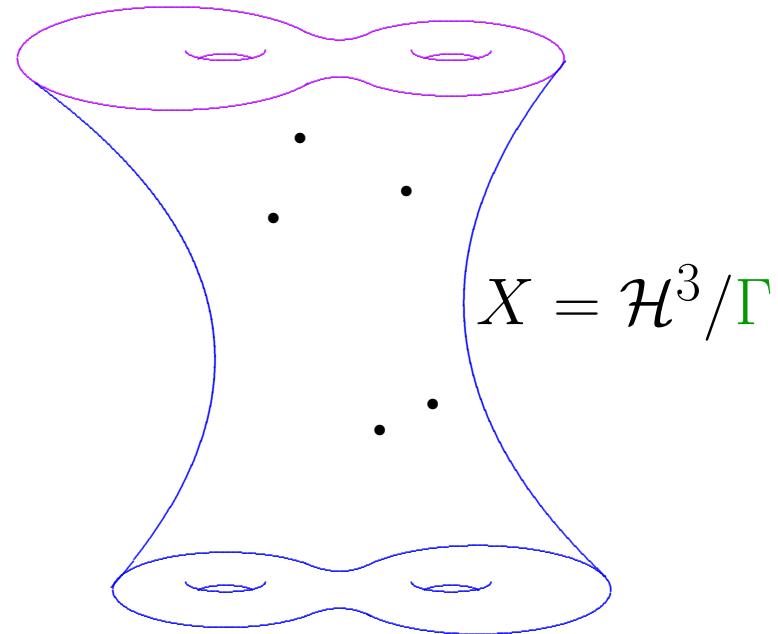
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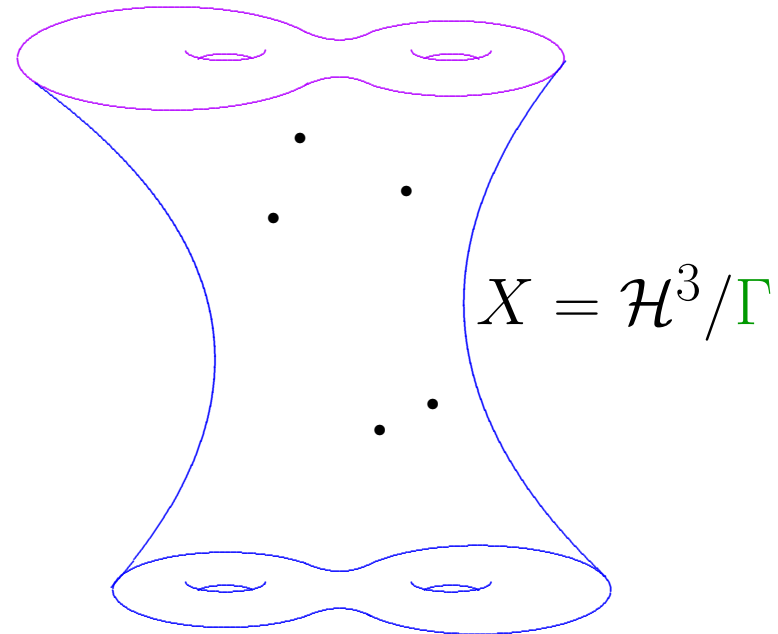
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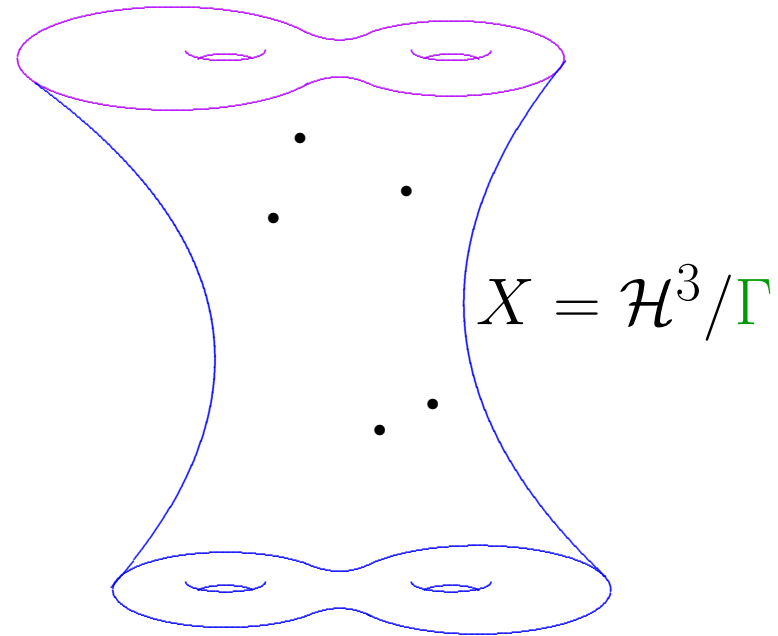
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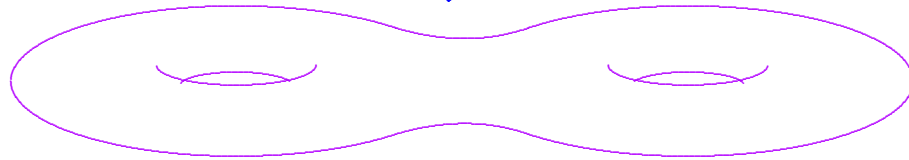
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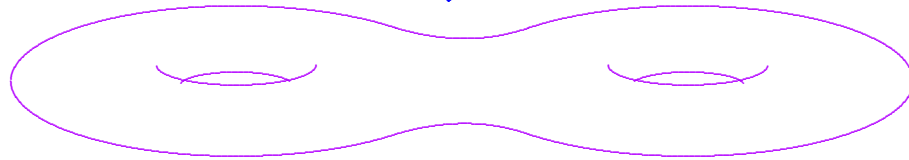
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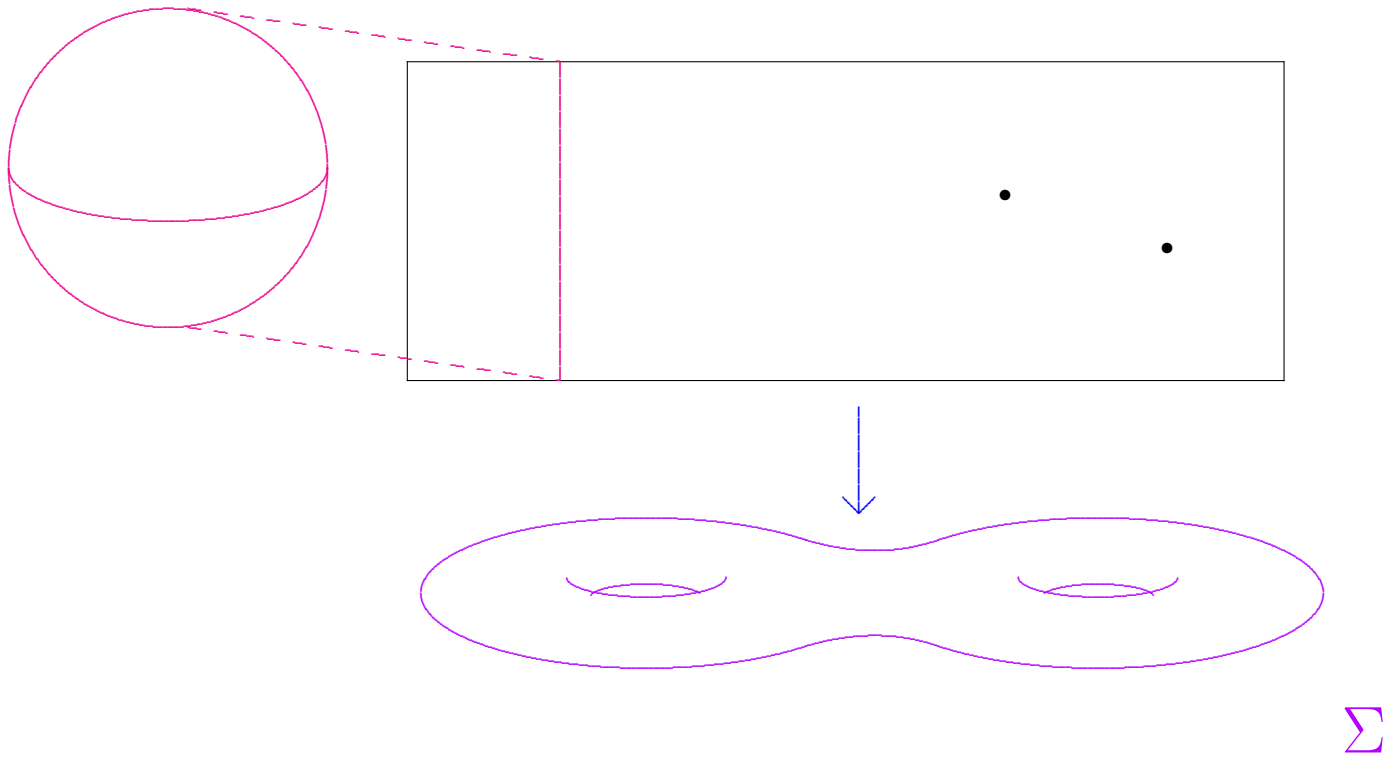
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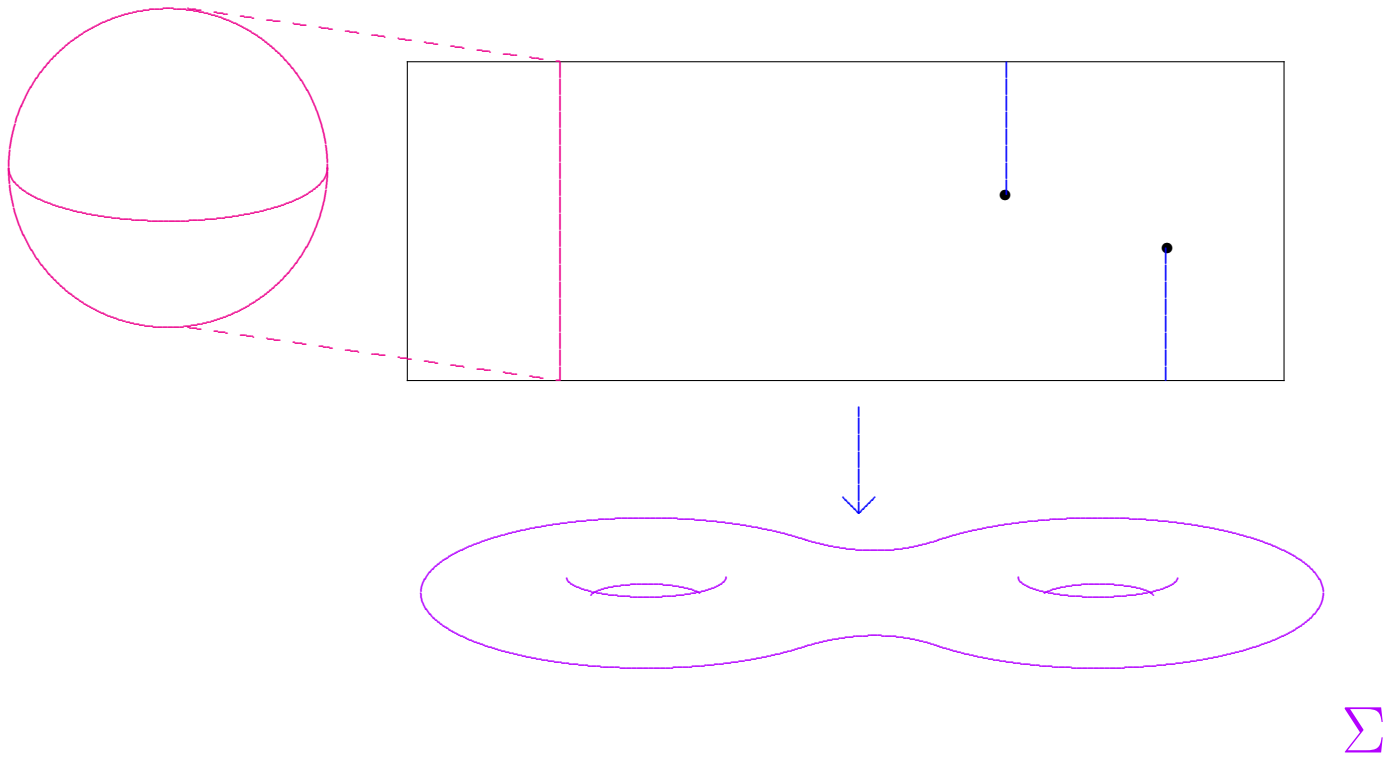
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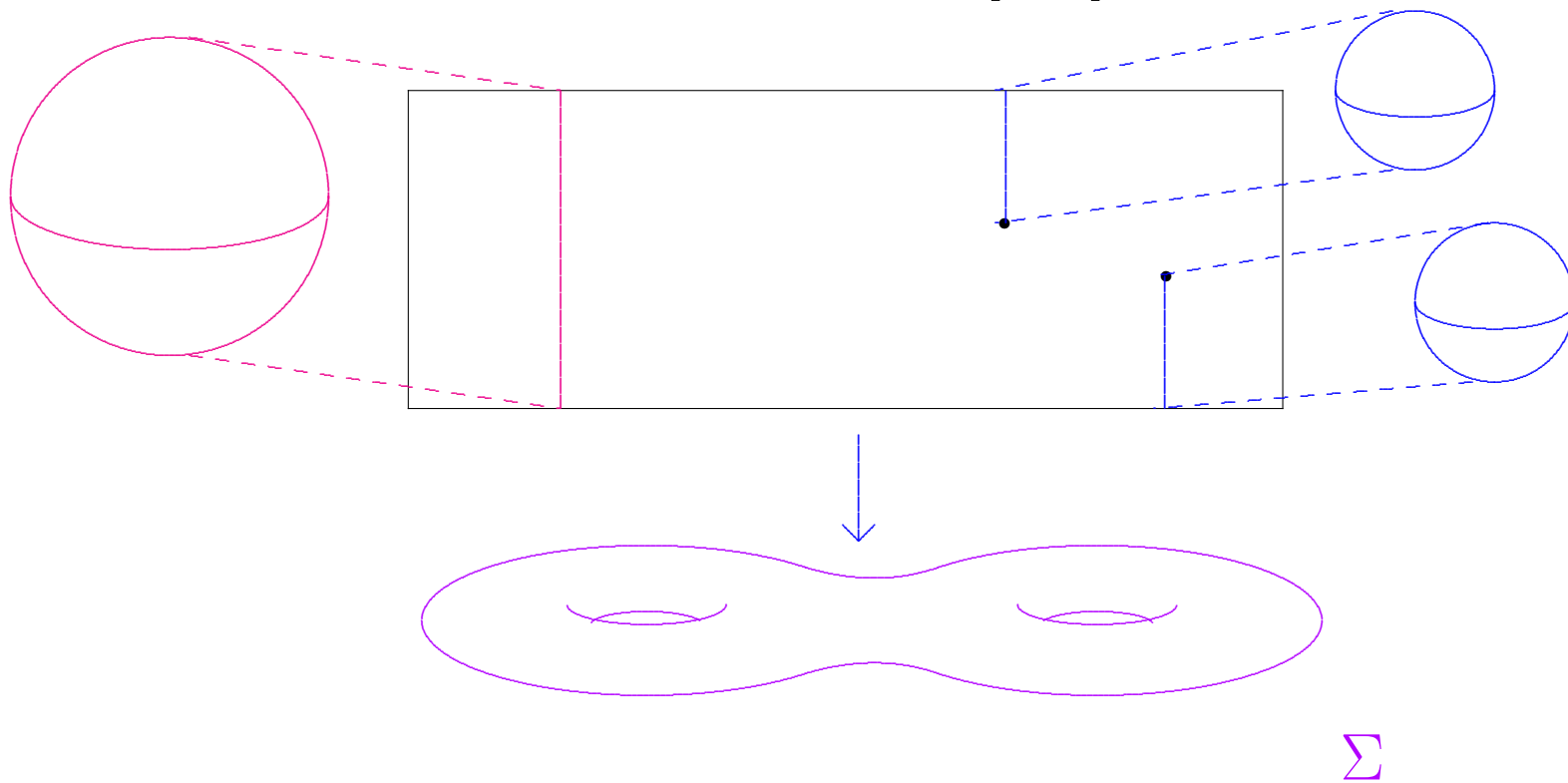


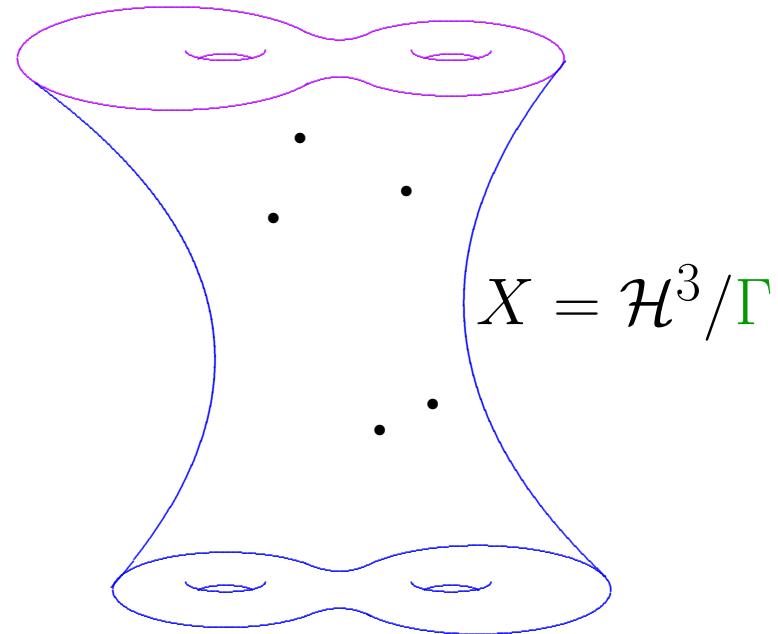


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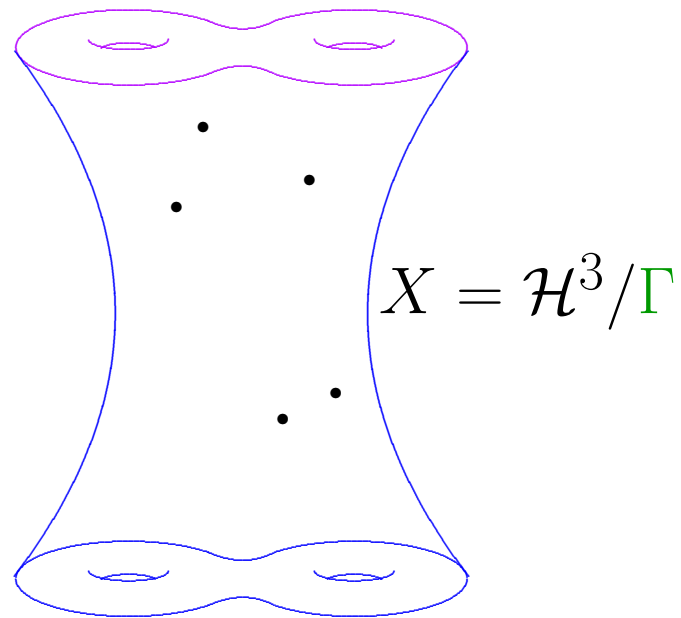


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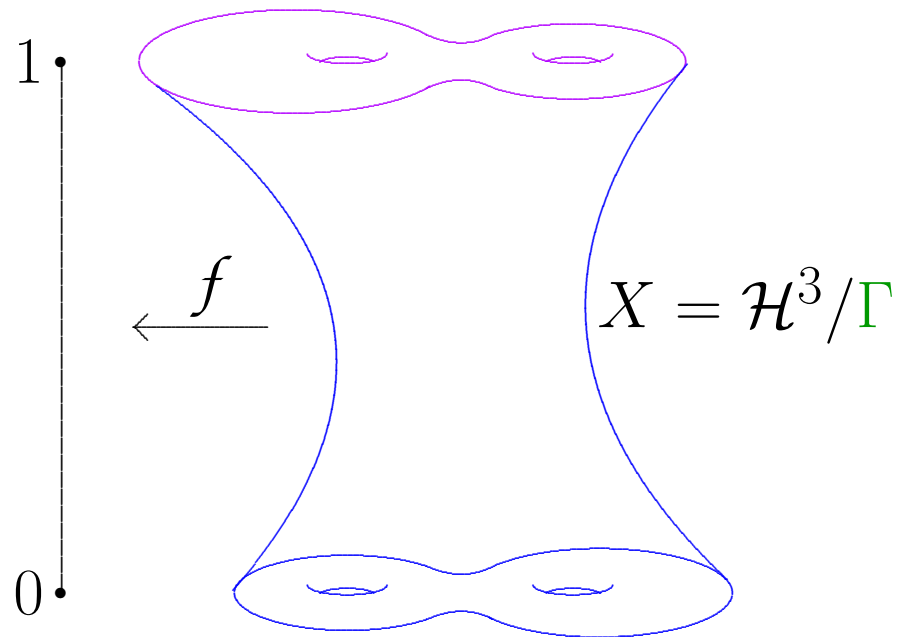


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Fuchsian case:  $(\Sigma \times S^2) \# k \overline{\mathbb{C}\mathbb{P}_2}$  scalar-flat Kähler



$\Gamma$  quasi-Fuchsian

$$\bar{X} \approx \Sigma \times [0, 1]$$

Tunnel-Vision function:

$$f : \bar{X} \rightarrow [0, 1]$$

$$\Delta f = 0$$

**Theorem.**

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Lemma.

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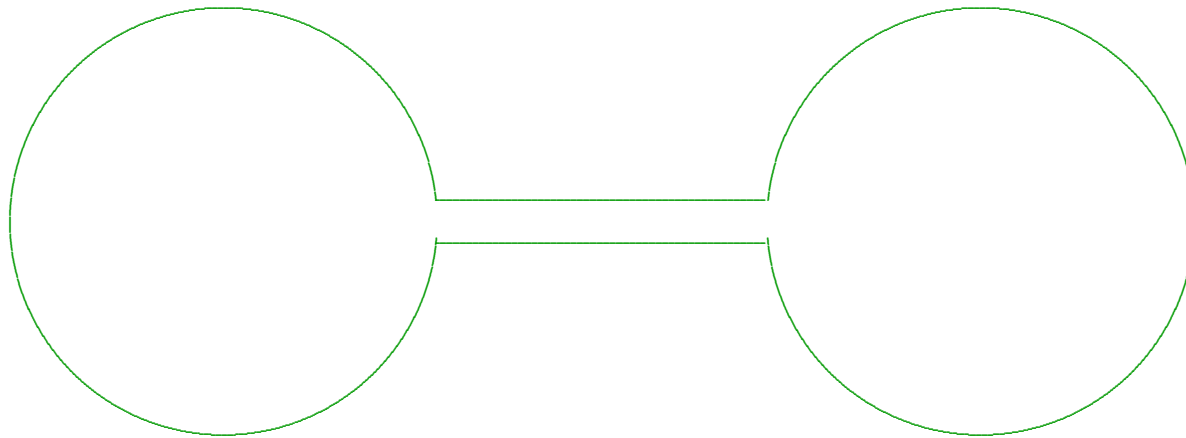
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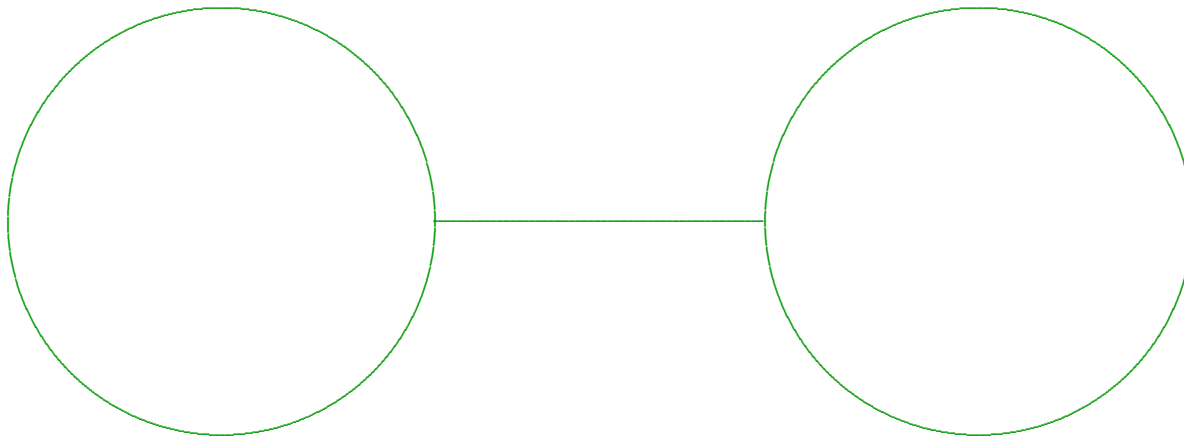
If  $\gamma$  is invariant under  $\zeta \mapsto -\zeta$ , and if  $g$  is even, we can also arrange for  $\Lambda(\Gamma)$  to also be invariant under reflection through the origin.

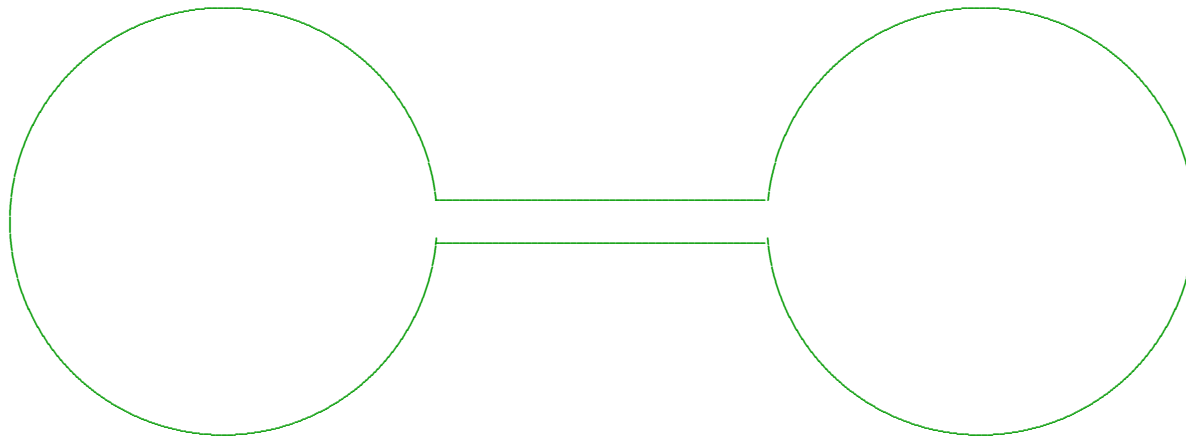
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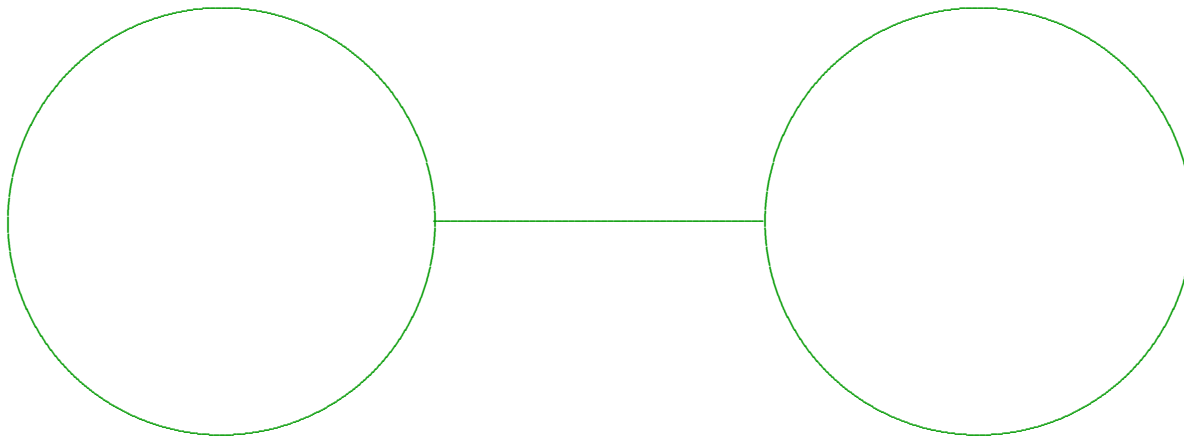
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Ahlfors-Bers: Quasi-conformal mappings

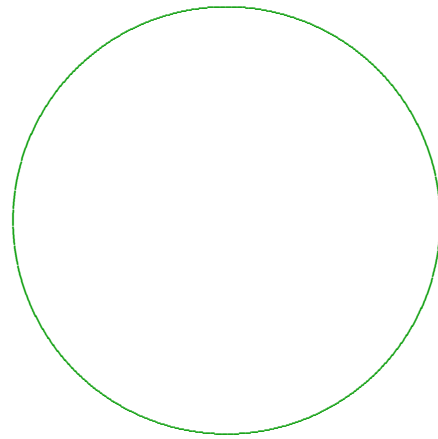
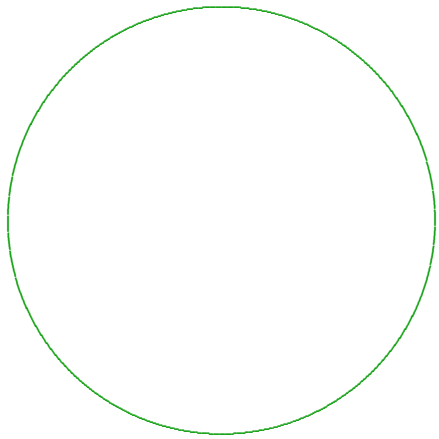


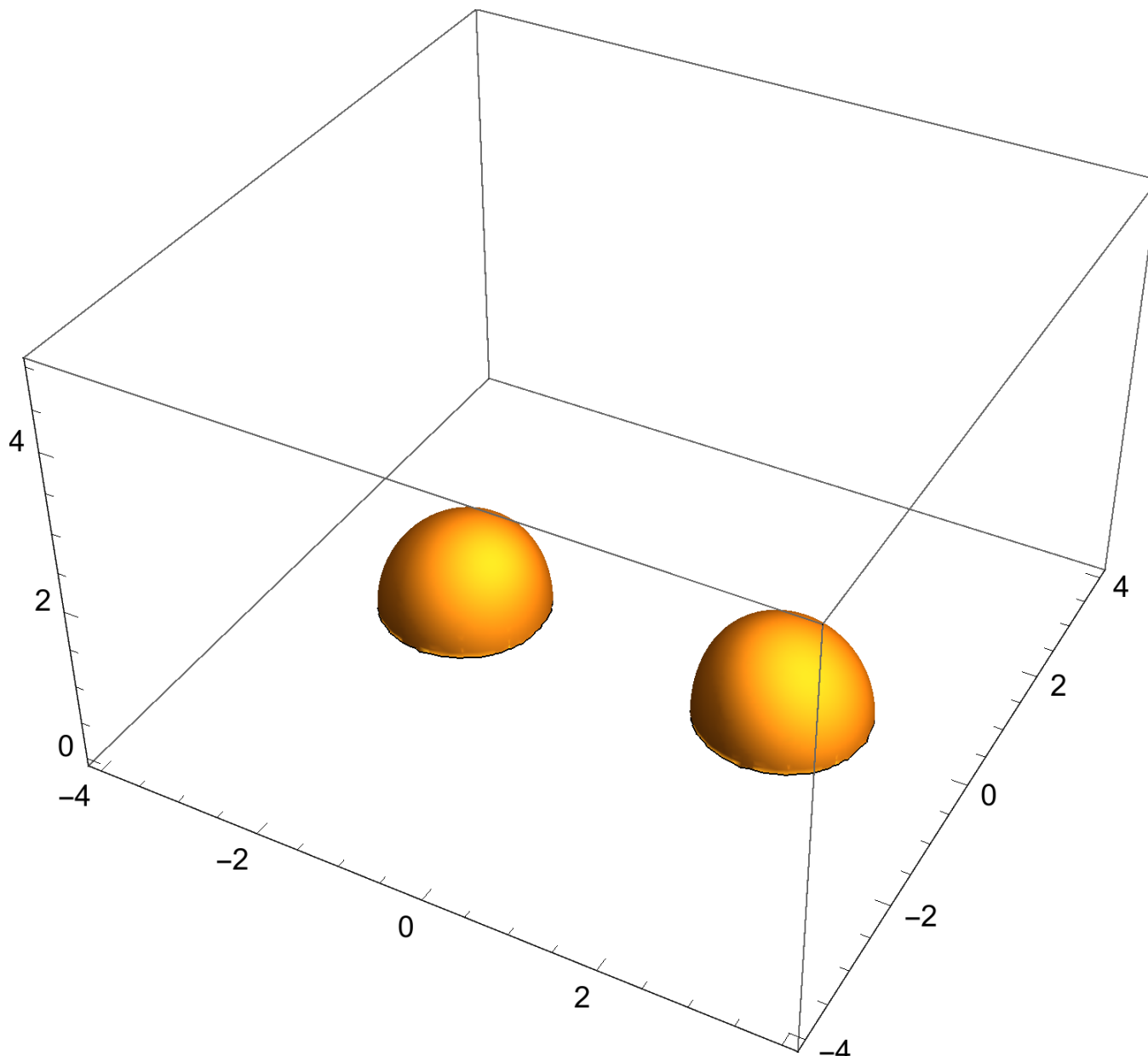


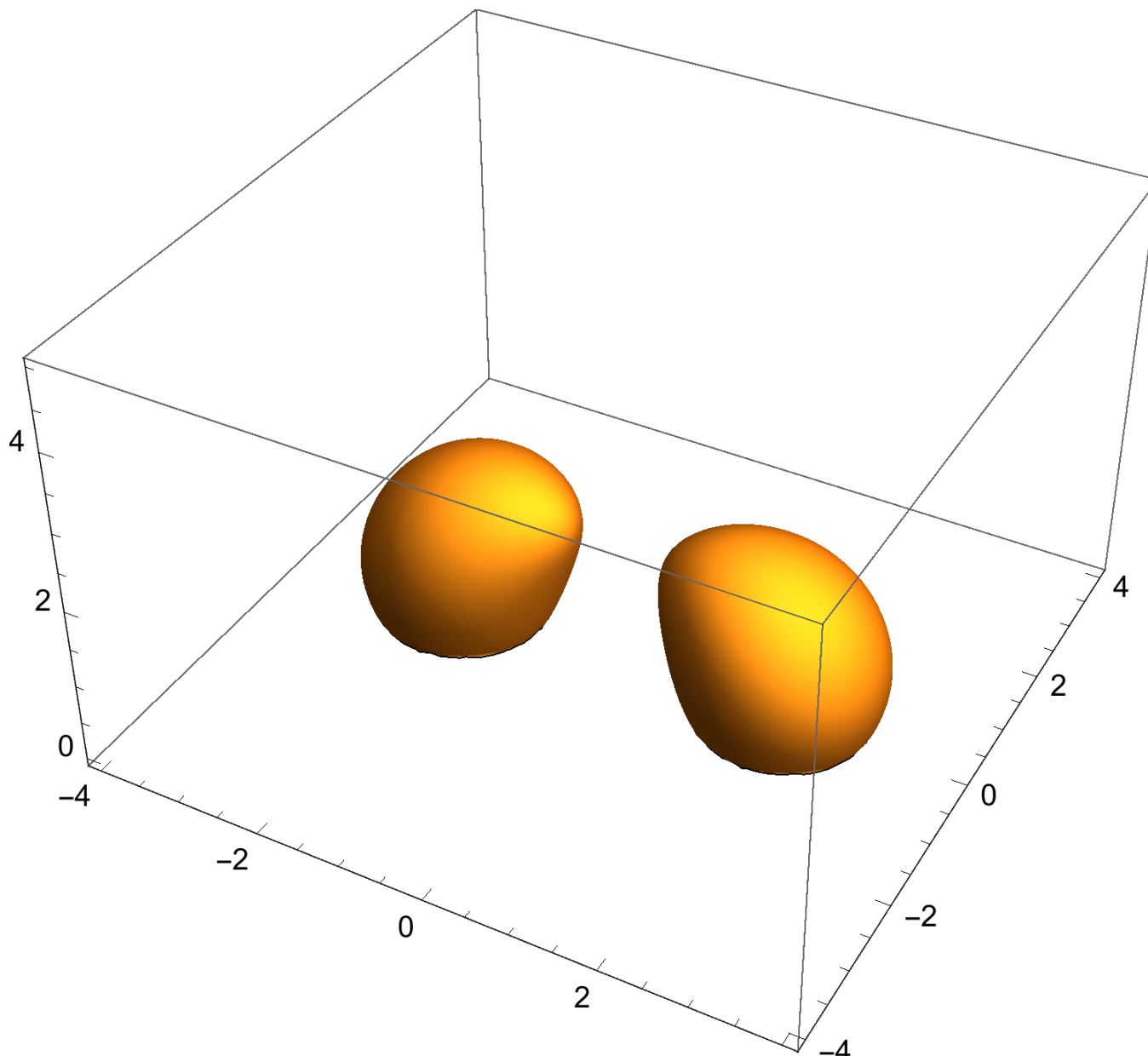


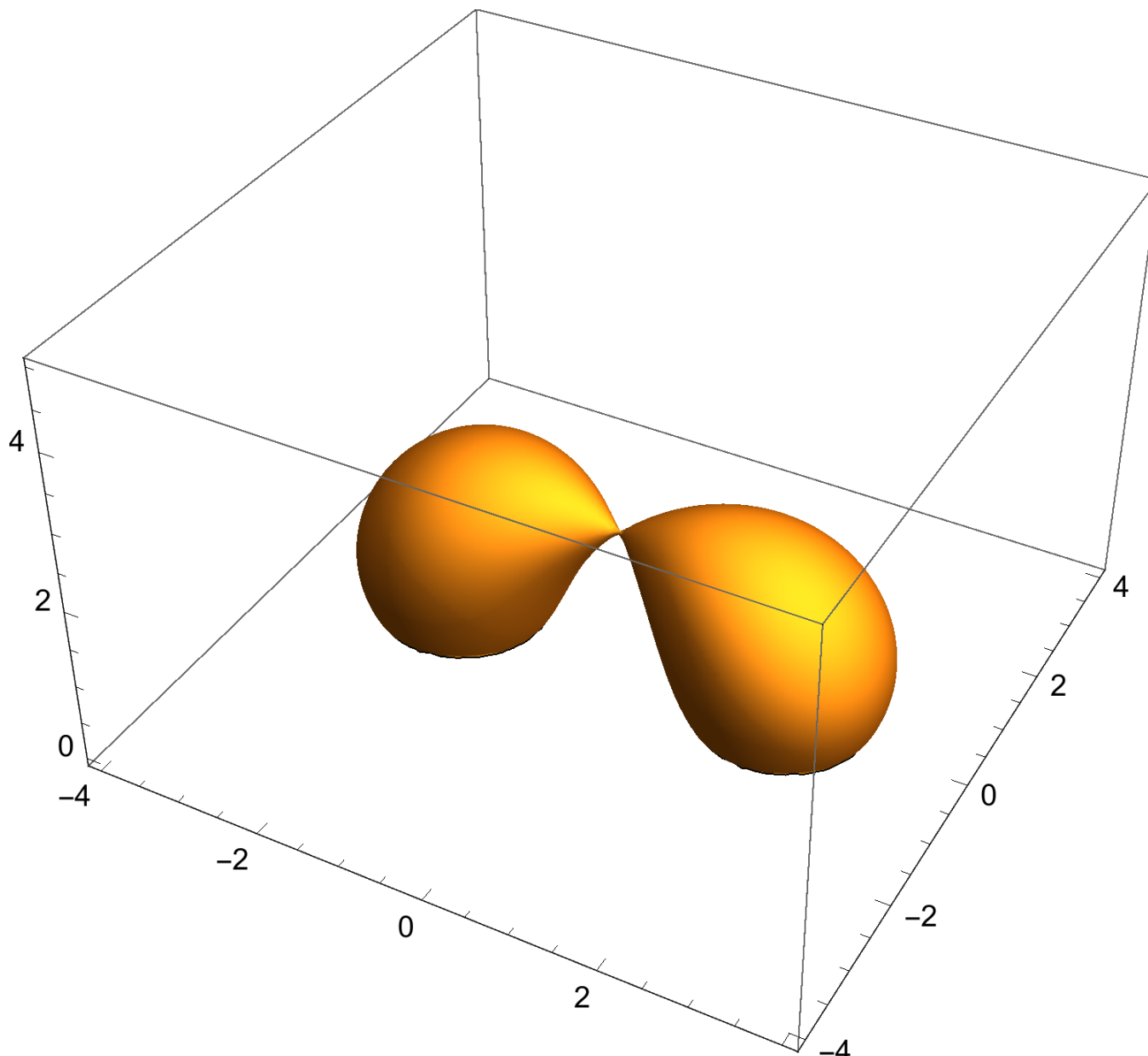


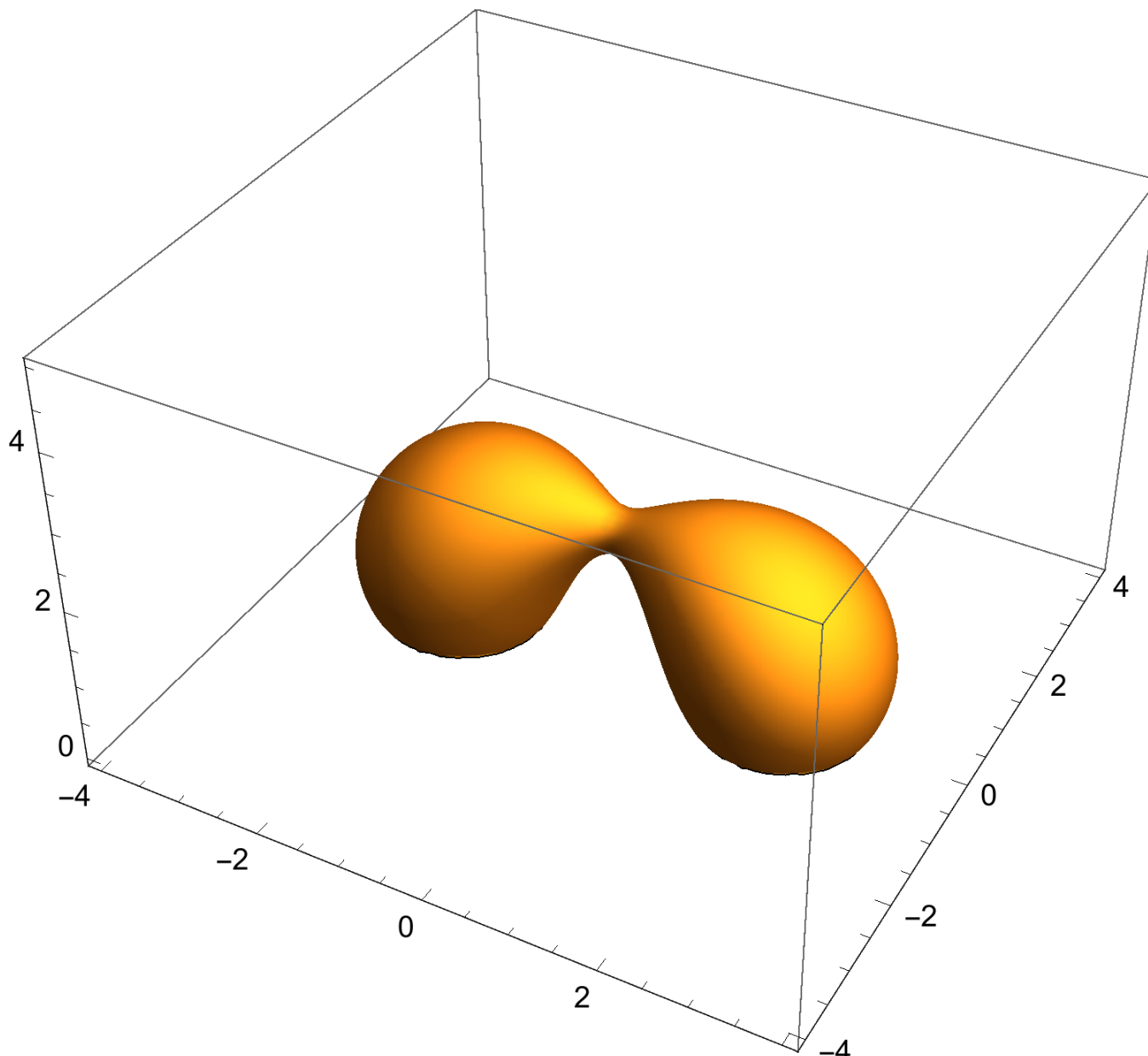


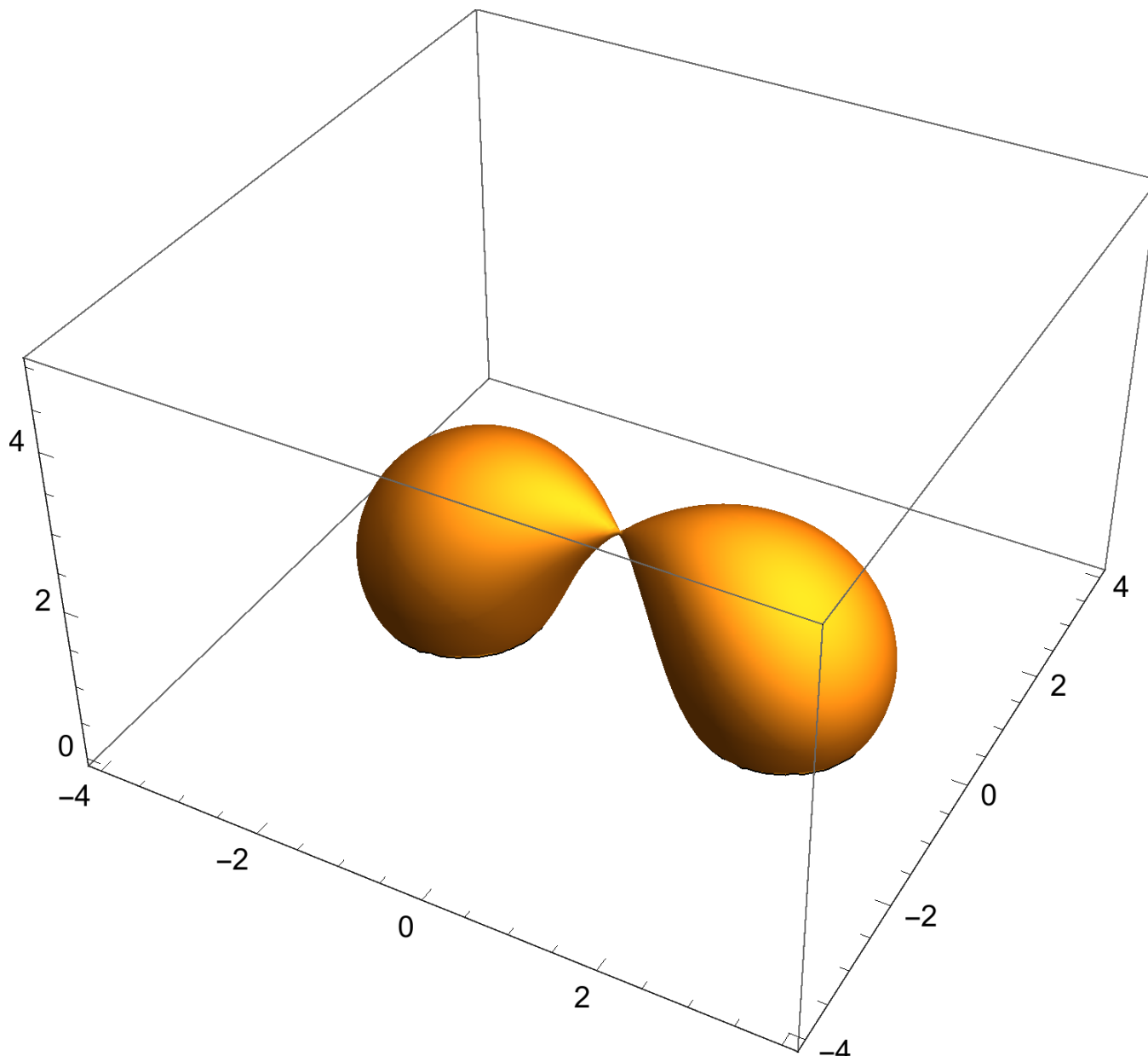


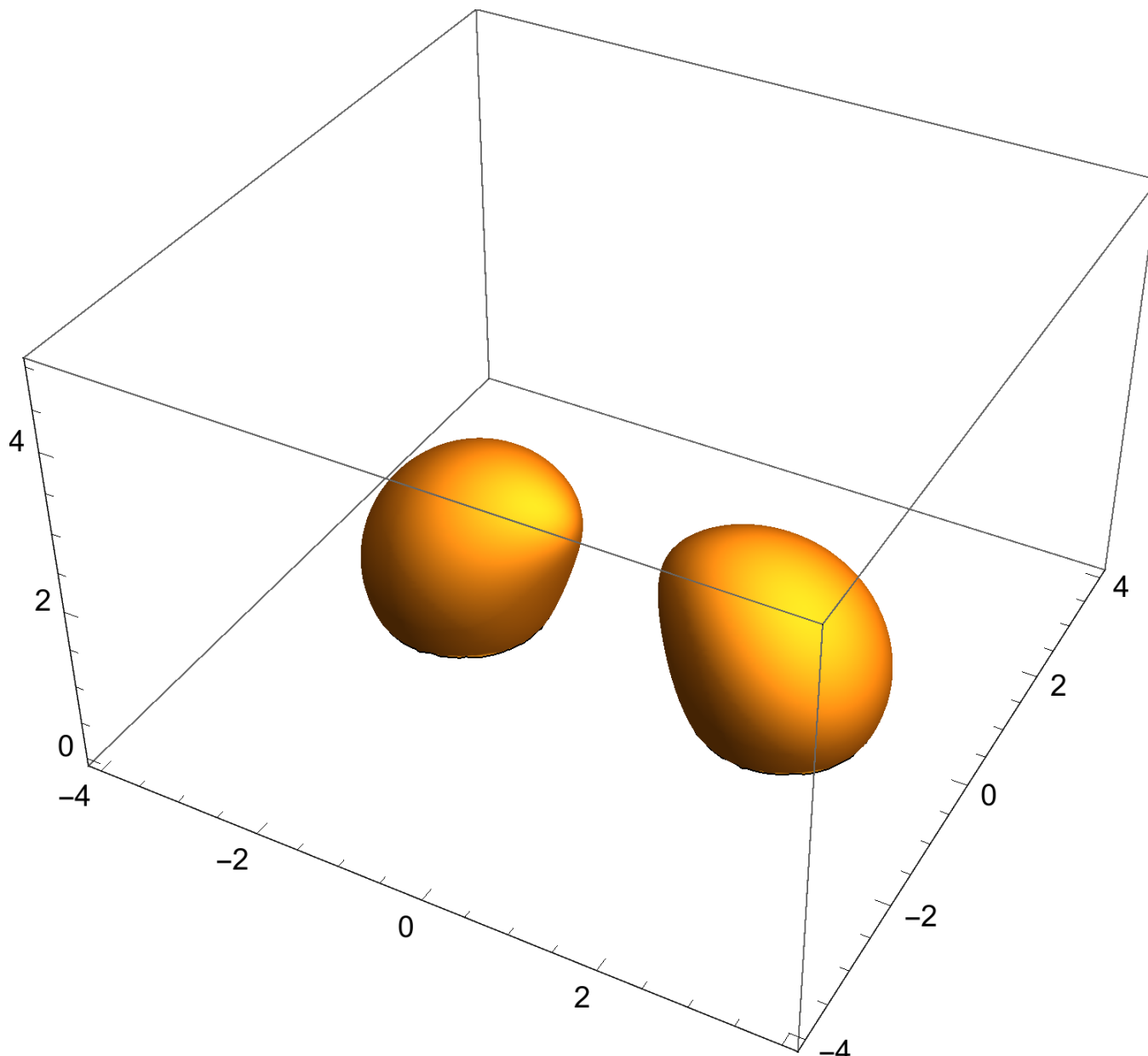


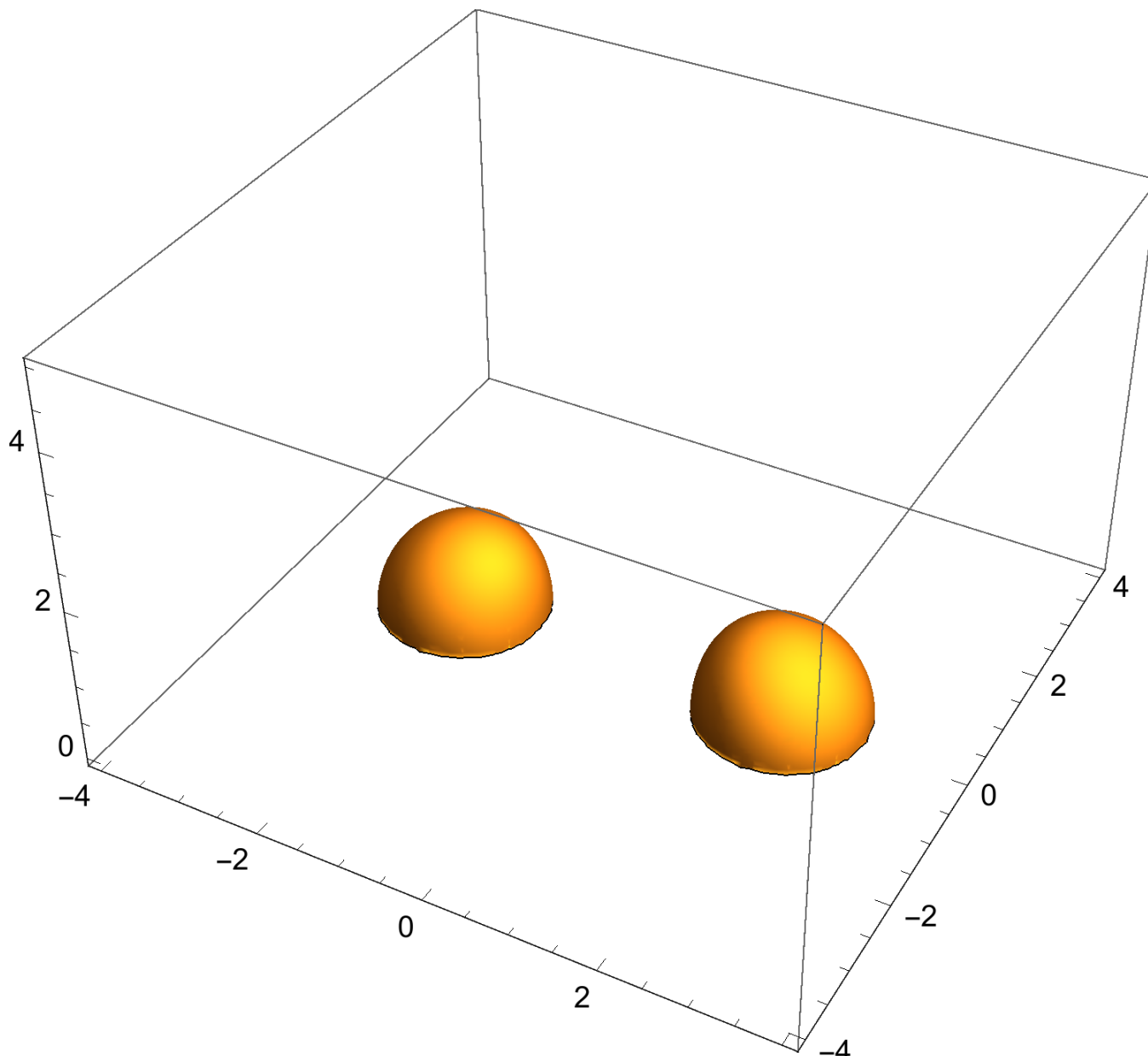




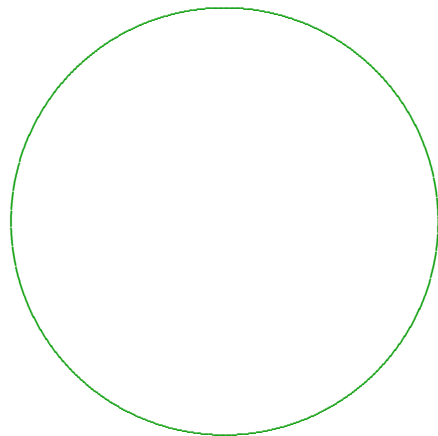
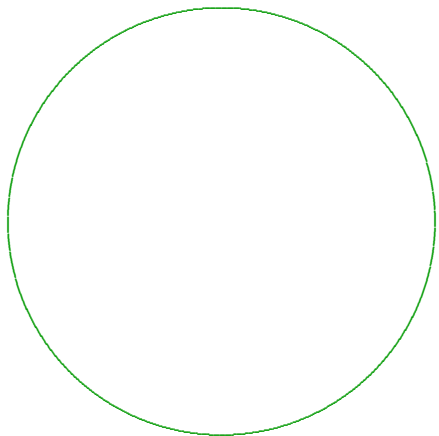


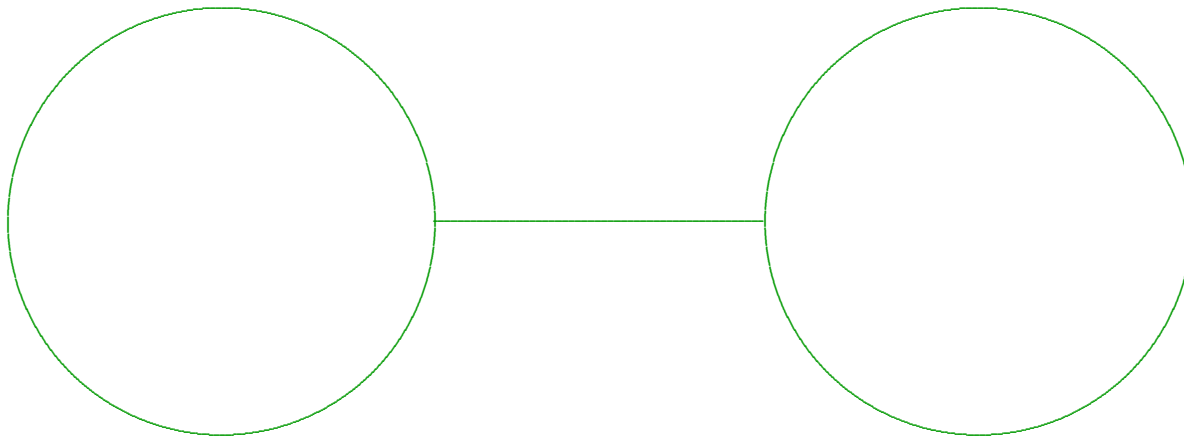


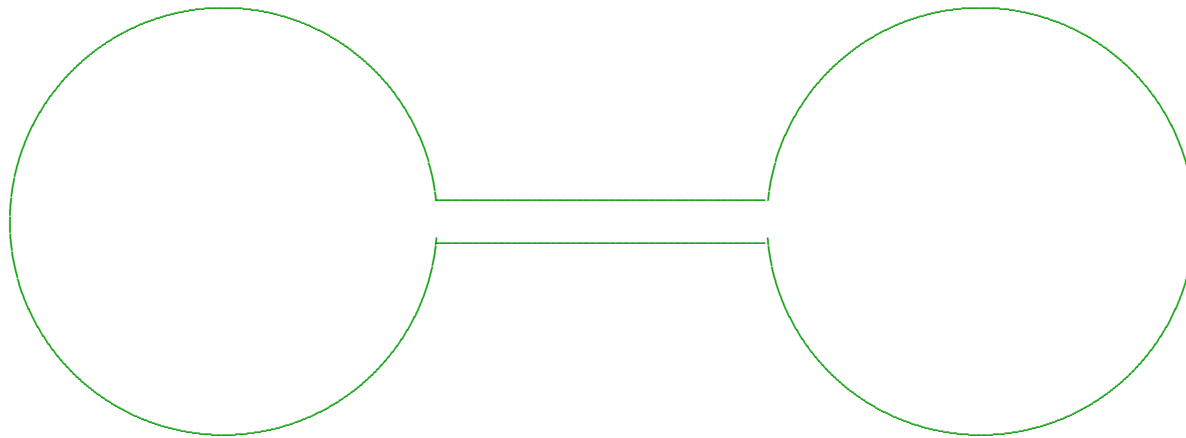


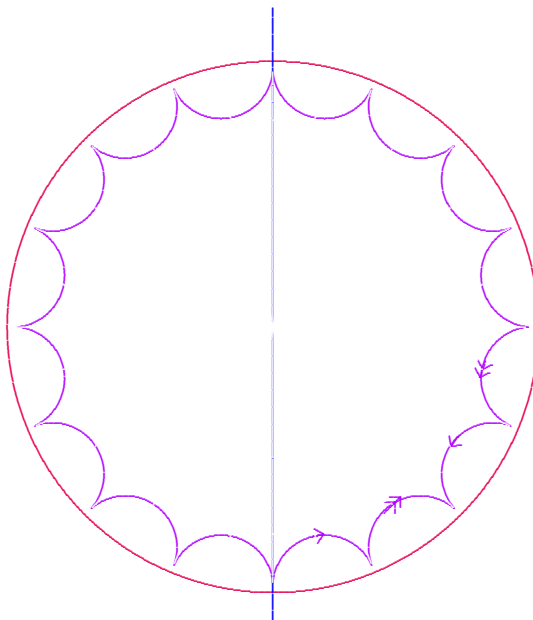
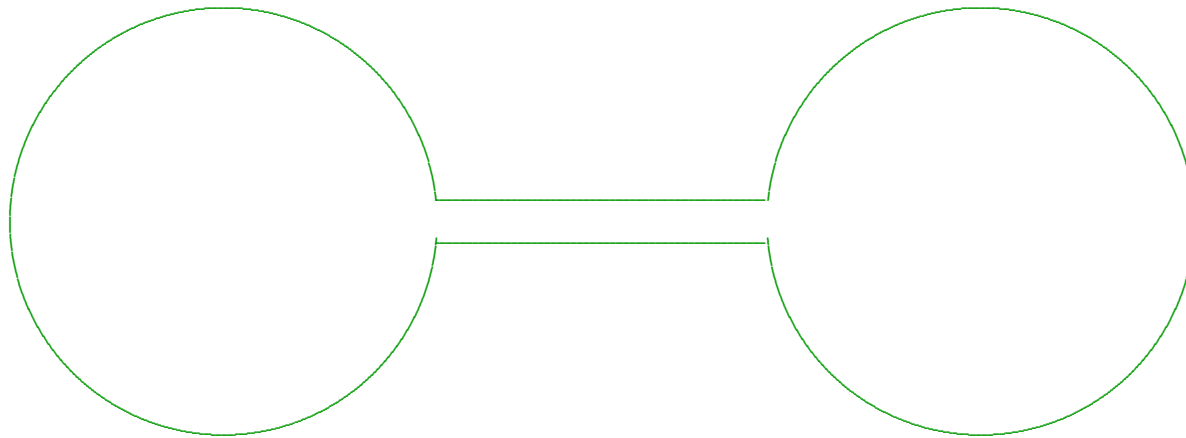


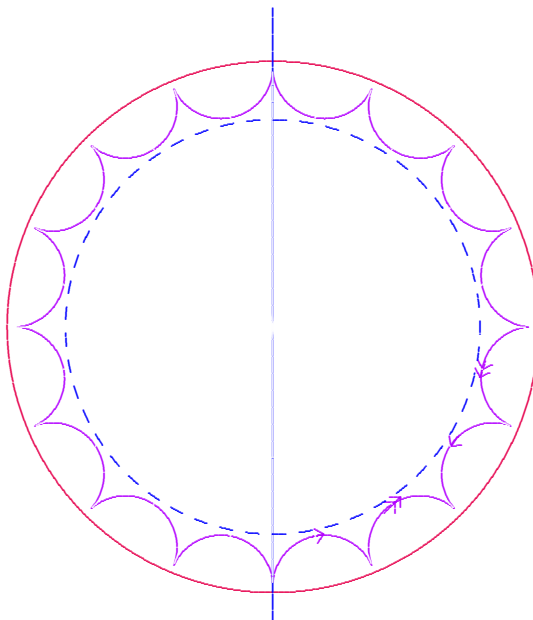
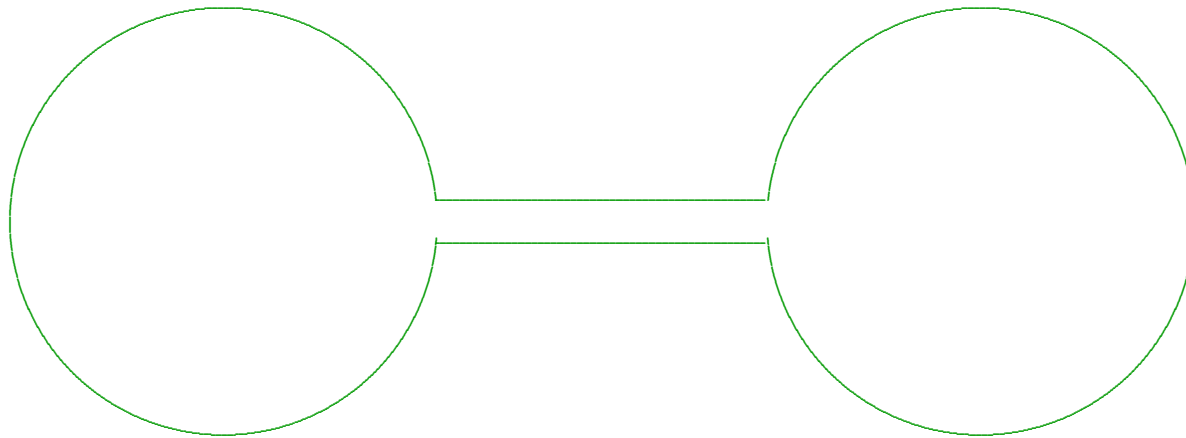


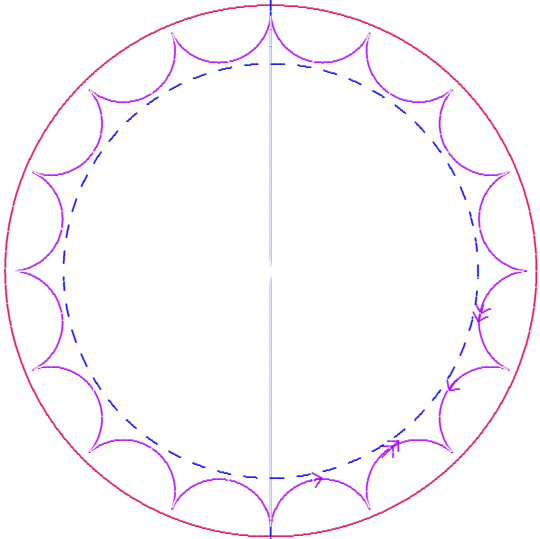
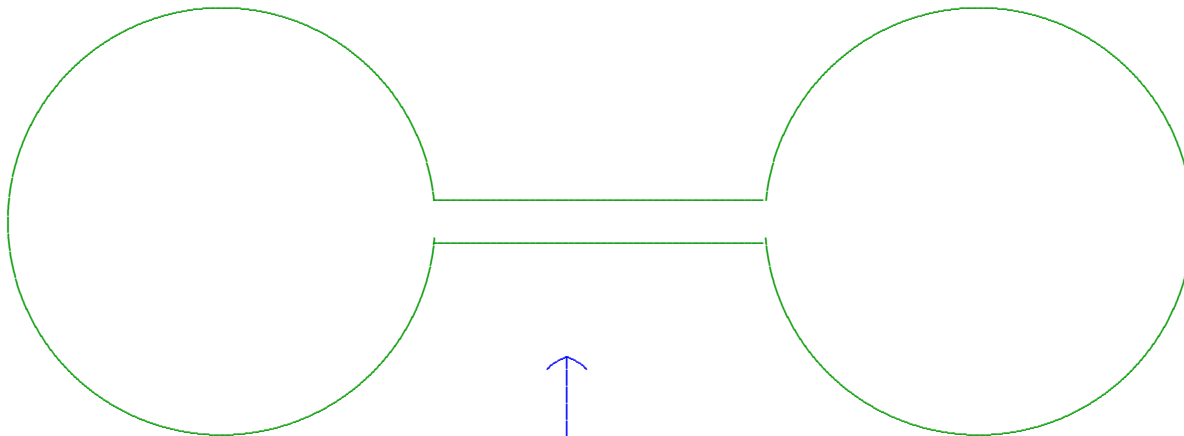


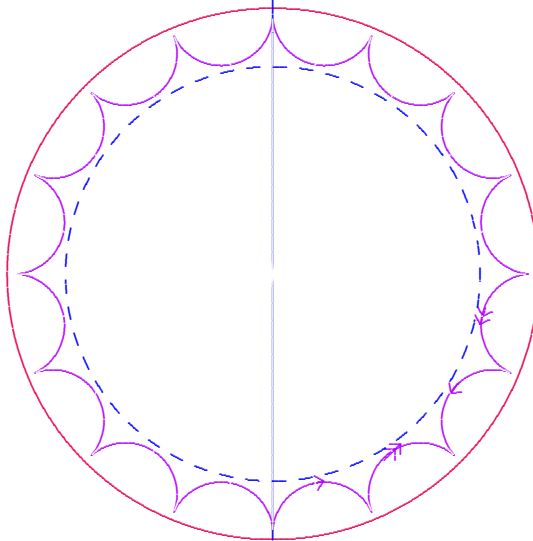
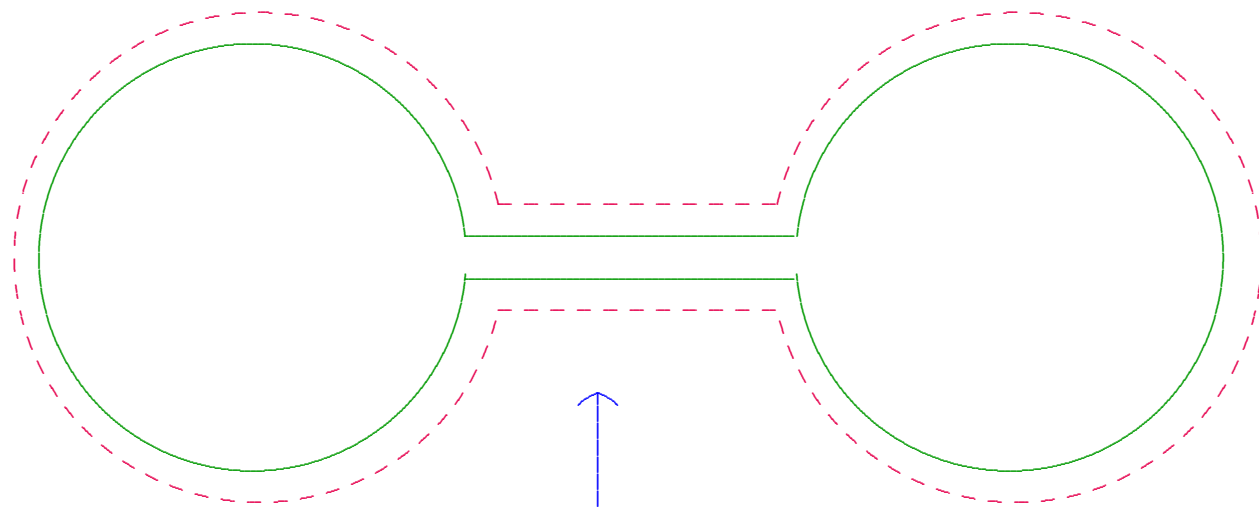


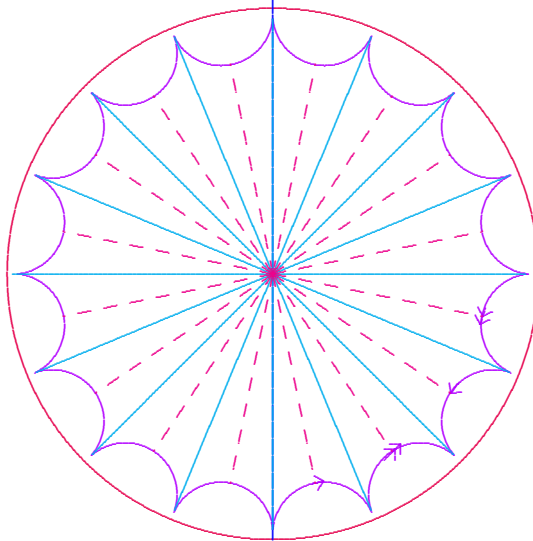
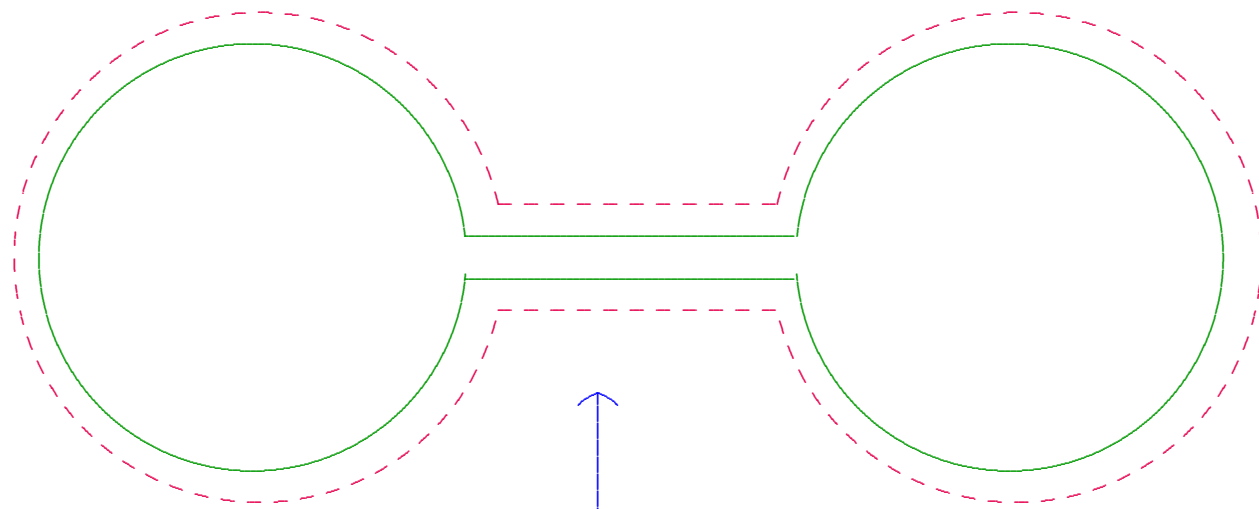




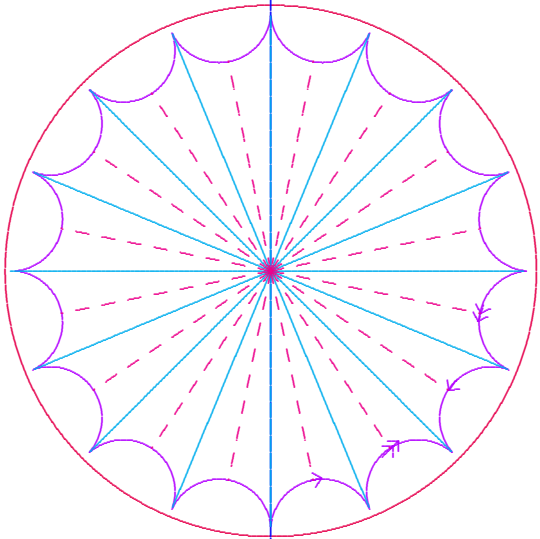
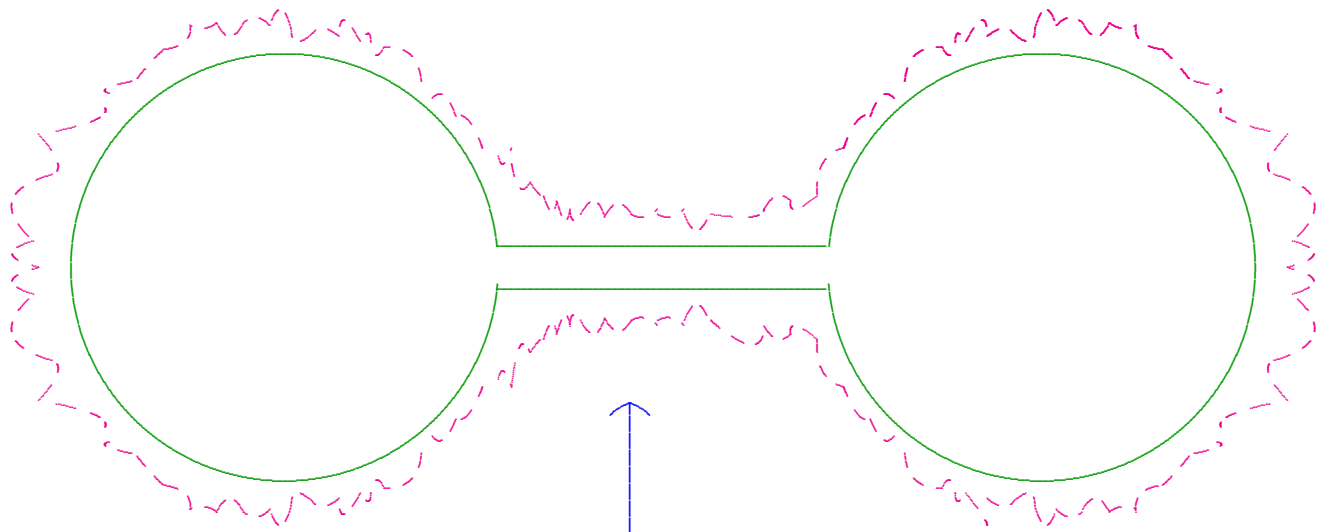








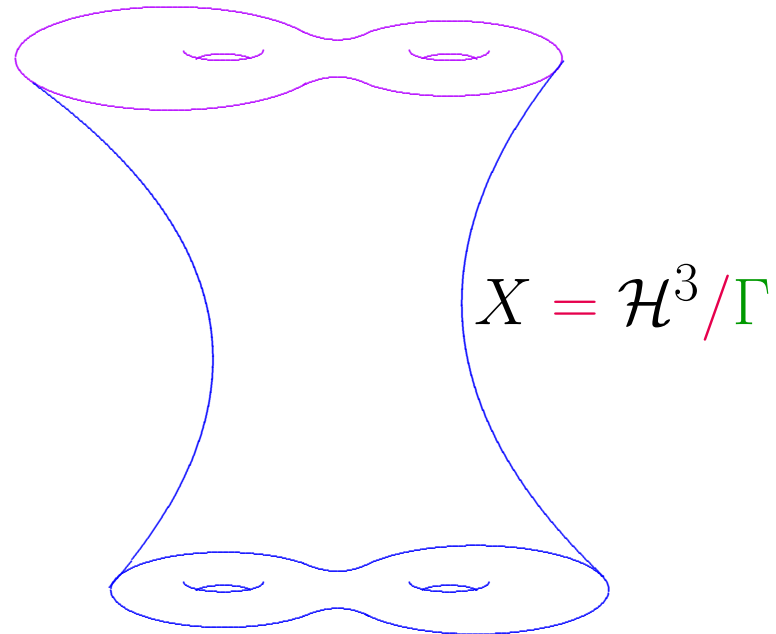




**Theorem A.** Consider 4-manifolds  $M = \Sigma \times S^2$ , where  $\Sigma$  compact Riemann surface of genus  $g$ .

Then  $\forall$  even  $g \gg 0$ ,  $\exists$  family  $[g_t]$ ,  $t \in [0, 1]$ , of locally-conformally-flat classes on  $M$ , such that

- $\exists$  scalar-flat Kähler metric  $g_0 \in [g_0]$ ; but
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Construction of conformally flat 4-manifolds:

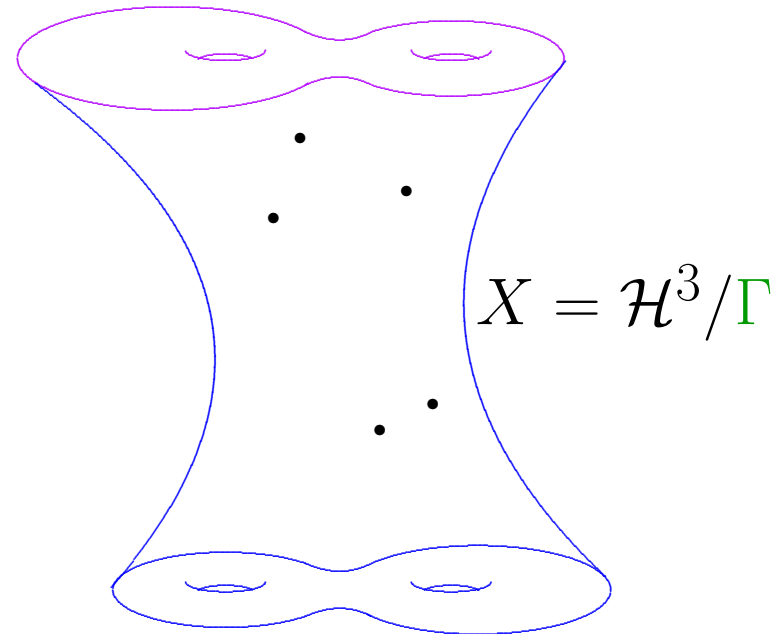
$$M = [\bar{X} \times S^1] / \sim$$

$$g = f(1 - f)[h + dt^2]$$

**Theorem B.** Fix an integer  $k \geq 2$ , and then consider the 4-manifolds  $M = (\Sigma \times S^2) \#^k \overline{\mathbb{C}\mathbb{P}}_2$ , where  $\Sigma$  compact Riemann surface of genus  $g$ .

Then  $\forall$  even  $g \gg 0$ ,  $\exists$  family  $[g_t]$ ,  $t \in [0, 1]$ , of anti-self-dual conformal classes on  $M$ , such that

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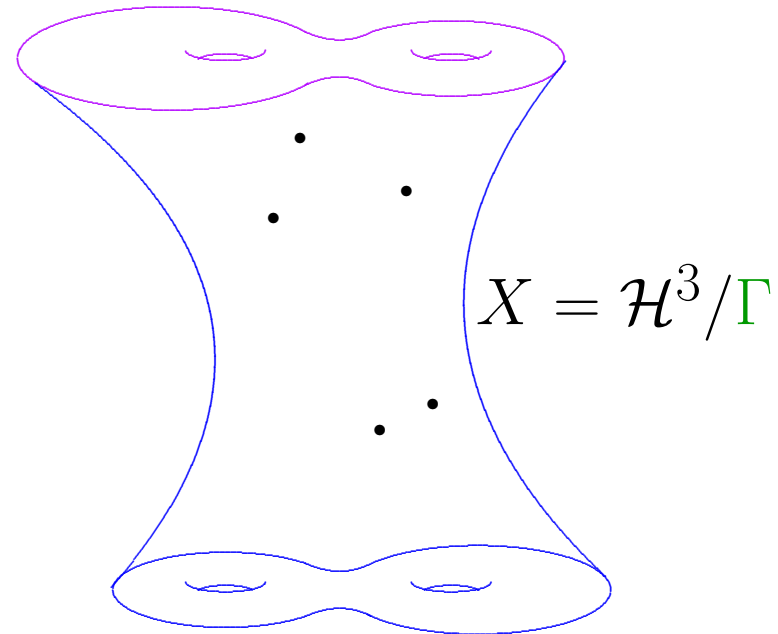
$$V = 1 + \sum_{j=1}^k G_j$$

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Merci aux organisateurs,  
et à l'École Normale Supérieure,  
de m'avoir invité!

