Kodaira Dimension

and the

Yamabe Problem,

Revisited

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Stony Brook University

Recent Advances on Scalar Curvature Problems
Simons Center for Geometry and Physics
Stony Brook, NY, June 28, 2022

Joint work with

Joint work with

Michael Albanese

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Michael Albanese
Université du Québec à Montréal

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to appear in
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and
Perspectives on Scalar Curvature,
Gromov and Lawson, editors.

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New results: non-Kähler case

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$$r = \lambda g$$

for some constant $\lambda \in \mathbb{R}$.

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where $V = \text{Vol}(M, g)$ inserted to make scale-invariant.

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Then restriction $\mathscr{E}|_\gamma$ is bounded below.

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Has s = constant.

Unique up to scale when $s \leq 0$.

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$$= \quad \text{only for round sphere.}$$

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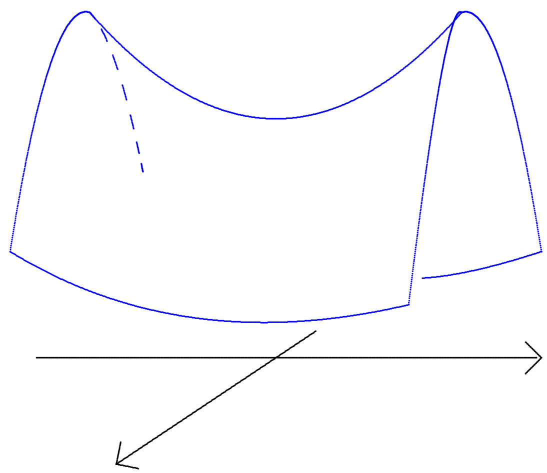

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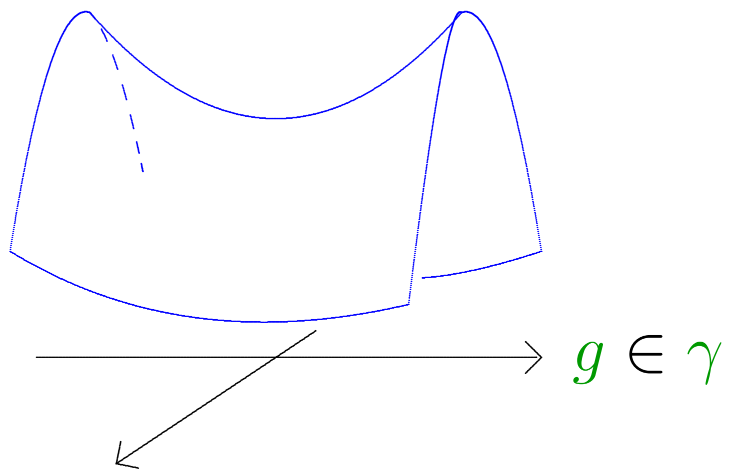

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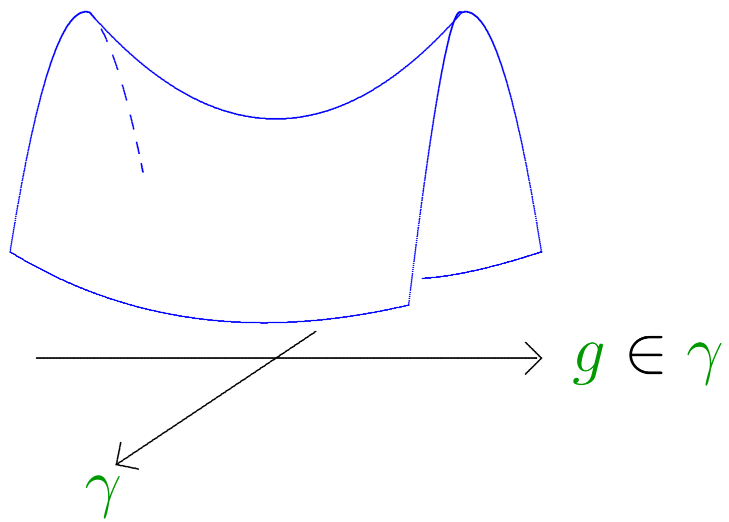

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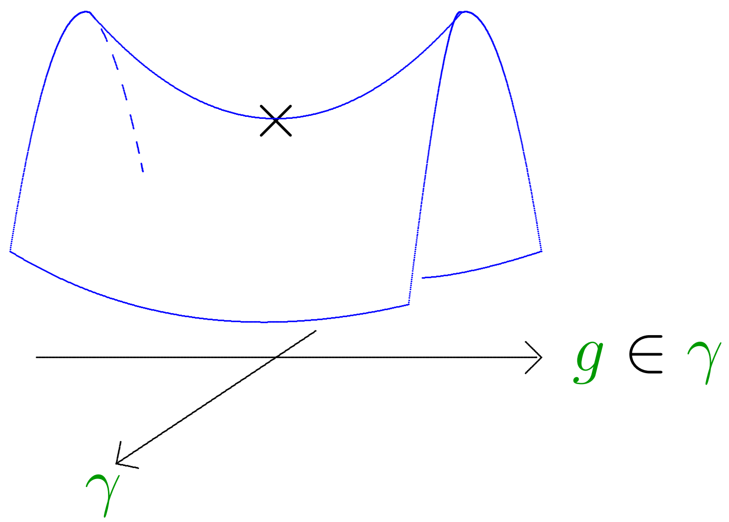

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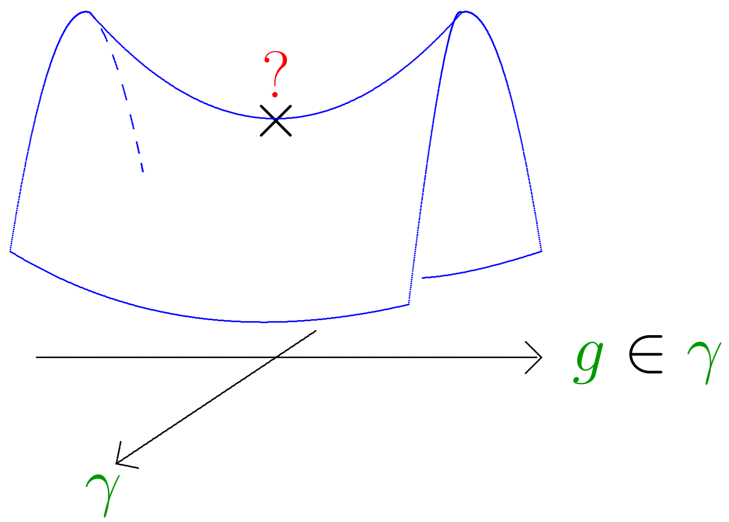

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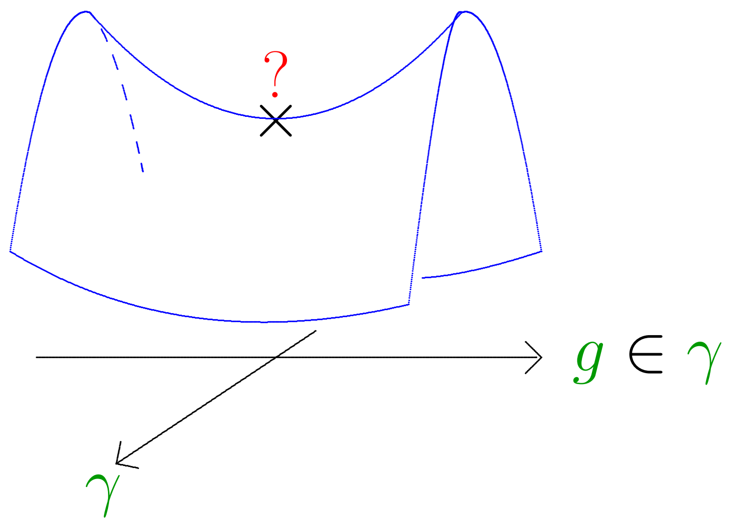

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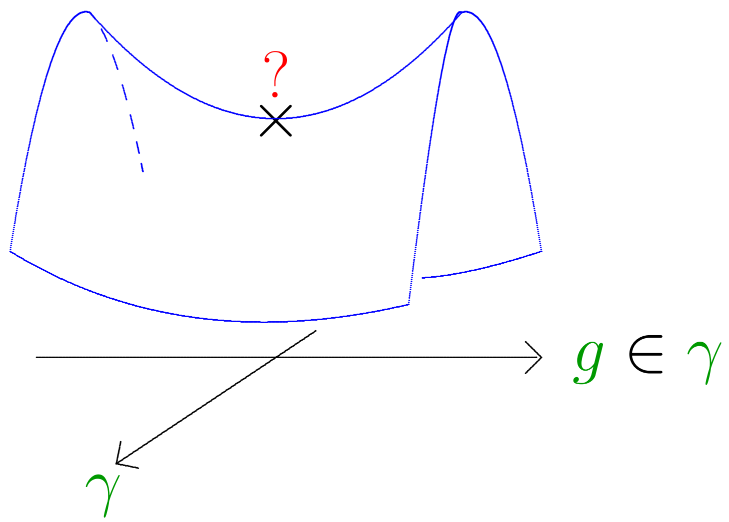

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R. Schoen ('87): "sigma constant"
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Problem. *Compute actual value of $\mathscr{Y}(M)$ for concrete, interesting manifolds.*

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Moreover, can choose M_j such that each $\mathscr{Y}(M_j)$ is realized by an Einstein metric g_j.

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By contrast, in complex dimension $m \geq 3$, Kod is not a diffeomorphism invariant, and has essentially nothing to do with $\mathscr{Y}(M)$.

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Today: what happens when $b_1(M)$ is odd?

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over maps defined by holomorphic sections of $K^{\otimes \ell}$.

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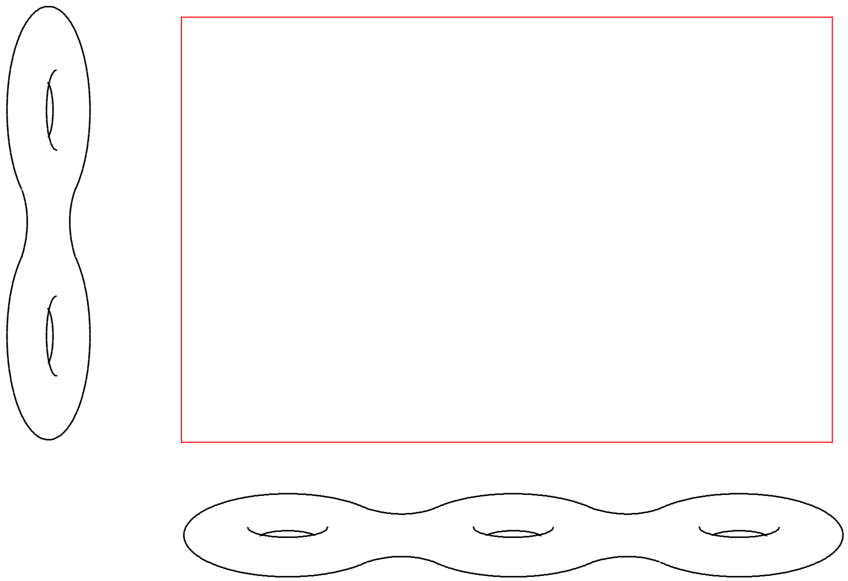

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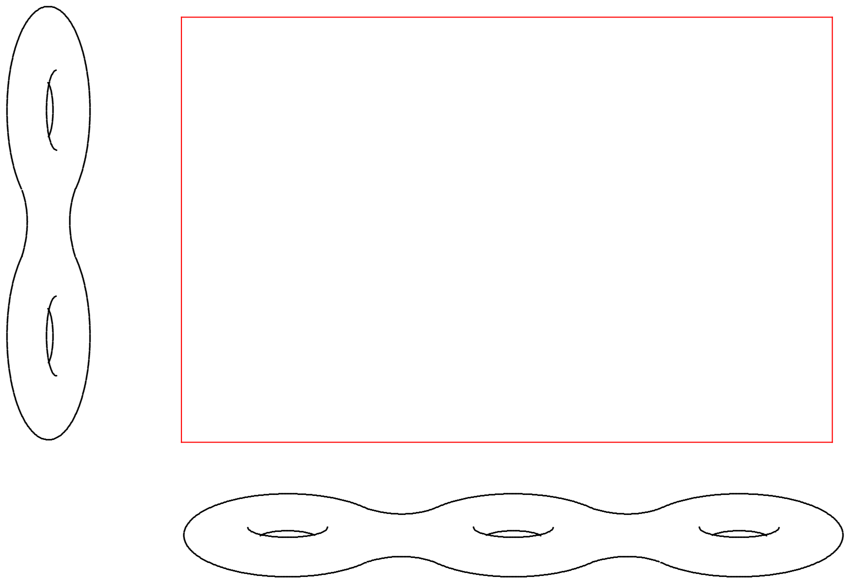
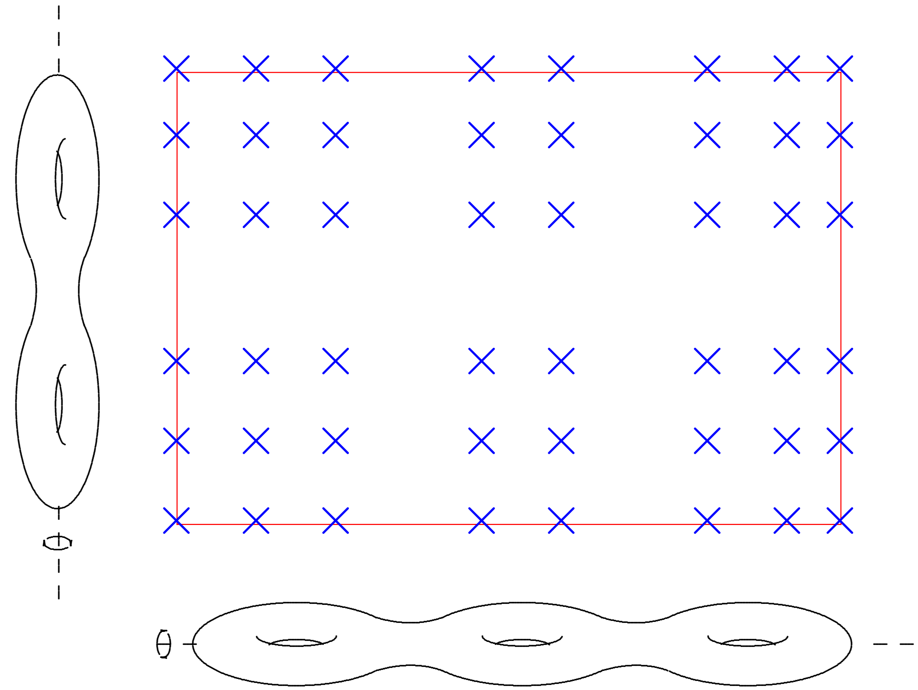


$$\mathrm{Kod}\,(\Sigma_1 \times \Sigma_2) = \mathrm{Kod}\,(\Sigma_1) + \mathrm{Kod}\,(\Sigma_2)$$

Examples. Simply connected examples:



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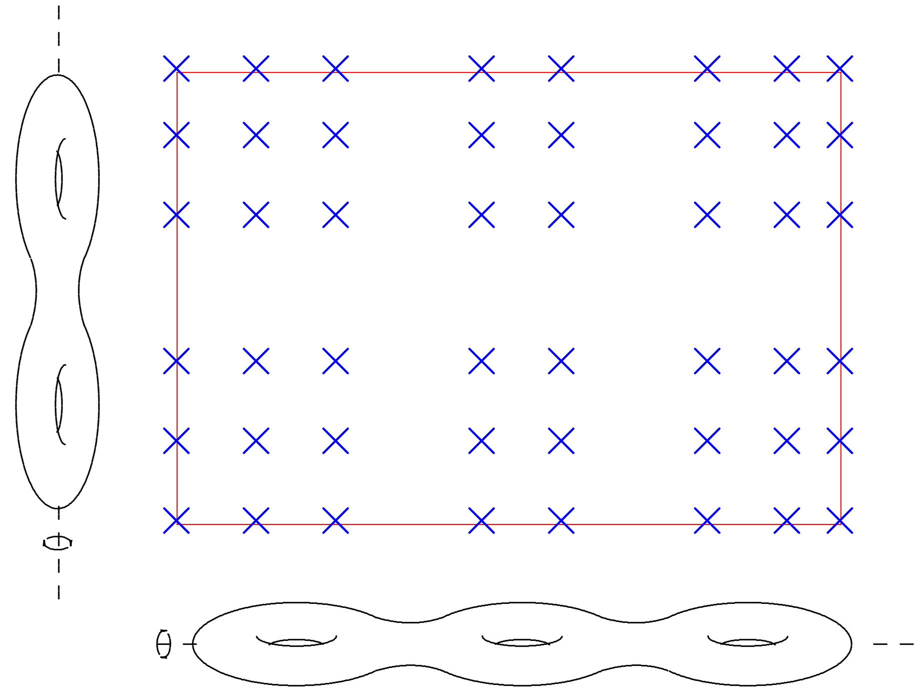

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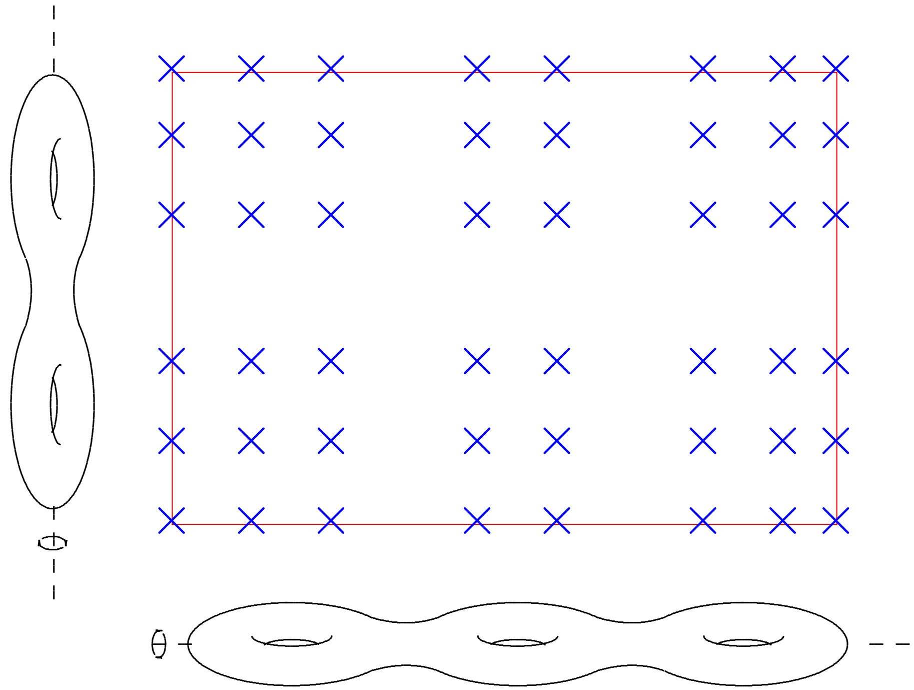

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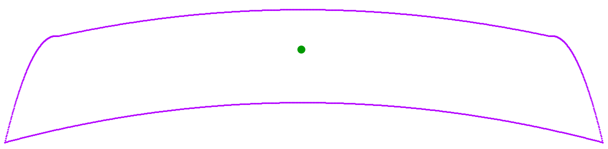

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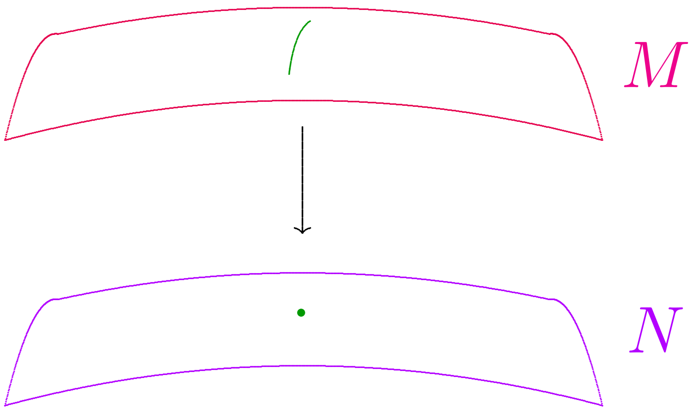

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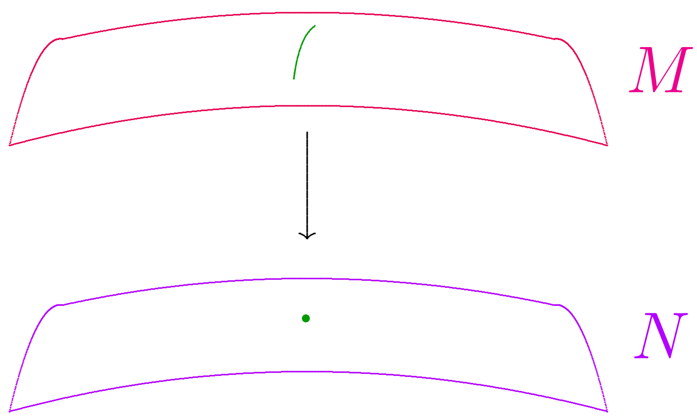

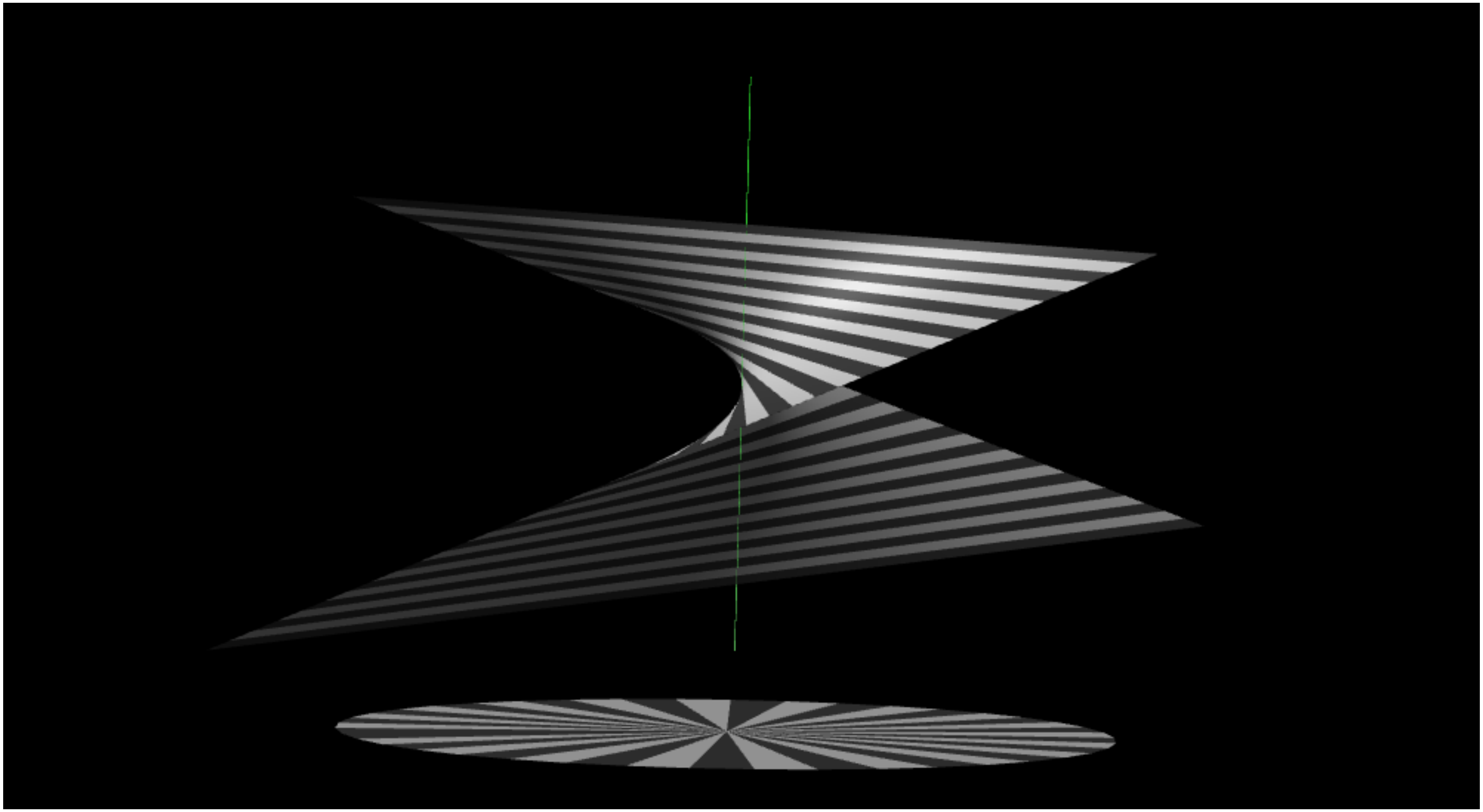

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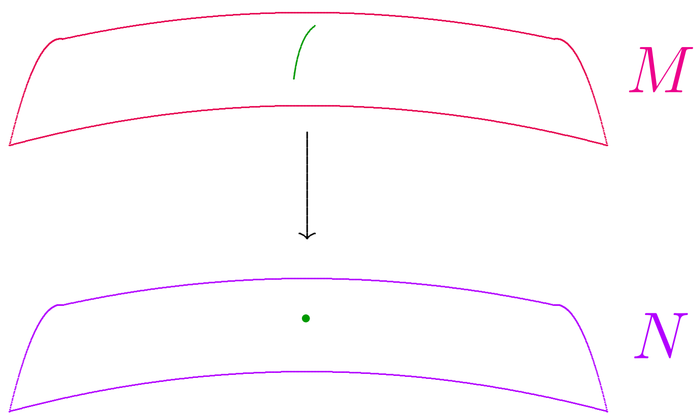

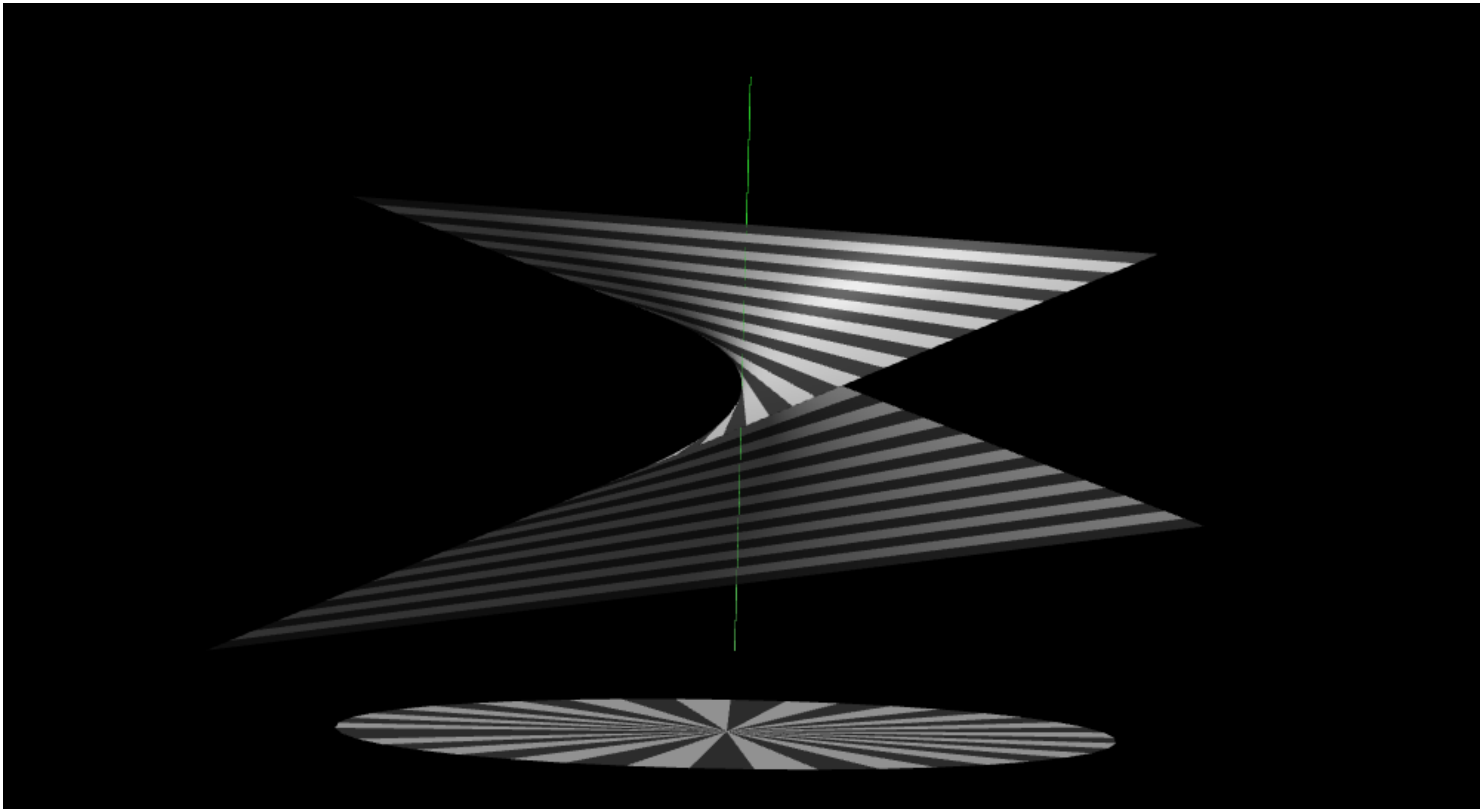

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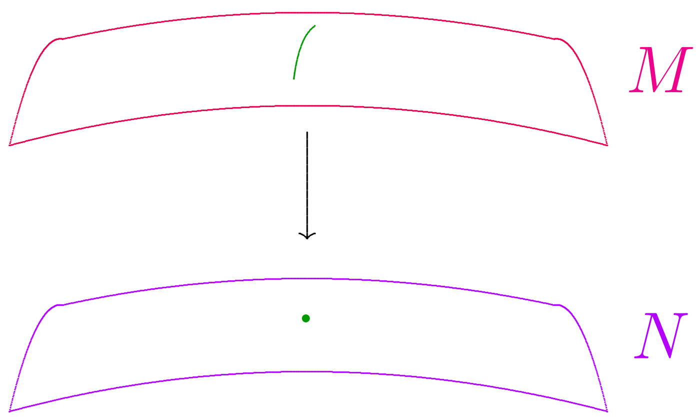

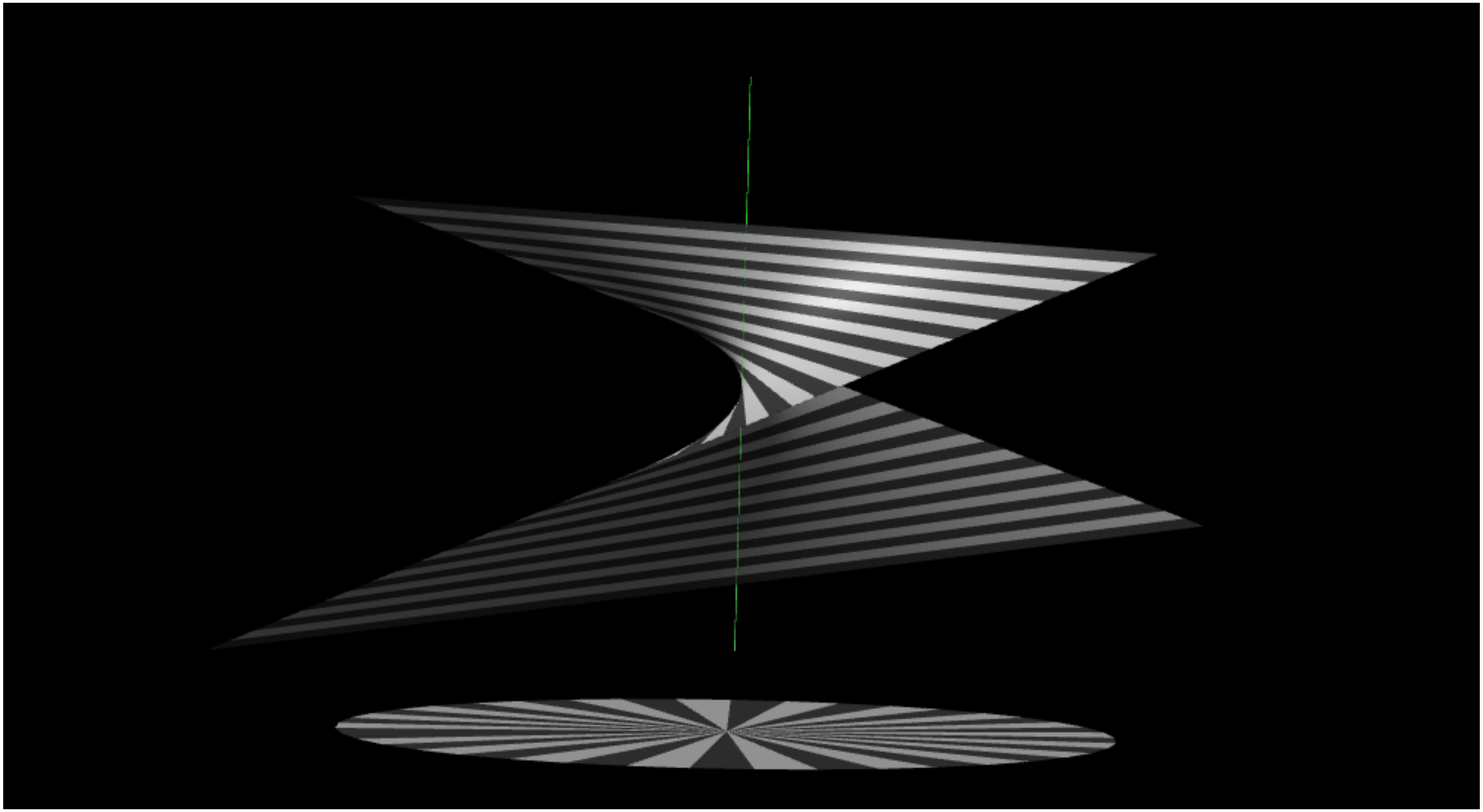

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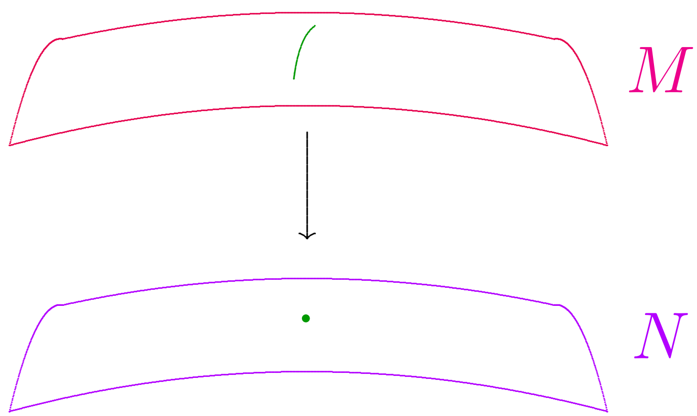

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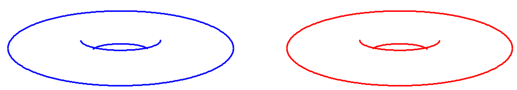

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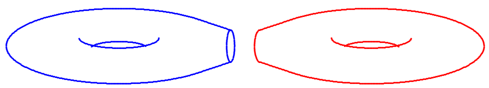

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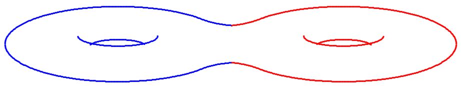

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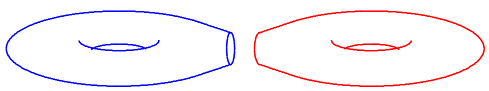

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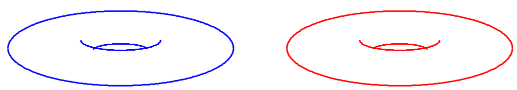

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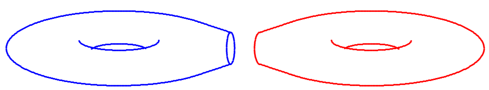

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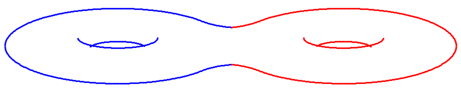

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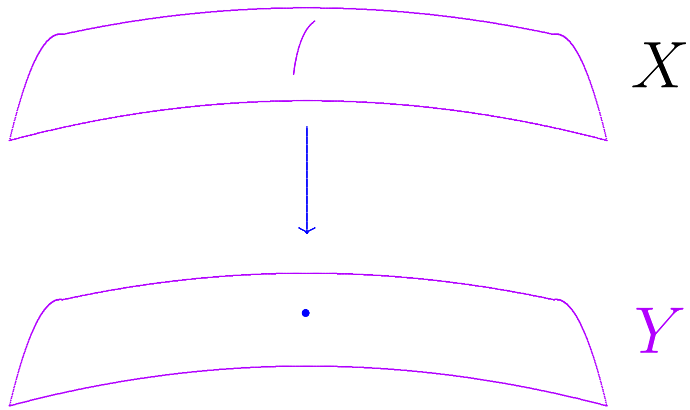

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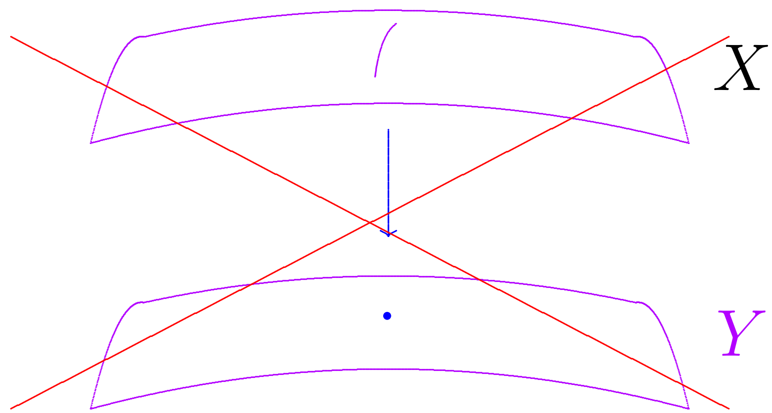

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"Fibration" allows singular fibers, so not fiber-bundle.

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$$\mathscr{Y}(M) > 0 \iff Kod(M, J) = -\infty,$$
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In fact, if X admits K-E metric, achieves $\mathscr{Y}(X)$.

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We'll see that this isn't so when Kod $= -\infty$!

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Missing piece:
Prove $\mathscr{Y}(M) \leq 0$ when Kod $= 1$ and b_1 is odd.

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$$\begin{aligned} \mathbb{V}_+ &= \Lambda^{0,0} \oplus \Lambda^{0,2} \\ \mathbb{V}_- &= \Lambda^{0,1} \end{aligned}$$

can formally be written as

$$\mathbb{V}_\pm = \mathbb{S}_\pm \otimes L^{1/2},$$

where $\mathbb{S}_\pm$ are left & right-handed spinor bundles.

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where $F_\theta{}^+ =$ self-dual part curvature of θ, and $\sigma : \mathbb{V}_+ \to \Lambda^+$ is a natural real-quadratic map,

$$|\sigma(\Phi)| = \frac{1}{2\sqrt{2}}|\Phi|^2.$$

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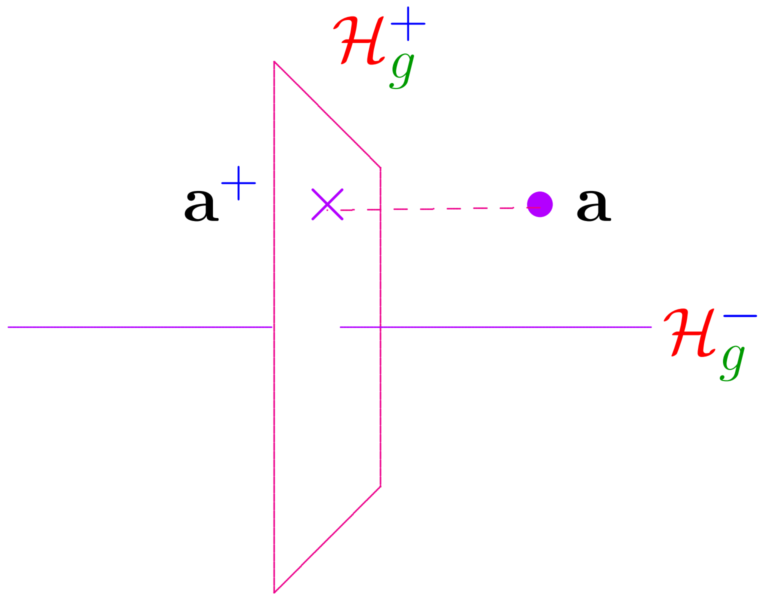

$$H^2(M, \mathbb{R})$$

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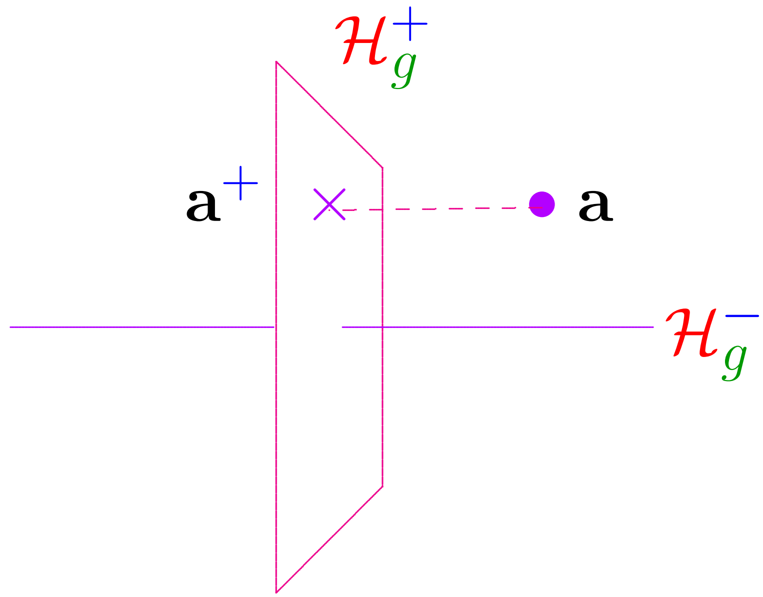

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However, with only a modicum of extra work, his method proves the existence of the following...

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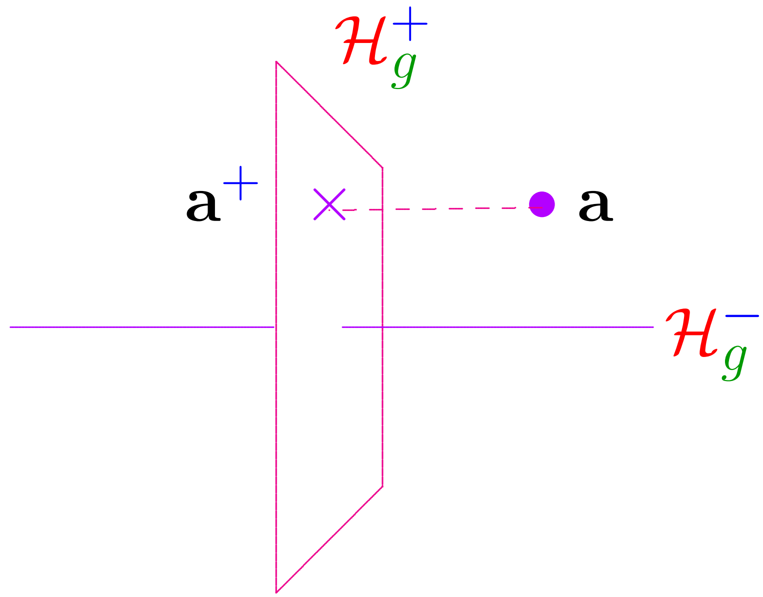

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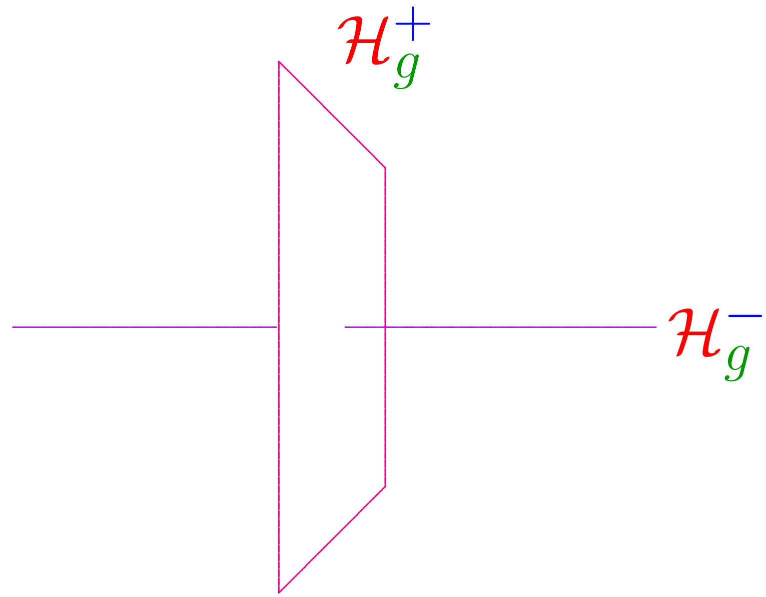

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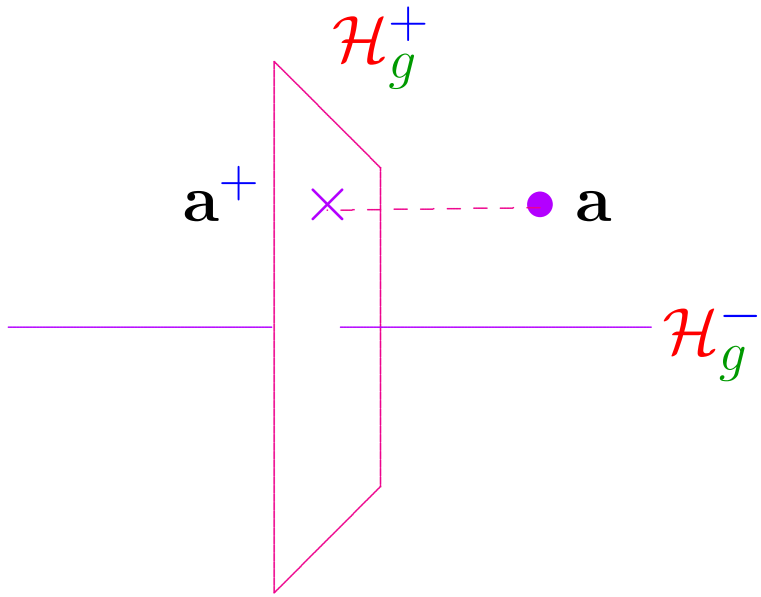

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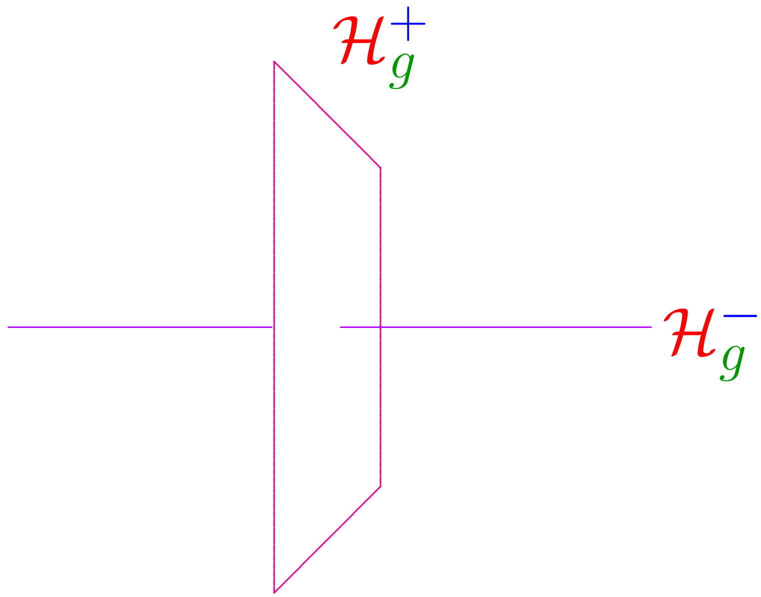

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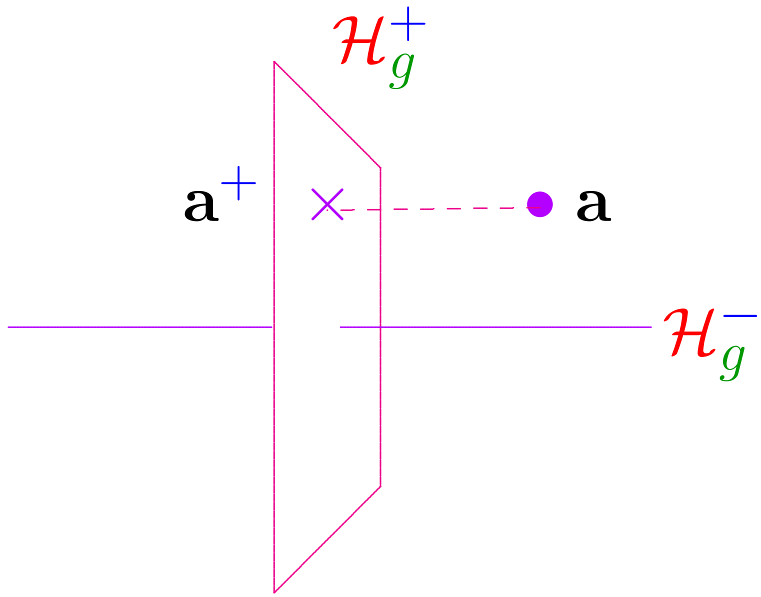

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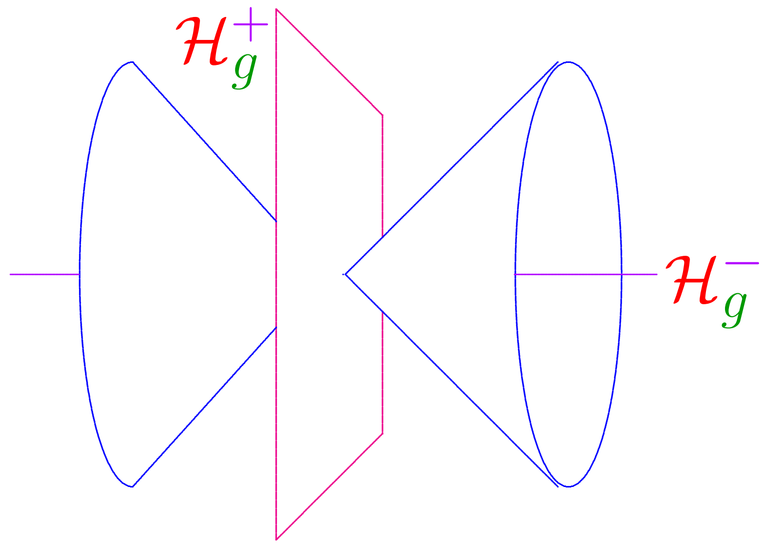

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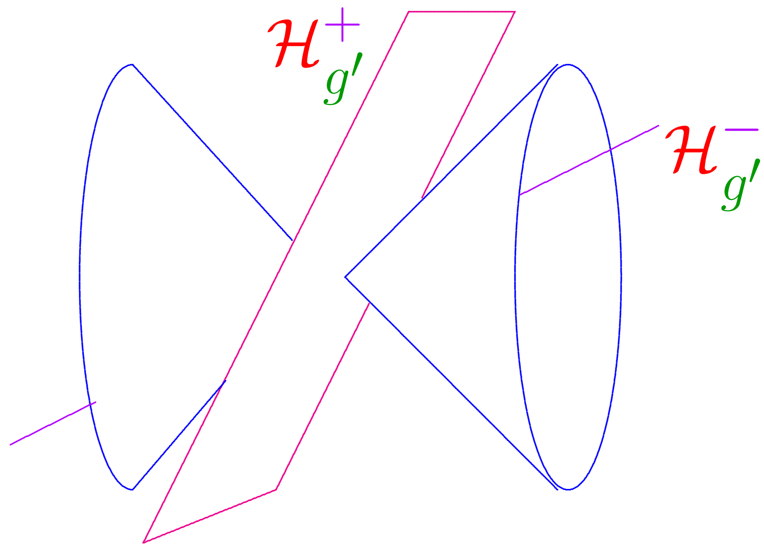

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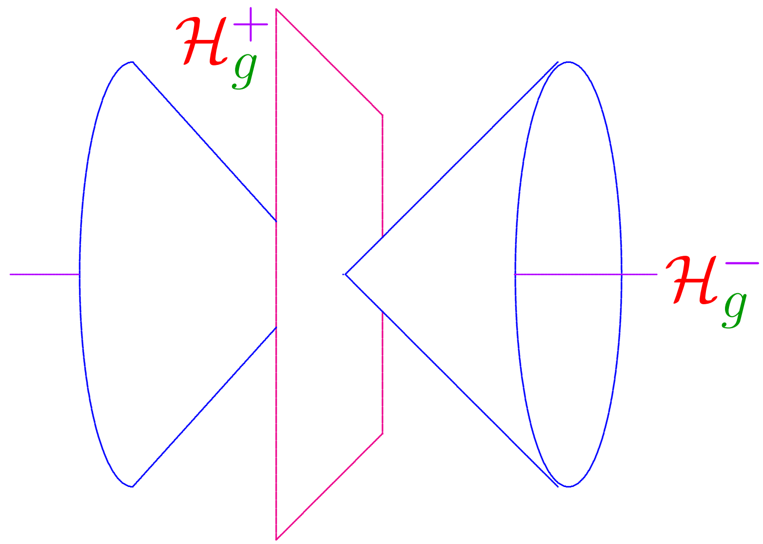

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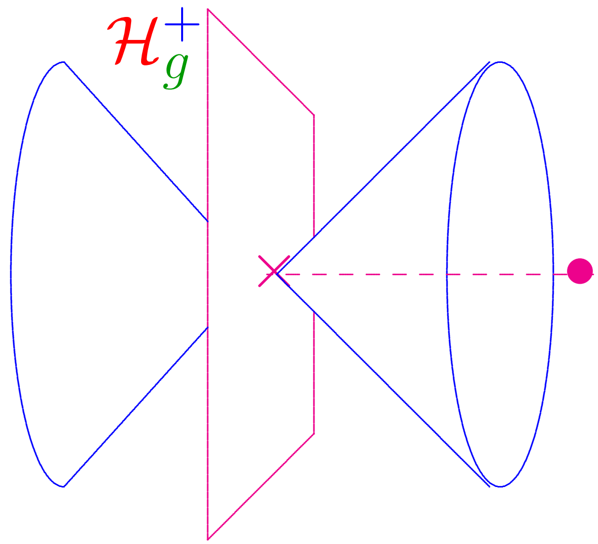

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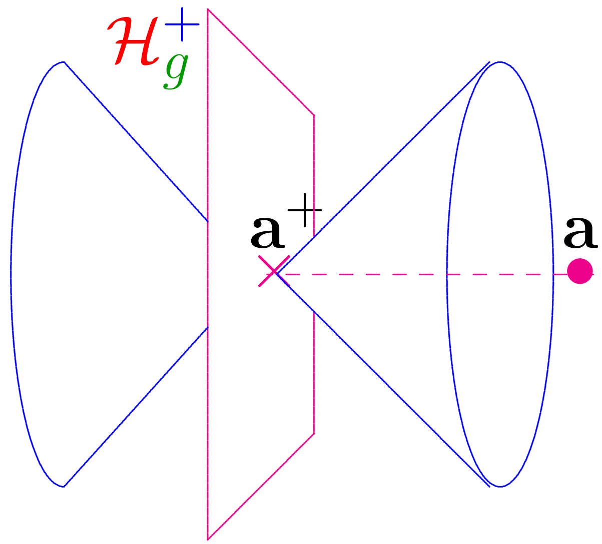

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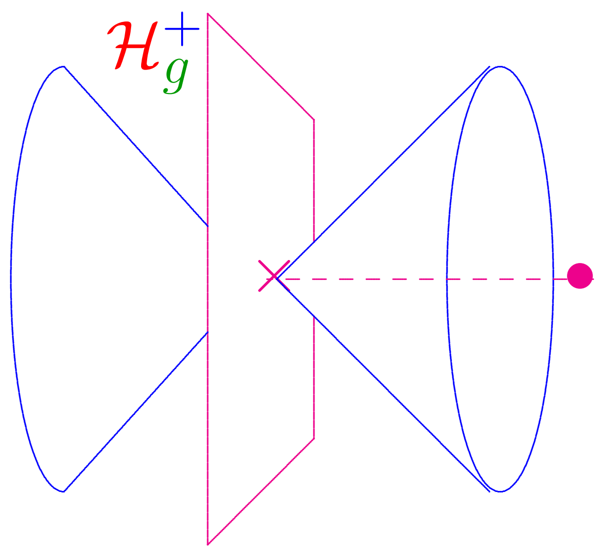

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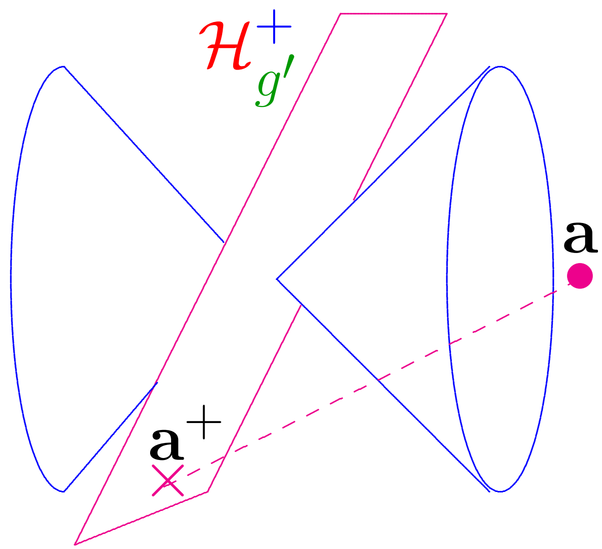

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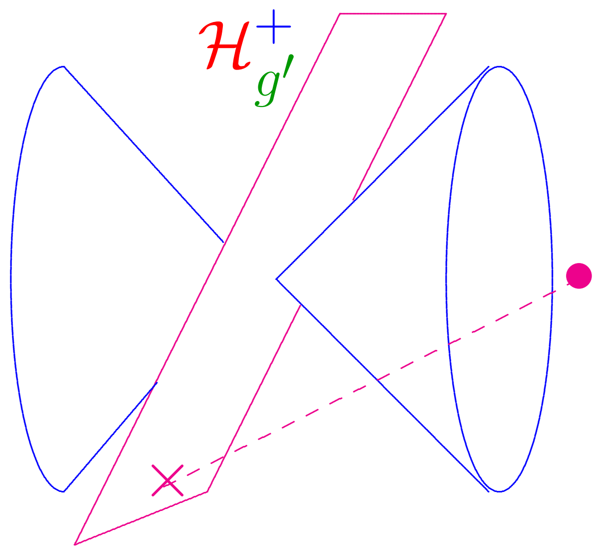

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$\mathscr{Y} > 0$ preserved under connected sums $(n \geq 3)$.

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On $M = X \# k\overline{\mathbb{CP}}_2$, mock-monopole $\mathbf{a} \in H^2(M, \mathbb{Z})/\text{torsion}$ must be non-zero, because pairing with Poincaré dual of the generator of $H_2(\overline{\mathbb{CP}}_2, \mathbb{Z})$ must be odd.

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$\mathscr{Y}(X) > 0 \Longrightarrow \mathscr{Y}(M) > 0$.

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Key Point: Brinzănescu '94 $\Longrightarrow$ minimal model X has unbranched covers diffeomorphic to $N \times S^1$, where $N \to \Sigma$ Chern-class-1 circle bundle over Σ of genus ≥ 2.

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Kronheimer constructs approximate solutions of the SW equations on a sequence of high-degree covers $\widetilde{M} \to M$, with error term uniformly bounded as the degree of the cover $\to +\infty$.

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Built from exact solutions on $(N \times \mathbb{R}) \# mk\overline{\mathbb{CP}}_2$, considered as a Riemannian manifold with conical ends and periodic interior geometry.

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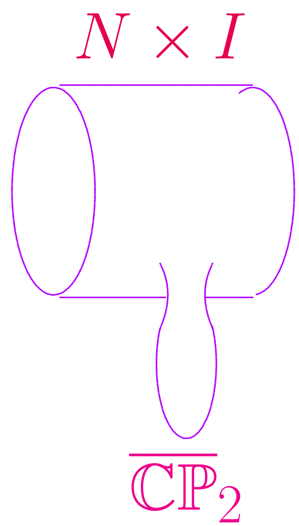

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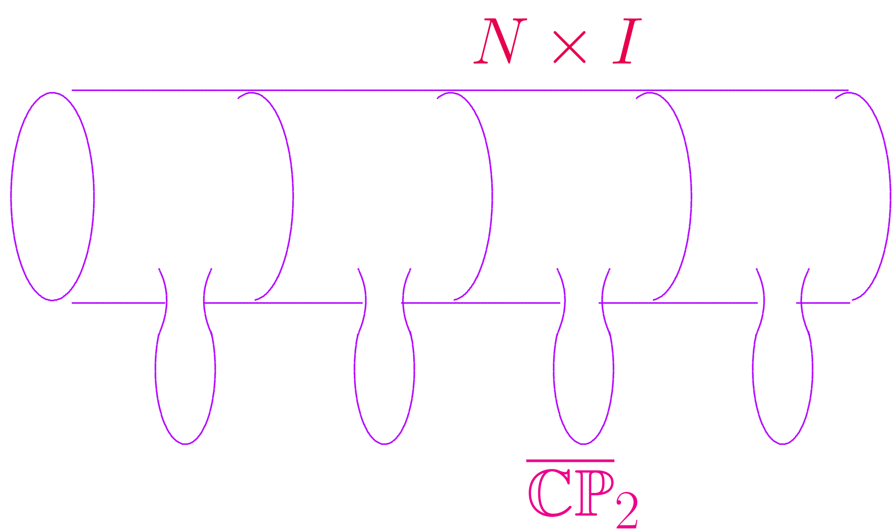

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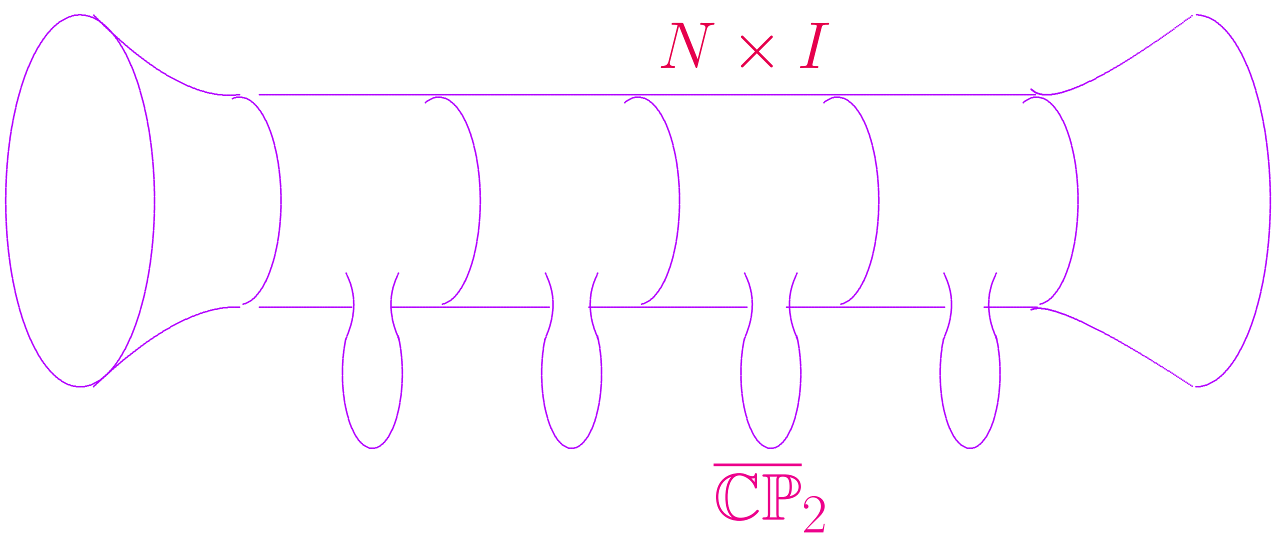

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In limit, one obtains desired inequality

$$\int_M (s_-)^2 \, d\mu_g \geq 32\pi^2 [\mathbf{a}^+]^2$$

for any Riemannian metric g on M.

Lemma C. *Let* (M, J) *be a compact complex surface with* b_1 *odd and* $Kod(M) = 1$. *Then* M *does not admit a Riemannian metric of positive scalar curvature.*

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$$\mathscr{Y}(M) = 0 \iff Kod(M, J) = 0 \text{ or } 1,$$

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For known classes of examples, sign of $\mathscr{Y}(M)$ is left unchanged by blowing up.

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However, this **Conjecture** is very difficult, and has only been proved with $b_2(M) \leq 3$.

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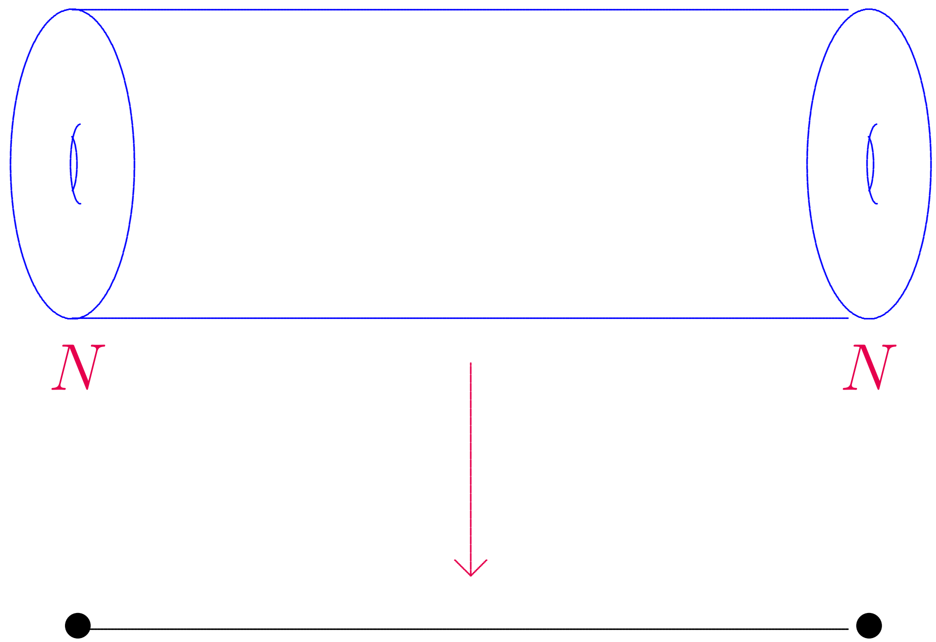

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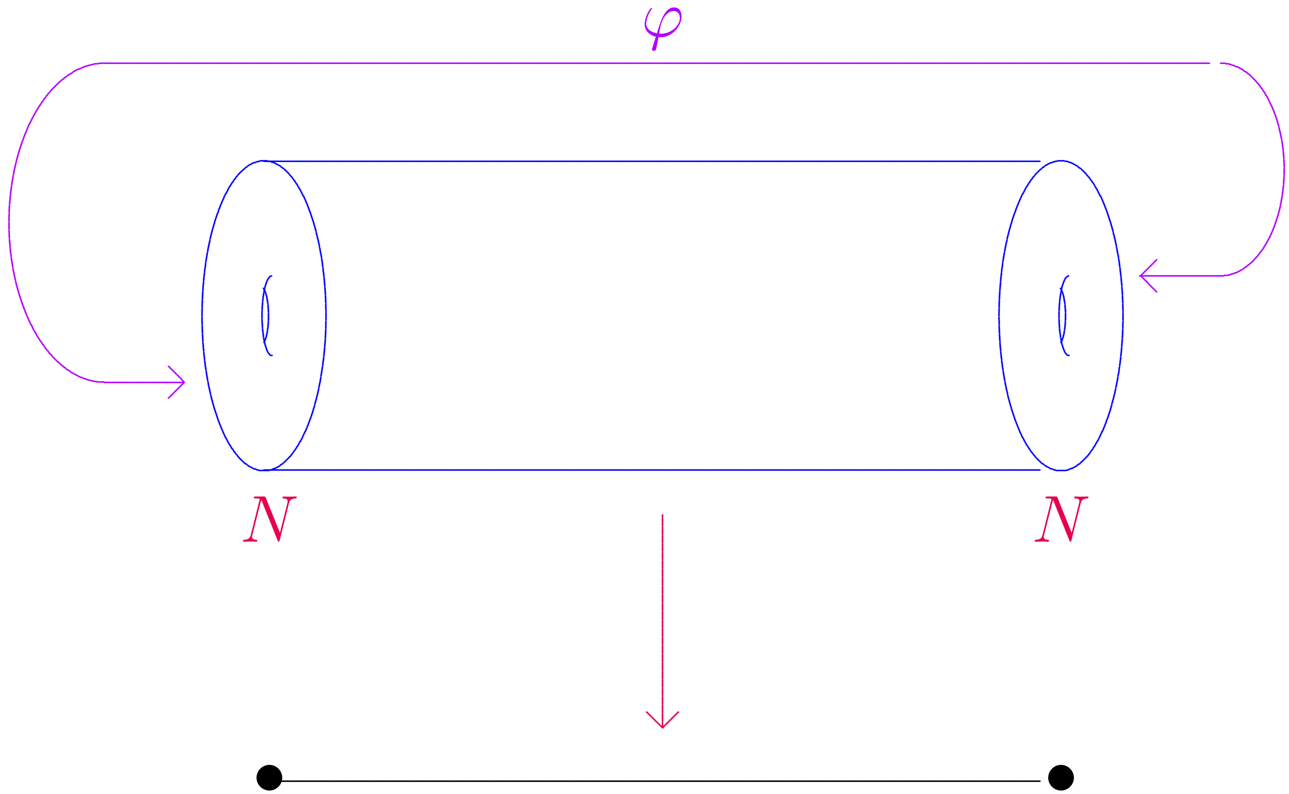

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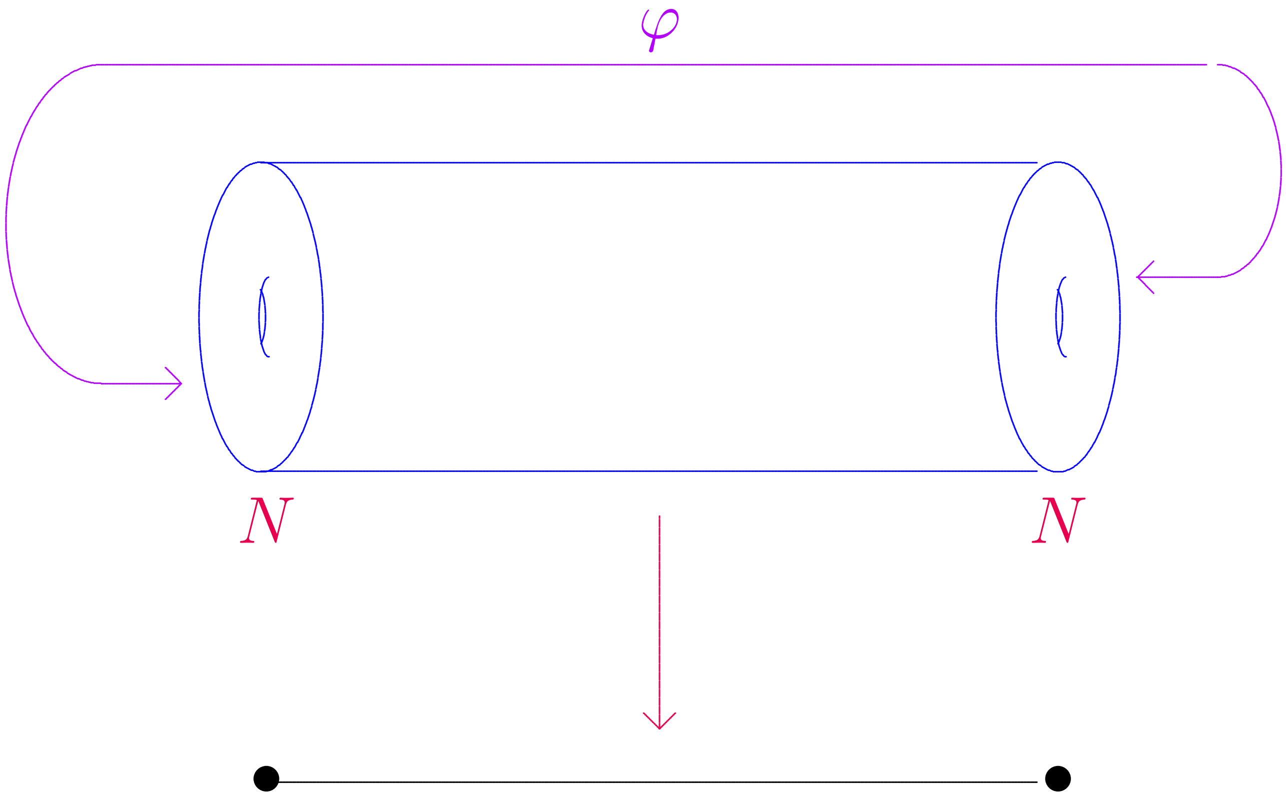

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