Mass in

Kähler Geometry

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Stony Brook University

Workshop on Mass in General Relativity
Simons Center for Geometry and Physics
Stony Brook, NY: March 30, 2018

Joint work with

Joint work with

Hans-Joachim Hein
Fordham University

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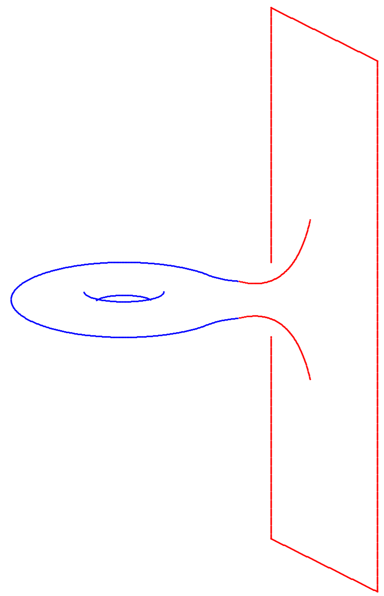
(M^n, g) complete non-compact Riemannian n-manifold.



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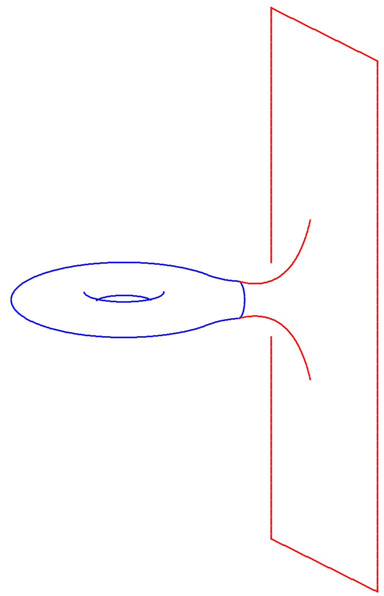

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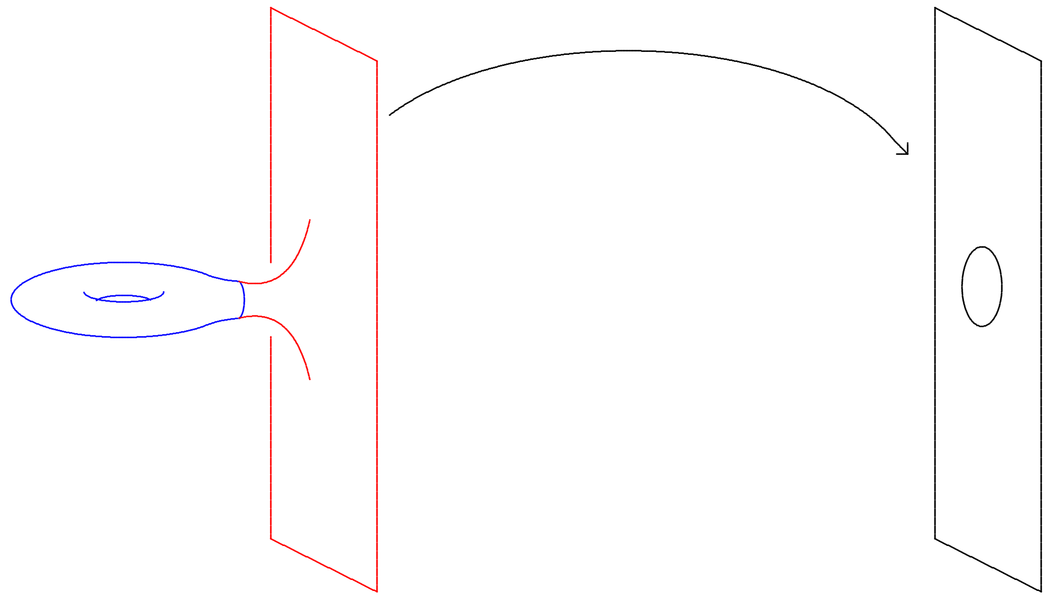

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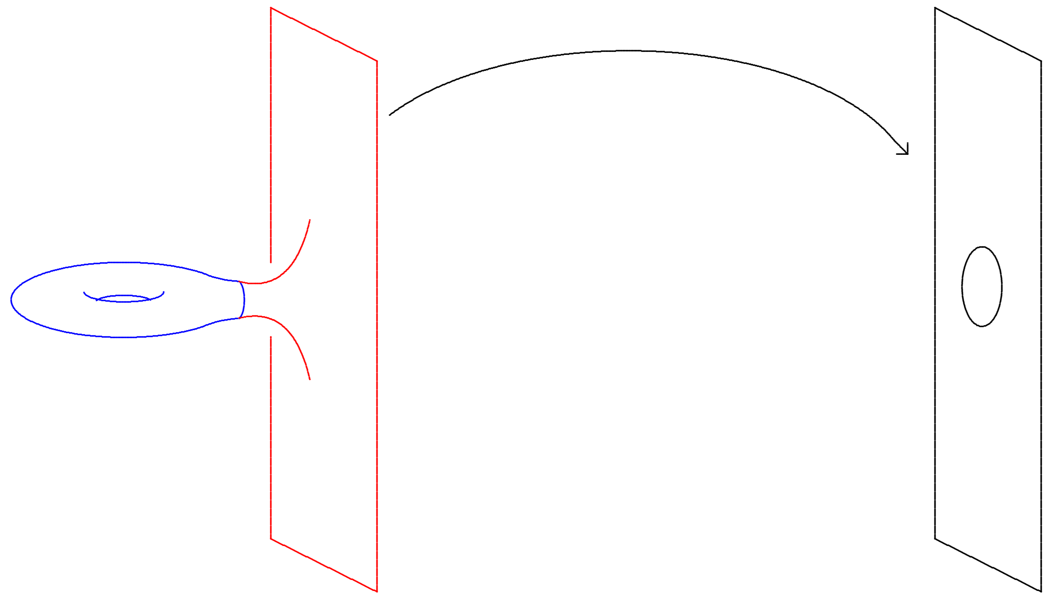

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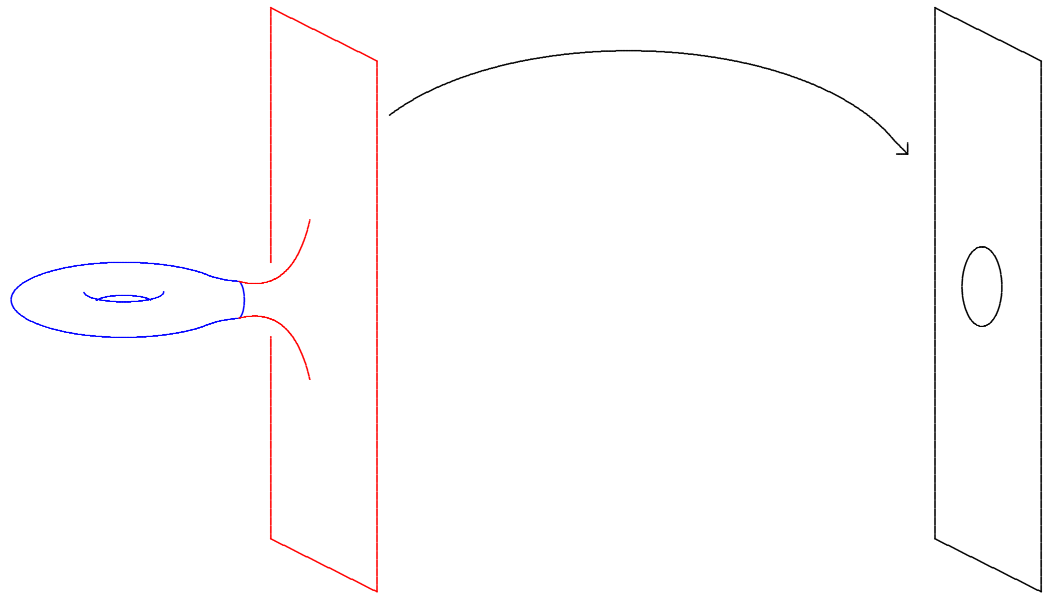

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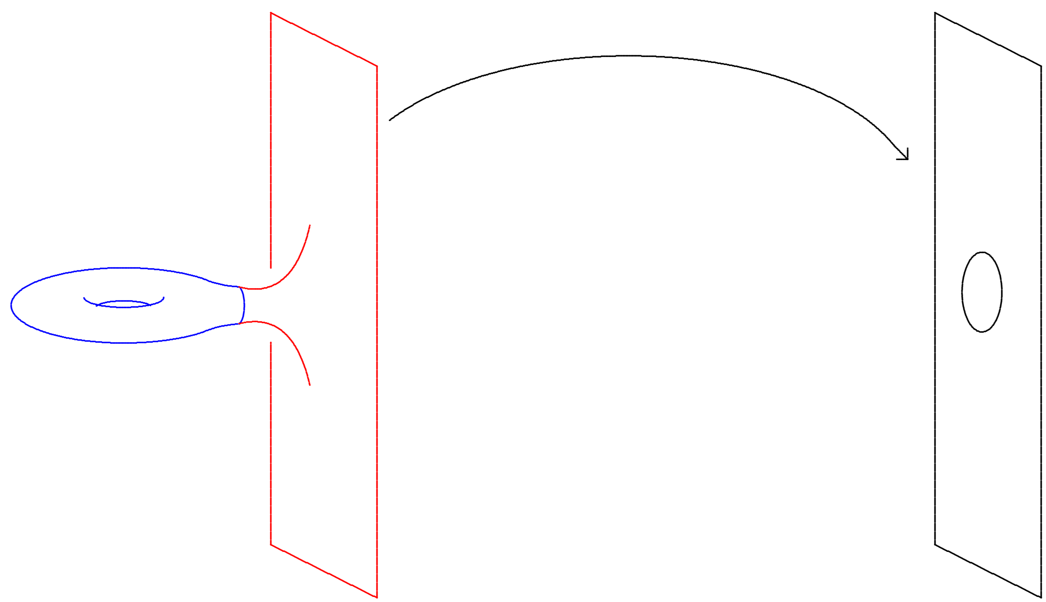

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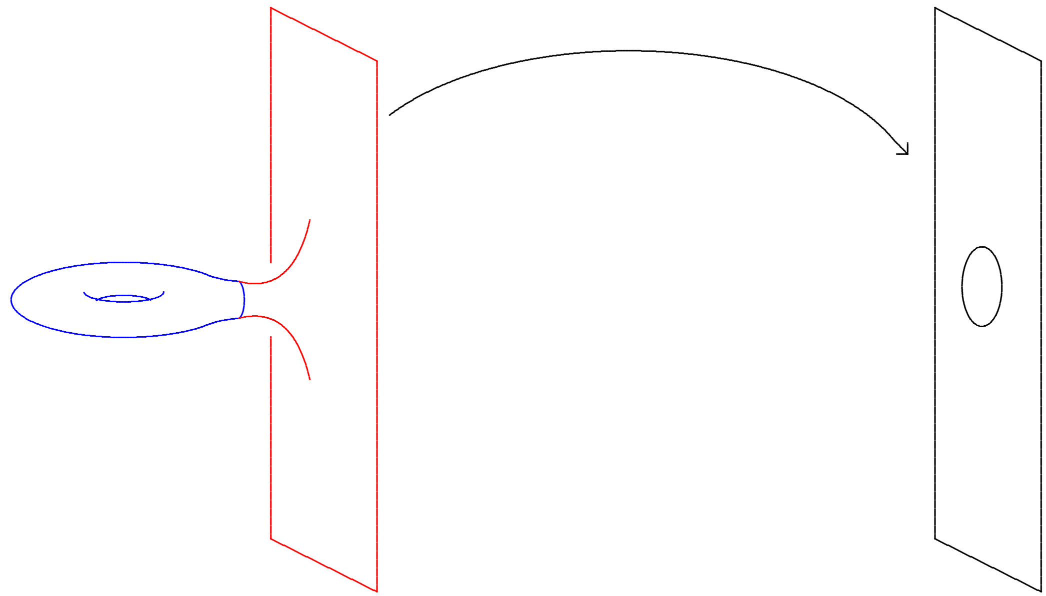

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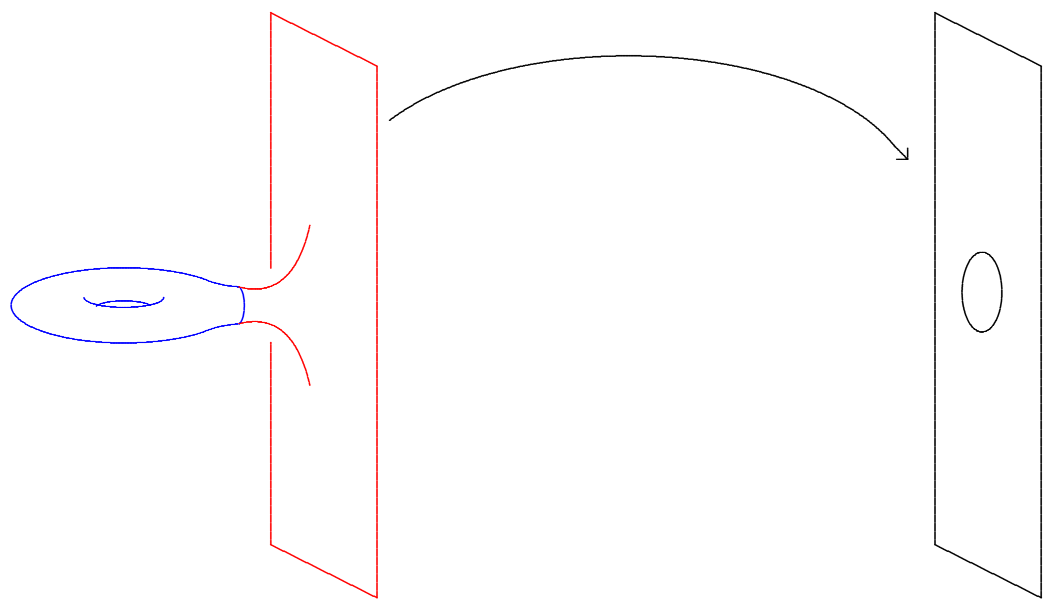

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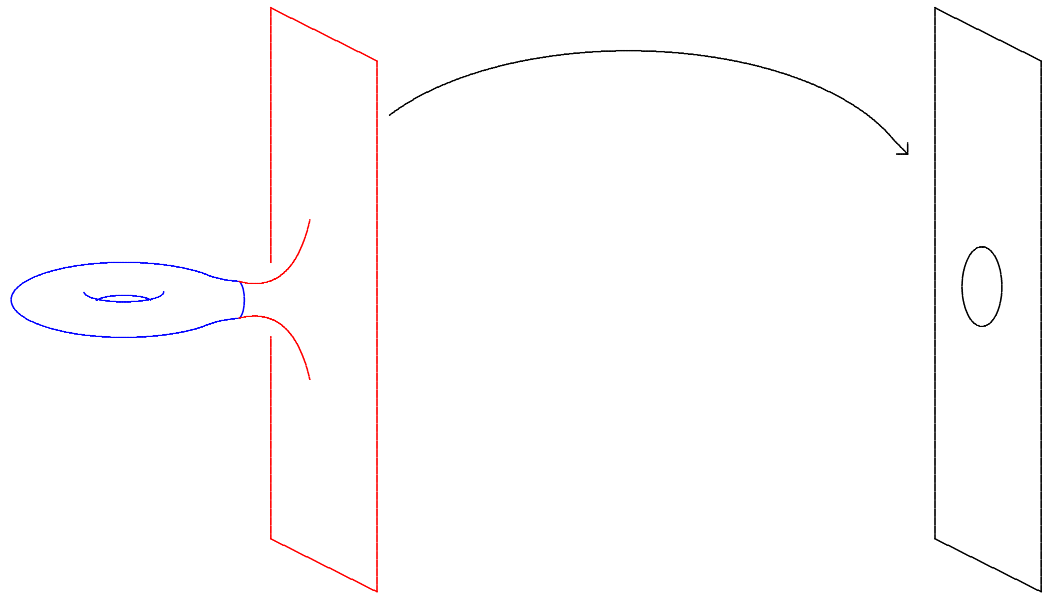

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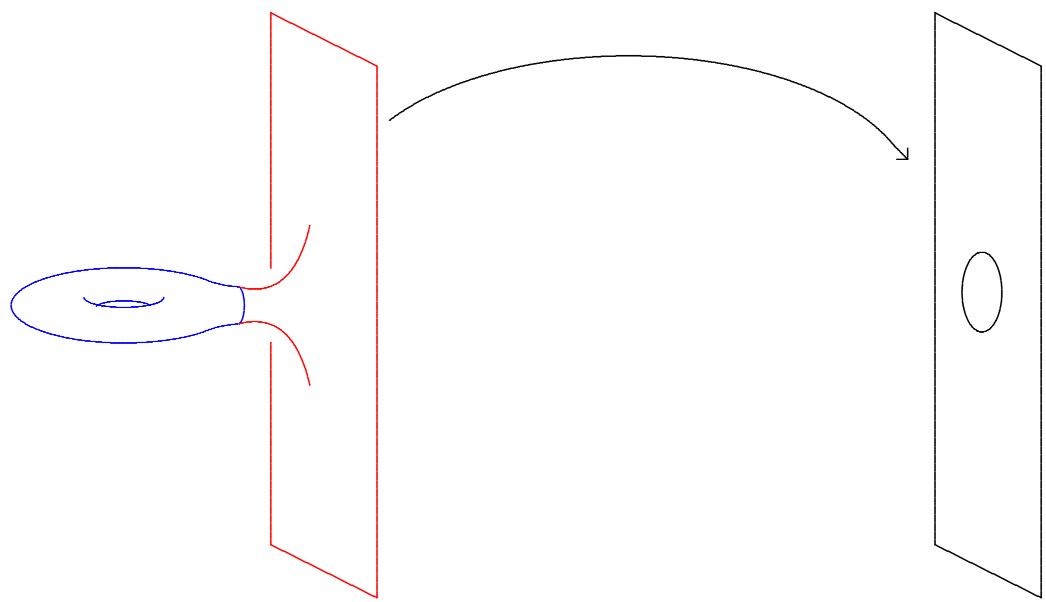

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Get result even with appropriate fall-off to Euclidean. . .

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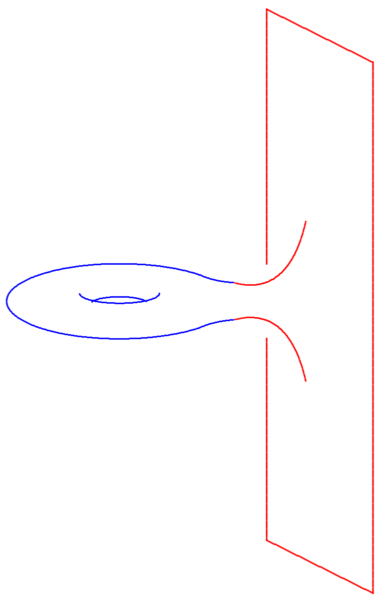

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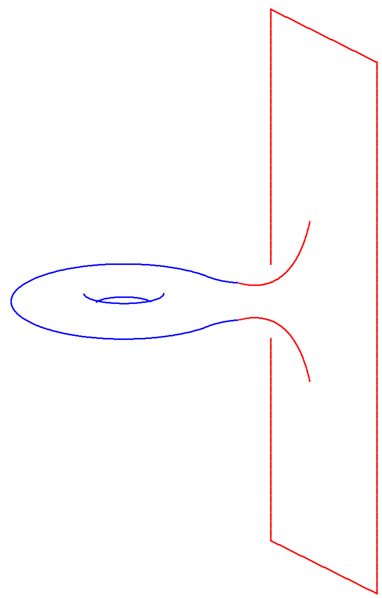

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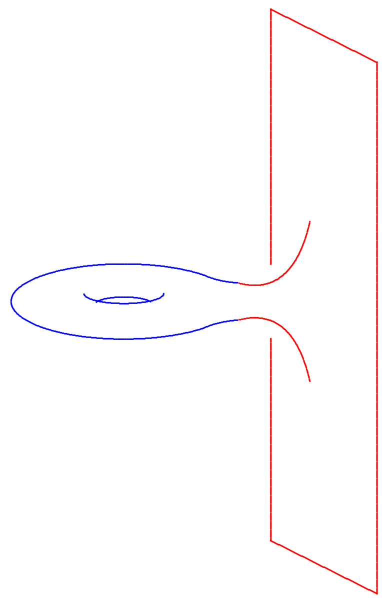

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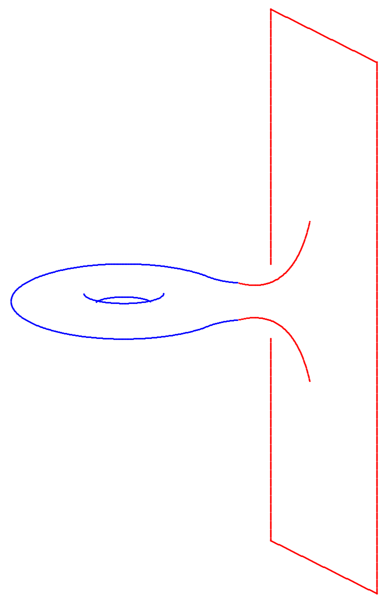

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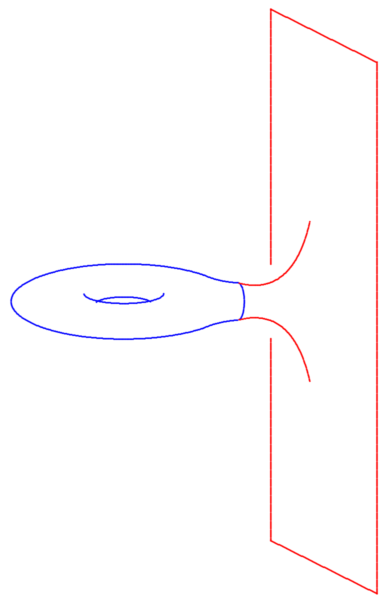

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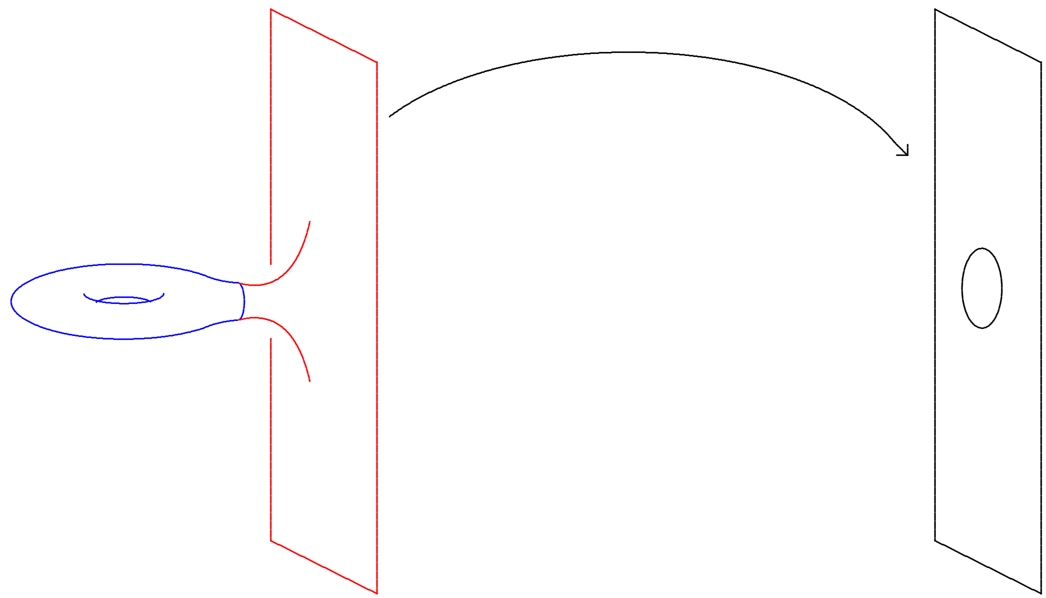

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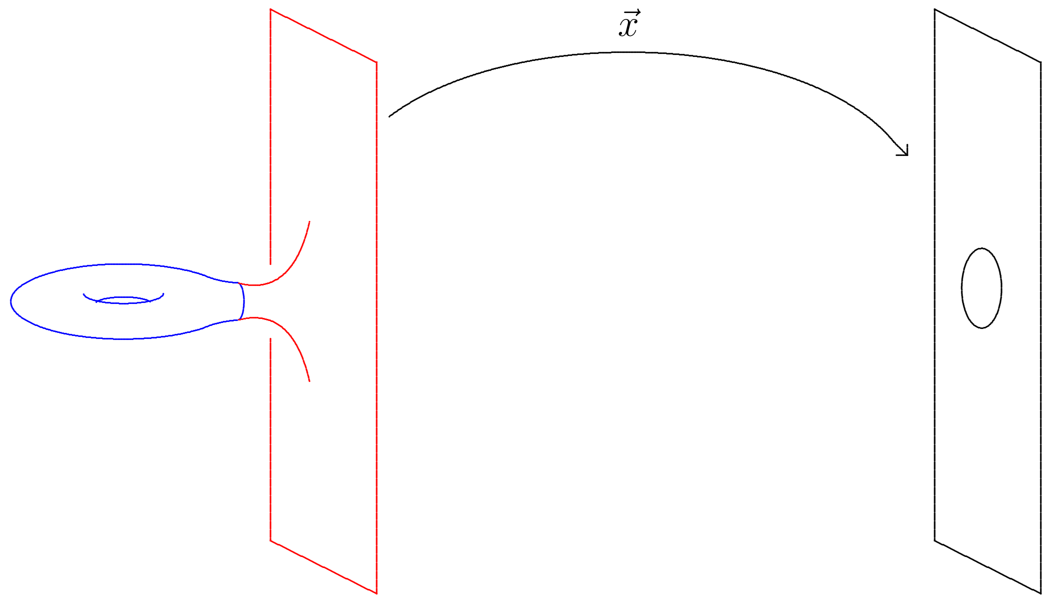

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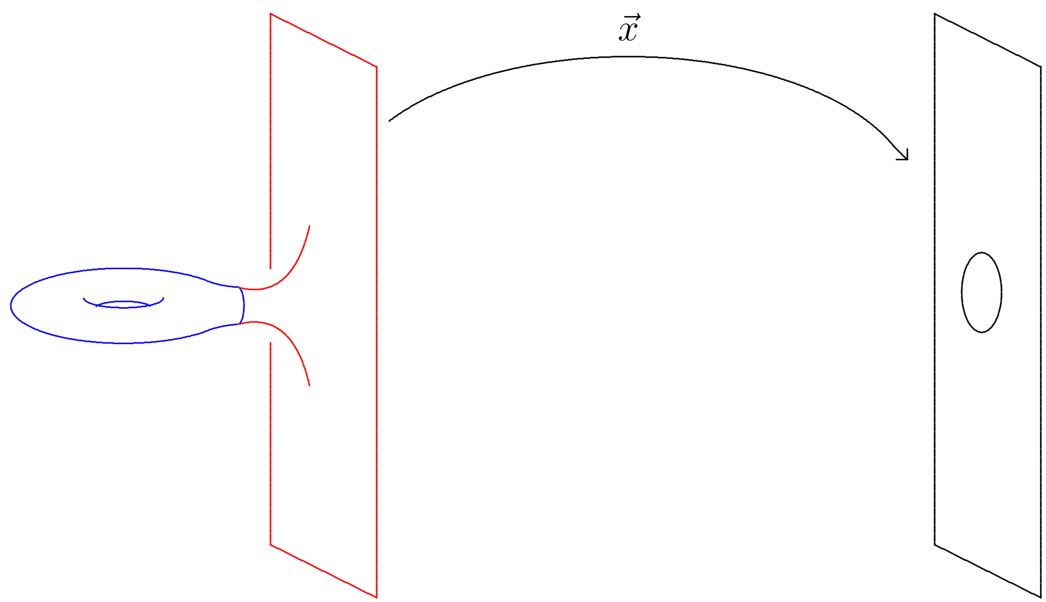

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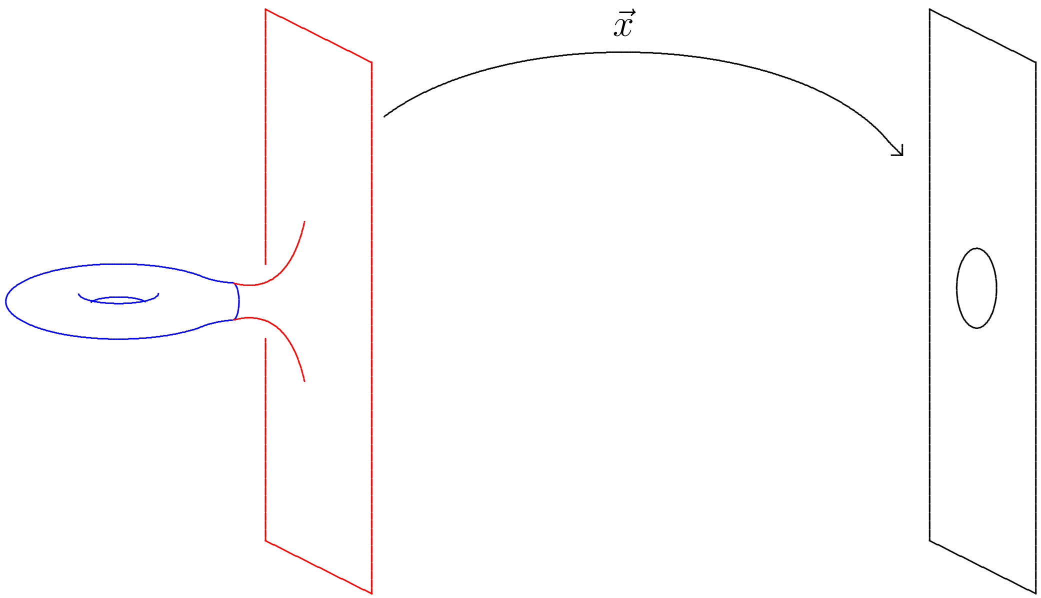

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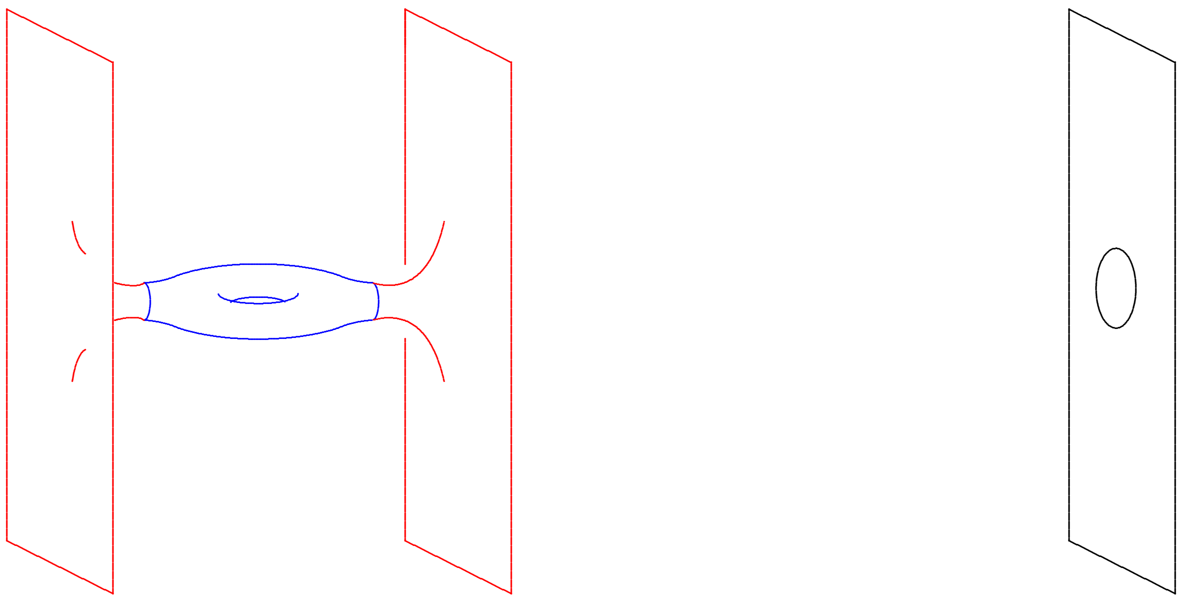

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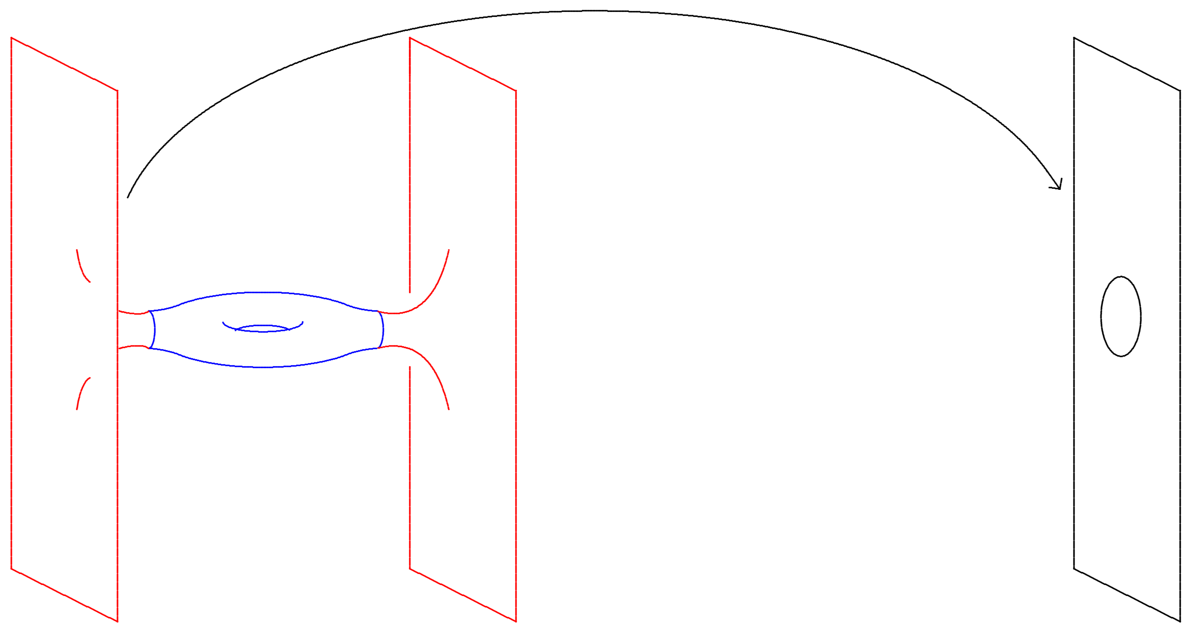

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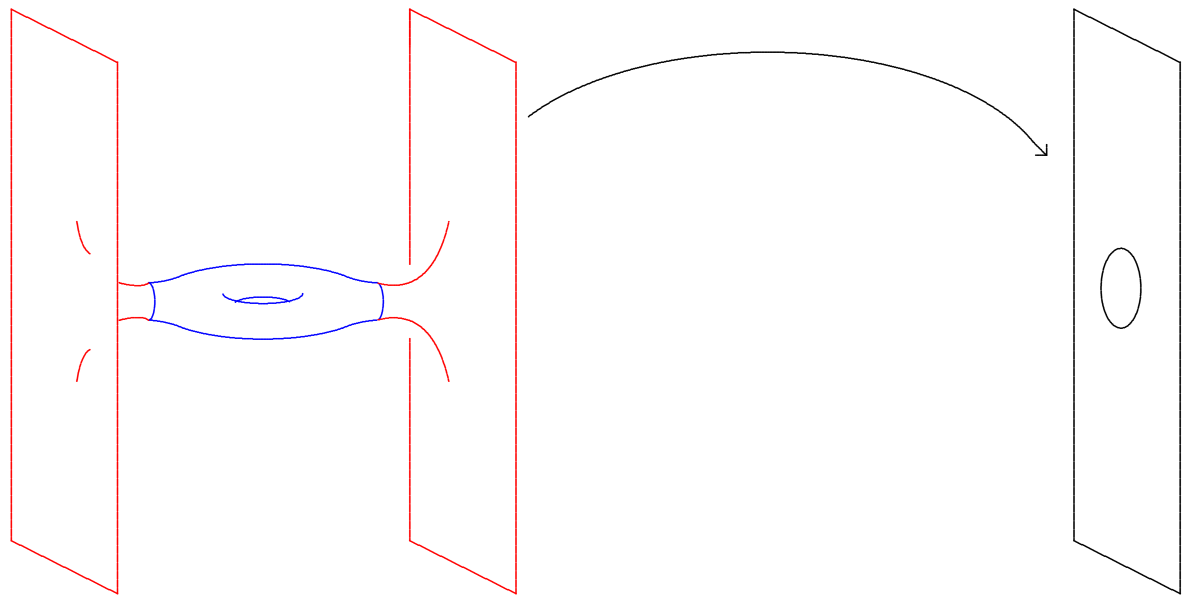

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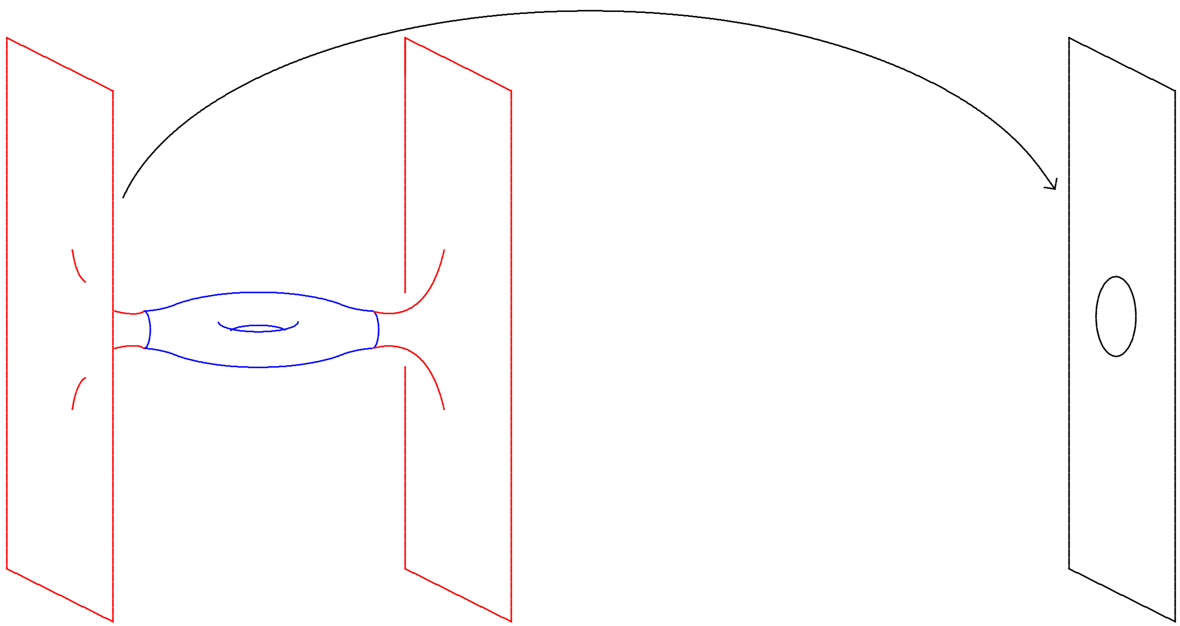

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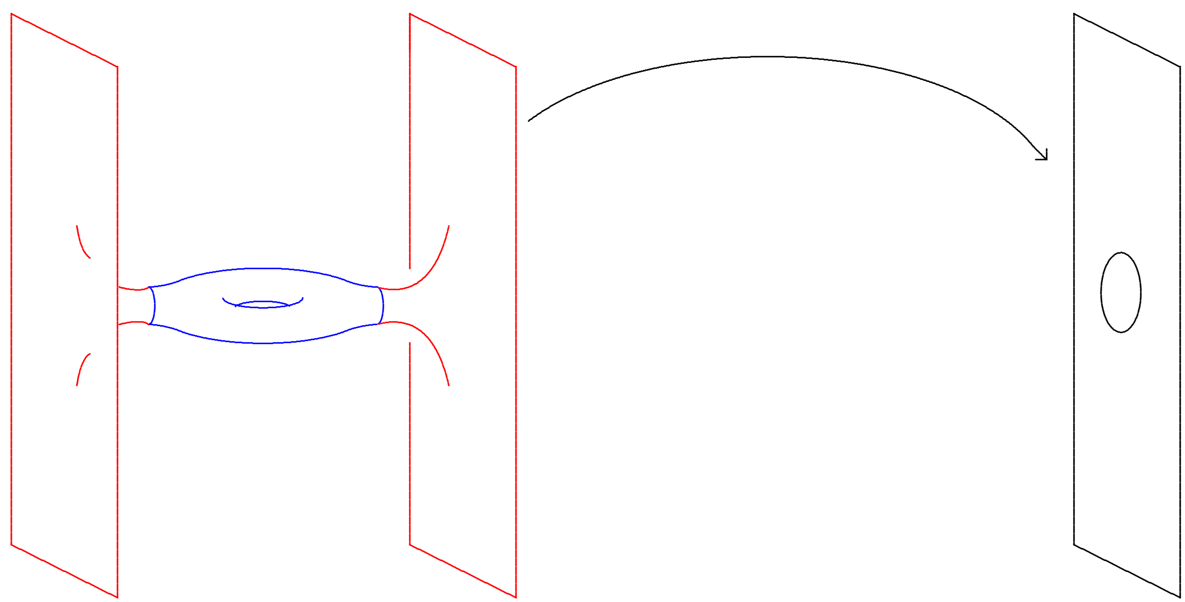

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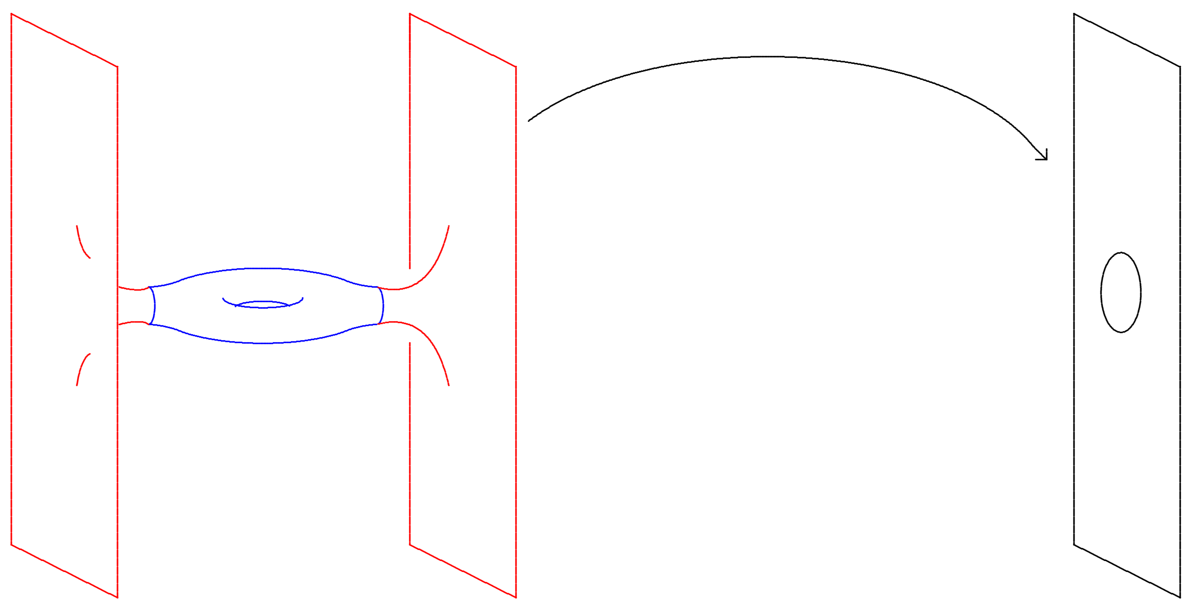

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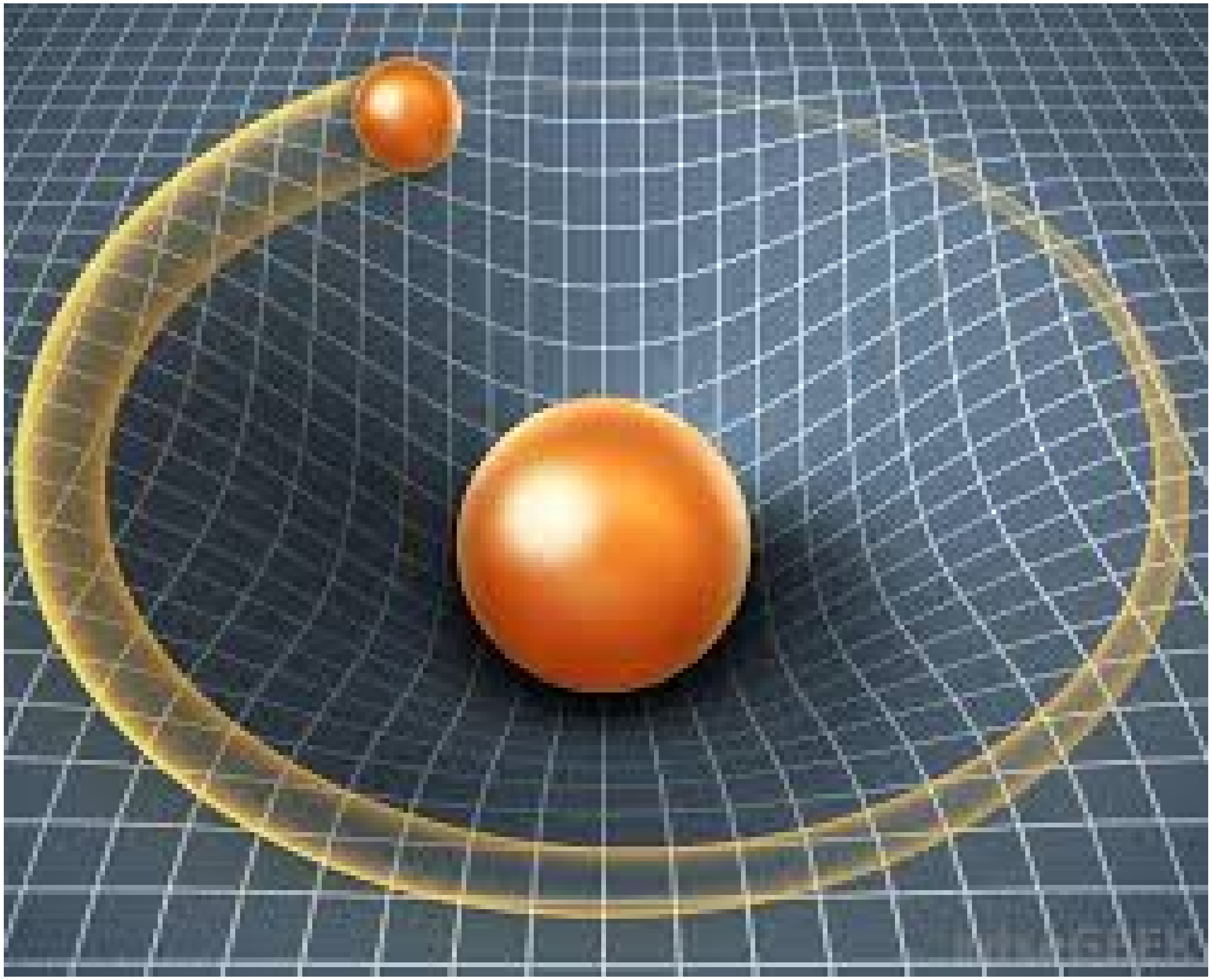

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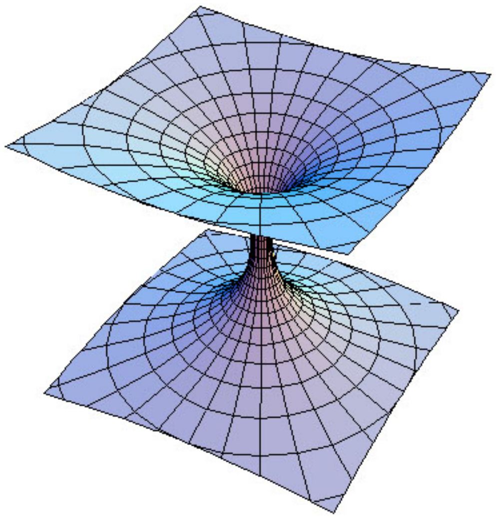

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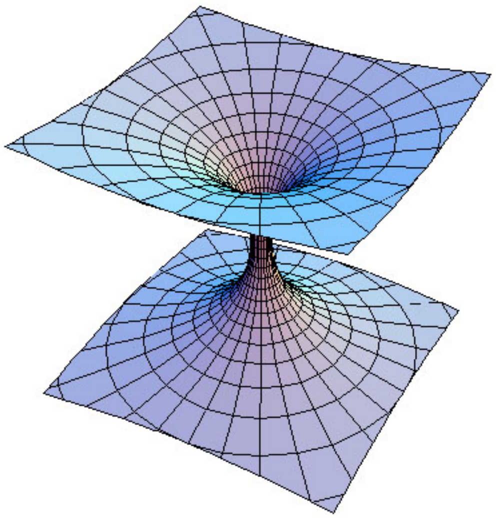

Motivation:

When $n = 3$, ADM mass in general relativity.

Reads off "apparent mass" from strength of the gravitational field far from an isolated source.

In any dimension, reproduces "mass" of $t = 0$ hypersurface in $(n+1)$-dimensional Schwarzschild

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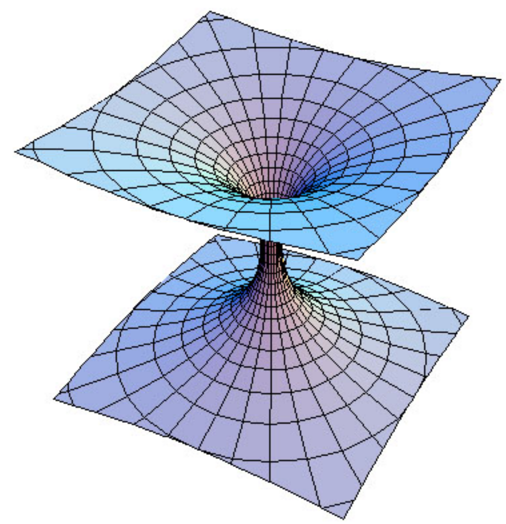

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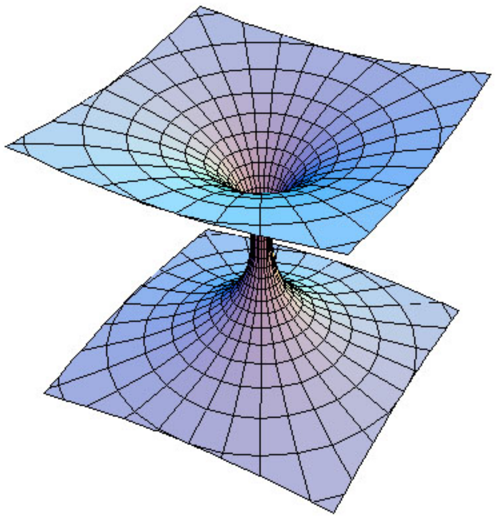

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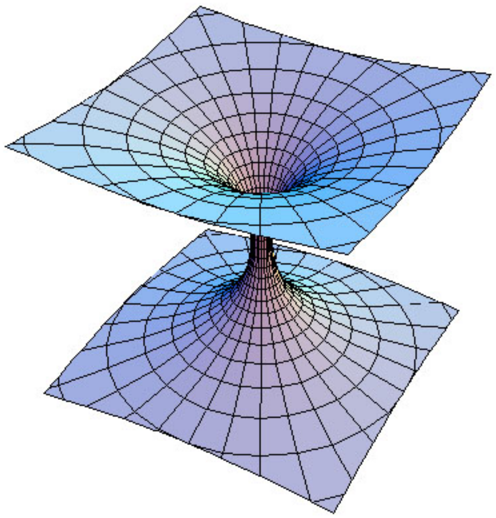

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$$g = \left(1 + \frac{m/2}{r^{n-2}}\right)^{4/(n-2)} \left[\sum (dx^j)^2\right]$$

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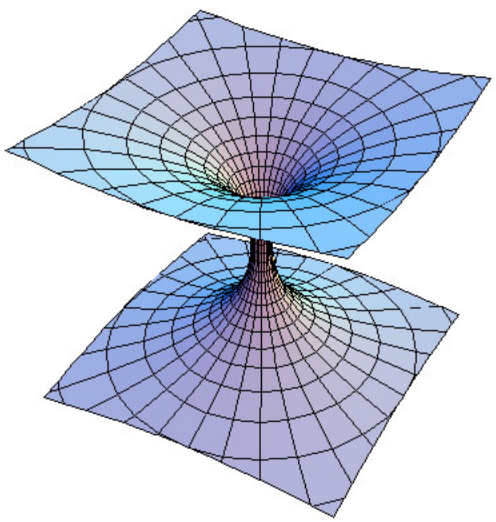

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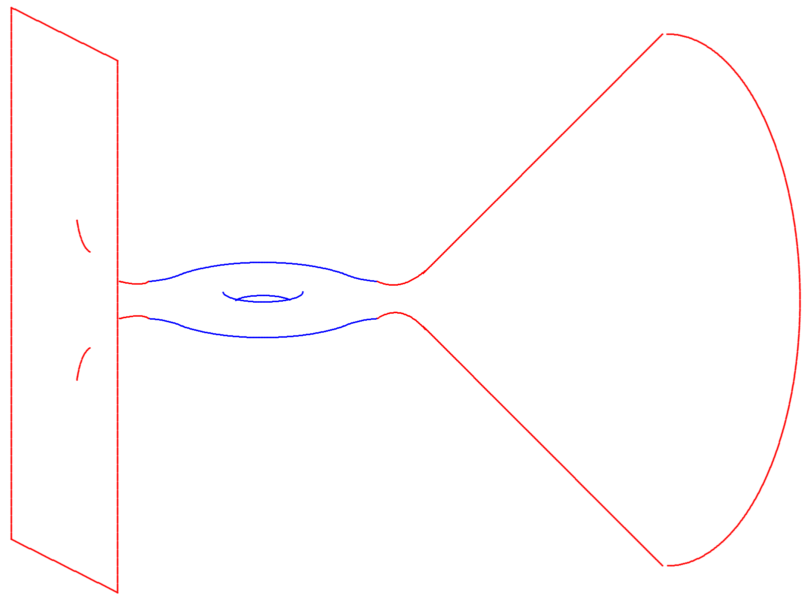
$$g_{jk} = \left(1 + \frac{2m}{(n-2)r^{n-2}}\right)\delta_{jk} + \cdots$$

A Generalization. . .

Definition. *Complete, non-compact* n*-manifold* (M^n, g) *is* asymptotically locally Euclidean



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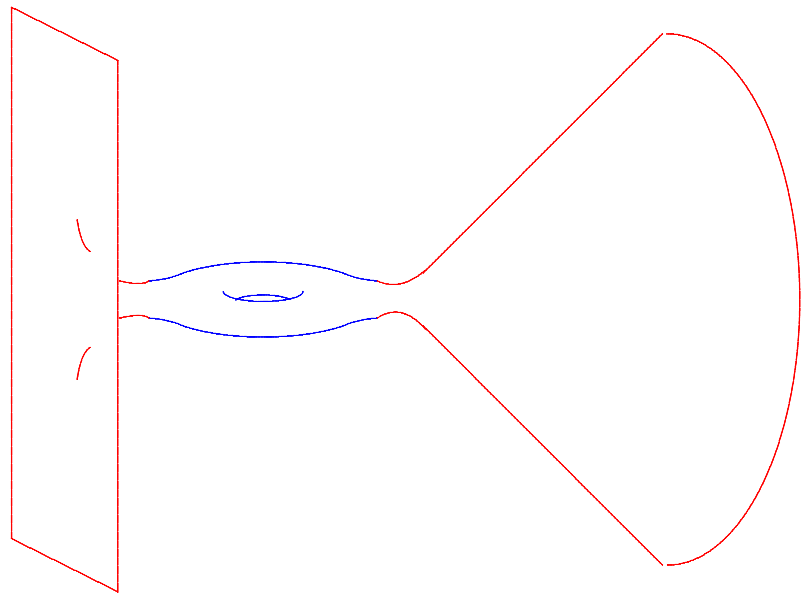

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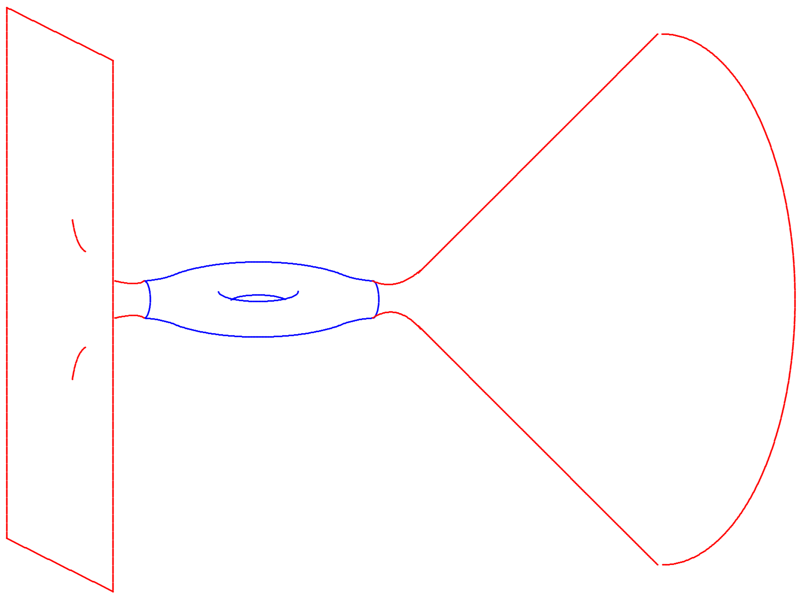

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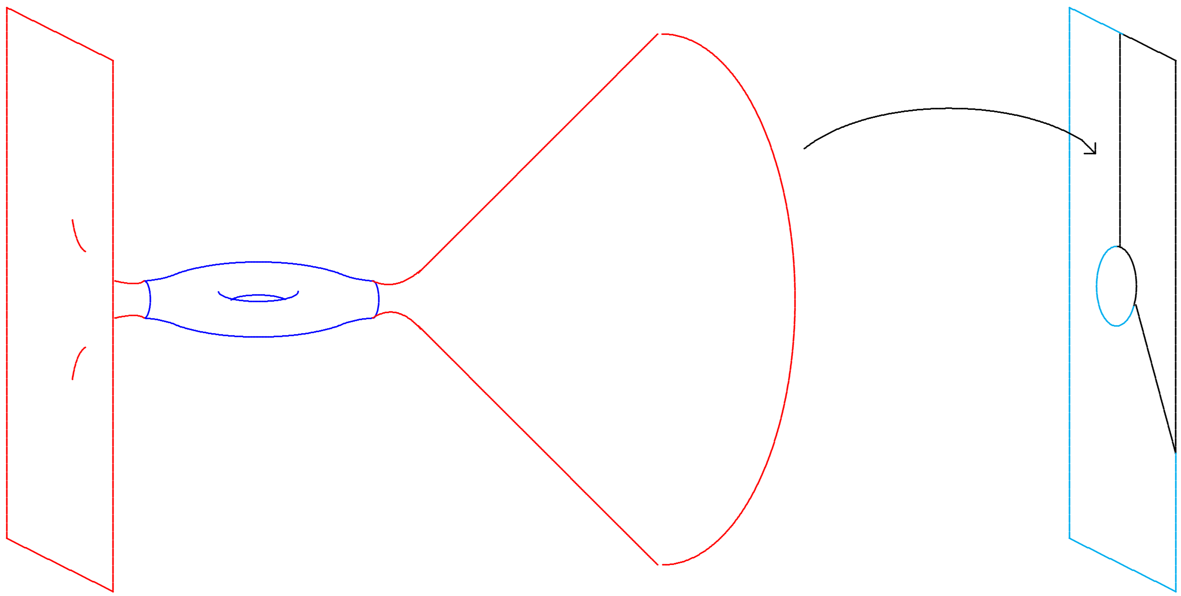

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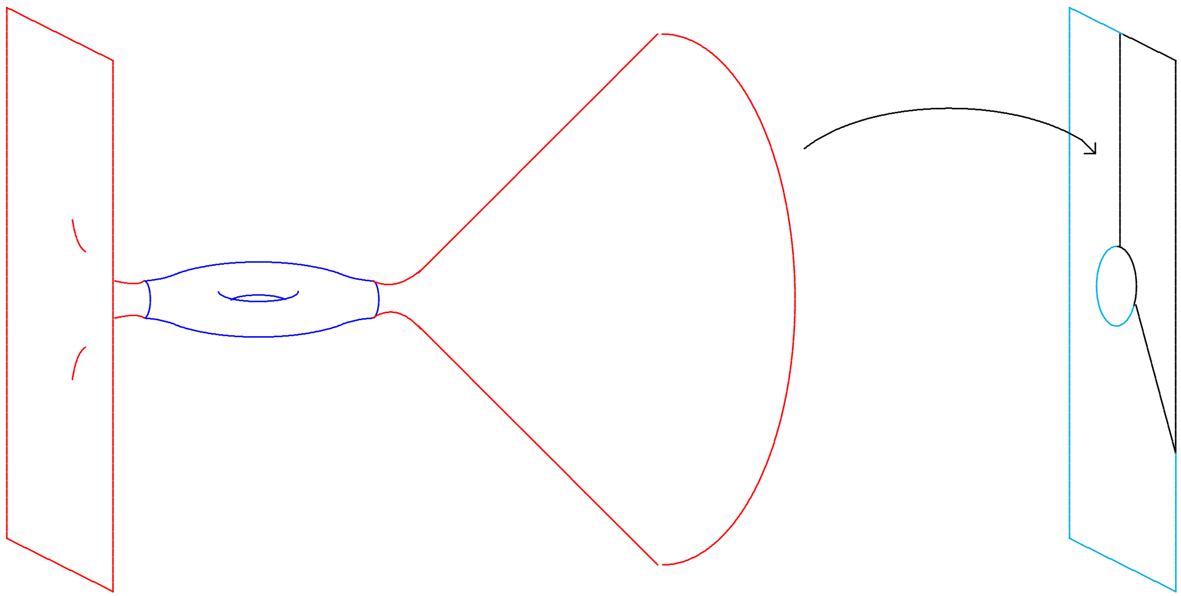

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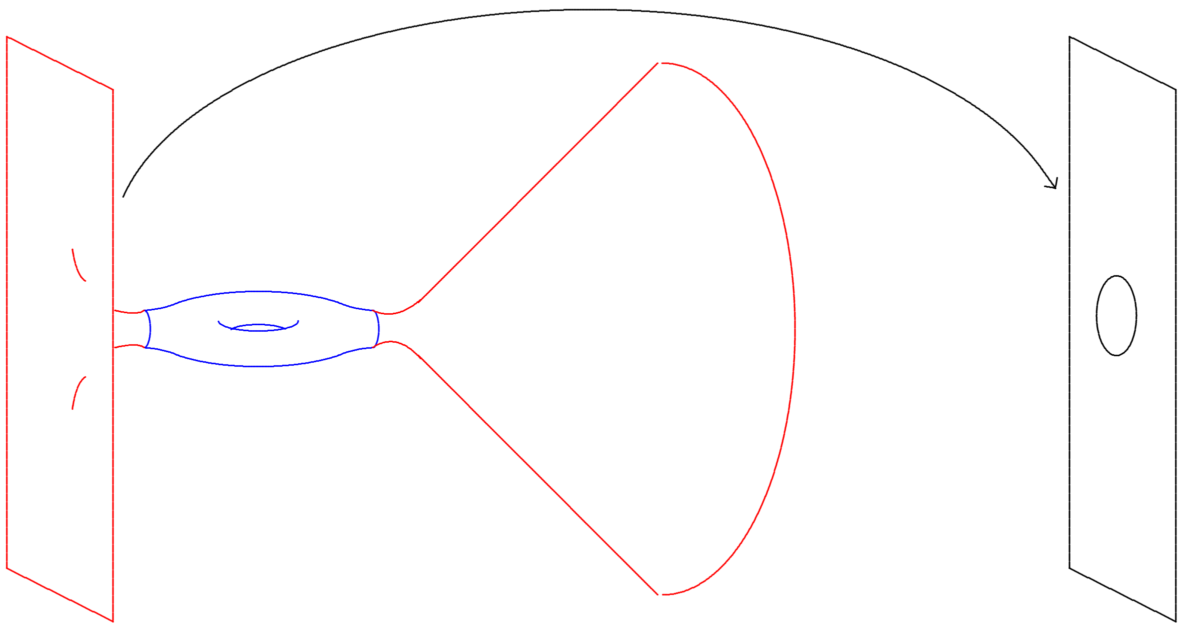

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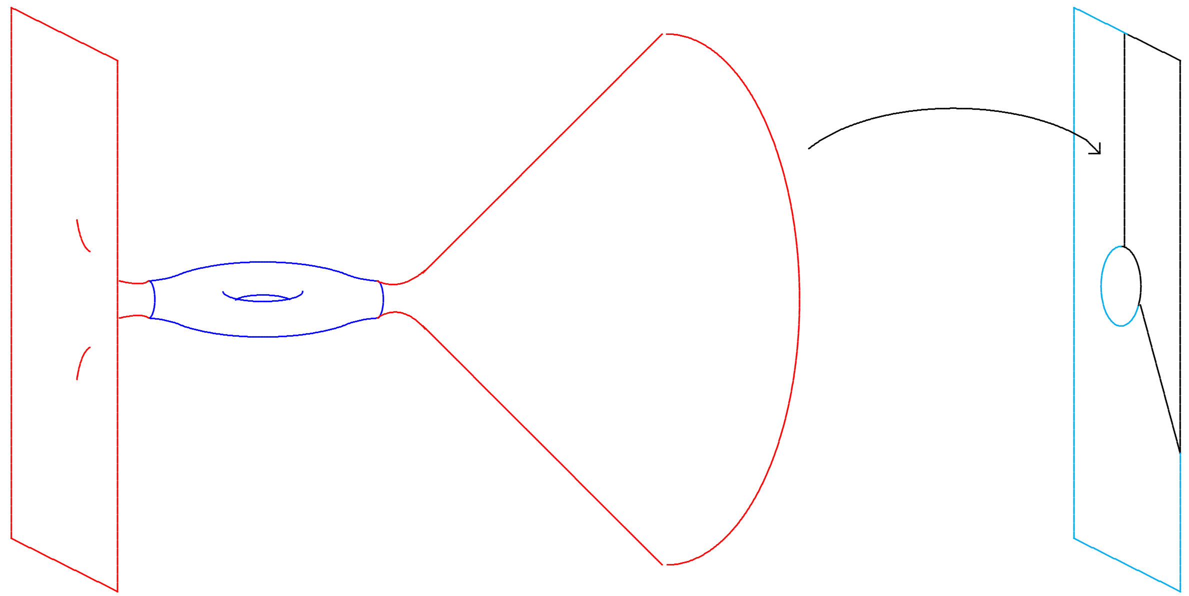

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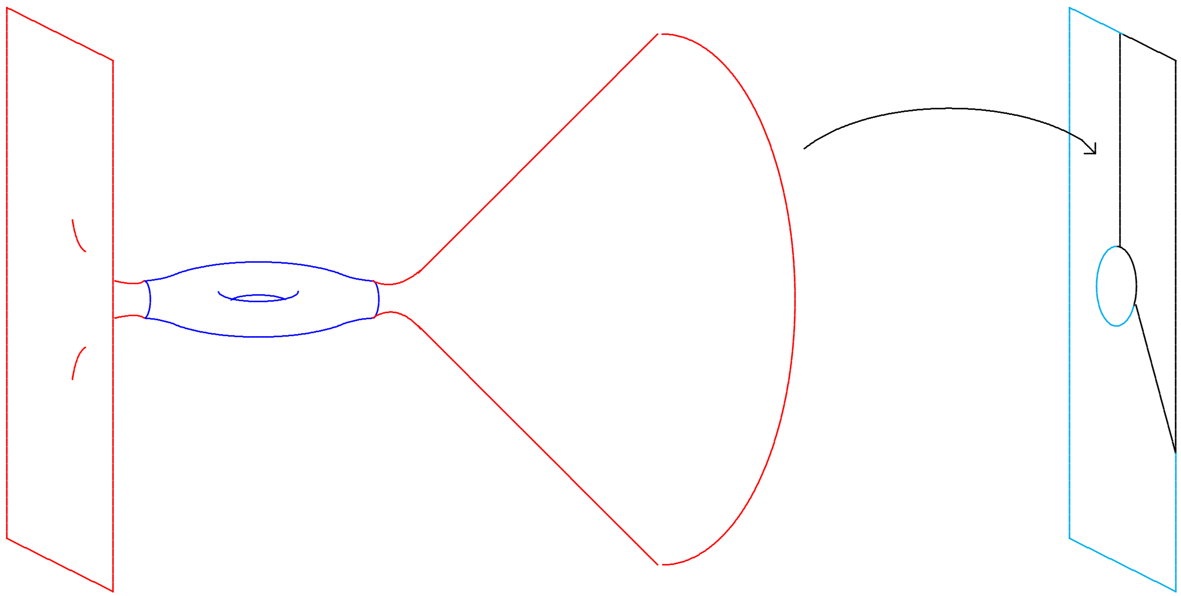

$$g_{jk} = \delta_{jk} + O(|x|^{1-\frac{n}{2}-\varepsilon})$$

$$g_{jk,\ell} = O(|x|^{-\frac{n}{2}-\varepsilon}), \quad s \in L^1$$

Why consider ALE spaces?

Key examples:

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By contrast, any Ricci-flat AE manifold must be flat, by the Bishop-Gromov inequality…

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The G-H metrics are hyper-Kähler, and were soon rediscovered independently by Hitchin.

(M^n, g):

holonomy

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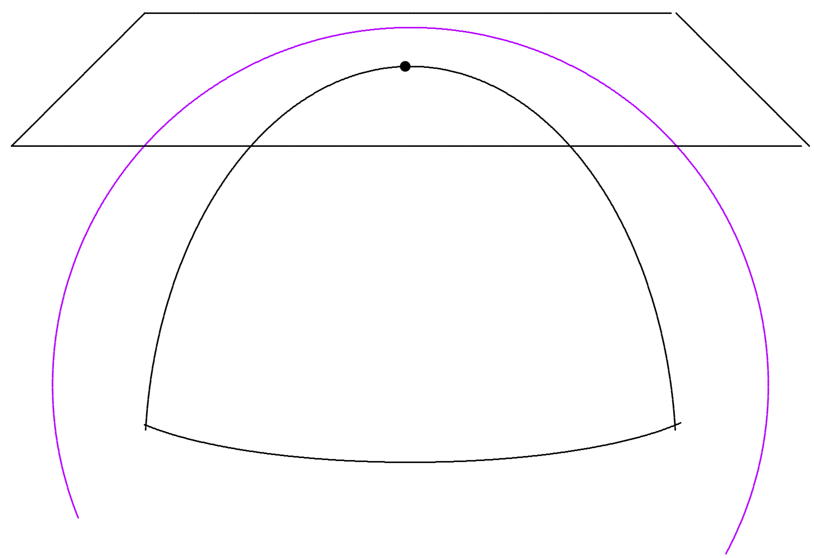

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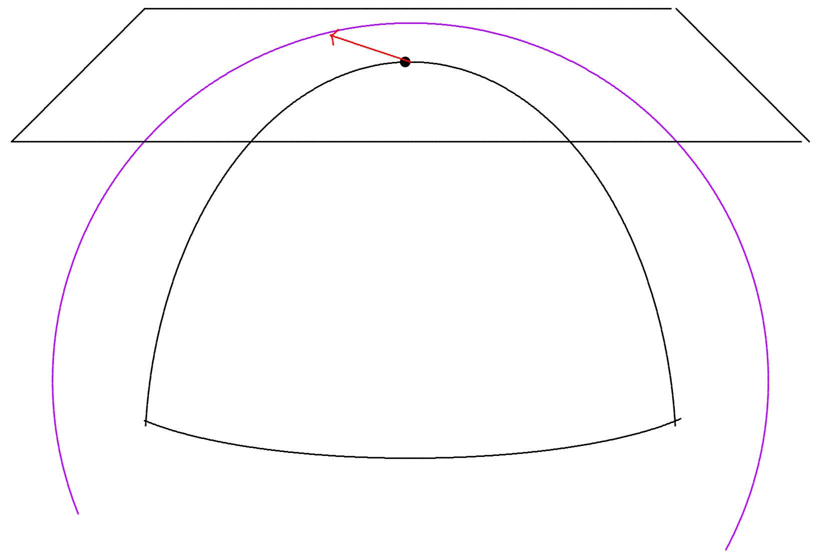

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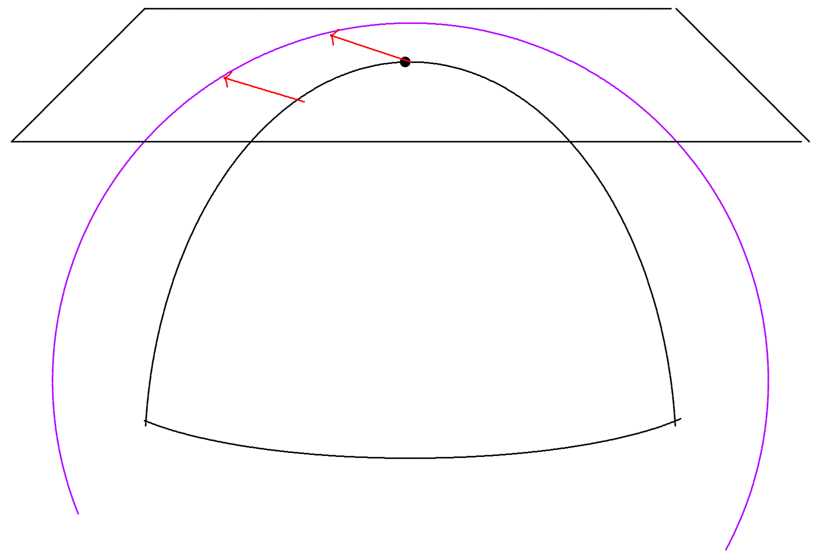

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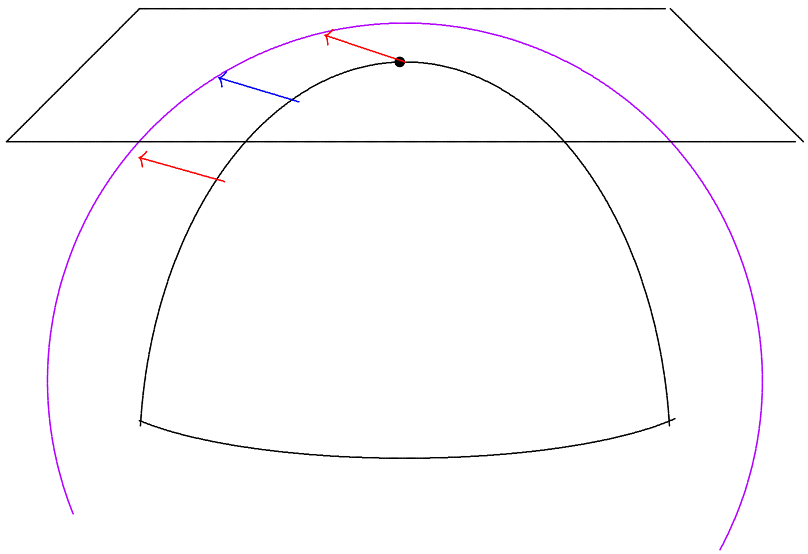

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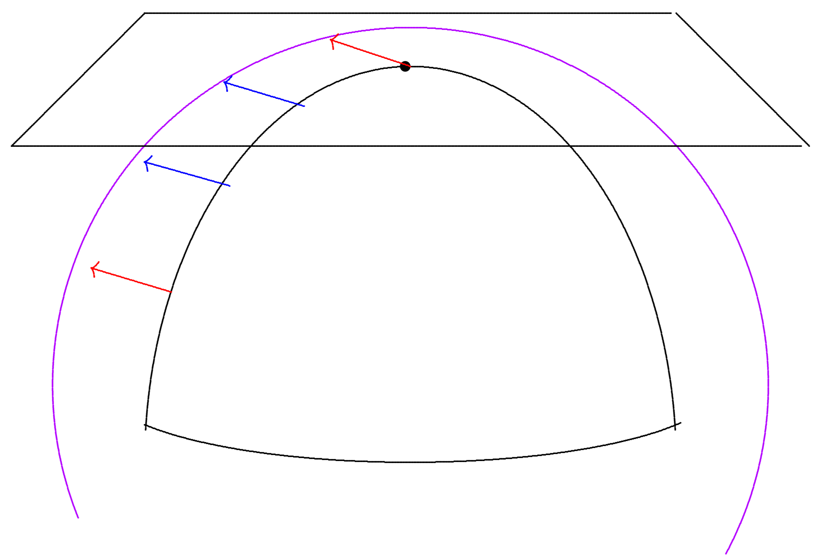

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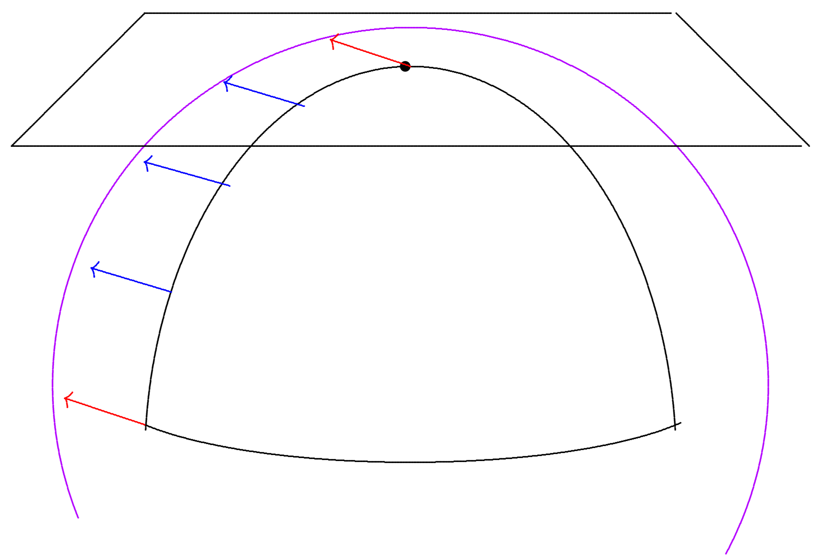

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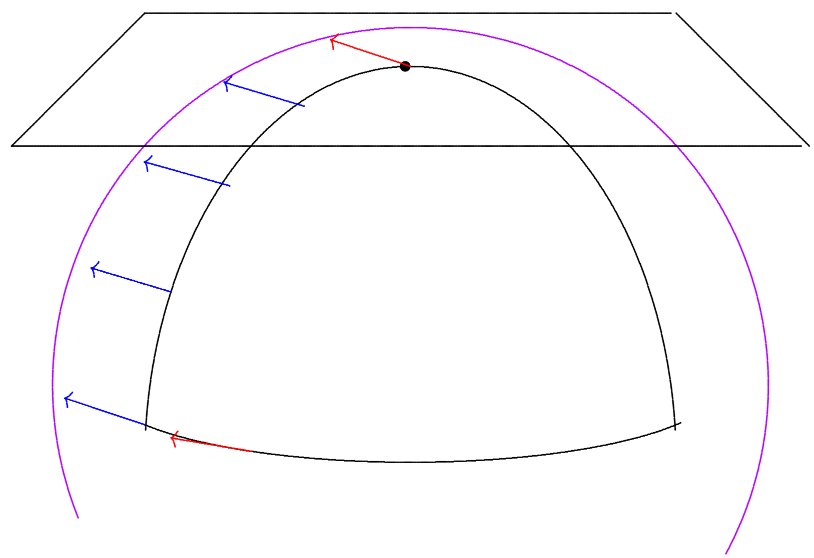

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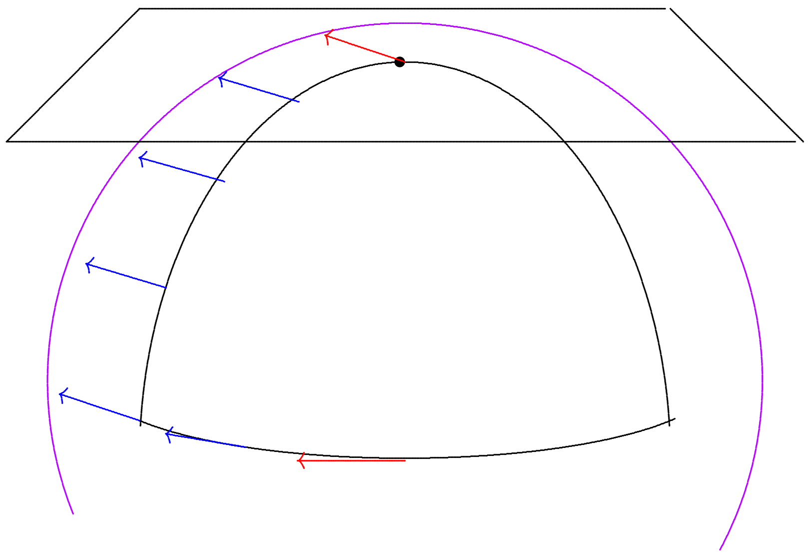

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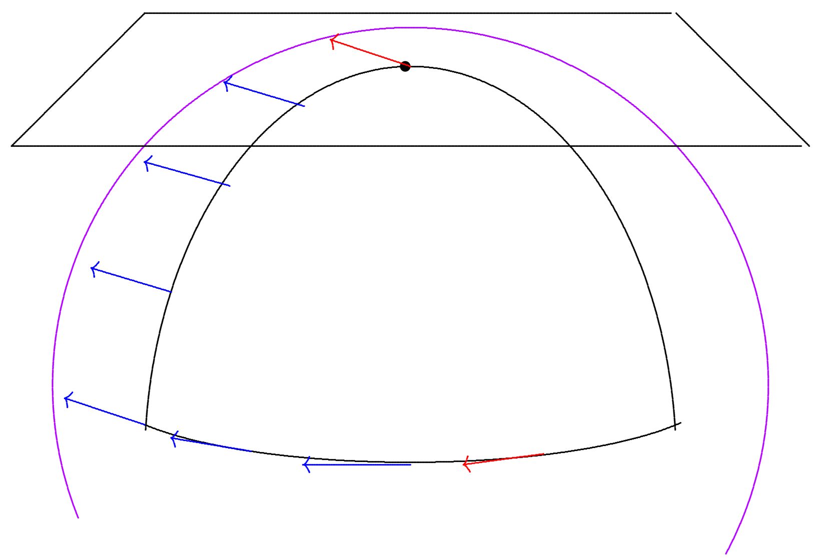

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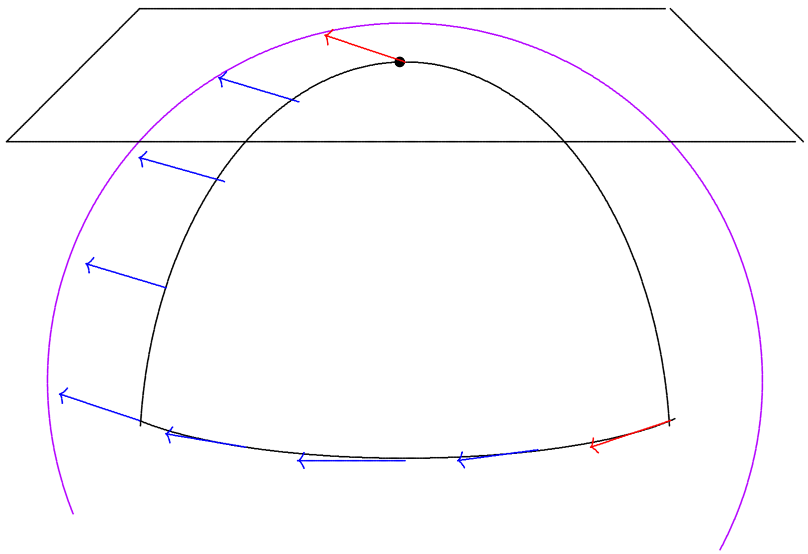

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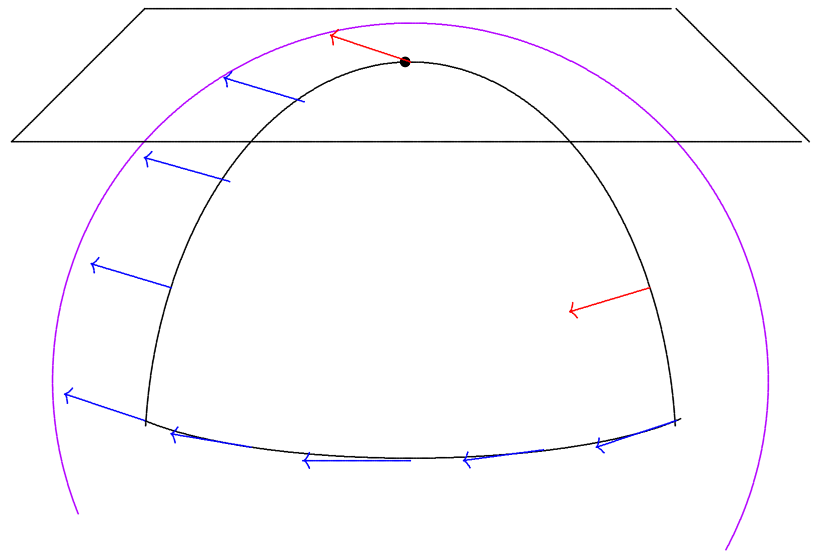

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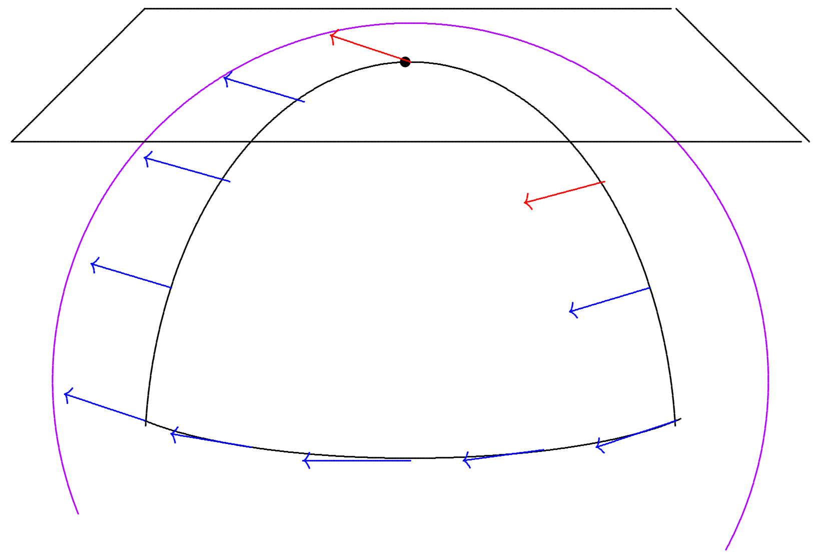

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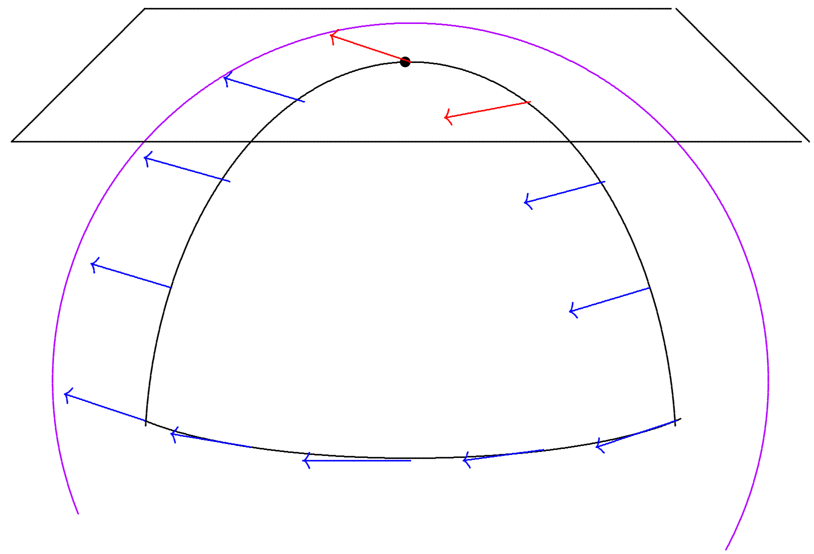

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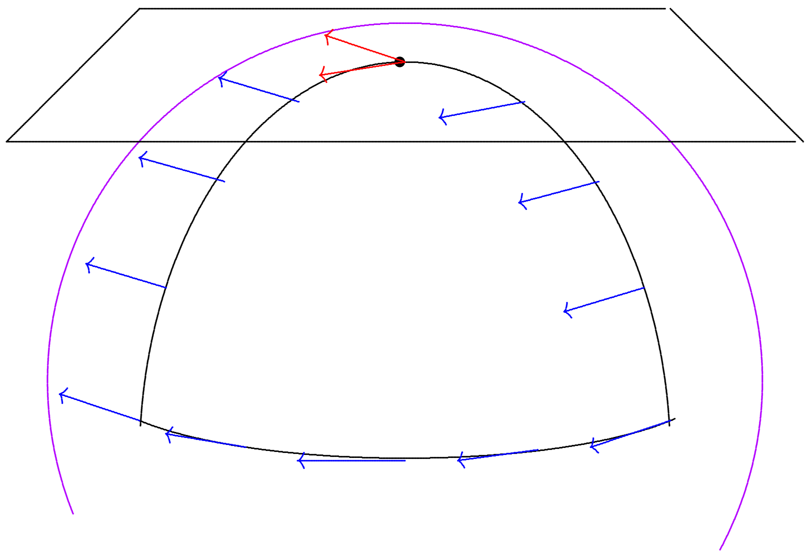

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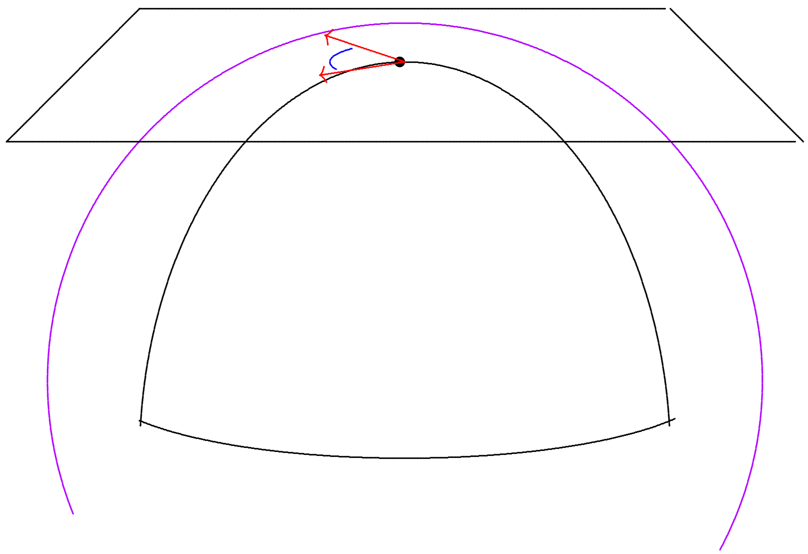
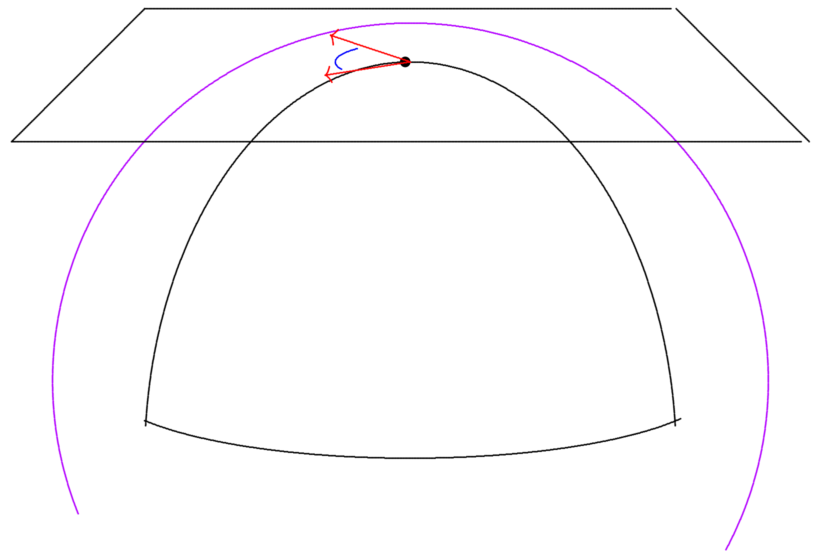
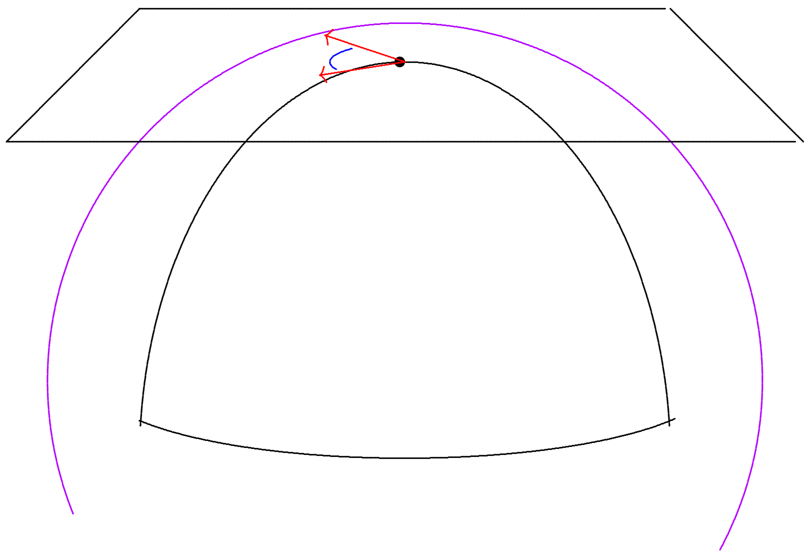


(M^n, g):   holonomy $\subset \mathbf{O}(n)$



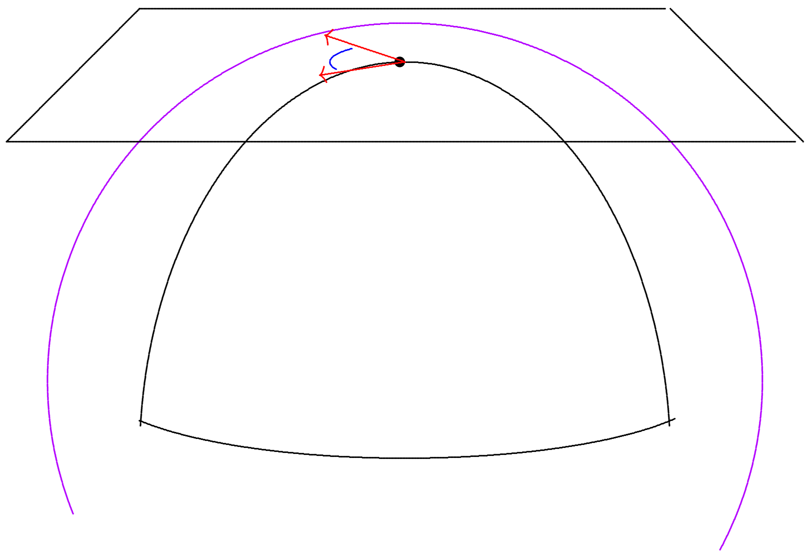
Kähler metrics:

(M^{2m}, g): holonomy



Kähler metrics:

(M^{2m}, g) Kähler $\Longleftrightarrow$ holonomy $\subset \mathbf{U}(m)$



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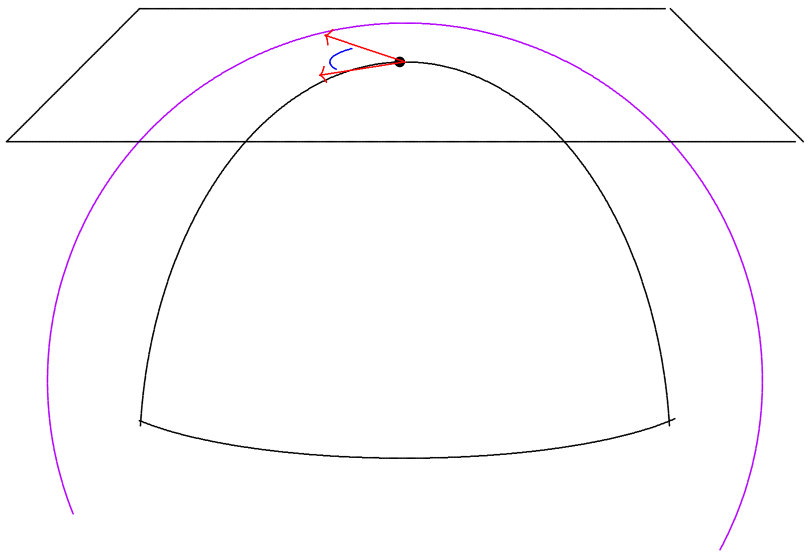

$$\mathbf{U}(m) := \mathbf{O}(2m) \cap \mathbf{GL}(m, \mathbb{C})$$

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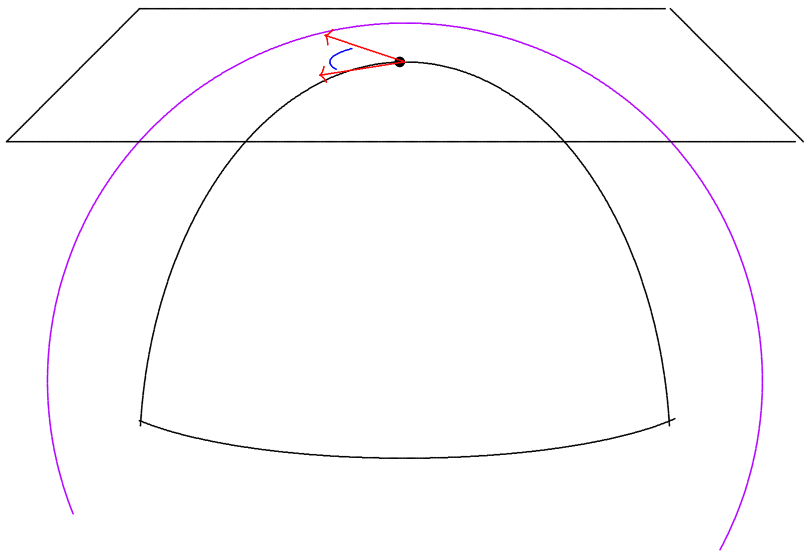

Makes tangent space a complex vector space!

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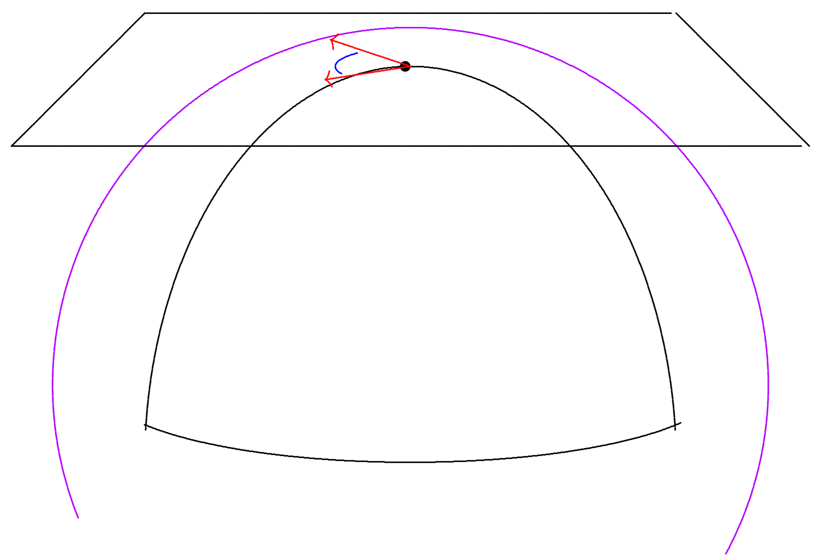

Makes tangent space a complex vector space!

$$J : TM \to TM, \qquad J^2 = -\text{identity}$$

"almost-complex structure"

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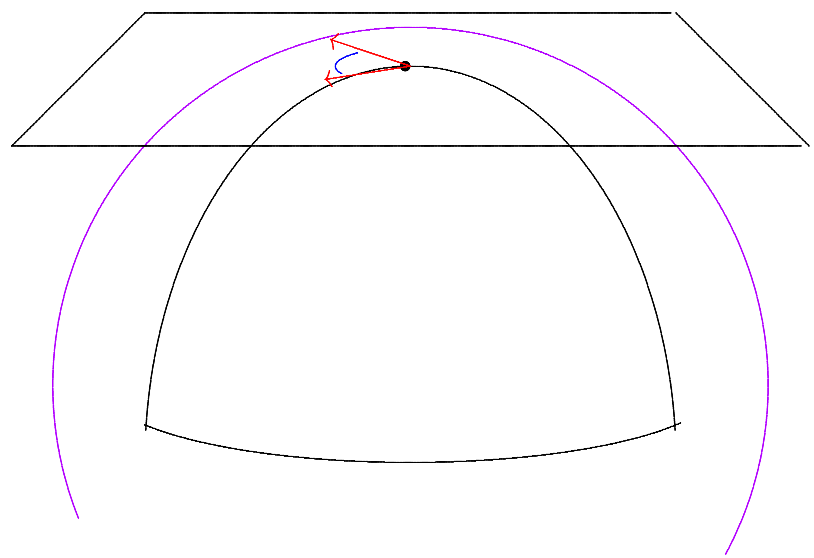

Makes tangent space a complex vector space!

Invariant under parallel transport!

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$\iff \exists$ almost complex-structure J with $\nabla J = 0$ and $g(J\cdot, J\cdot) = g$.

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$$d\omega = 0$$

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$$[\omega] \in H^2(M)$$

"Kähler class"

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$\Longleftrightarrow$ In local complex coordinates $(z^1, \ldots, z^m)$,

$$g = \sum_{j,k=1}^{m} \frac{\partial^2 f}{\partial z^j \partial \bar{z}^k} \left[dz^j \otimes d\bar{z}^k + d\bar{z}^k \otimes dz^j \right]$$

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$$\omega = i \sum_{j,k=1}^{m} \frac{\partial^2 f}{\partial z^j \partial \bar{z}^k} \, dz^j \wedge d\bar{z}^k$$

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Kähler magic:

$$r = -\sum_{j,k=1}^{m} \frac{\partial^2}{\partial z^j \partial \bar{z}^k} \log \det[g_{p\bar{q}}] \left[dz^j \otimes d\bar{z}^k + d\bar{z}^k \otimes dz^j\right]$$

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Kähler magic:

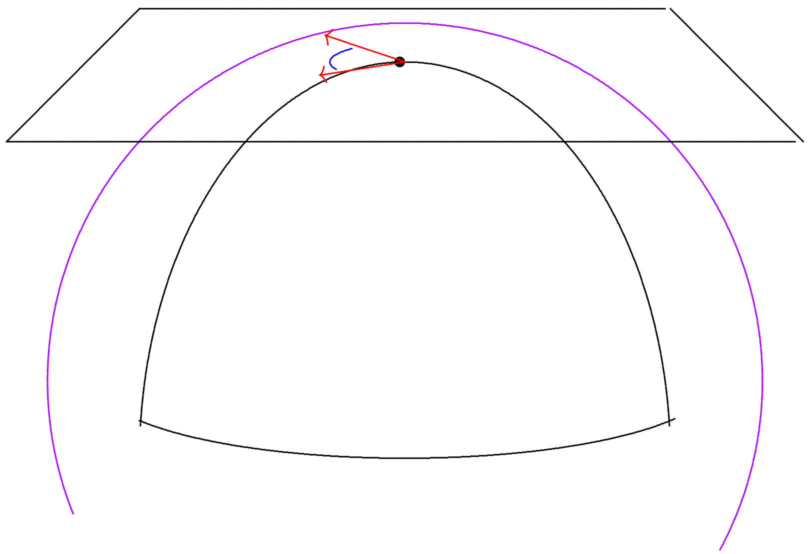
If we define the Ricci form by

$$\rho = r(J\cdot, \cdot)$$

then $i\rho$ is curvature of canonical line bundle $\Lambda^{m,0}$.

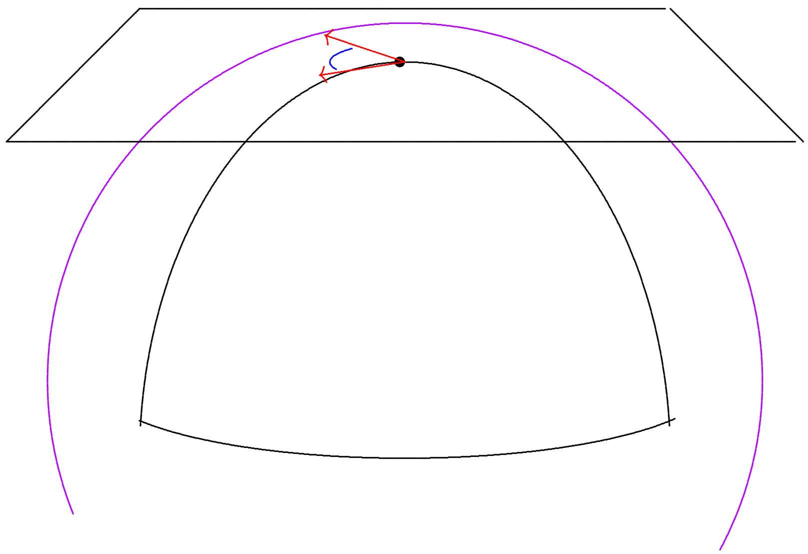
Kähler metrics:

(M^{2m}, g): holonomy



Kähler metrics:

(M^{2m}, g): Ricci-flat Kähler $\Longleftarrow$ holonomy $\subset \mathbf{SU}(m)$



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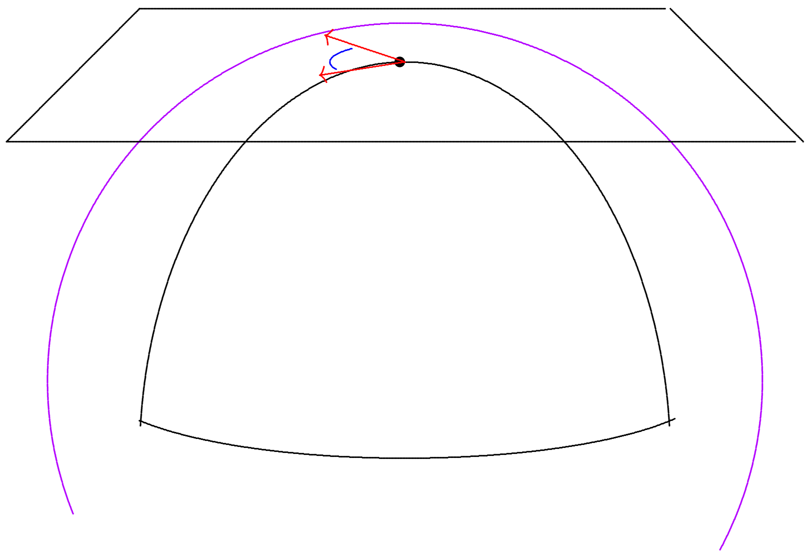
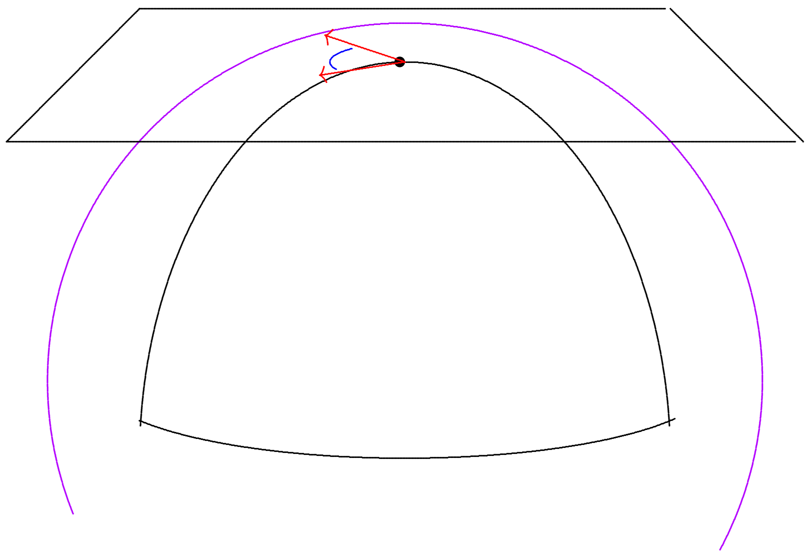


$\mathbf{SU}(m) \subset \mathbf{U}(m)$: $\quad \{A \mid \det A = 1\}$

Kähler metrics:

(M^{2m}, g): Ricci-flat Kähler $\Longleftarrow$ holonomy $\subset \mathbf{SU}(m)$



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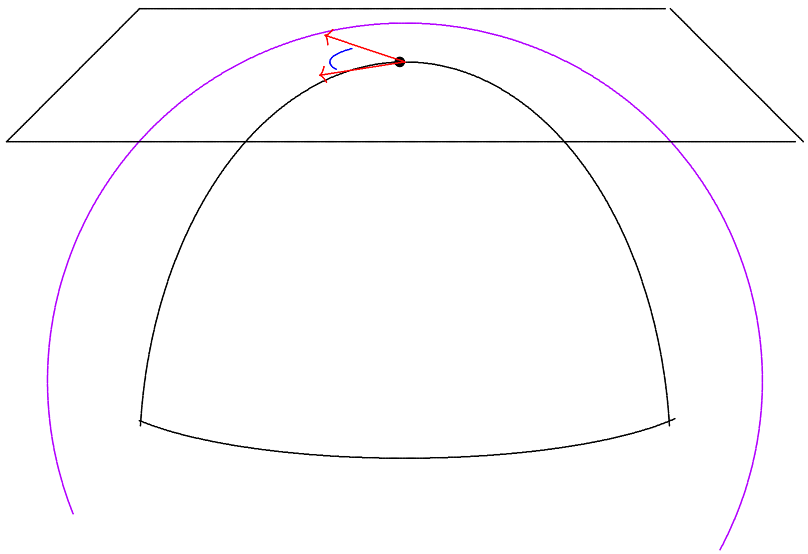
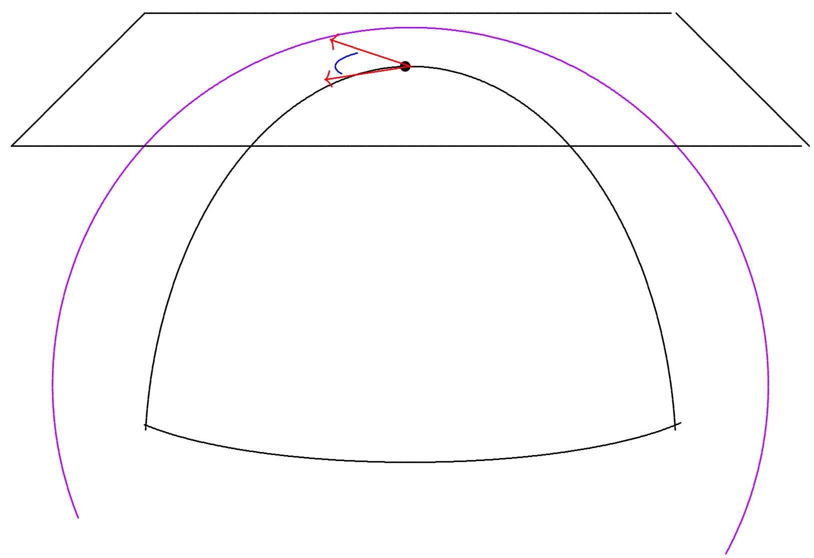


if M is simply connected.

Hyper-Kähler metrics:

$(M^{4\ell}, g)$

holonomy



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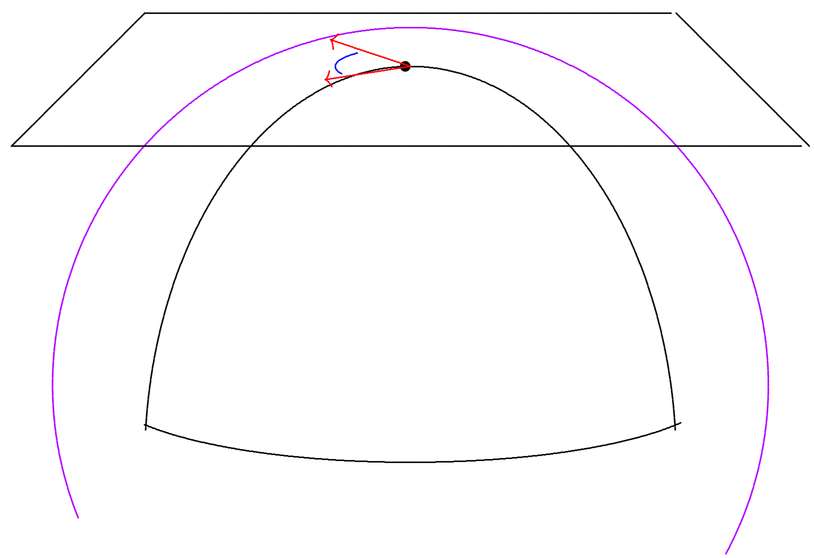

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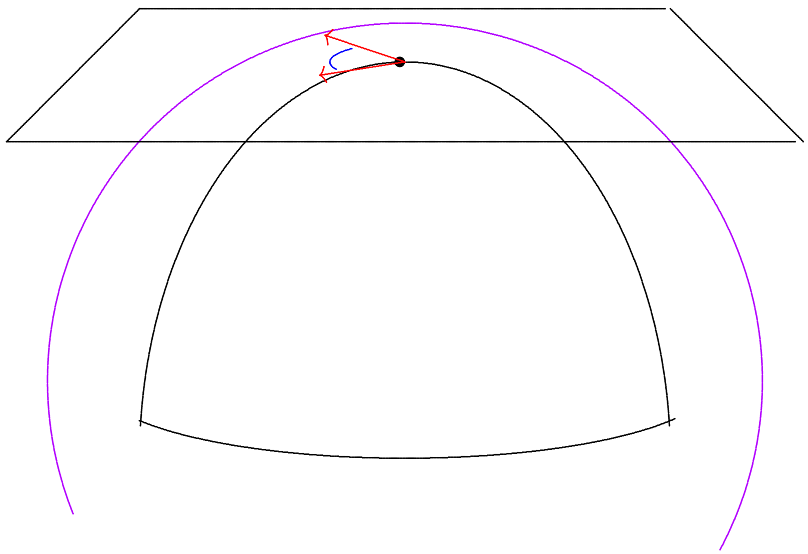

$$\mathbf{Sp}(\ell) := \mathbf{O}(4\ell) \cap \mathbf{GL}(\ell, \mathbb{H})$$

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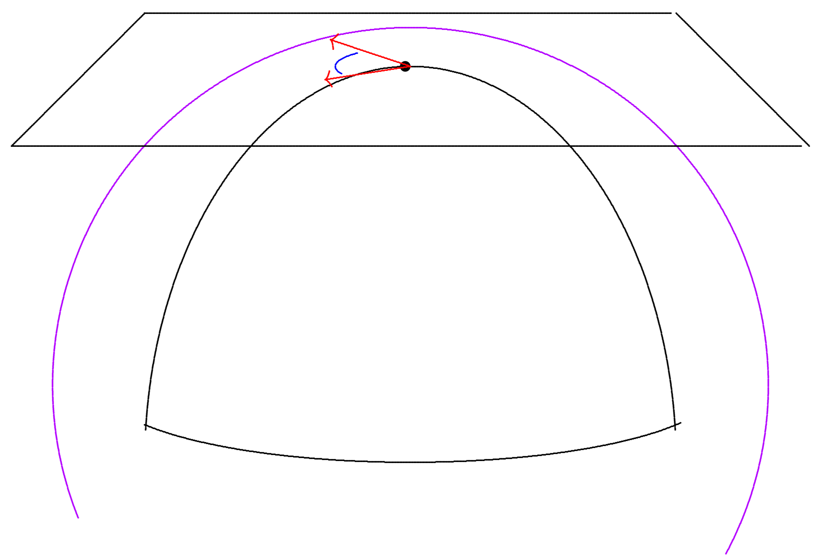

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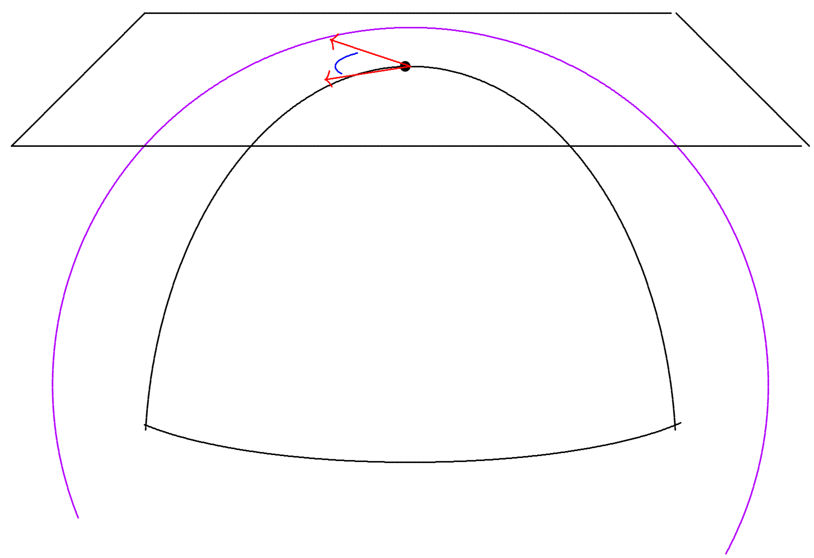

$$\mathbf{Sp}(\ell) \subset \mathbf{SU}(2\ell)$$

in many ways!

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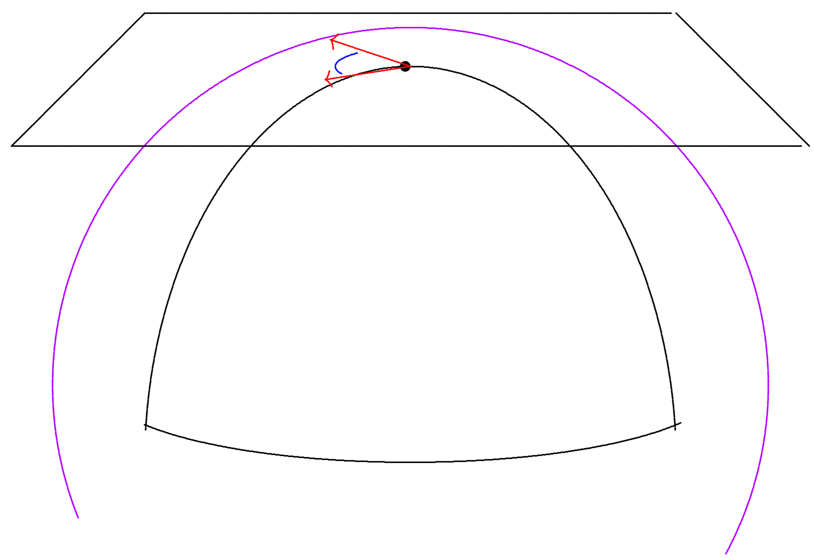

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in many ways! (For example, permute i, j, $k \ldots$)

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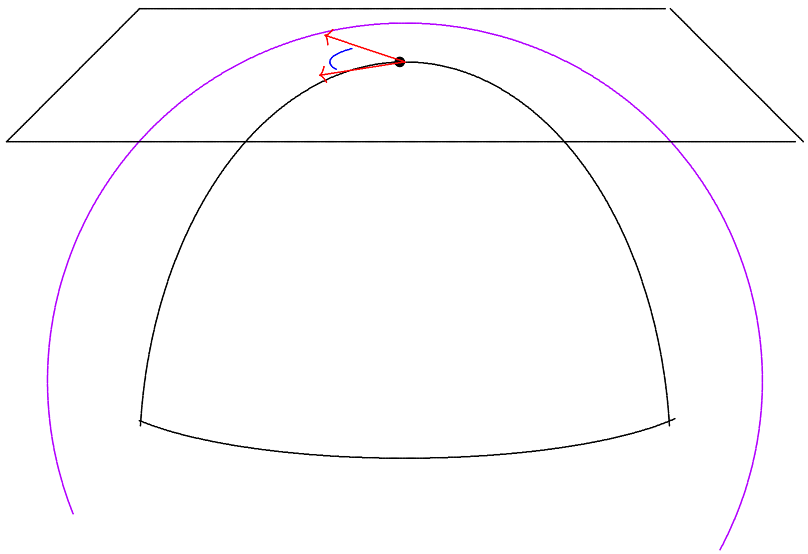

$\mathbf{Sp}(\ell) \subset \mathbf{SU}(2\ell)$

Ricci-flat and Kähler,

for many different complex structures!

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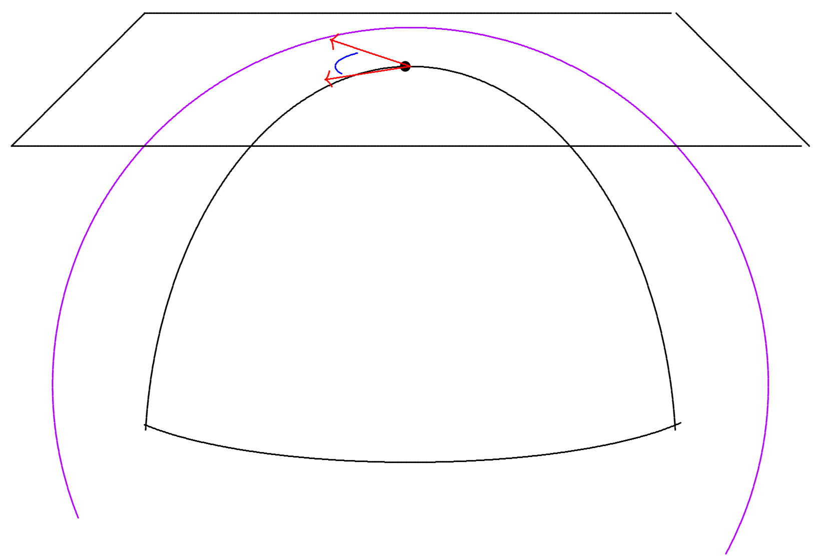

$\mathbf{Sp}(\ell) \subset \mathbf{SU}(2\ell)$

Hyper-Kähler metrics:

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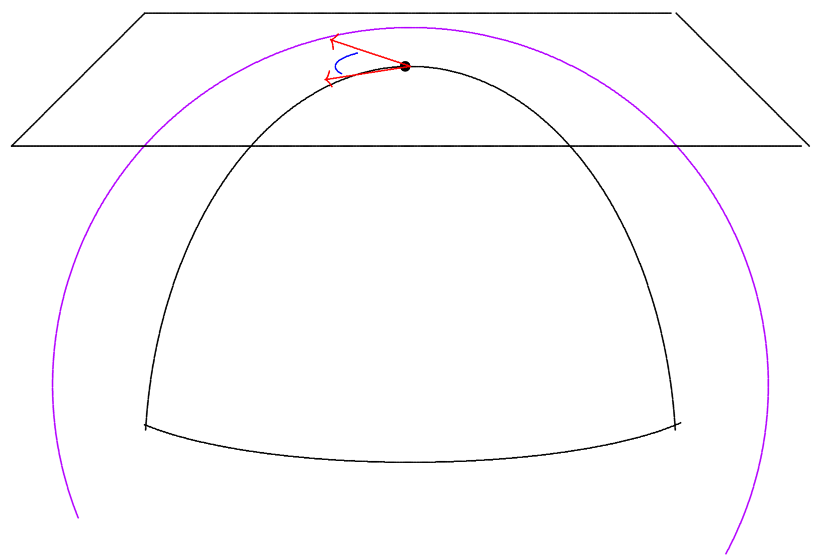

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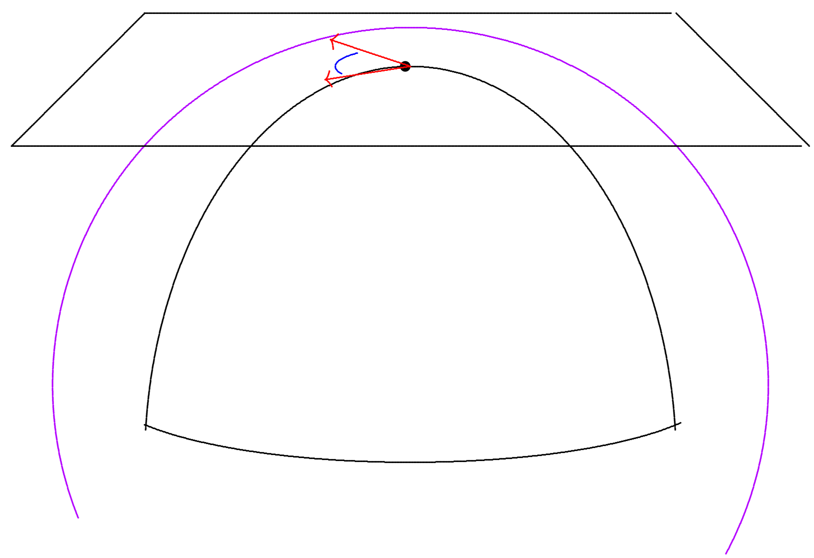

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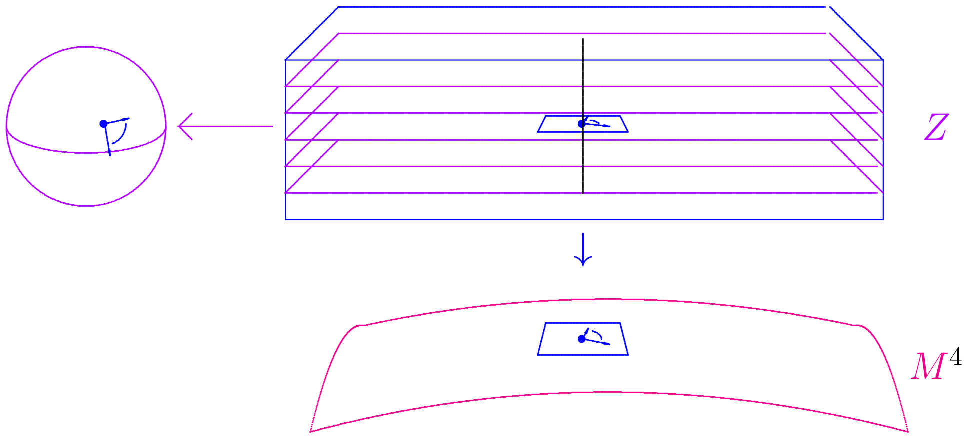
When (M^4, g) simply connected:

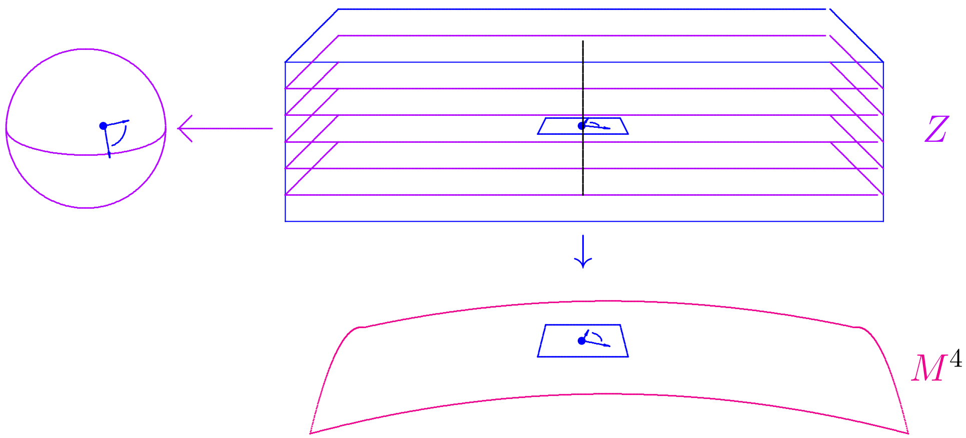
hyper-Kähler $\Longleftrightarrow$ Ricci-flat Kähler.

All these complex structures can be repackaged



All these complex structures can be repackaged as

Penrose Twistor Space (Z, J),



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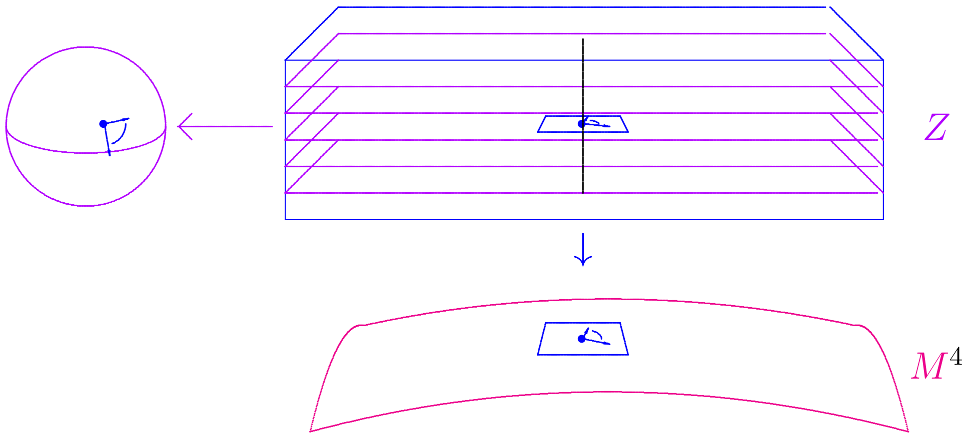

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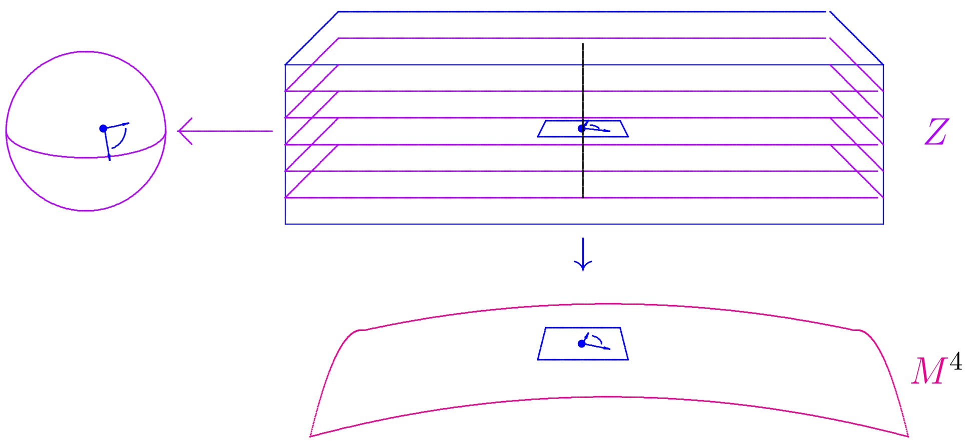

Riemannian non-linear graviton construction.

Key examples:

Term ALE coined by Gibbons & Hawking, 1979.

They wrote down various explicit Ricci-flat ALE 4-manifolds they called gravitational instantons.

Their examples have just one end, with

$$\Gamma \cong \mathbb{Z}_\ell \subset \mathbf{SU}(2) \subset \mathbf{O}(4).$$

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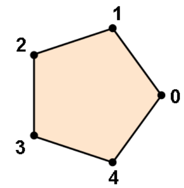
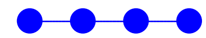
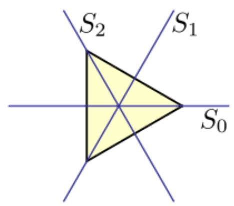
This conjecture was proved by Kronheimer, 1986.

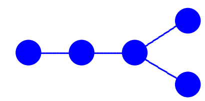


$\mathbb{Z}_{k+1} \longleftrightarrow A_k$

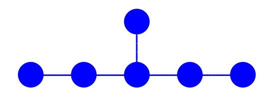


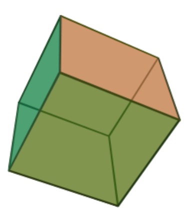


$\mathrm{Dih}^*_{k-2} \longleftrightarrow D_k$

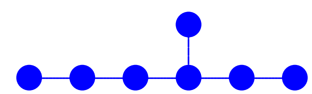


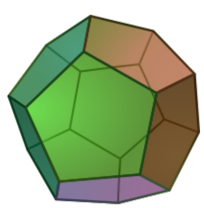
$T^* \longleftrightarrow E_6$



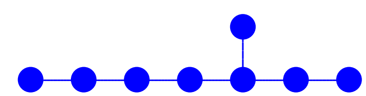


$O^* \longleftrightarrow E_7$





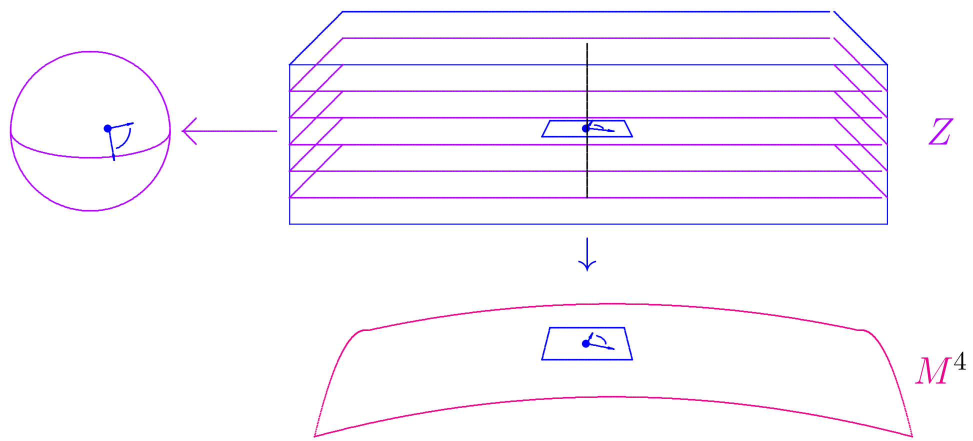
$I^* \longleftrightarrow E_8$



All these complex structures can be repackaged as

Penrose Twistor Space (Z, J),

which is a complex 3-manifold.



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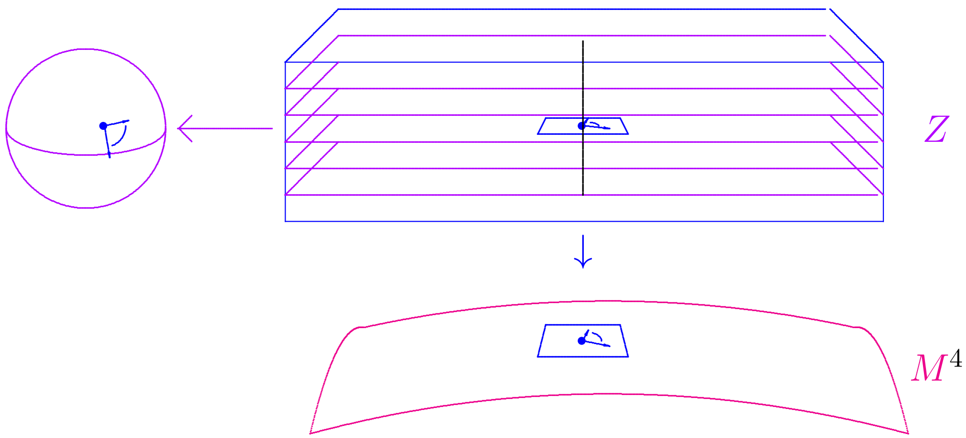

But similar for scalar-flat Kähler surfaces (M^4, g, J)!

Any scalar-flat Kähler surface (M^4, g, J) has a

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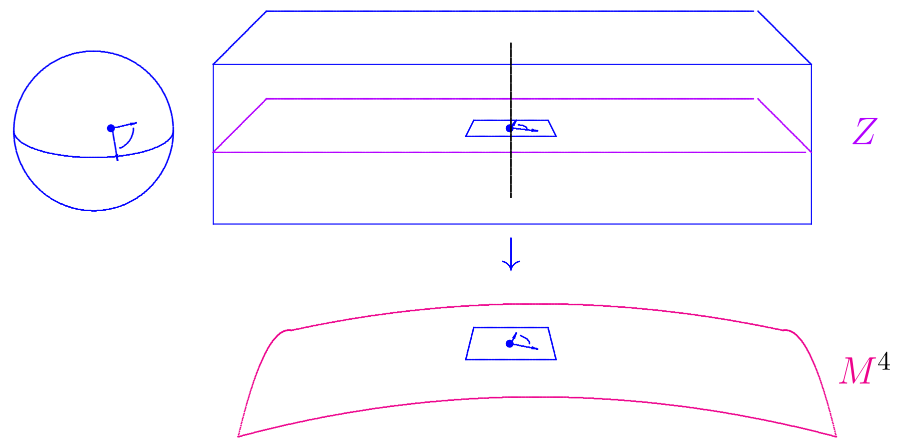

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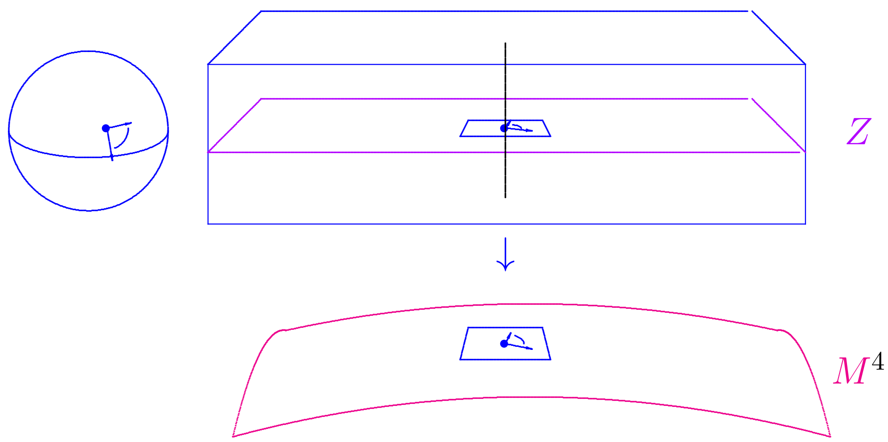

The construction of scalar-flat Kähler surfaces and the study of their twistor spaces was a main focus of my own work during the decade 1985-1994.

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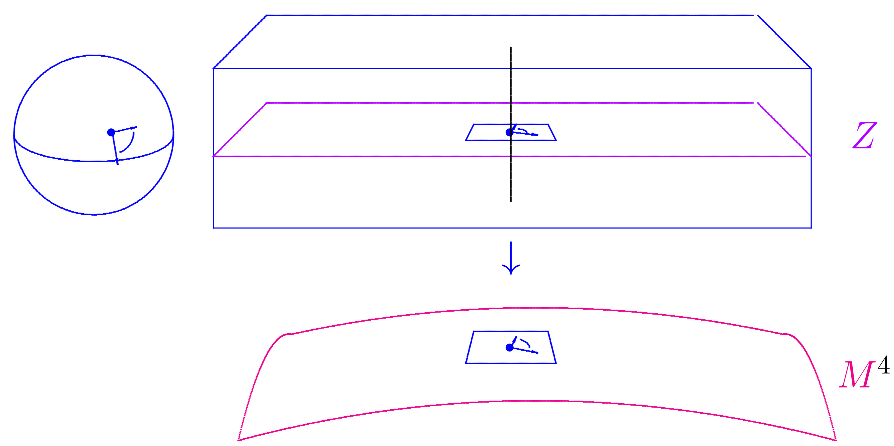

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Constructed ALE examples on line bundles $L \to \mathbb{CP}_1$ with $c_1 < 0$, and on their blow-ups.

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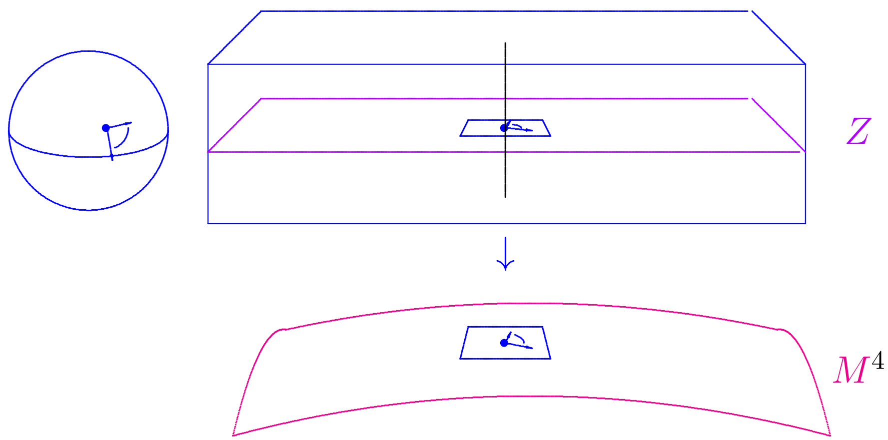

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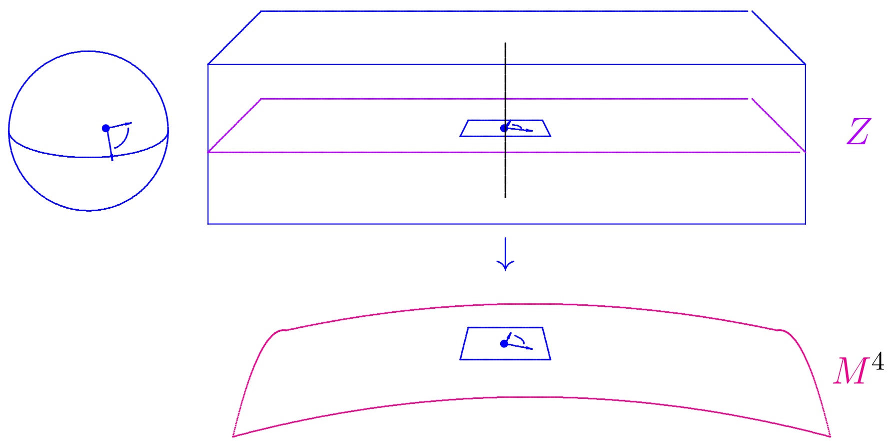

These ALE spaces arise naturally in the study of compact Einstein or Bach-flat 4-manifolds as bubbling modes for sequences of metrics.

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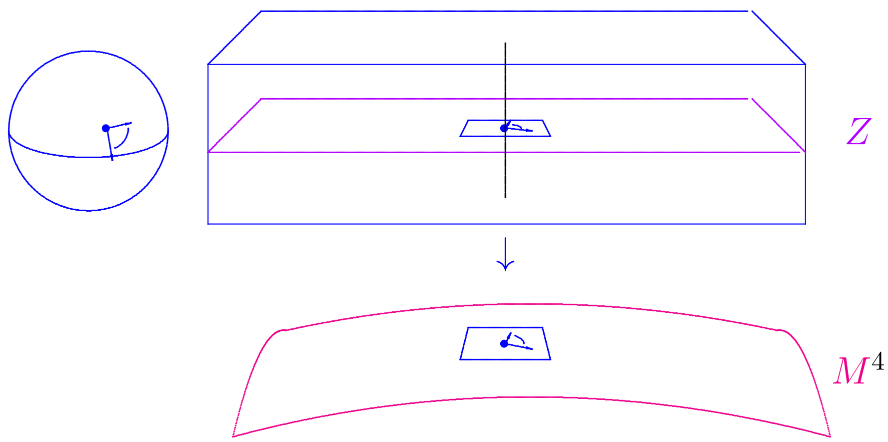

Lots more ALE scalar-flat Kähler surfaces now known:

Any scalar-flat Kähler surface (M^4, g, J) has a

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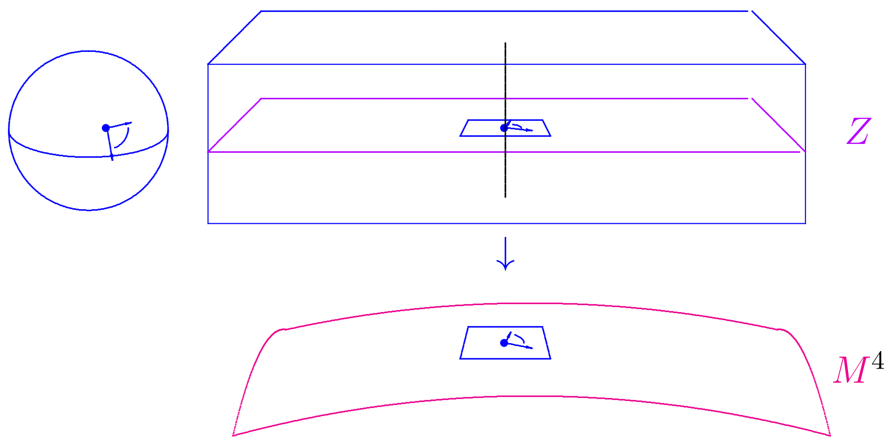

Lots more ALE scalar-flat Kähler surfaces now known:

Joyce, Calderbank-Singer, Lock-Viaclovsky. . .

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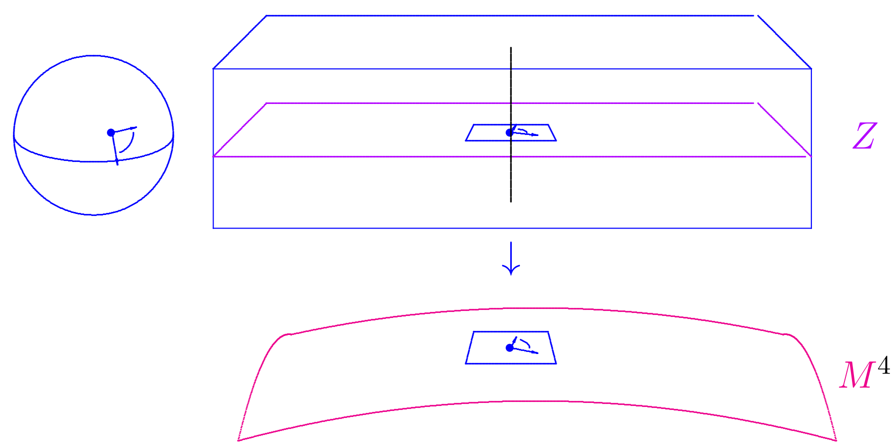

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Every possible $\Gamma \subset \mathbf{U}(2)$ occurs.

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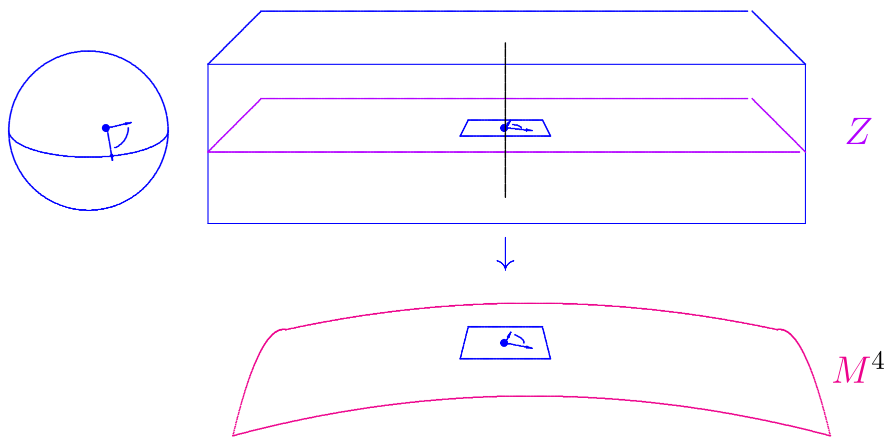
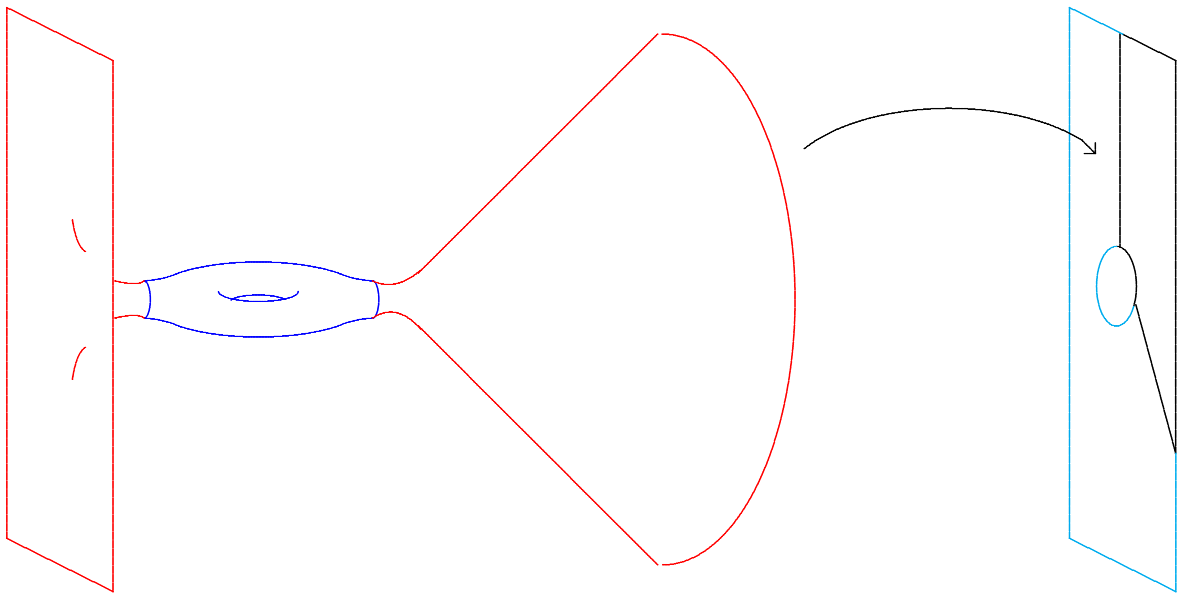


Lots more ALE scalar-flat Kähler surfaces now known:

Joyce, Calderbank-Singer, Lock-Viaclovsky...

But full classification remains an open problem.

Definition. *Complete, non-compact* n*-manifold* (M^n, g) *is* asymptotically locally Euclidean *(ALE) if* $\exists$ *compact set* $K \subset M$ *such that* $M - K \approx \coprod_i (\mathbb{R}^n - D^n)/\Gamma_i$, *where* $\Gamma_i \subset \mathbf{O(n)}$, *such that*



$$g_{jk} = \delta_{jk} + O(|x|^{1-\frac{n}{2}-\varepsilon})$$

$$g_{jk,\ell} = O(|x|^{-\frac{n}{2}-\varepsilon}), \quad s \in L^1$$

Mass still meaningful in this context. . .

Definition. *The mass (at a given end) of an ALE n-manifold is defined to be*

$$\mathfrak{m}(M, g) := \lim_{\varrho \to \infty} \frac{\Gamma(\frac{n}{2})}{4(n-1)\pi^{n/2}} \int_{\Sigma(\varrho)} \left[g_{ij,i} - g_{ii,j}\right] \nu^j \alpha_E$$

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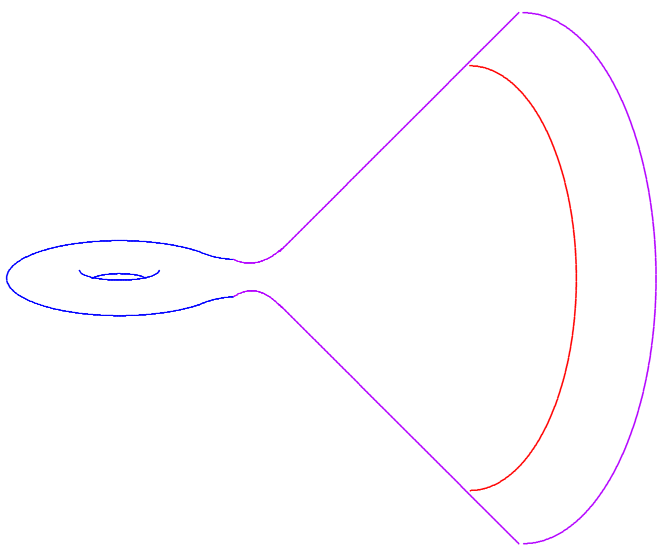
where

- $\Sigma(\varrho) \approx S^{n-1}/\Gamma_i$



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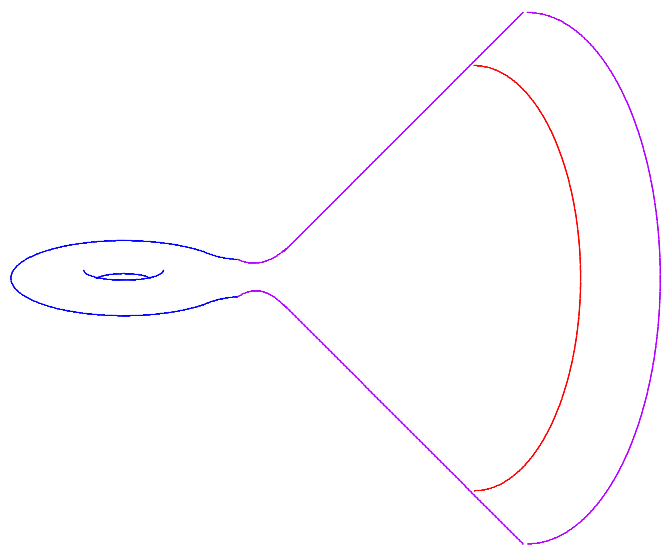

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Bartnik/Chruściel (1986): With weak fall-off conditions, the mass is well-defined & coordinate independent.

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Chruściel-type fall-off:

$$g_{jk} - \delta_{jk} \in C^1_{-\tau}, \quad \tau > \frac{n-2}{2}$$

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Bartnik-type fall-off:

$$g_{jk} - \delta_{jk} \in W^{2,q}_{-\tau}, \quad \tau > \frac{n-2}{2}, \quad q > n$$

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Bartnik-type fall-off $\Longrightarrow$

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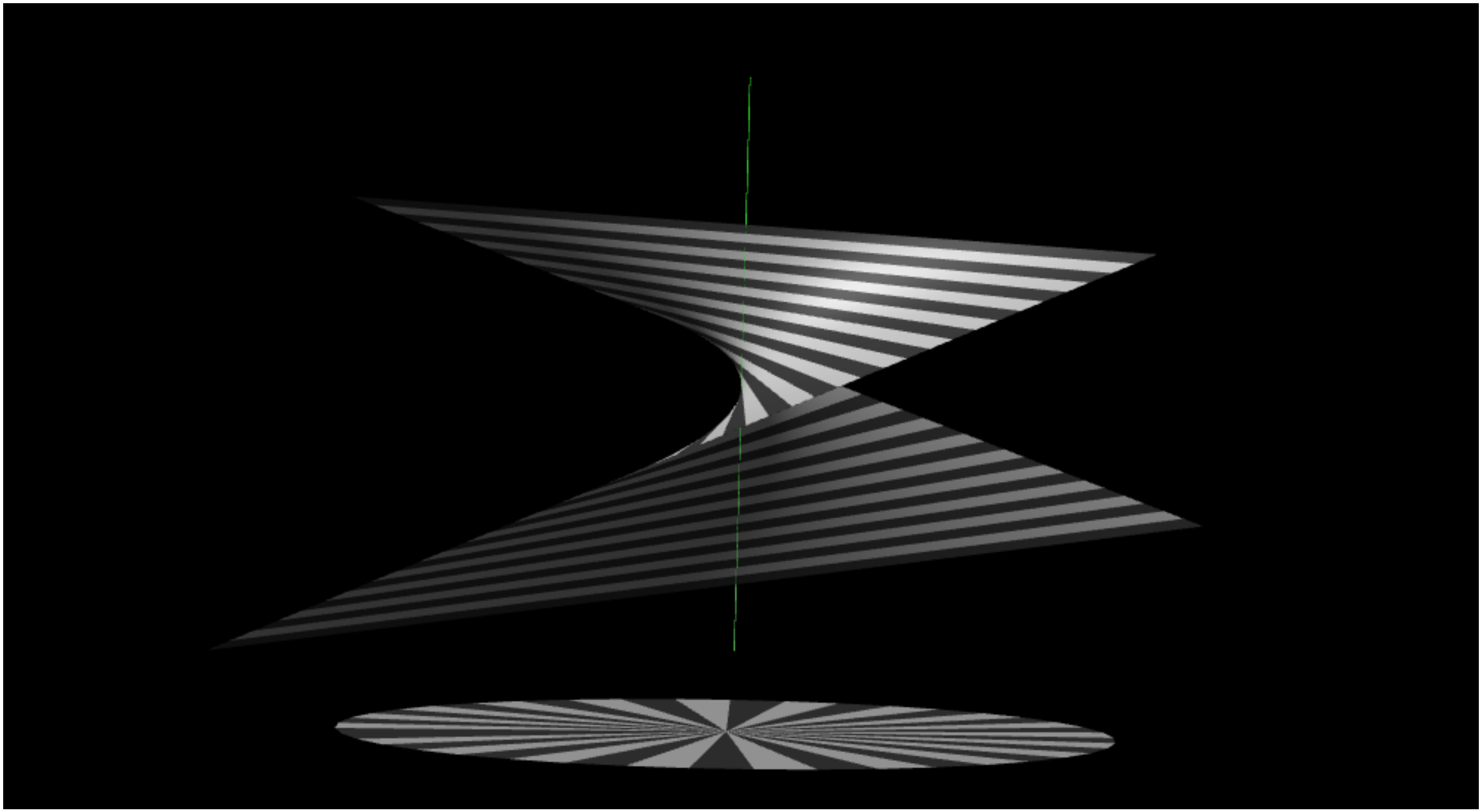

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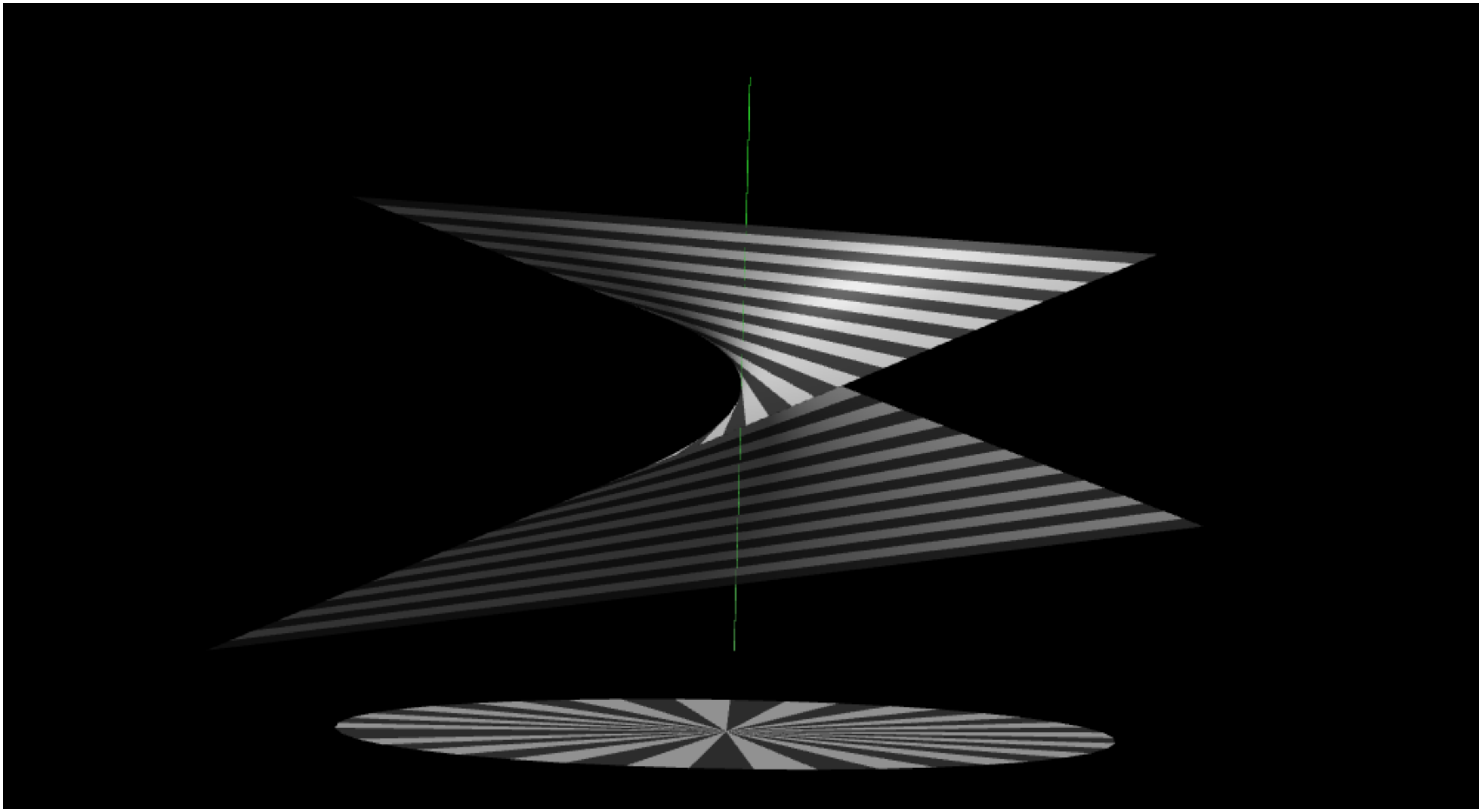

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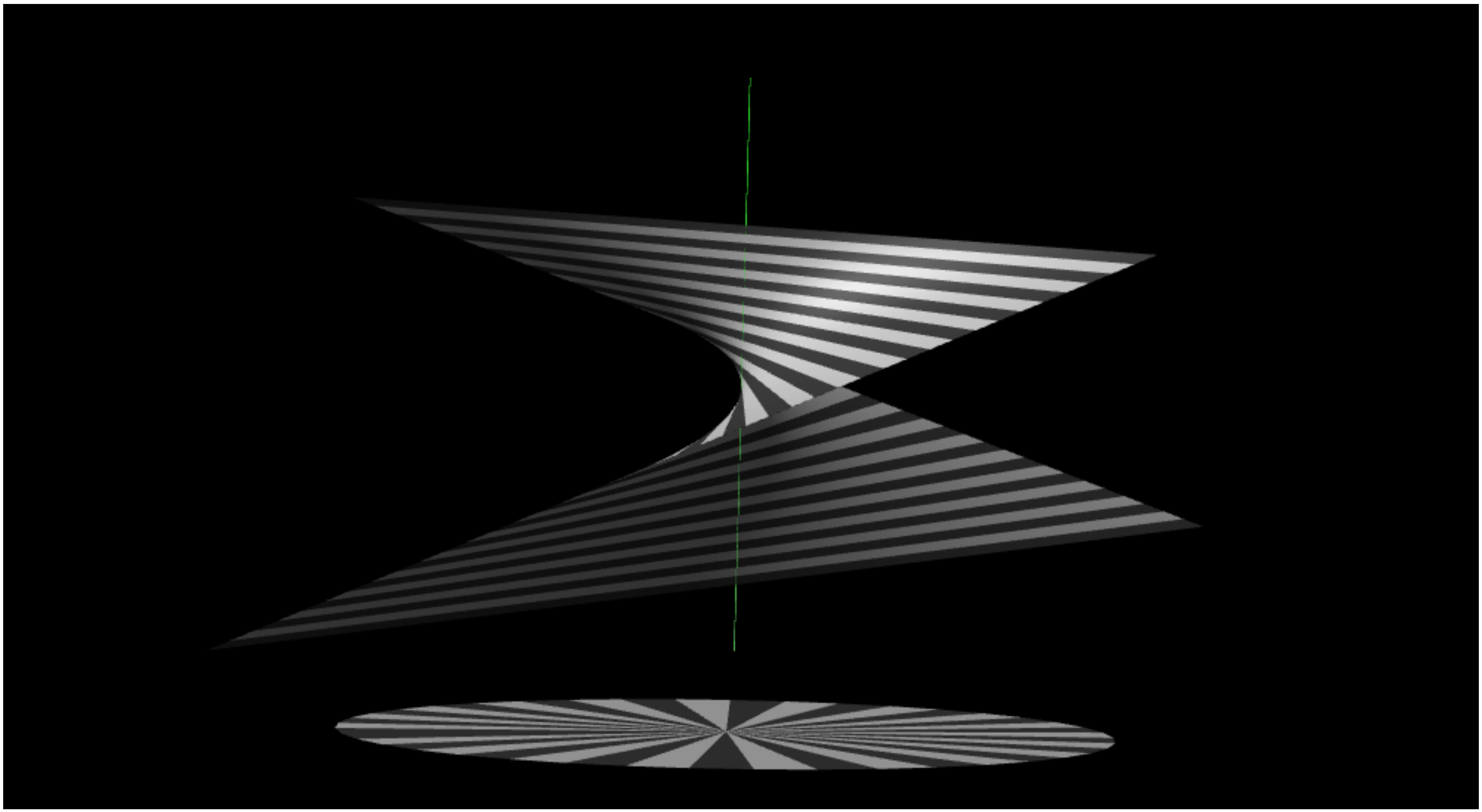

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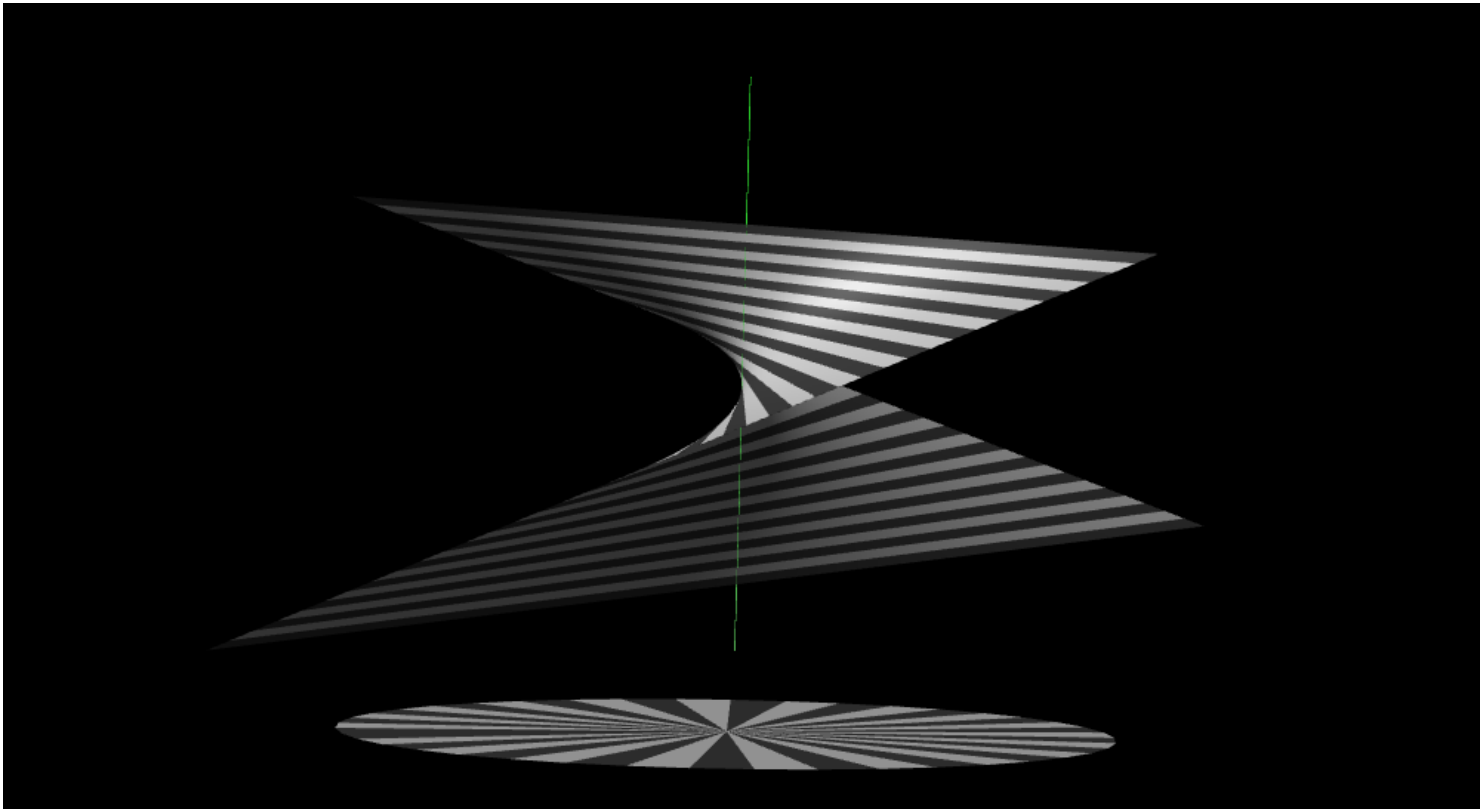

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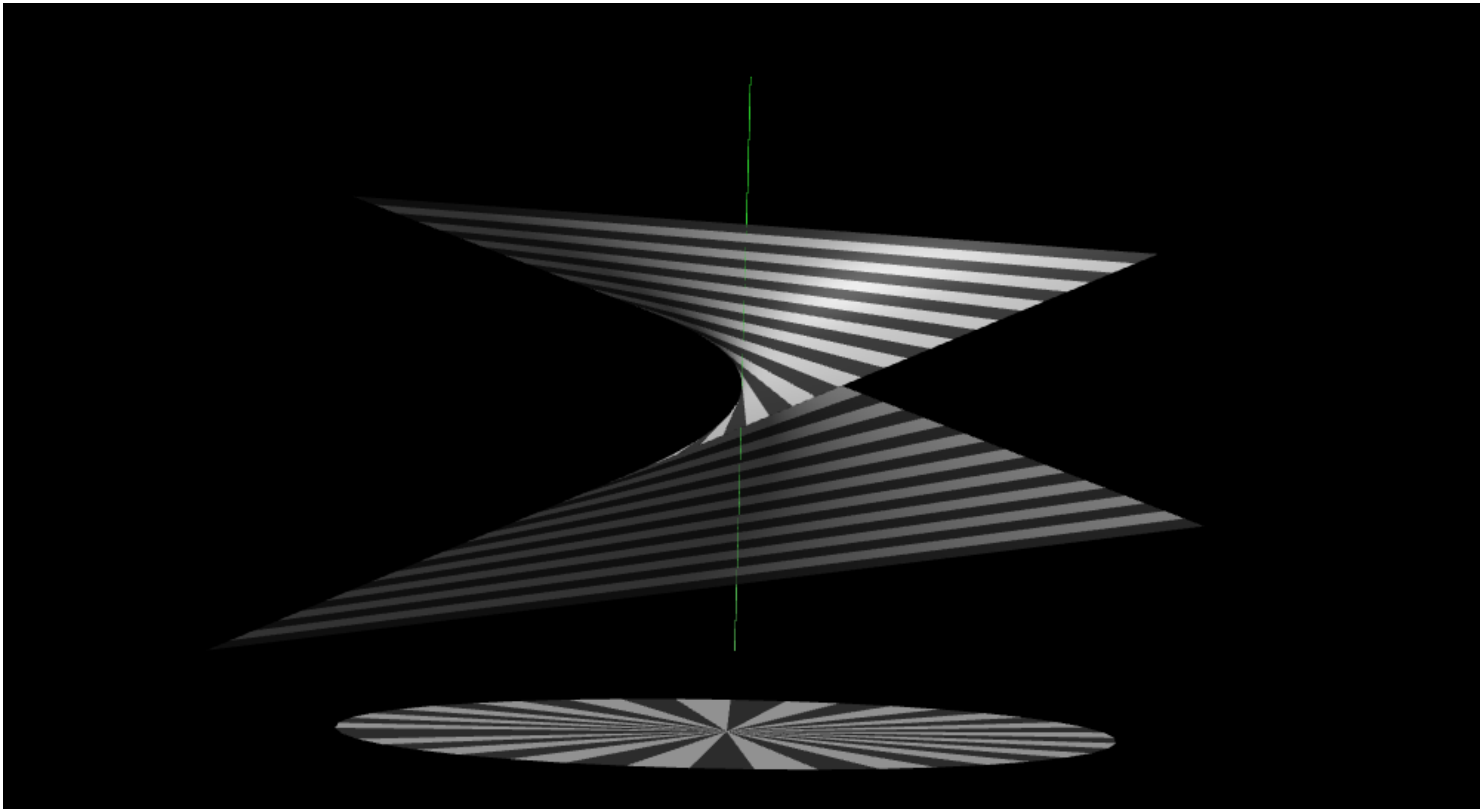

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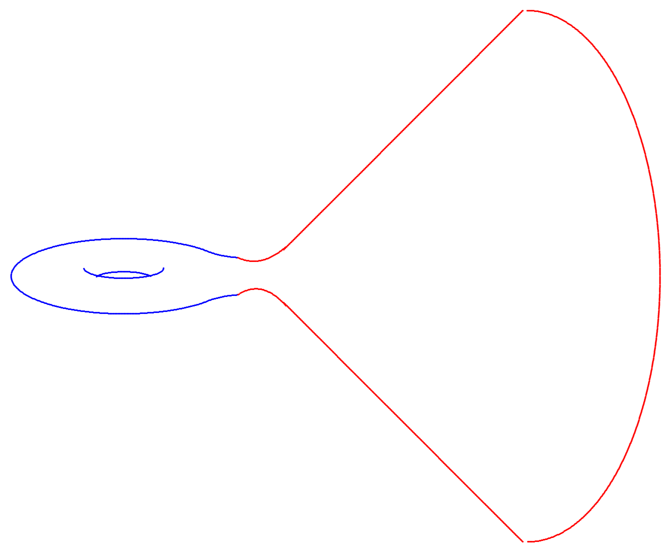

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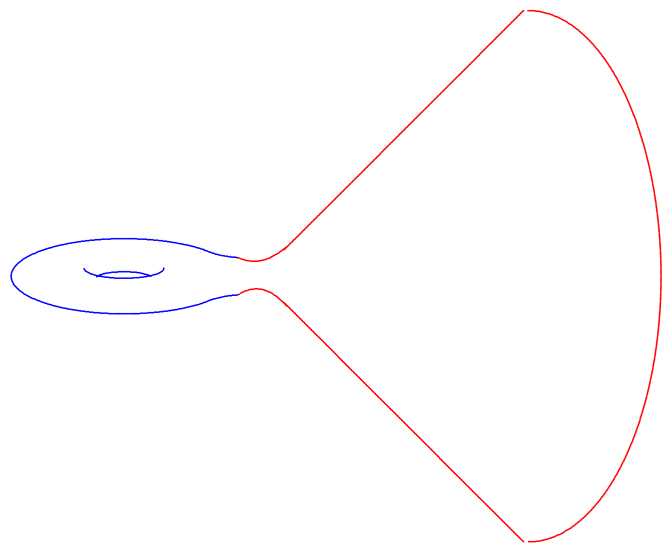

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In fact, we will see that there is an explicit formula for the mass in terms of these data!

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Non-minimal resolutions typically admit families of such metrics for which the mass can be continuously deformed from negative to positive.

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So the mass is a "boundary correction" to the topological formula for the total scalar curvature.

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So **Theorem A** is an immediate consequence!

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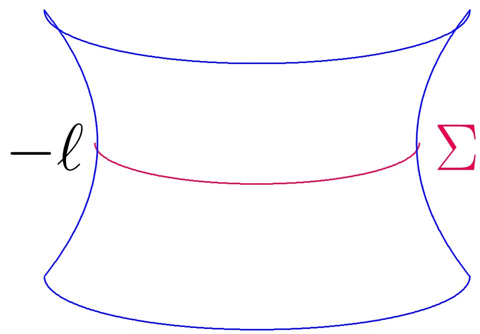
Example. Blow up Chern class $-\ell$ line bundle over $\mathbb{CP}_1$ at k points on zero section Σ.



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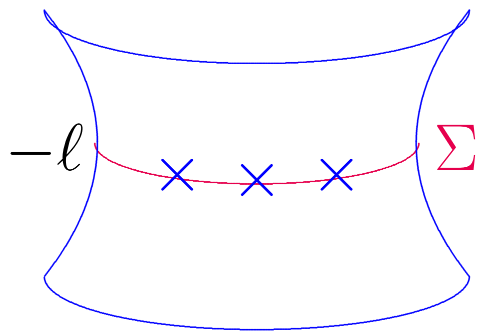

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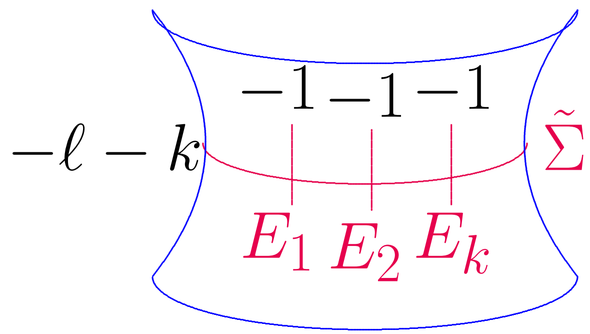

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$$g^{jk}\left(g_{j\ell,k} - g_{jk,\ell}\right)\nu^\ell \alpha_E = -\star d \log\left(\sqrt{\det g}\right) + O(\varrho^{-3-\varepsilon}).$$

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However, since $s=0$,

$$d(\theta\wedge\omega) = \rho\wedge\omega = \frac{s}{4}\omega^2 = 0.$$

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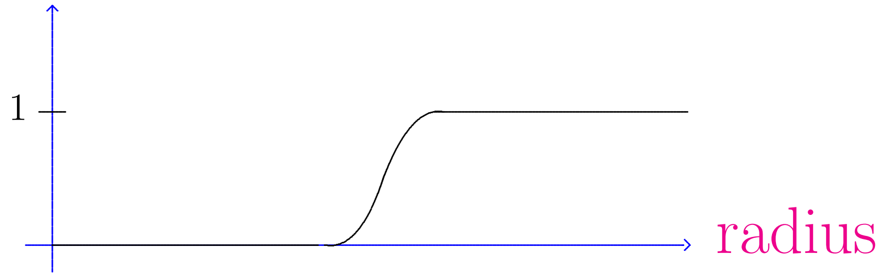

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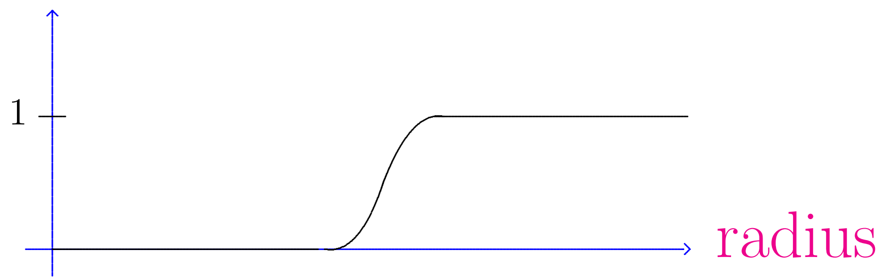

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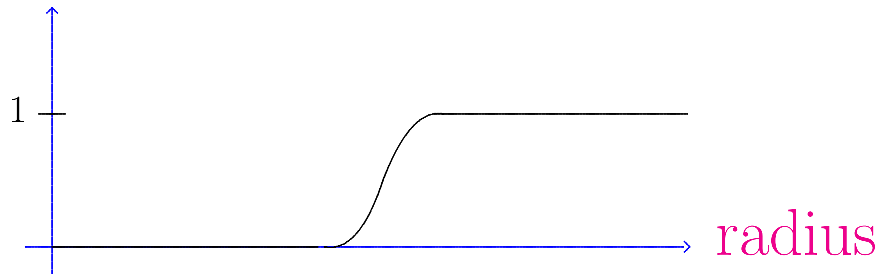

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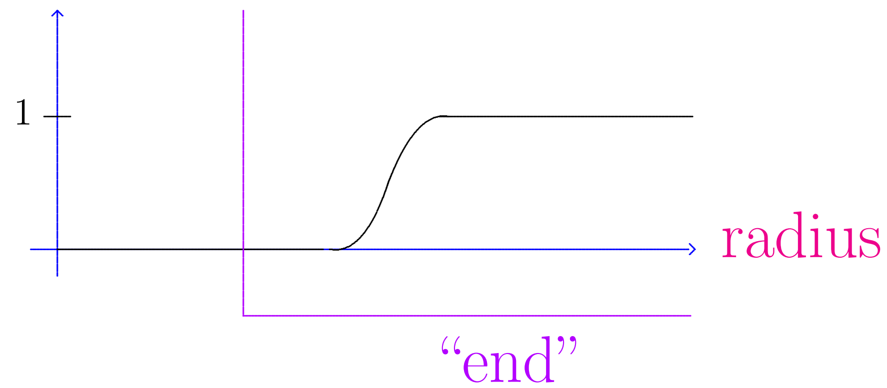

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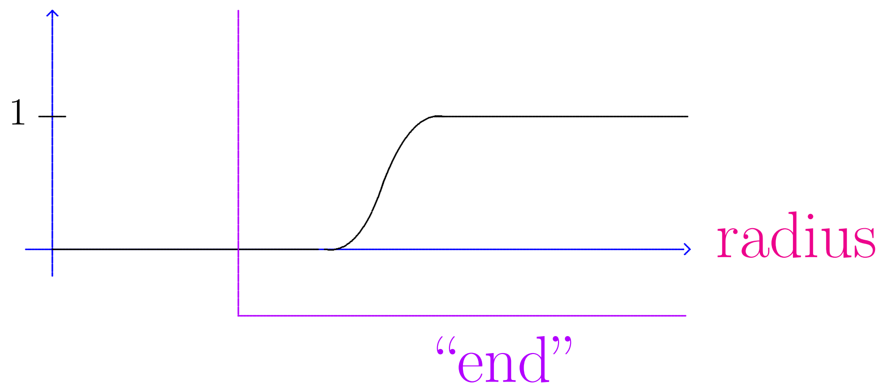

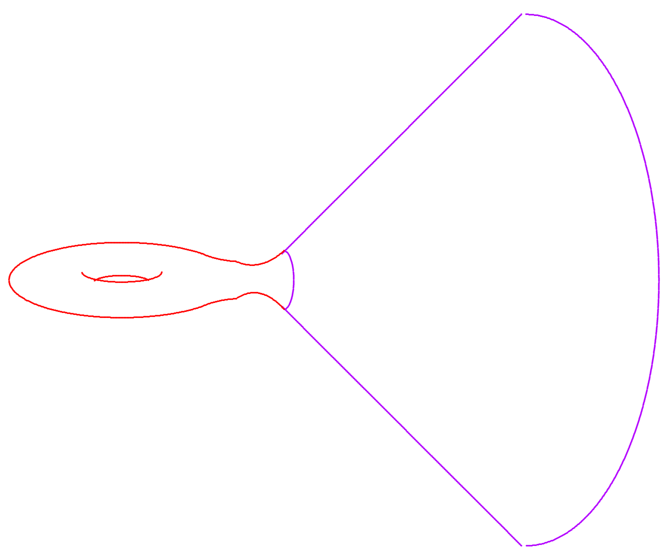

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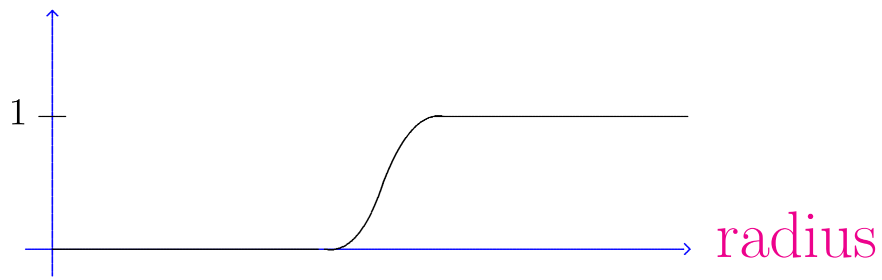

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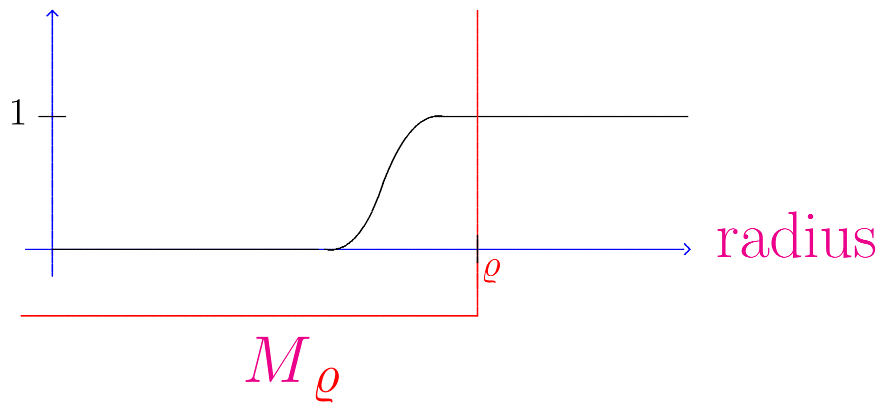

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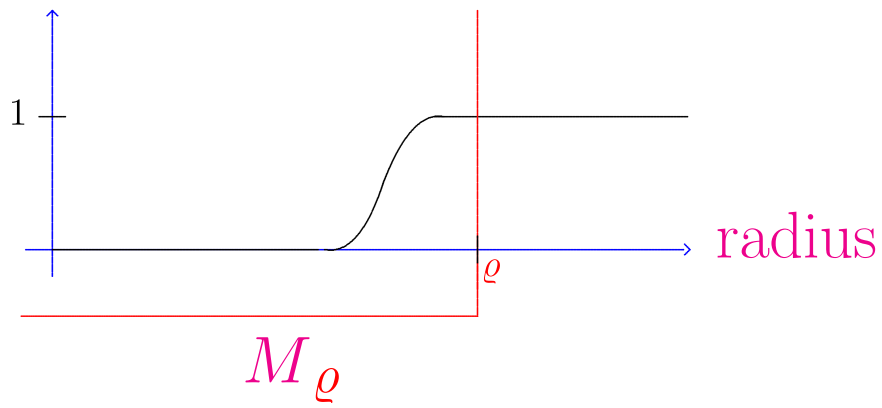

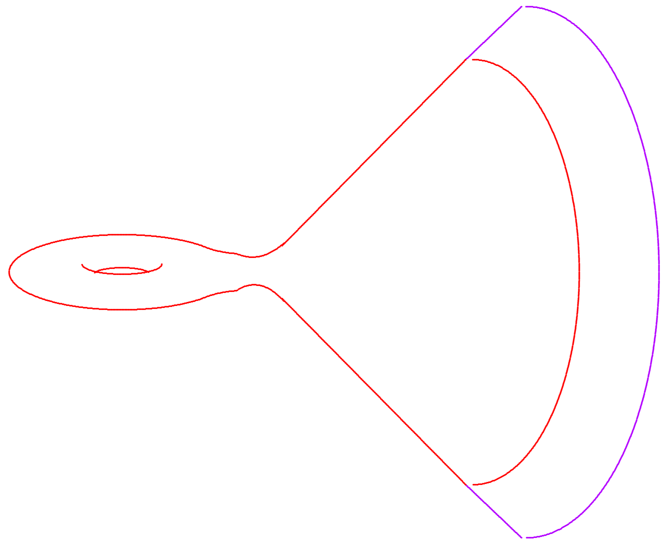

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Compactly supported, because $d\theta = \rho$ near infinity.

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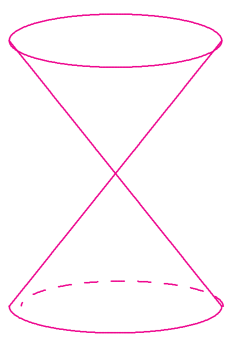

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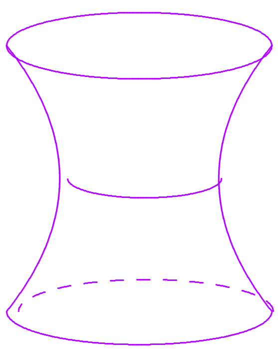

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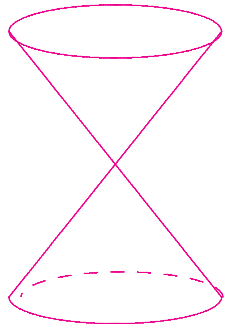

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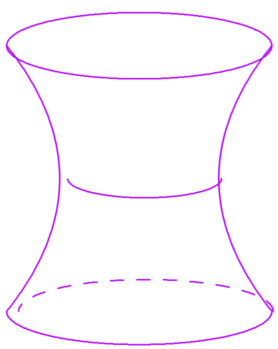

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$$J = J_0 + O(\varrho^{-3}), \qquad \nabla J = O(\varrho^{-4})$$

in suitable asymptotic coordinates adapted to g.

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This has some interesting consequences. . .

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Proof actually shows something stronger!

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so the mass formula implies the claim.

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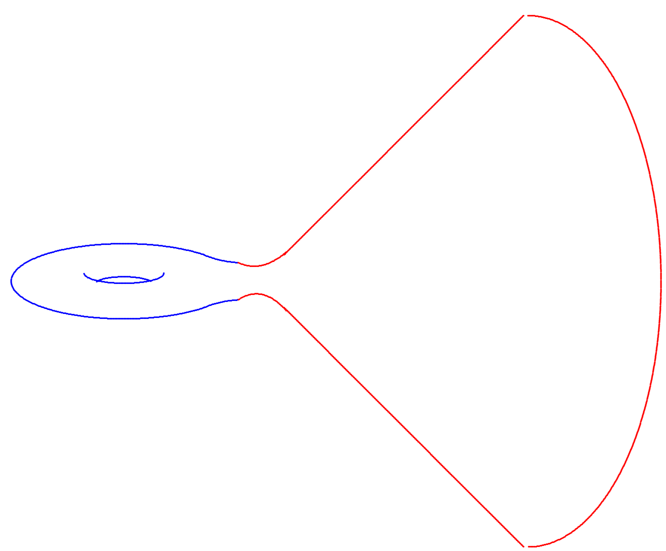

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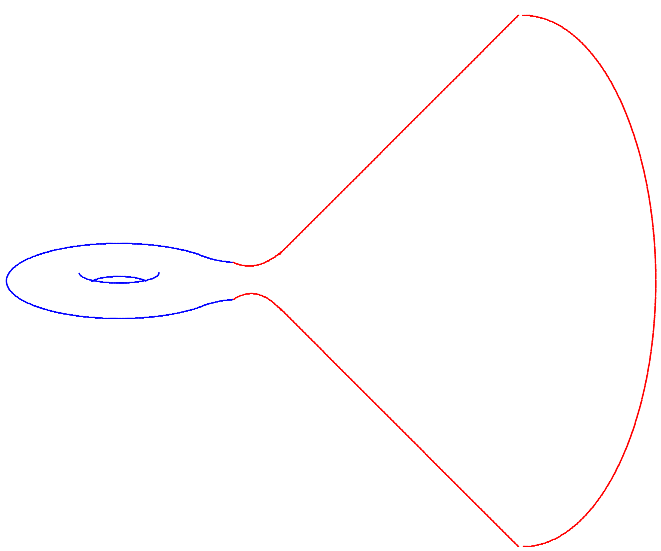

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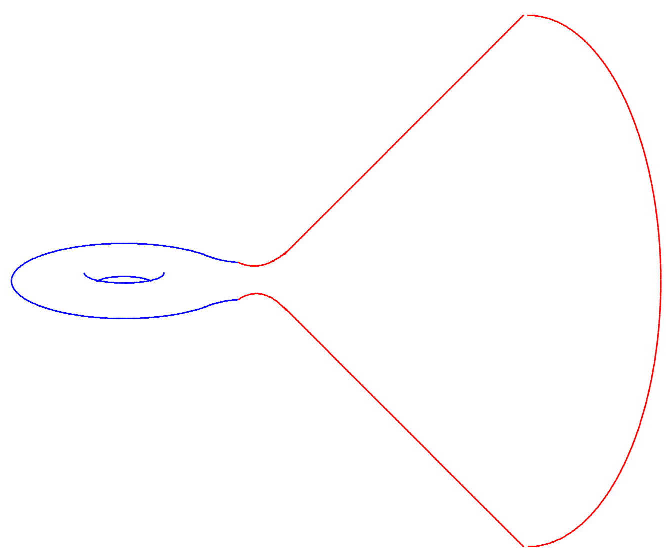

Thanks for coming!

Happy Easter!

A Joyous Passover!

Safe Travels!