Einstein Metrics,

Harmonic Forms, &

Symplectic Four-Manifolds

Claude LeBrun
Stony Brook University

Sardegna, 2014/9/20

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"...the greatest blunder of my life!"

— A. Einstein, to G. Gamow

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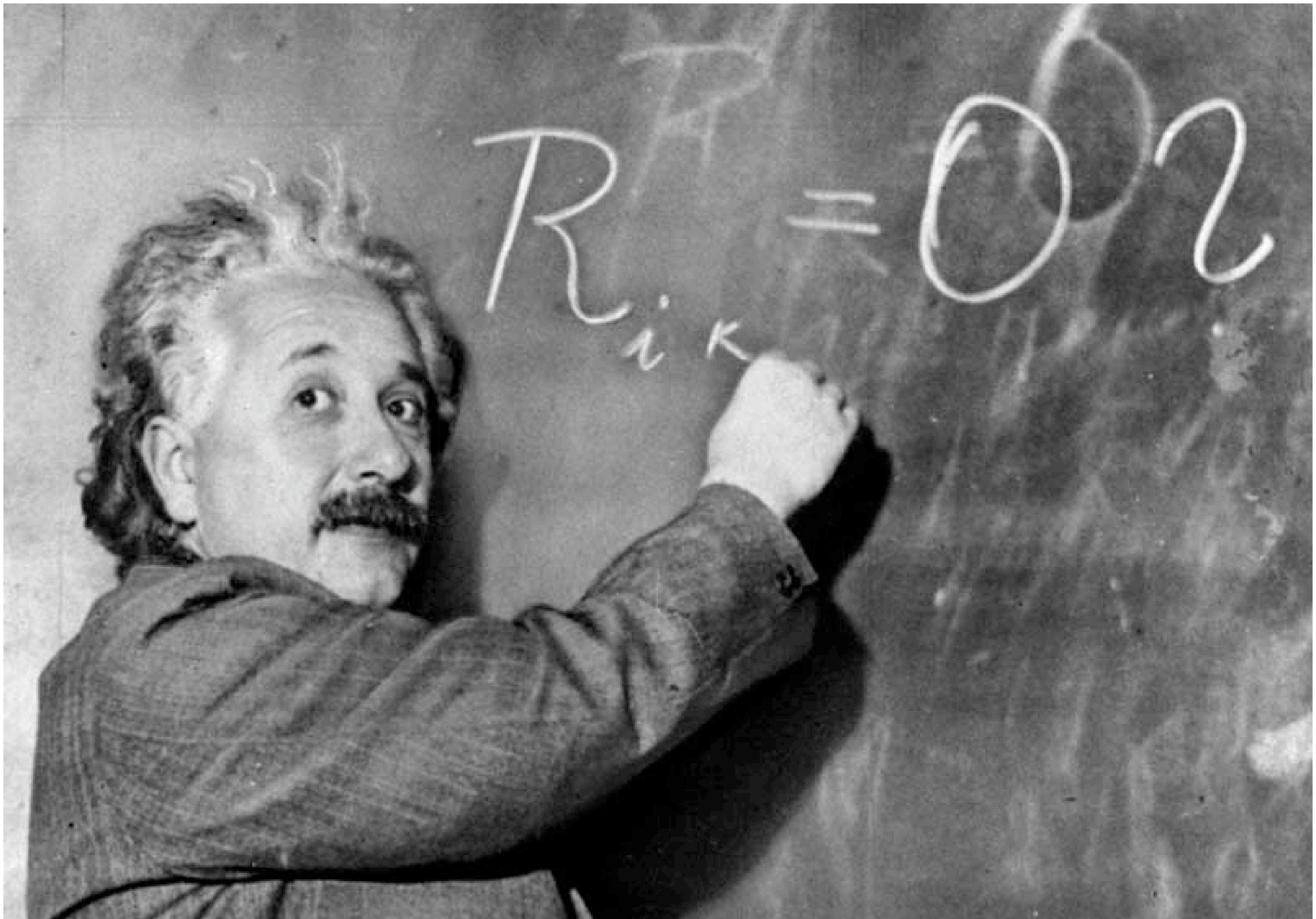

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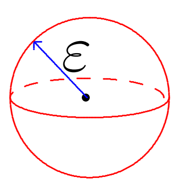
$$\frac{\mathrm{vol}_g(B_\varepsilon(p))}{c_n \varepsilon^n} = 1 - s\, \frac{\varepsilon^2}{6(n+2)} + O(\varepsilon^4)$$



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Perhaps reasonable in other dimensions?

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When $n \geq 4$, situation is more encouraging...

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One key question:

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One key question:

Does enough rigidity really hold in dimension four to make this a genuine geometrization?

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Some Suggestive Questions. *If (M^4, ω) is a symplectic 4-manifold, when does M^4 admit an Einstein metric h (unrelated to ω)?*

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Some Suggestive Questions. *If* (M^4, ω) *is a symplectic* 4*-manifold, when does* M^4 *admit an Einstein metric* h *(unrelated to* ω*)? What if we also require* $\lambda \geq 0$*?*

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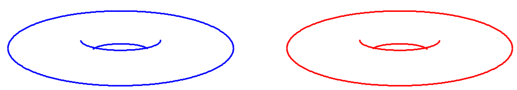

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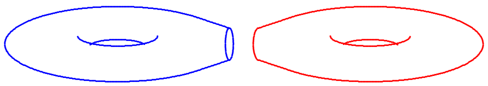

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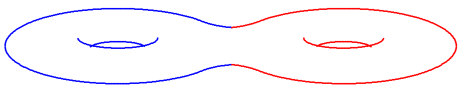

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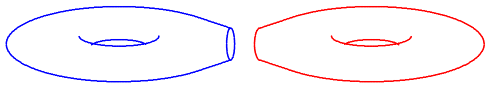

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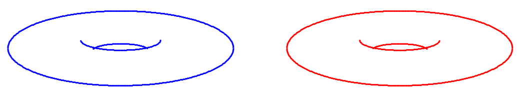

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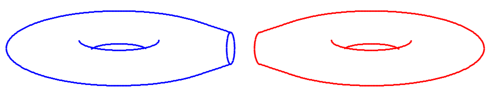

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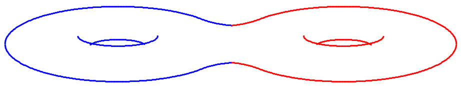

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Del Pezzo surfaces,
K3 surface, Enriques surface,
Abelian surface, Hyper-elliptic surfaces.

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Every Einstein metric is Ricci-flat Kähler.

Moduli space $\mathscr{E}(M)$ connected!

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Know an Einstein metric on each manifold.

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- These form a connected family, mod diffeos; and
- No other Einstein metrics belong to $\mathscr{U}$!

Formulation will depend on. . .

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Λ^+ self-dual 2-forms.
Λ^- anti-self-dual 2-forms.

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s = scalar curvature

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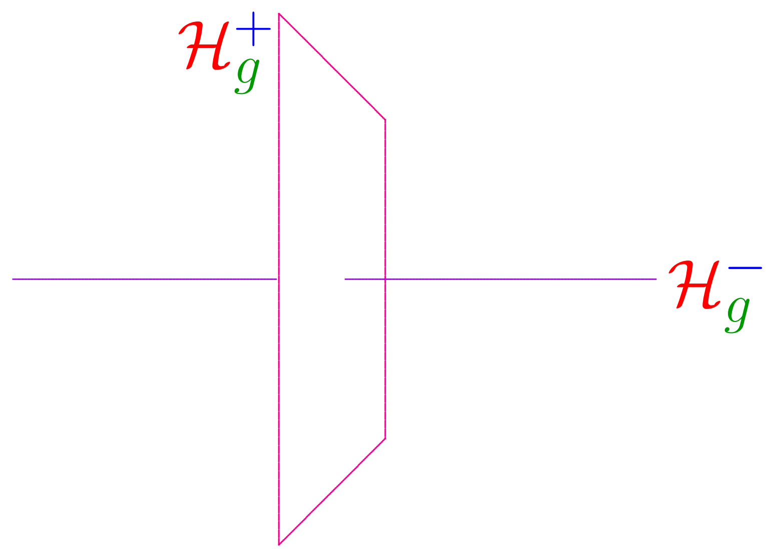
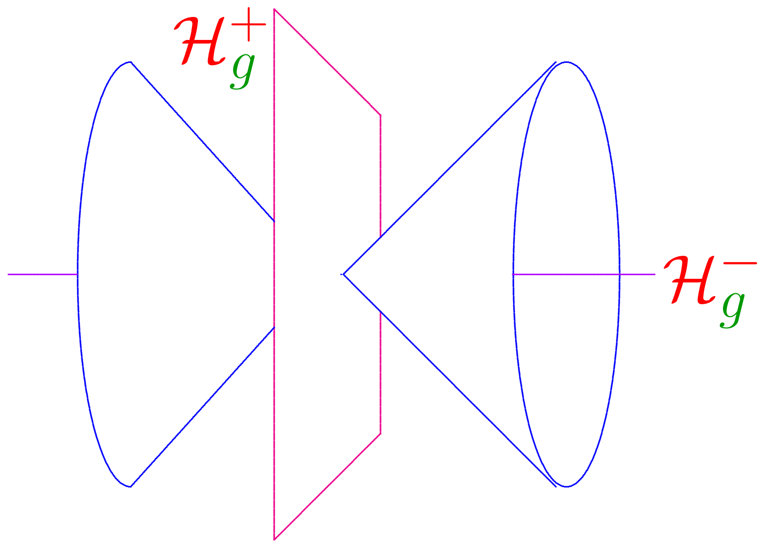


$$H^2(M,\mathbb{R})$$



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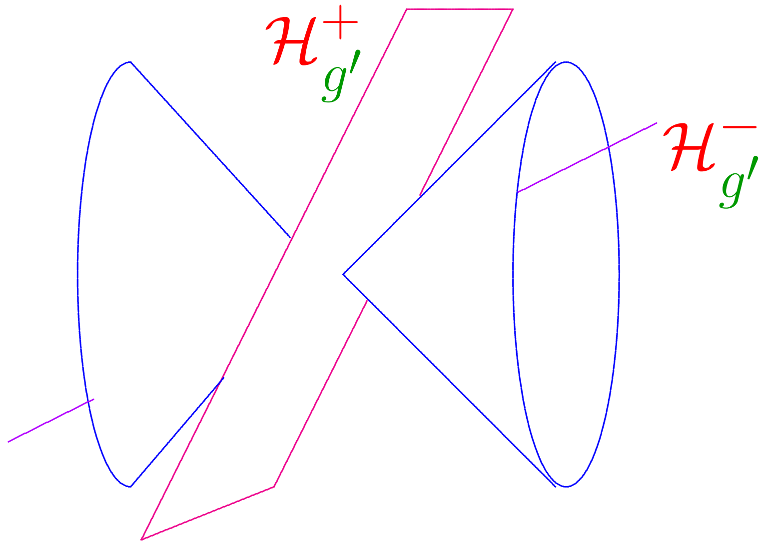

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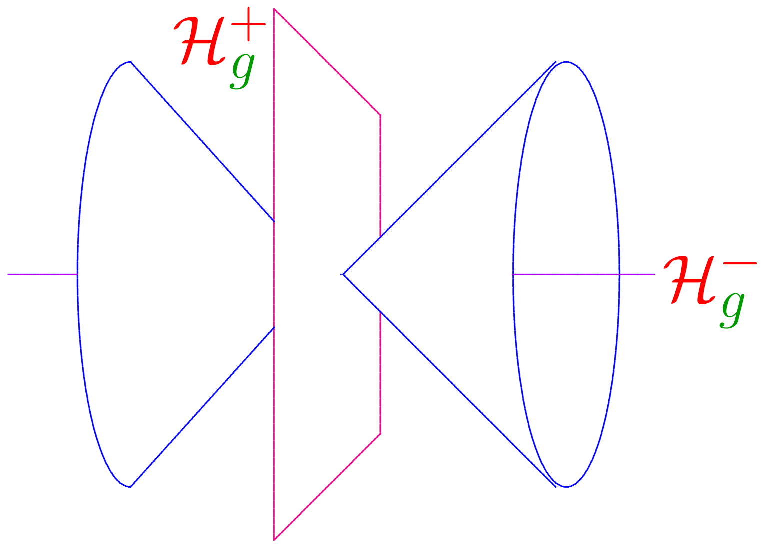

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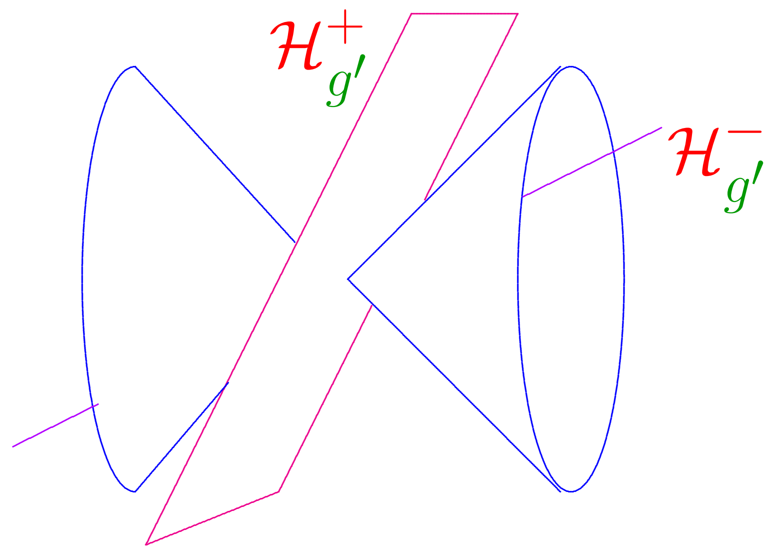

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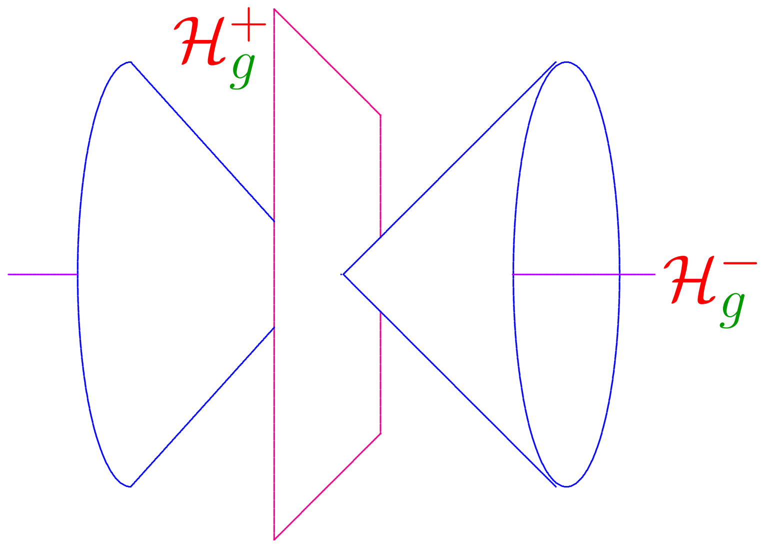

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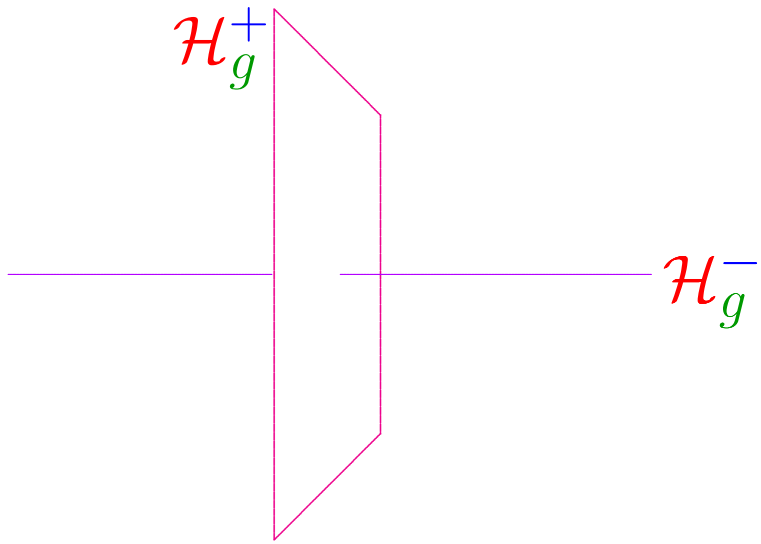

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Kähler if the 2-form

$$\omega = h(J\cdot, \cdot)$$

is closed:

$$d\omega = 0.$$

But we do not assume this!

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Only two metrics arise in non-Kähler case!

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"Riemannian Goldberg-Sachs Theorem."

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Einstein Hermitian metrics with $\lambda > 0$?

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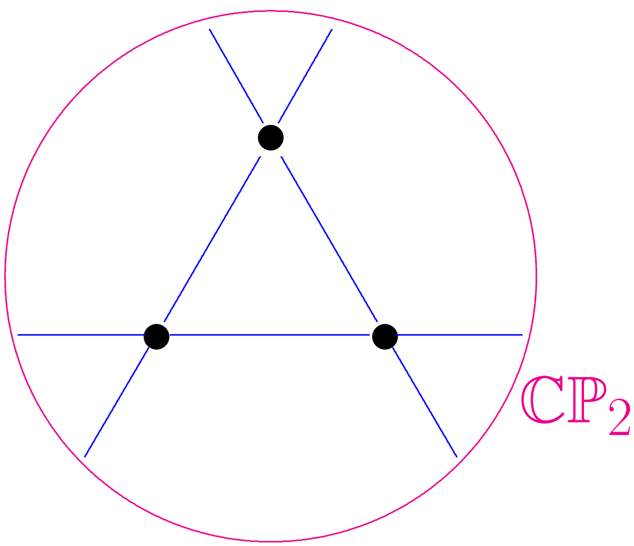

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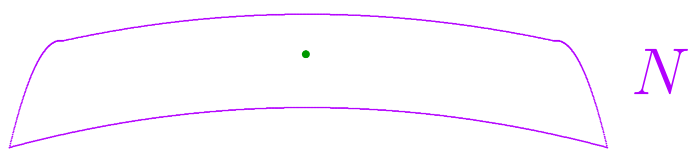

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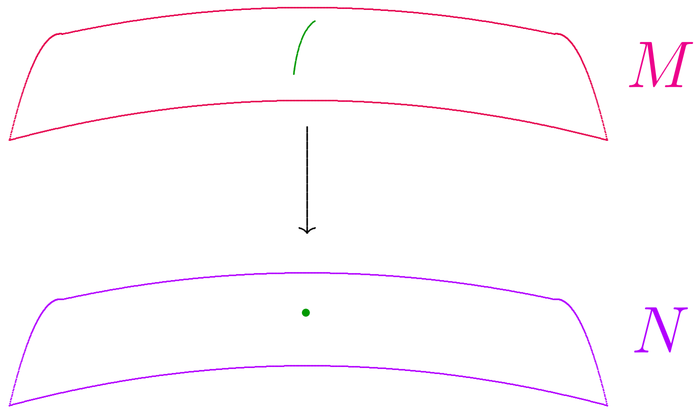

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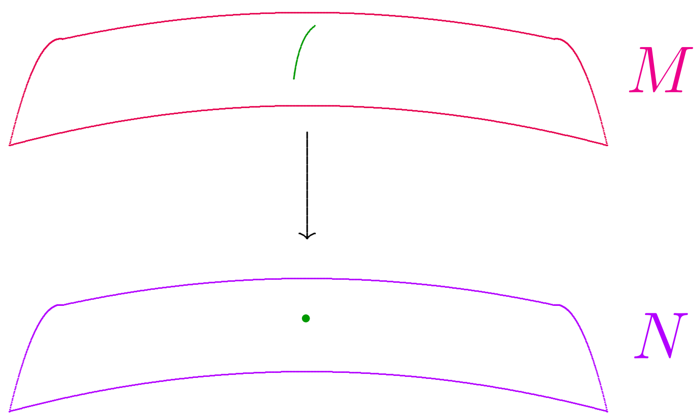
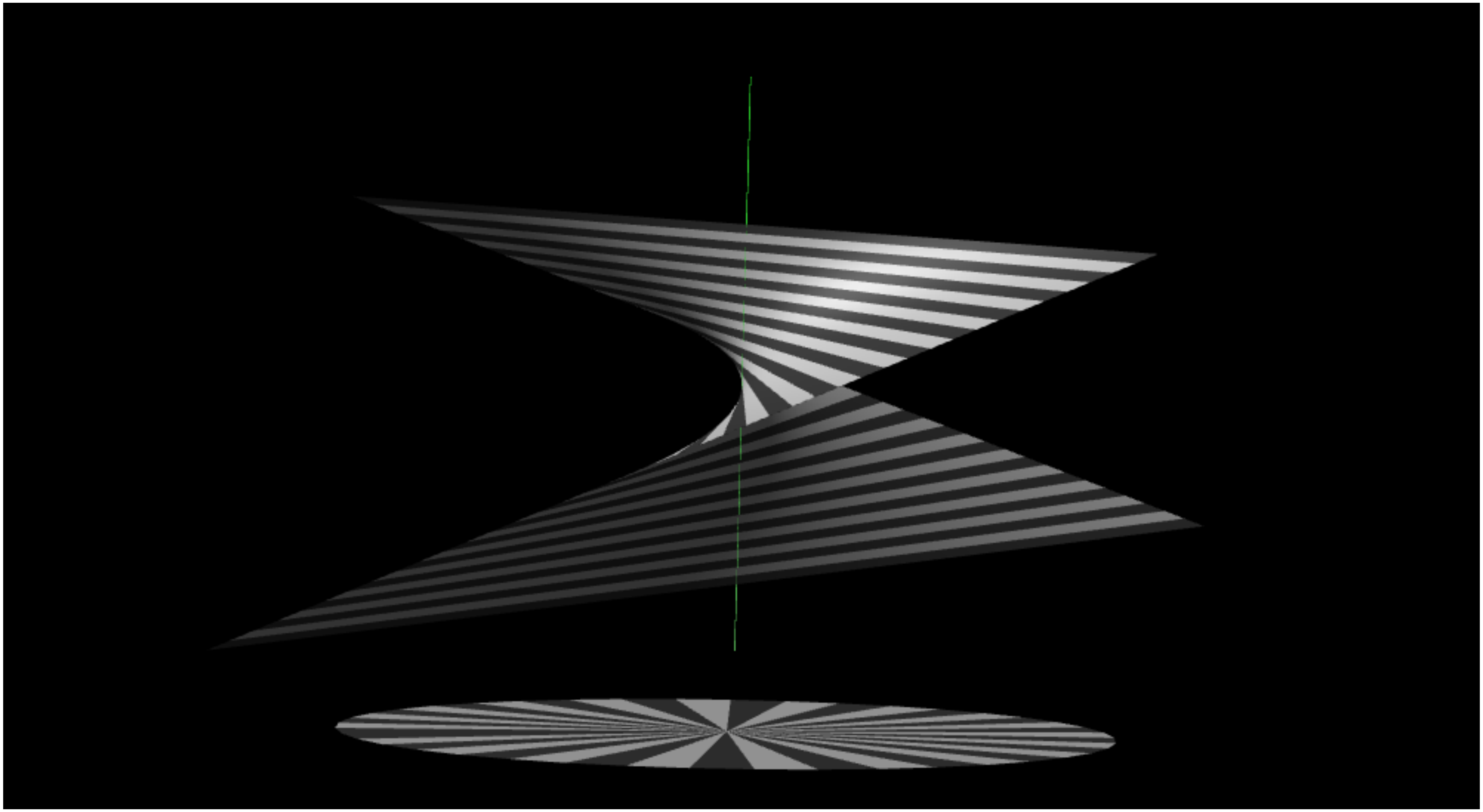
in which added $\mathbb{CP}_1$ has normal bundle $\mathcal{O}(-1)$.





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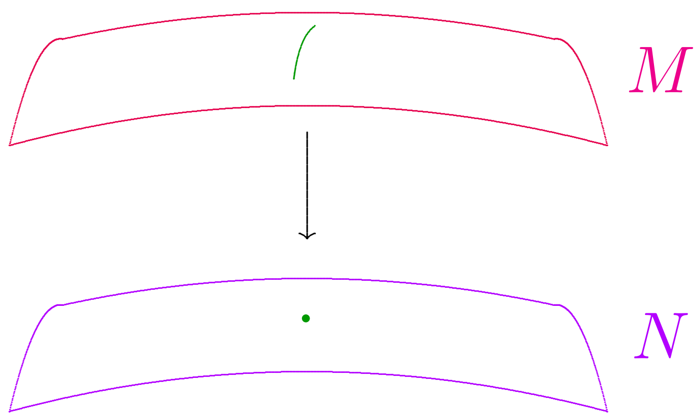

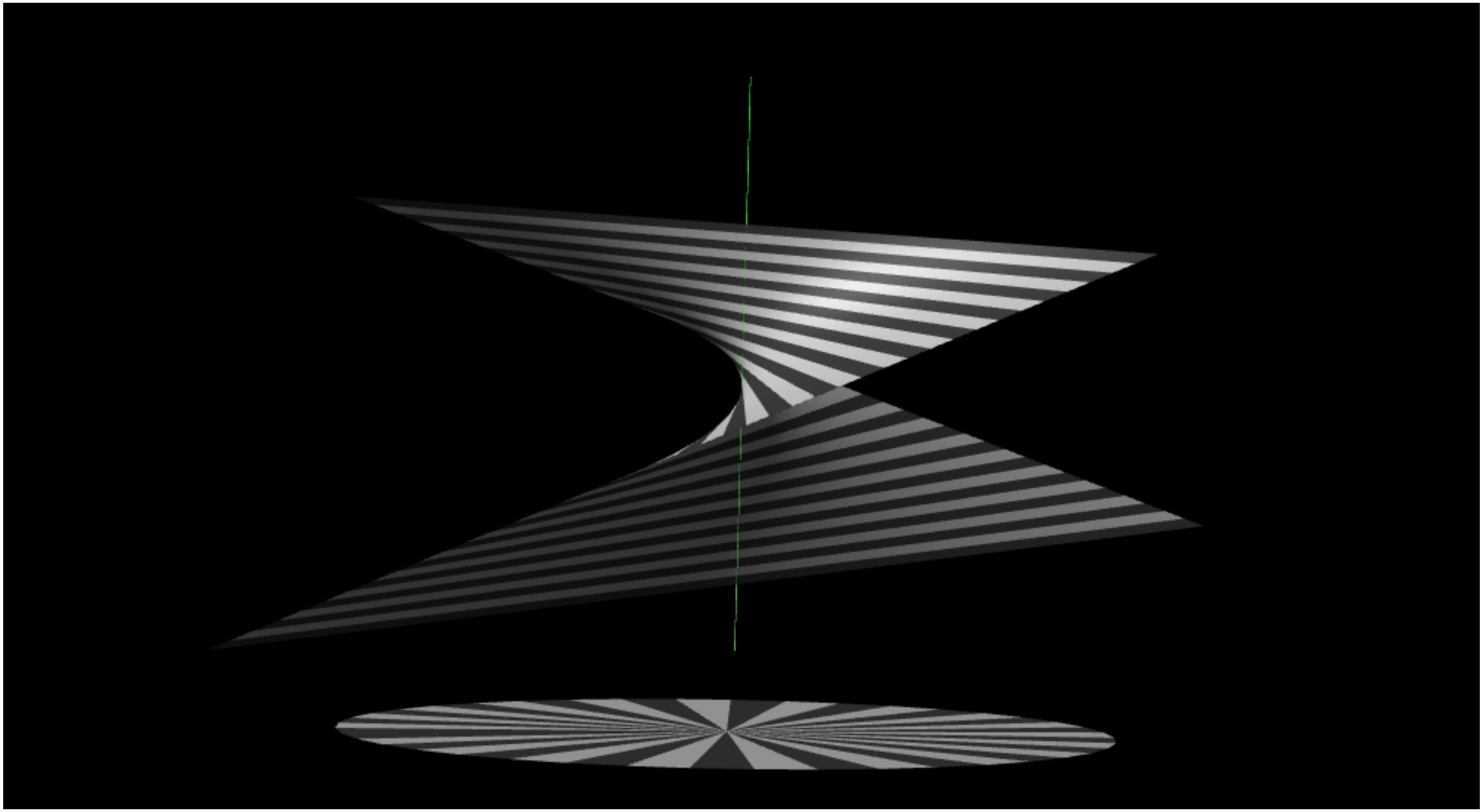

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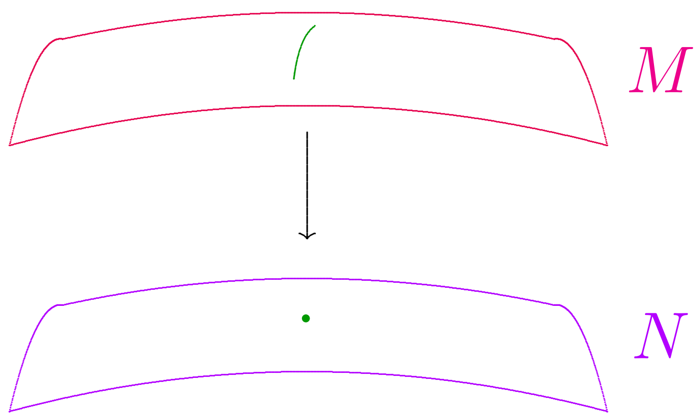

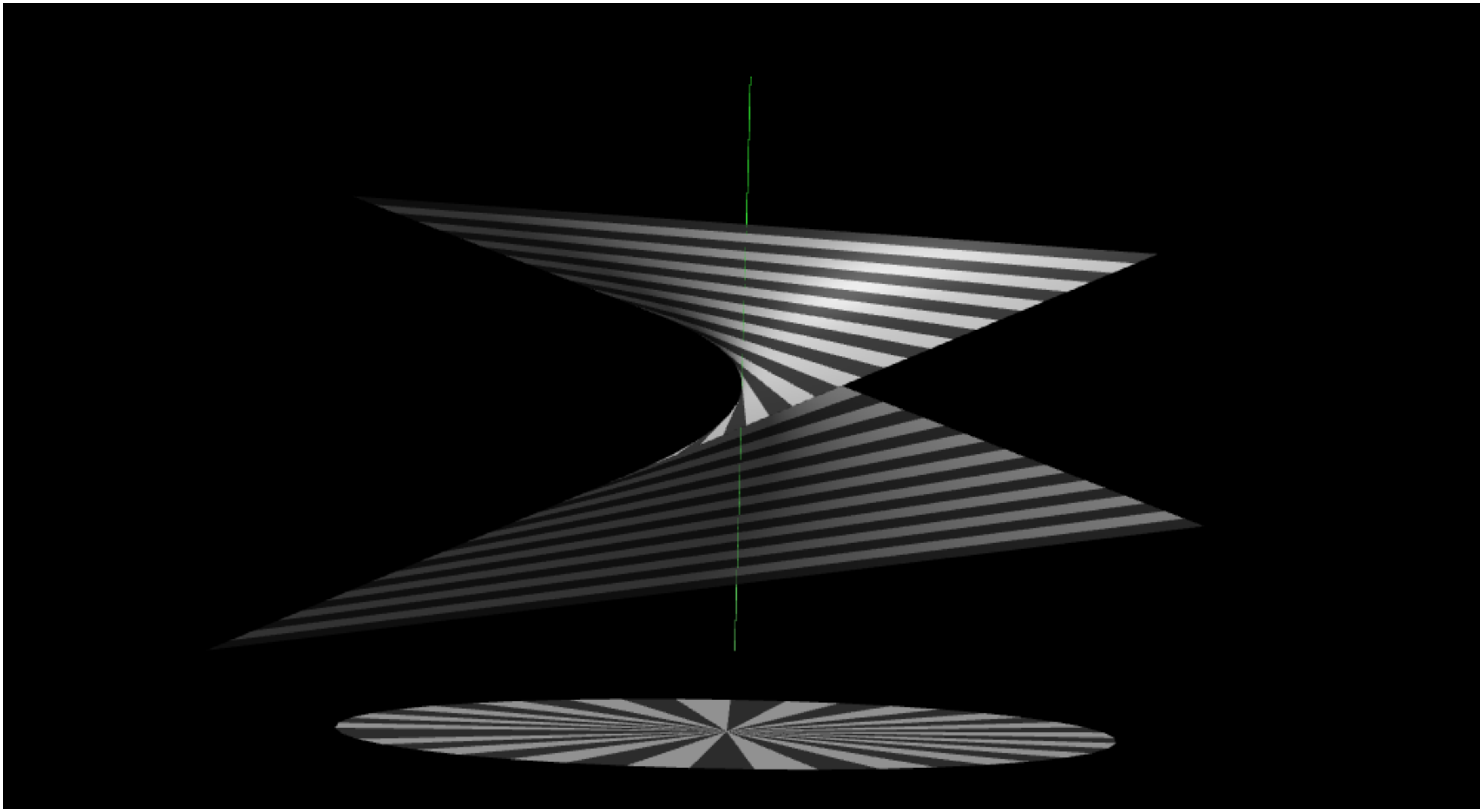

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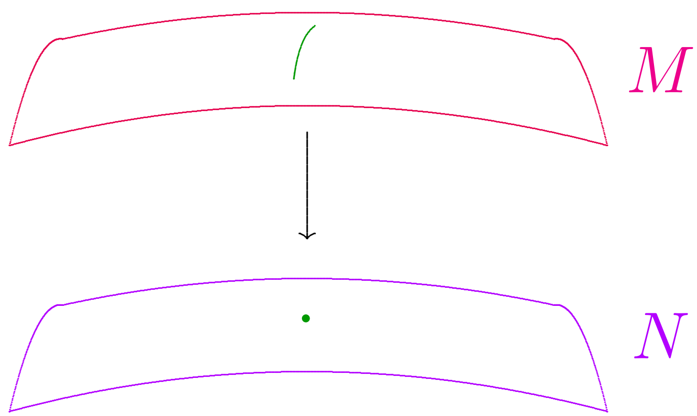
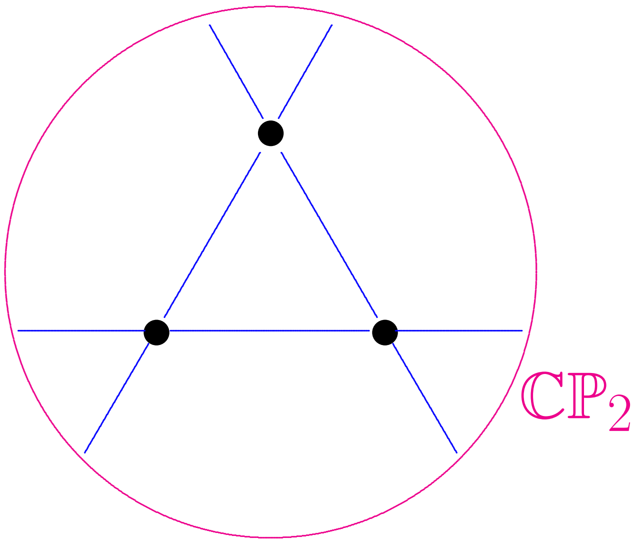


Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.



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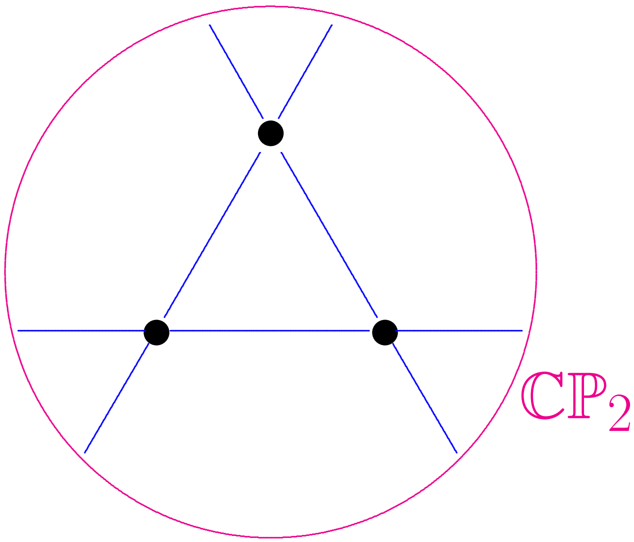

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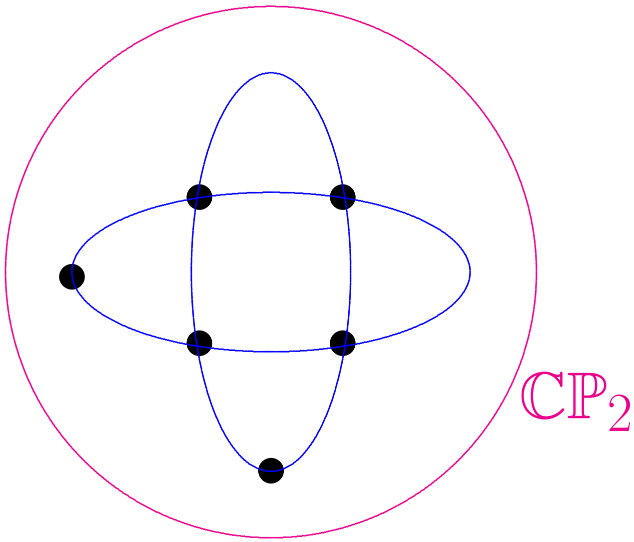

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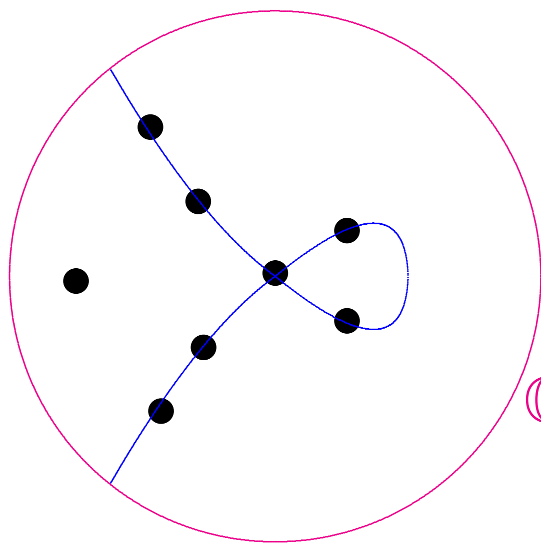

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$$\text{Kähler} \quad \Longrightarrow \quad W_+ = \begin{pmatrix} -\frac{s}{12} & & \\ & -\frac{s}{12} & \\ & & \frac{s}{6} \end{pmatrix}$$

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Conversely, all these are of positive symplectic type.

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Corollary. $\mathscr{E}^+_\omega(M)$ *is exactly one connected component of* $\mathscr{E}(M)$.

Method of Proof.

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$$\boxed{0 = \nabla^{*}\nabla(f W^{+}) + \frac{s}{2} f W^{+} - 6 f W^{+} \circ W^{+} + 2 f |W^{+}|^2 I}$$

Equation $\delta W^+ = 0$ not conformally invariant!

If $h = f^2 g$ satisfies

$$\delta W^+ = 0$$

then g instead satisfies

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$$\boxed{0 = \nabla^*\nabla(fW^+) + \frac{s}{2} fW^+ - 6fW^+ \circ W^+ + 2f|W^+|^2 I}$$

for $fW^+ \in \mathrm{End}(\Lambda^+)$.

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Remark. If such metrics exist, $b_+(M) = 1$.

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Theorem A follows by restricting to Einstein case.

Theorem A. *Let* (M, h) *be a smooth compact* 4*-dimensional Einstein manifold. If* h *is of positive symplectic type, then it's a conformally Kähler, Einstein metric on a Del Pezzo surface* (M, J)*.*

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Corollary. $\mathscr{E}^+_\omega(M)$ *is exactly one connected component of* $\mathscr{E}(M)$*.*

Application to Almost-Kähler Geometry:

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(Special case of "Goldberg conjecture.")

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Helped motivate the discovery of **Theorem A**...

Tanti auguri, Stefano!

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E grazie agli organizzatori

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per un convegno così bello!