

Mass, Kähler Manifolds, &

Symplectic Geometry

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Stony Brook University

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arXiv: 1810.11417 [math.DG]

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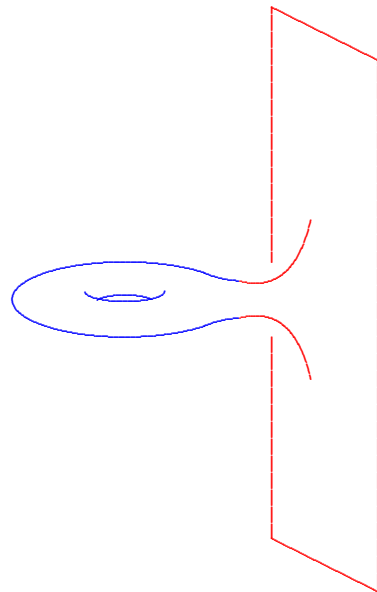
Builds on previous paper

Mass in Kähler Geometry

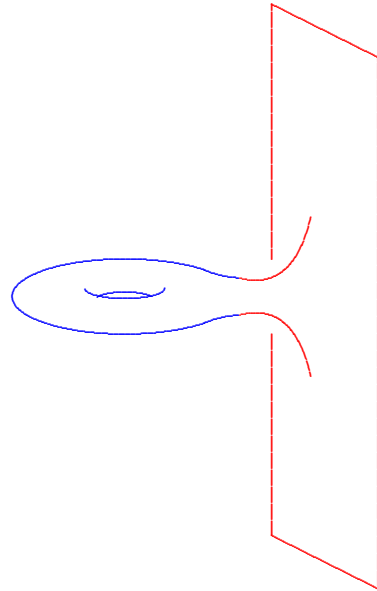
Comm. Math. Phys. 347 (2016) 621–653.

(Joint with Hans-Joachim Hein)

Definition. A complete, non-compact Riemannian n -manifold (M^n, g)

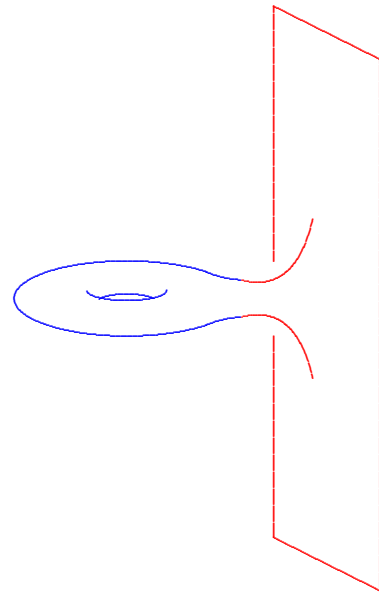


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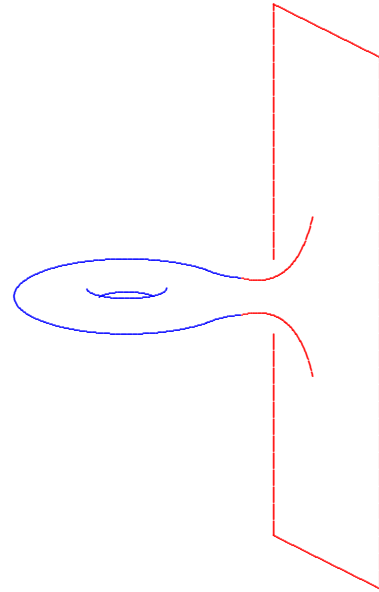
$$n \geq 3$$

Definition. A complete, non-compact Riemannian n -manifold (M^n, g) is called asymptotically Euclidean



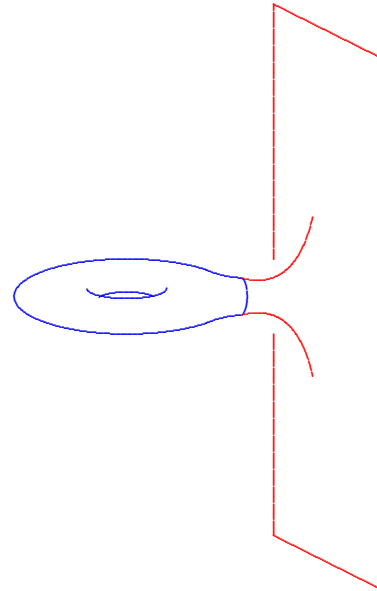
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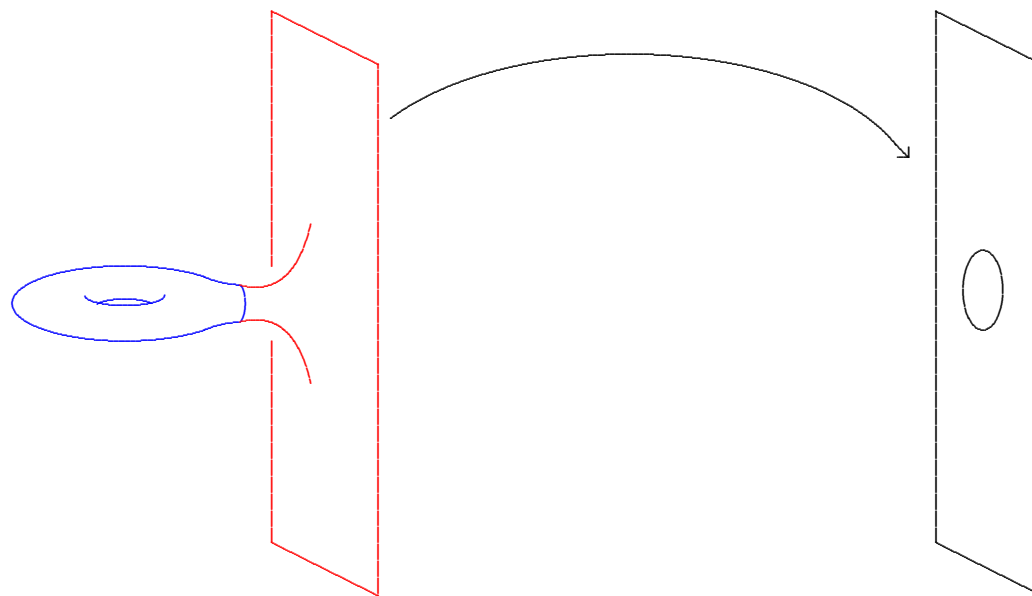


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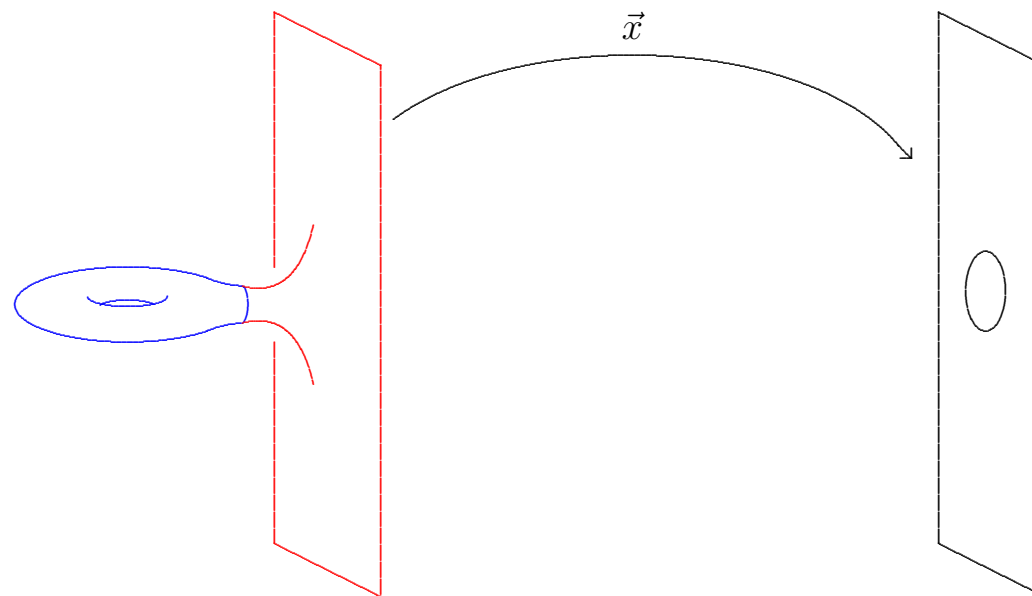
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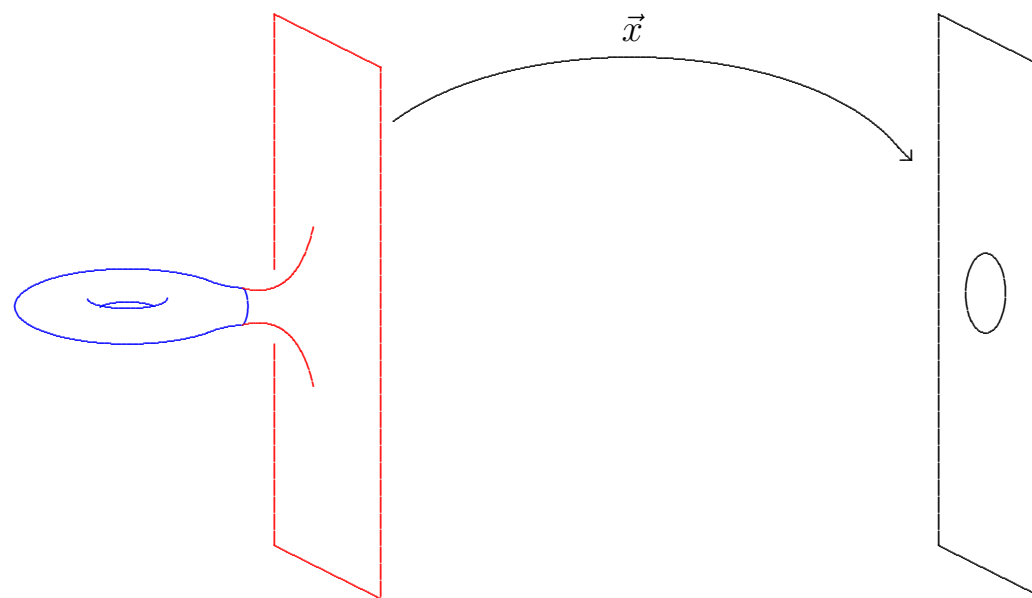


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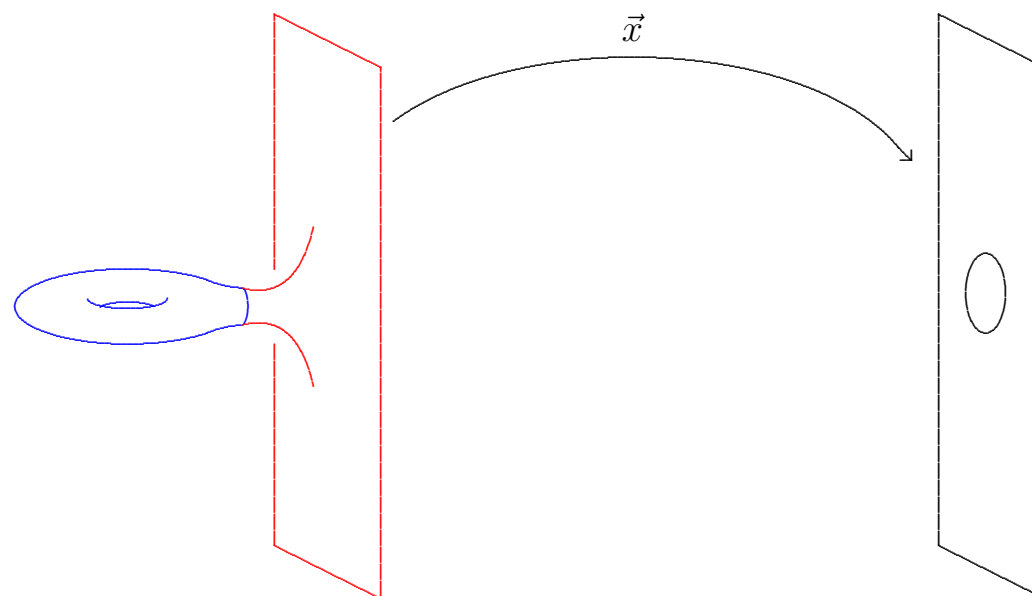
$$g_{jk} = \delta_{jk} + \text{terms that fall-off at infinity}$$

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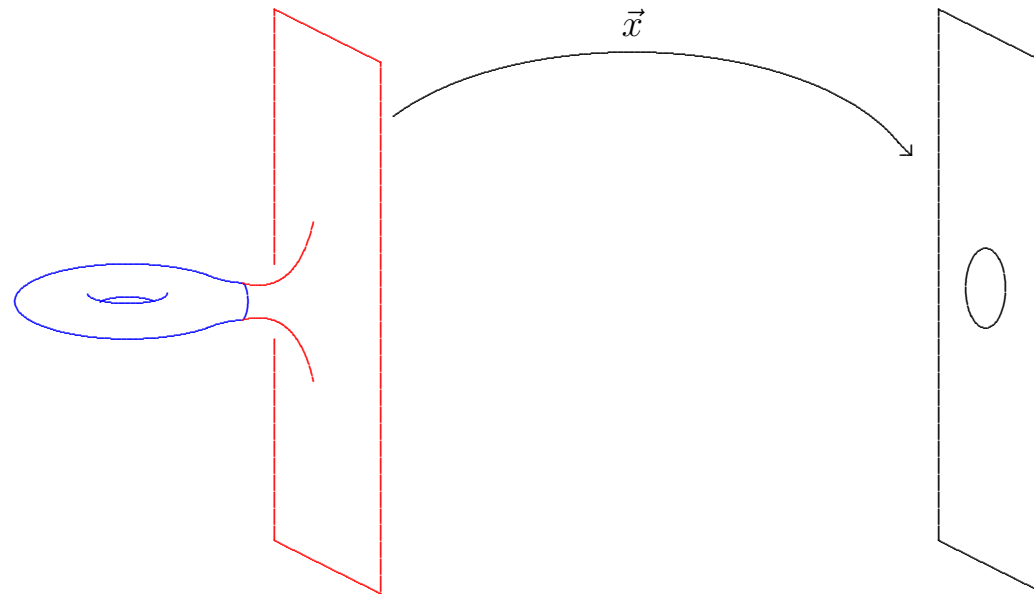
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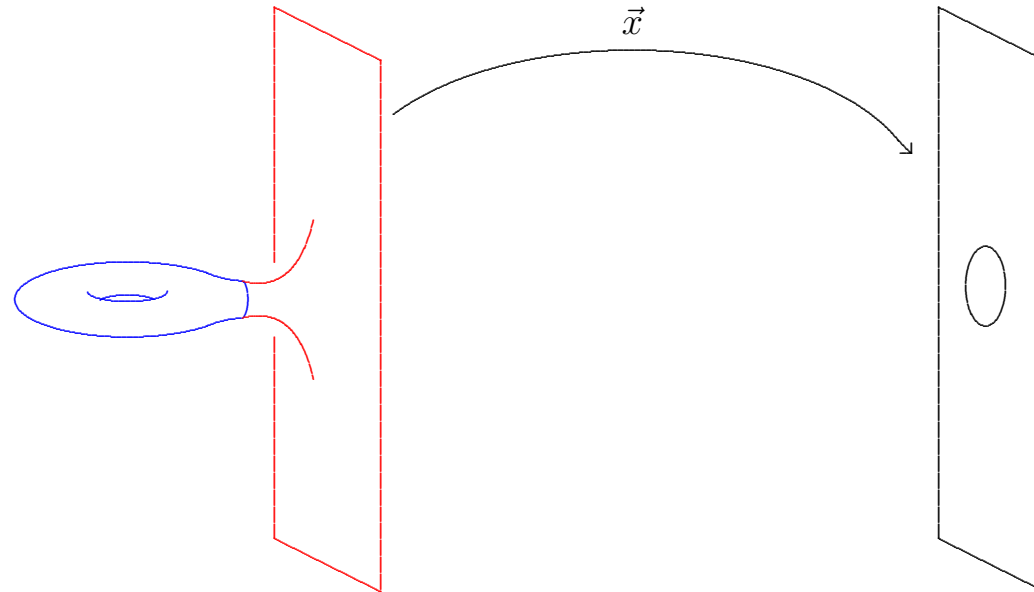
Chruściel-type fall-off:

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$$g_{jk} = \delta_{jk} + O(|x|^{1-\frac{n}{2}-\varepsilon})$$

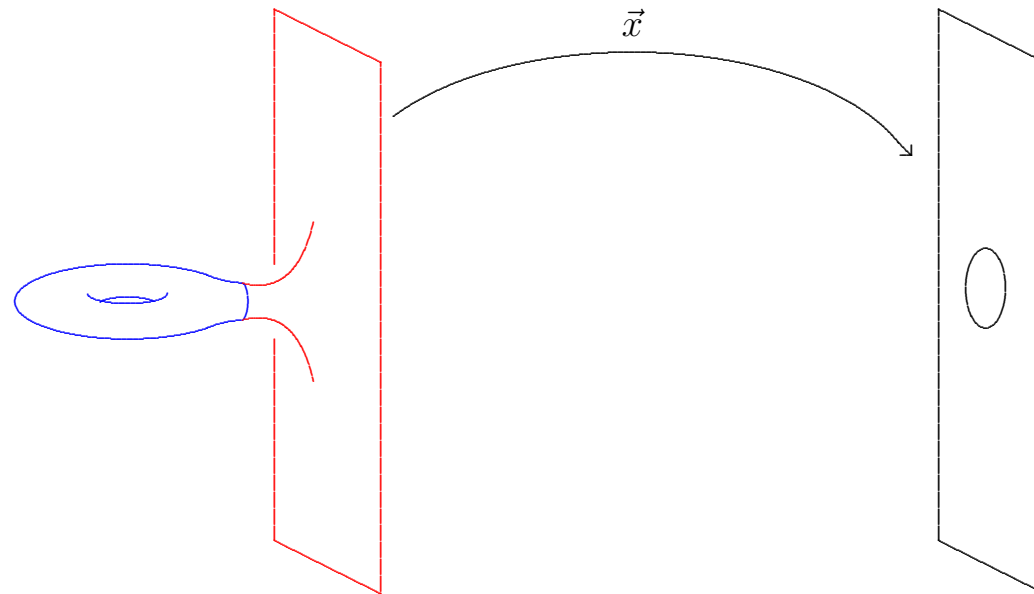
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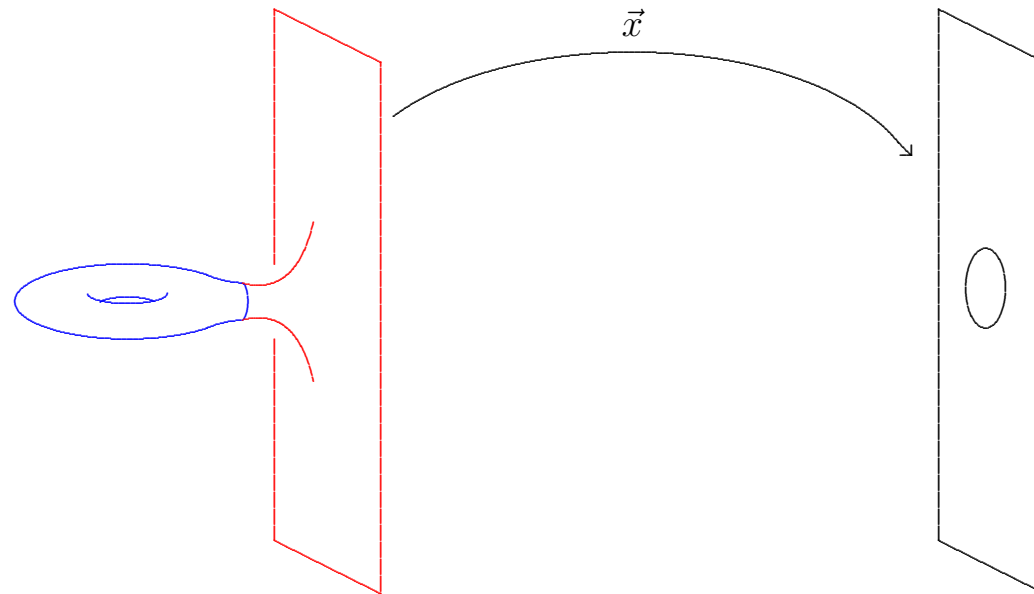
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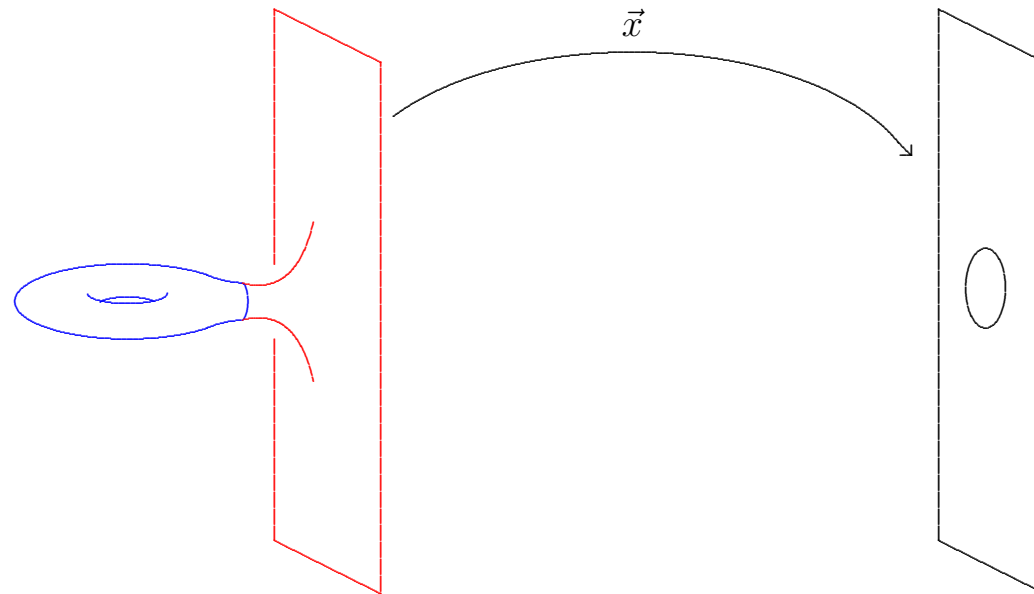
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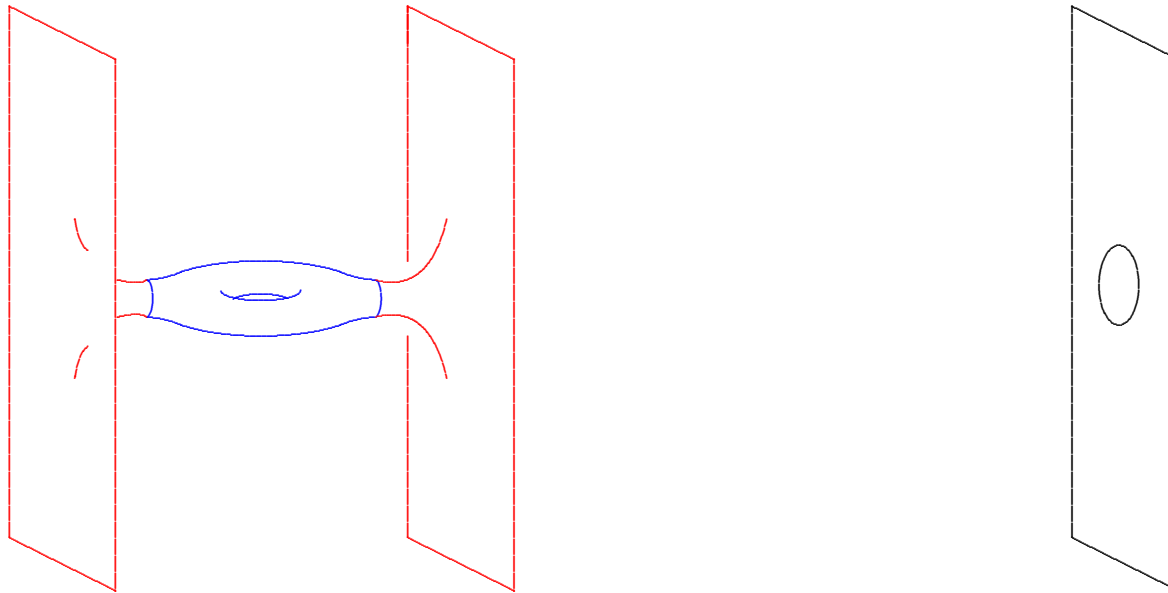
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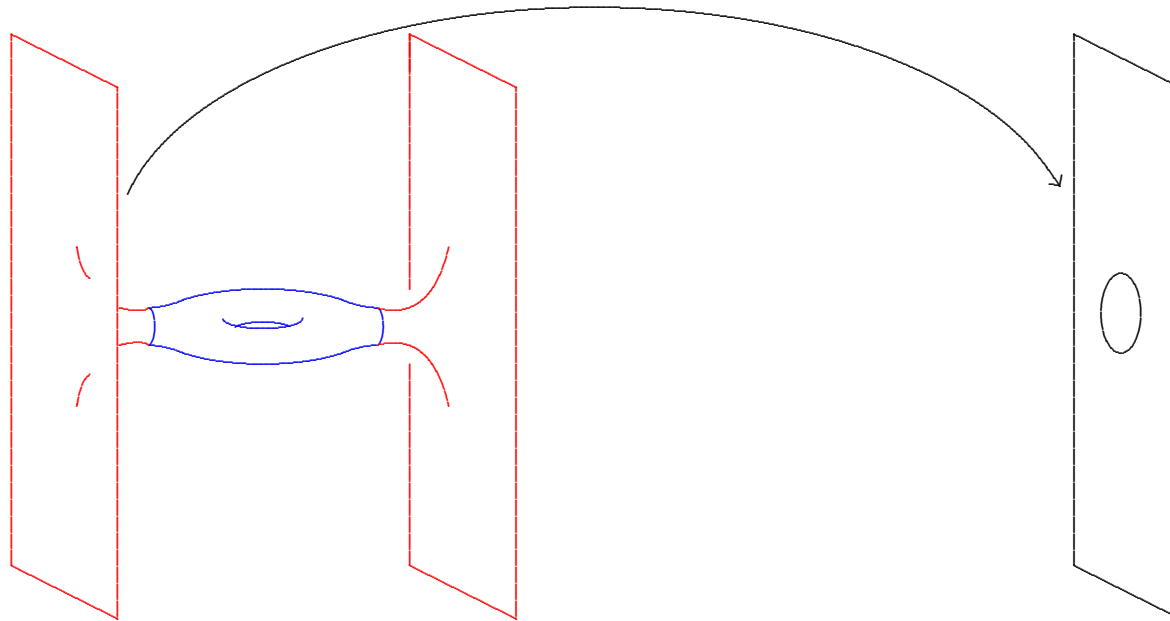
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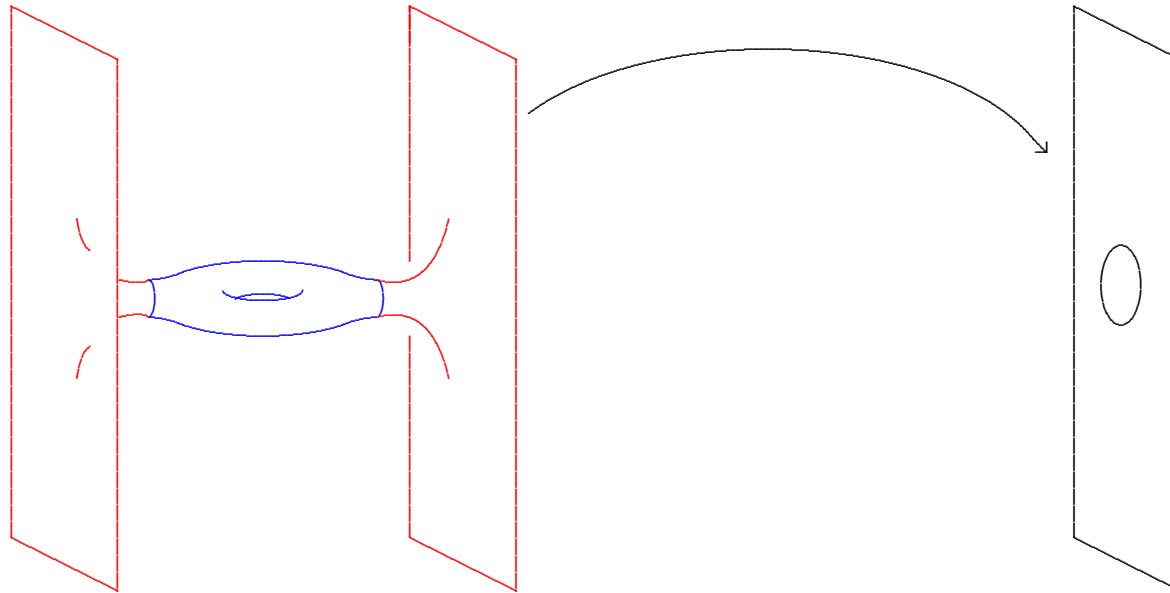
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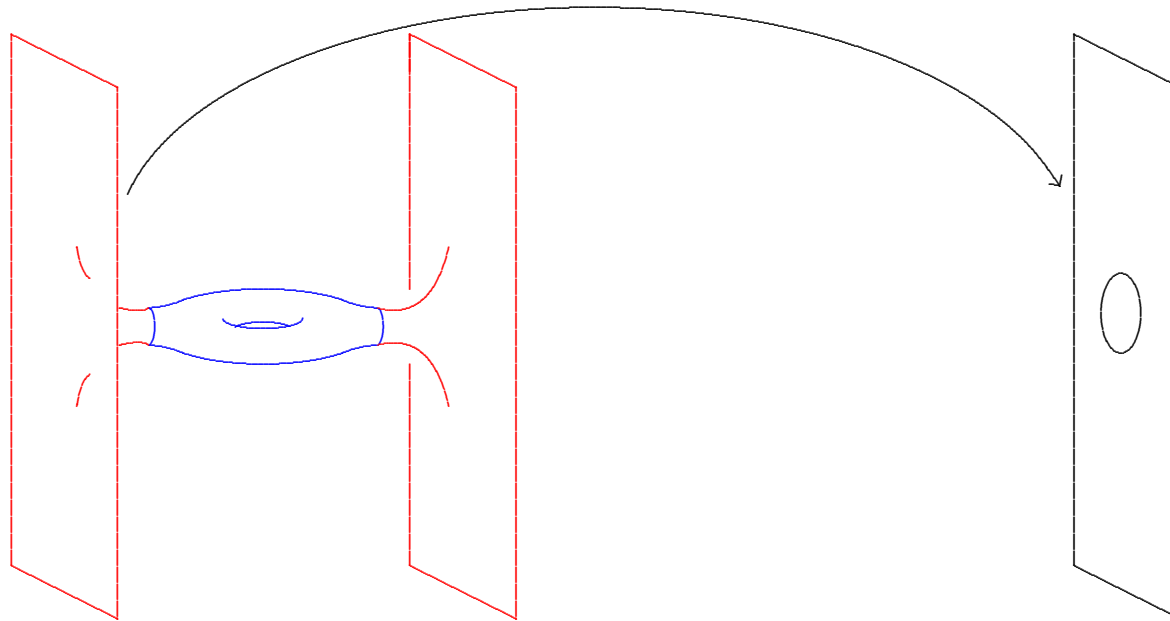
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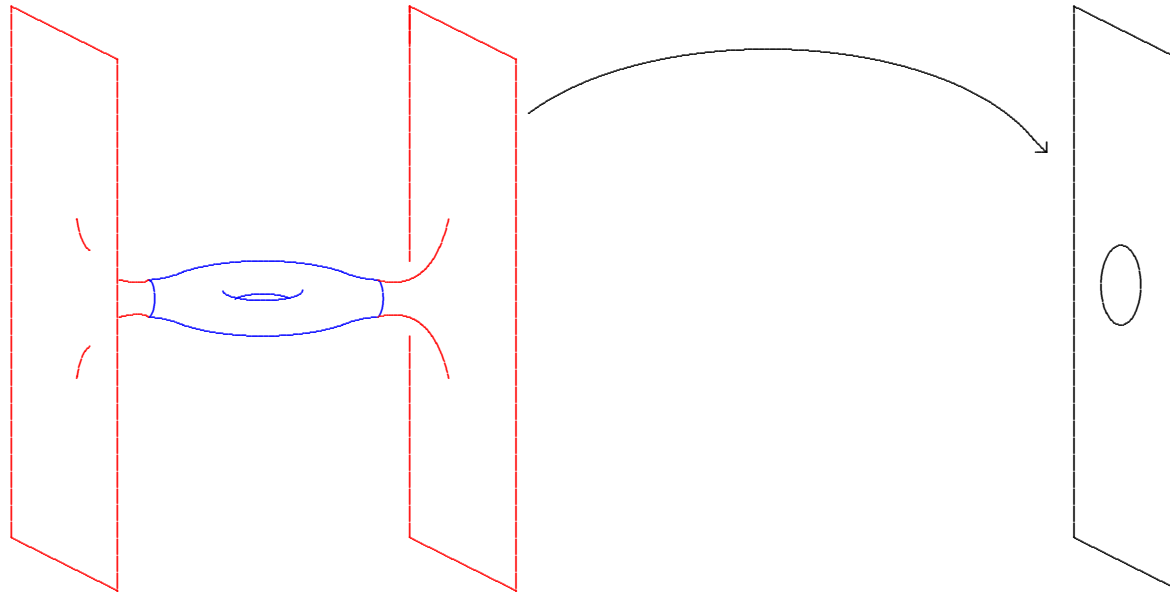
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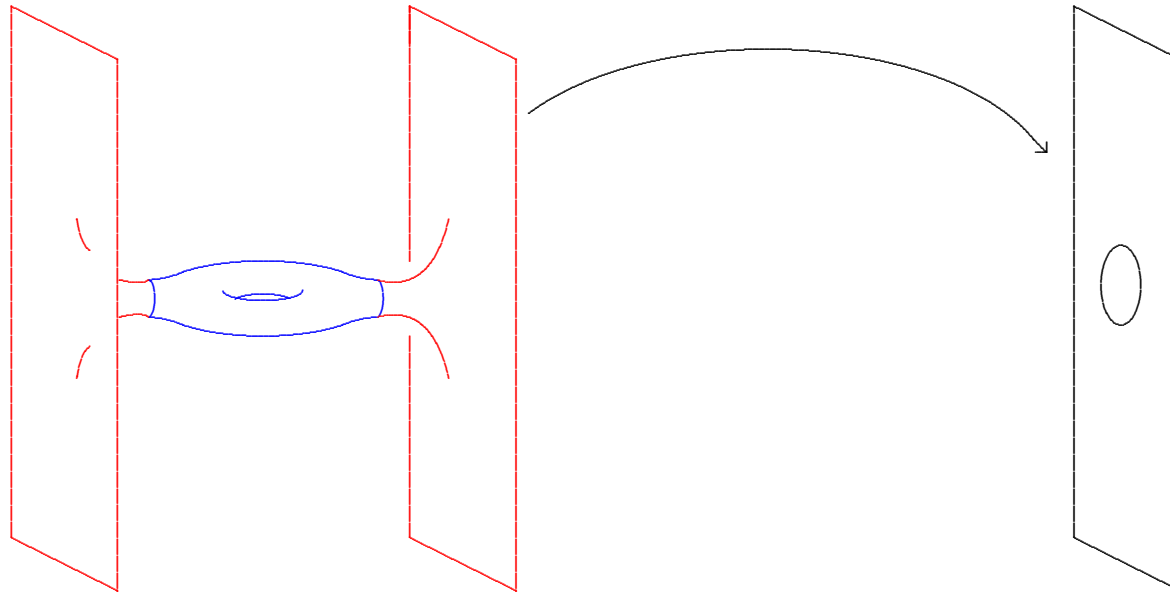
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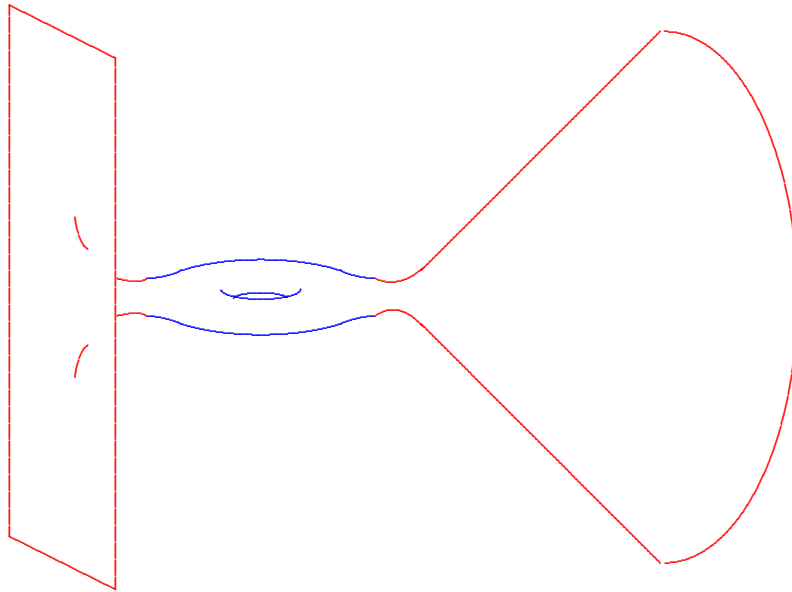
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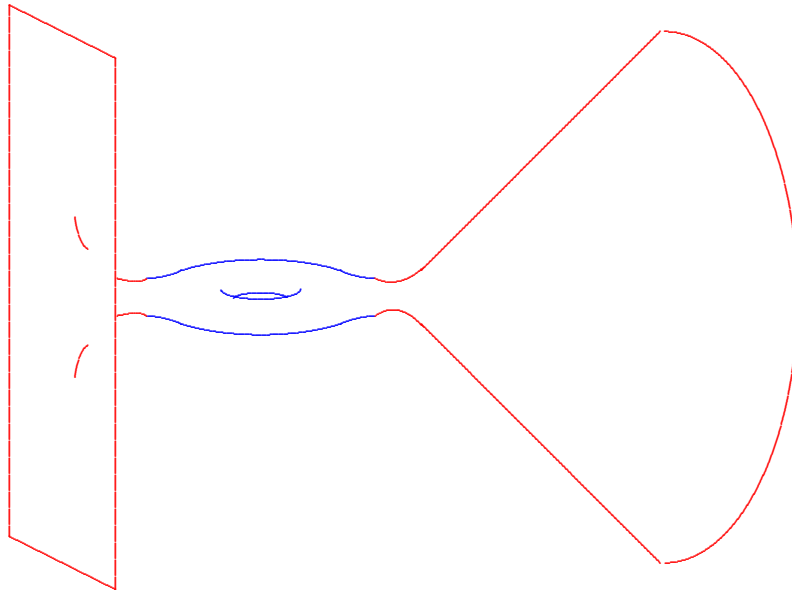
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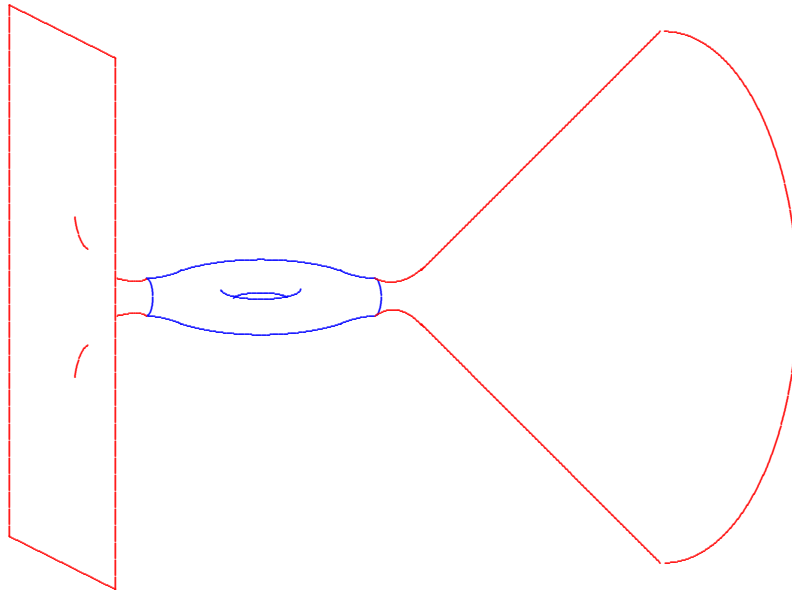
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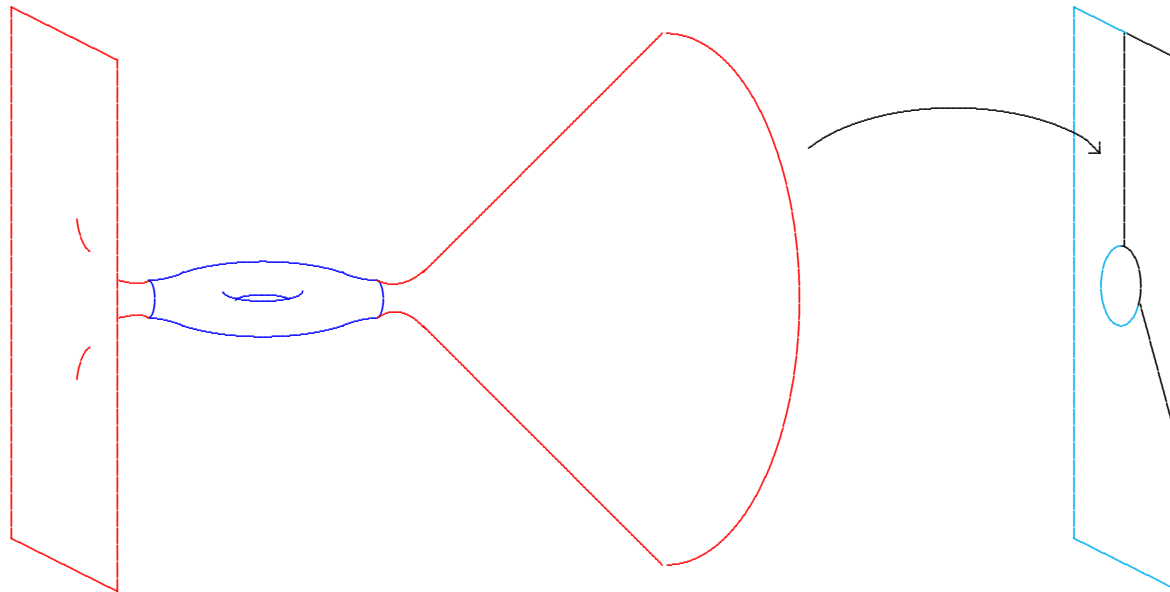
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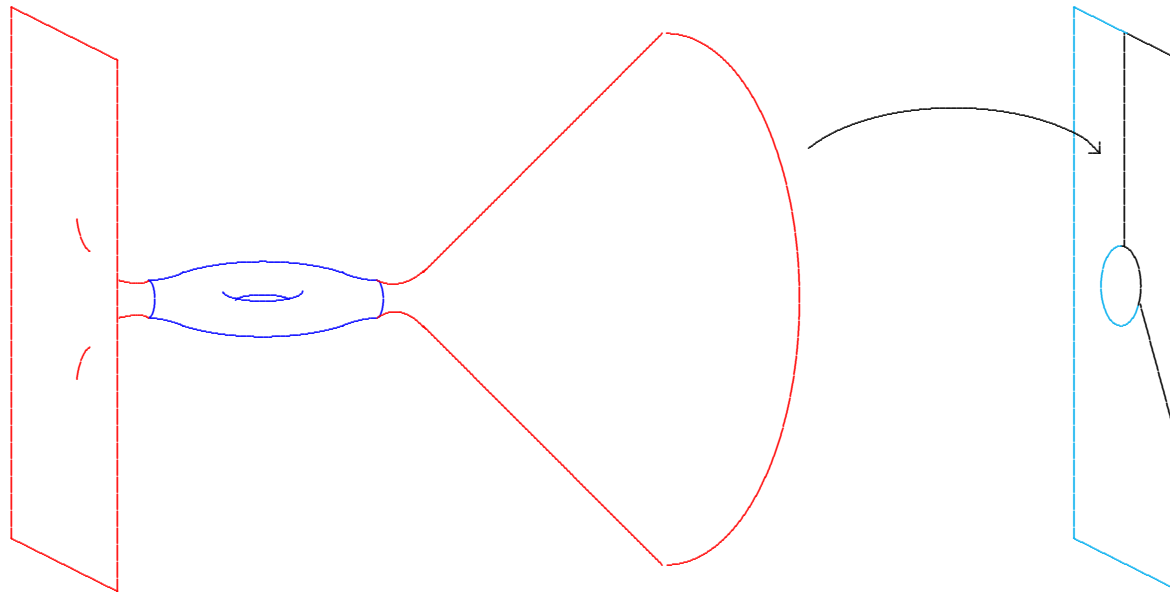
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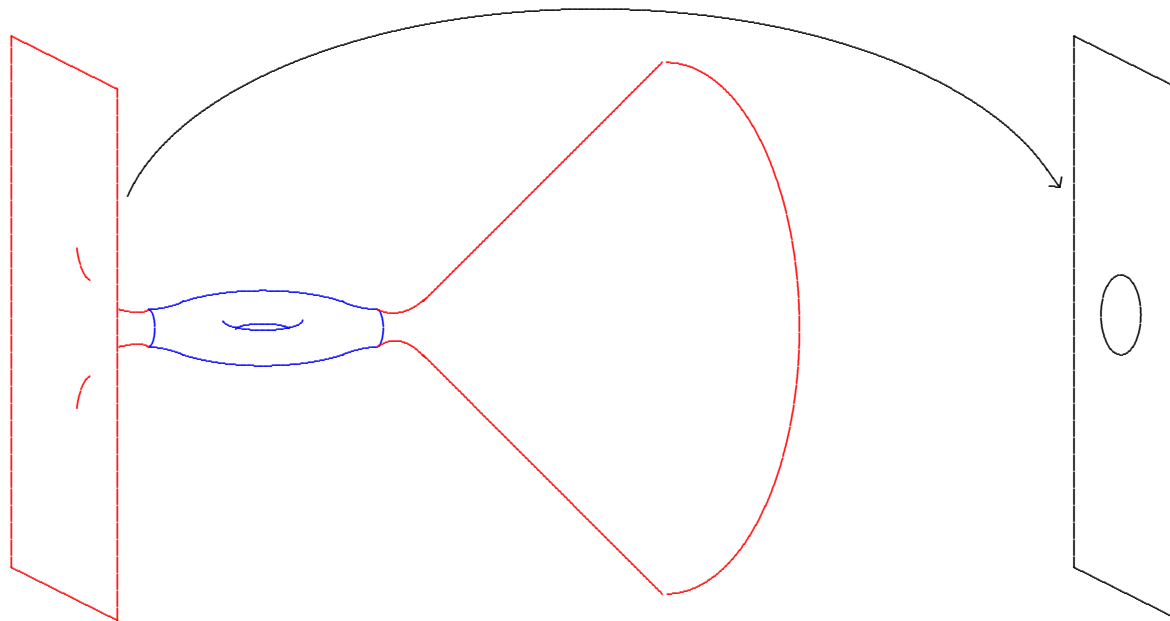
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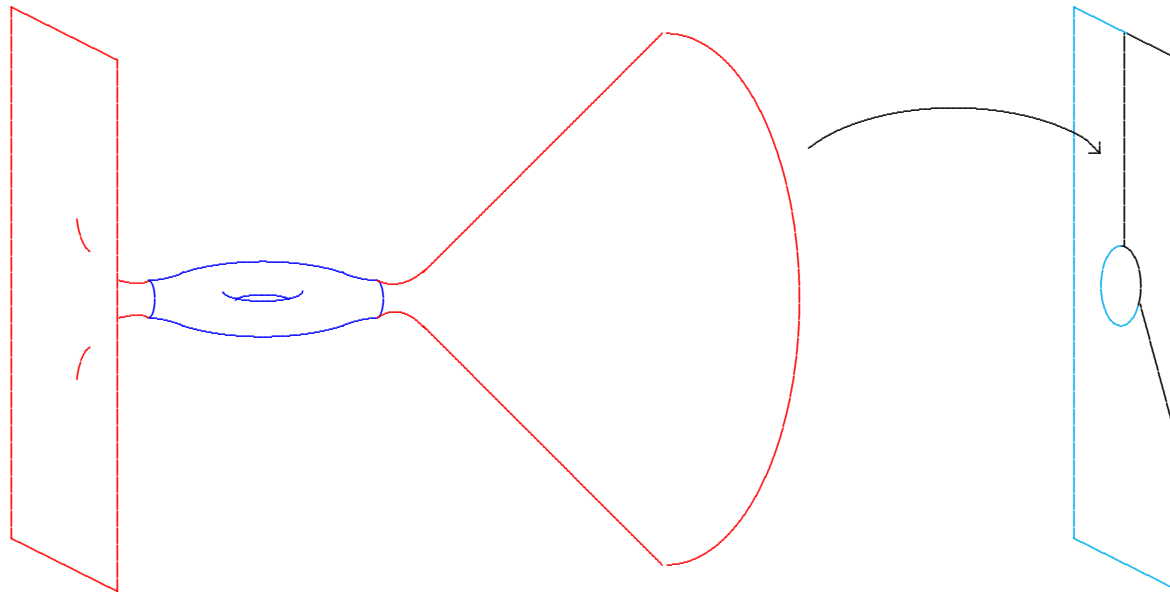
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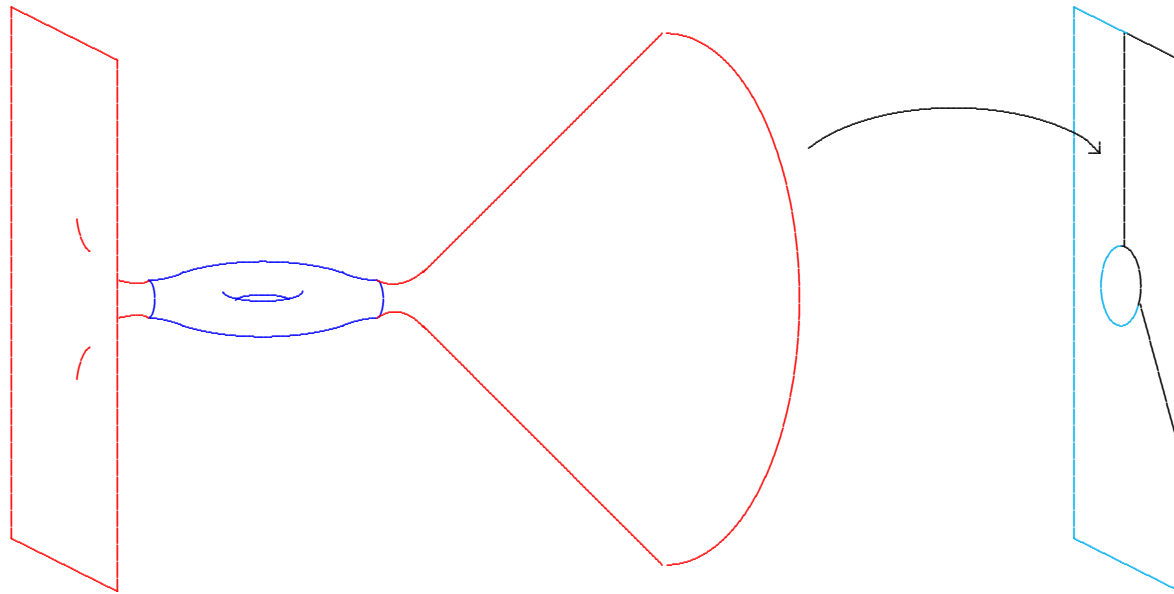
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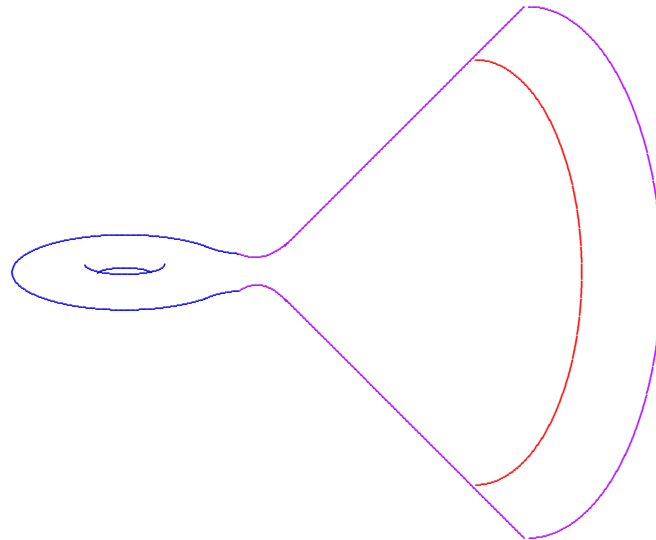
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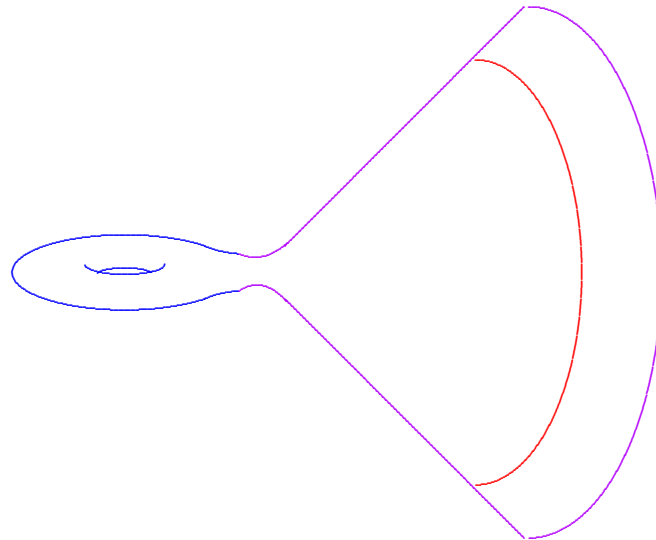


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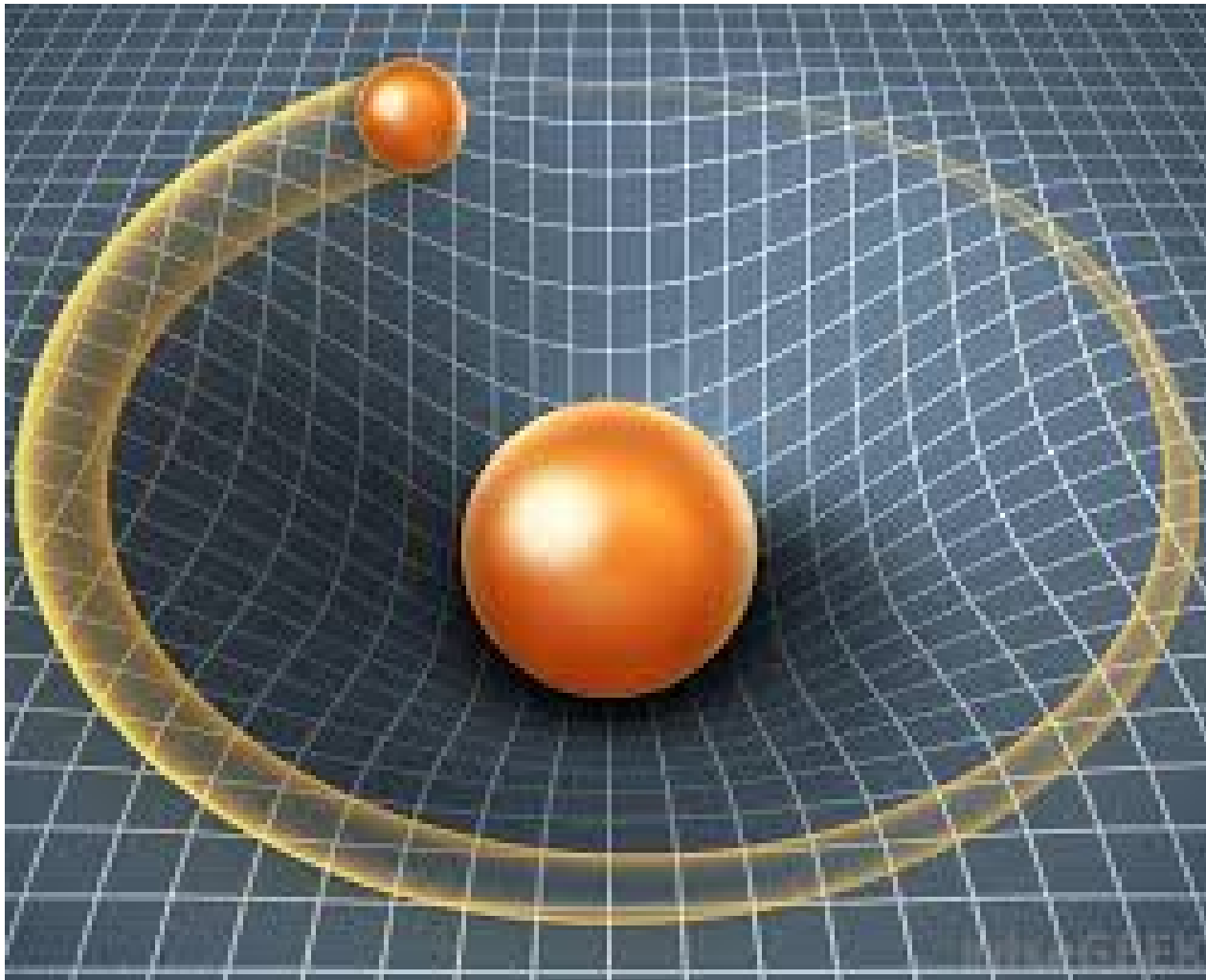
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Mass in Kähler Geometry

Comm. Math. Phys. 347 (2016) 621–653.

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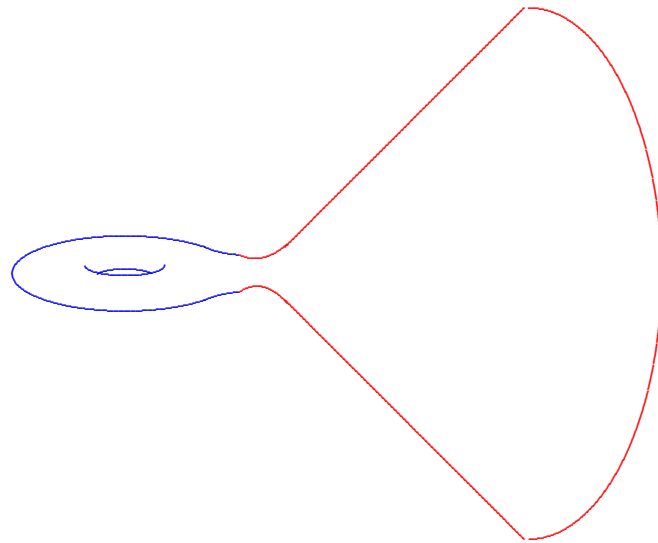
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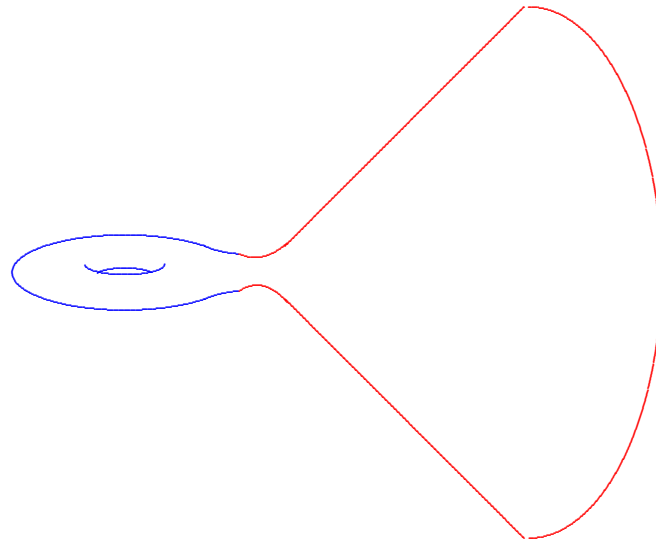
In particular, Chruściel fall-off suffices to imply all the following results:

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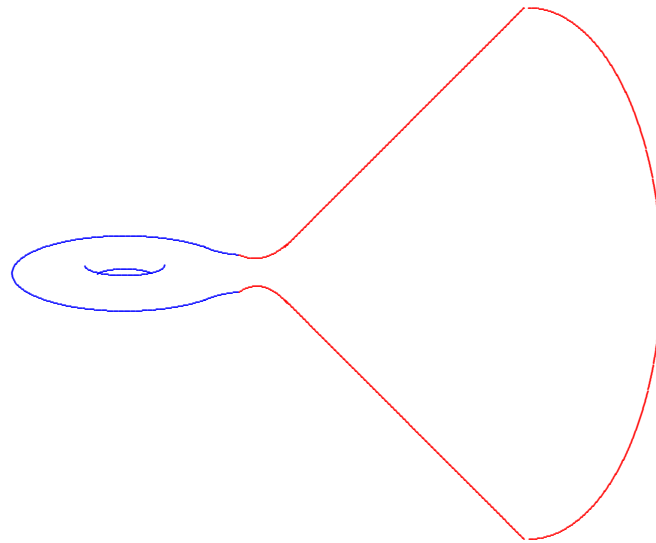


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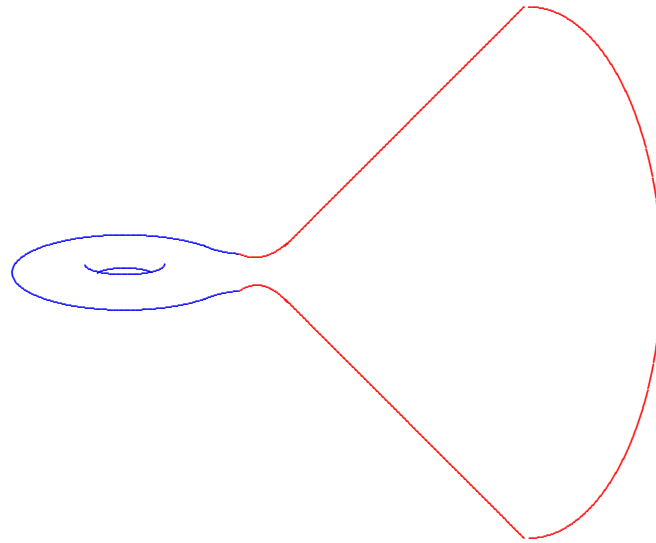


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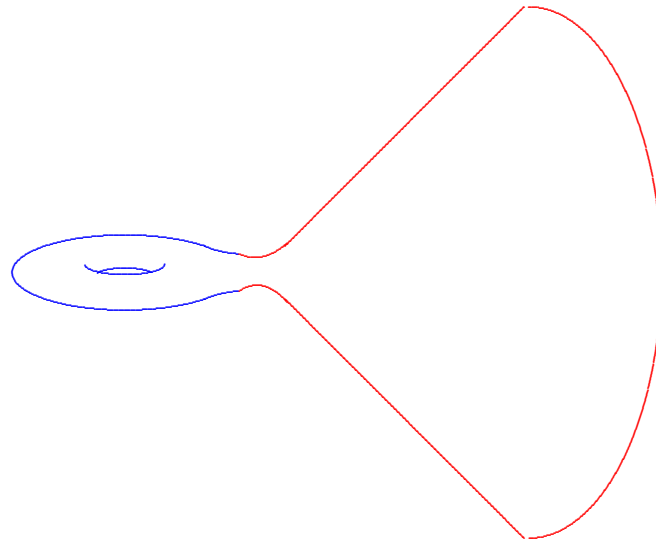


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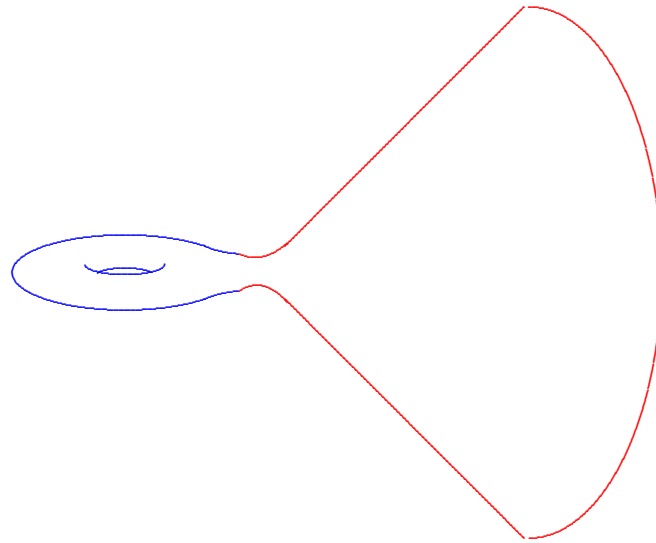
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Mass of an ALE Kähler manifold is unambiguous.

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Mass of an ALE Kähler manifold is unambiguous.

Does not depend on the choice of an end!

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New proof shows this follows from Chruściel fall-off.

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In particular, this shows mass is coordinate-invariant, without ever invoking Bartnik-Chruściel!

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This has an interesting corollary...

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Now use Bishop-Gromov inequality.

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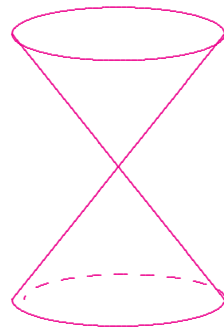
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Example: Eguchi-Hanson.

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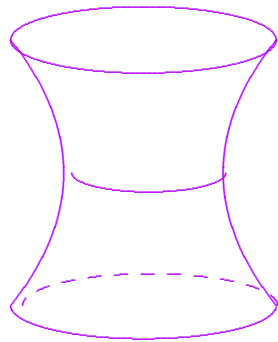
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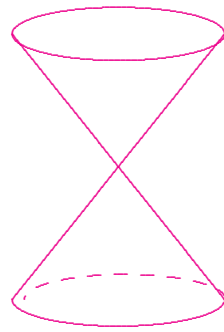
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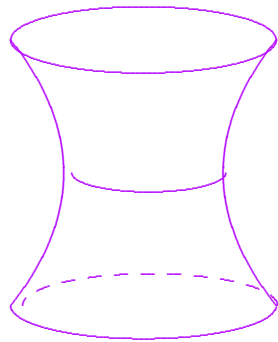
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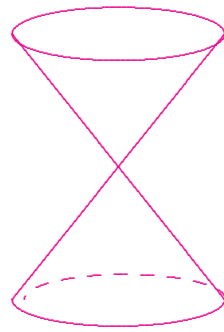
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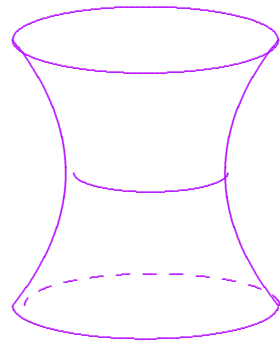
Example: Honda metrics.



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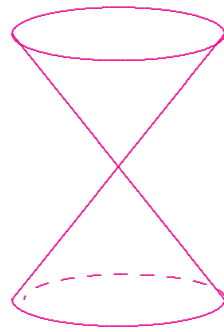
Example: Honda metrics. Scalar-flat Kähler.



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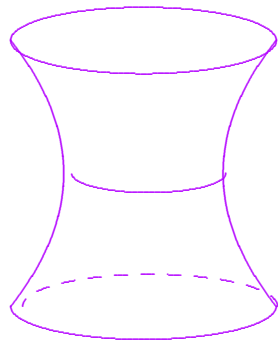
Example: Honda metrics. Non-hyper-Kähler.



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Example: Honda metrics. Deform $\mathcal{O}(-3) \rightarrow \mathbb{C}P_1$.



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Fortunately, however, the symplectic structure is always standard at infinity!

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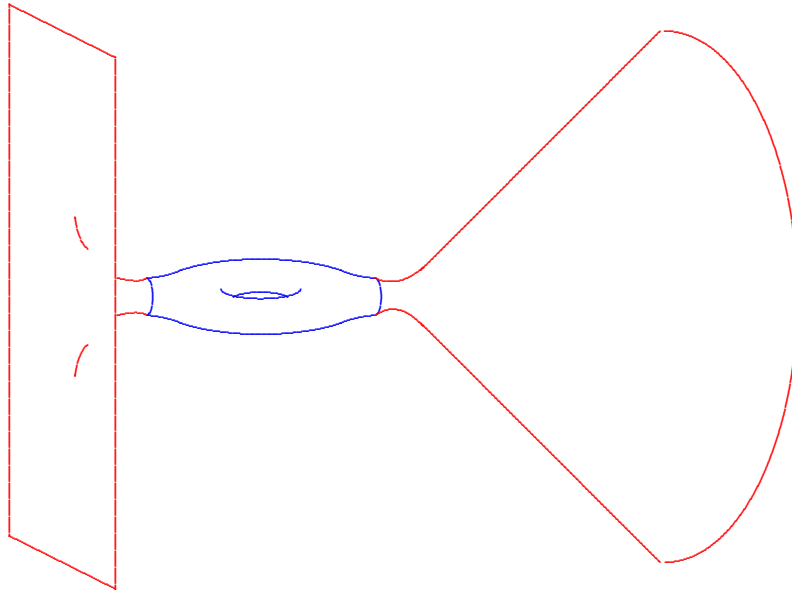
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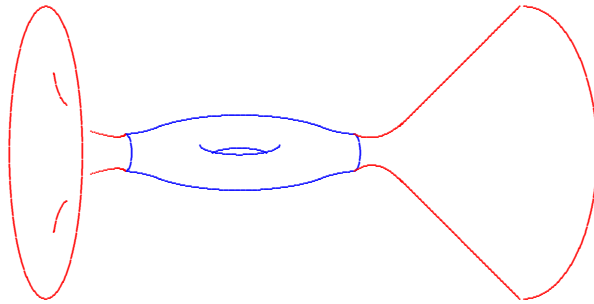
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Quantitative version of Moser stability argument...

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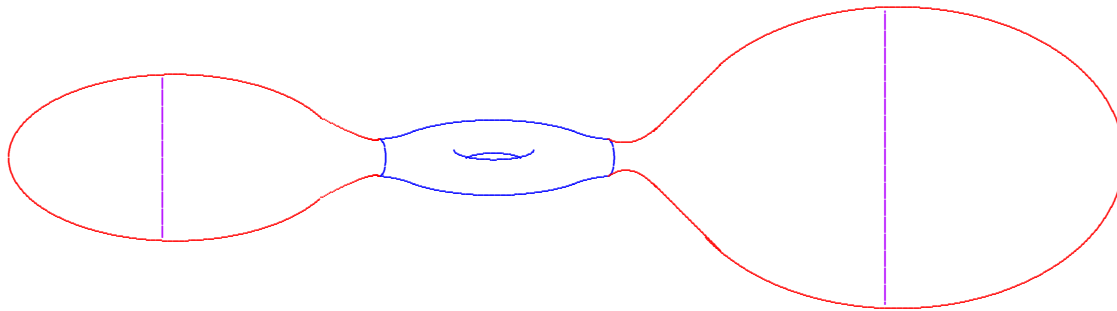


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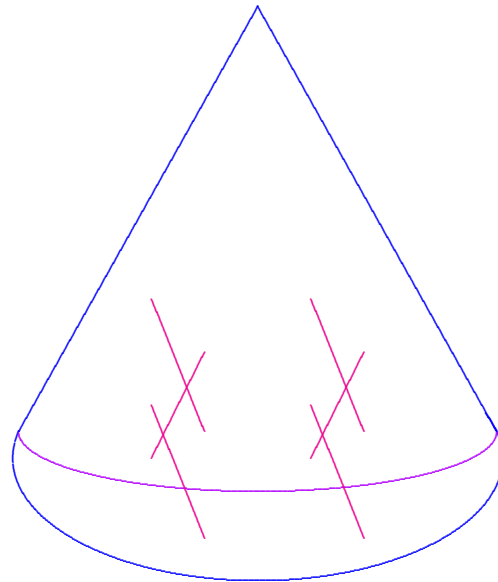
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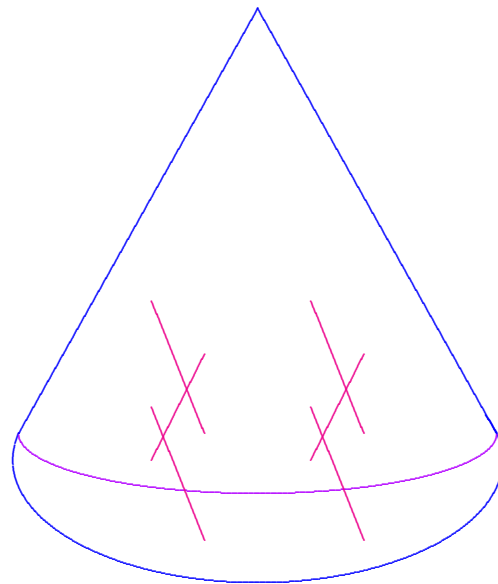
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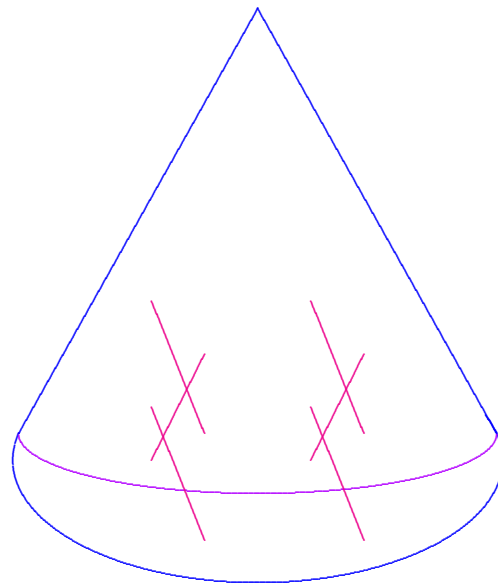


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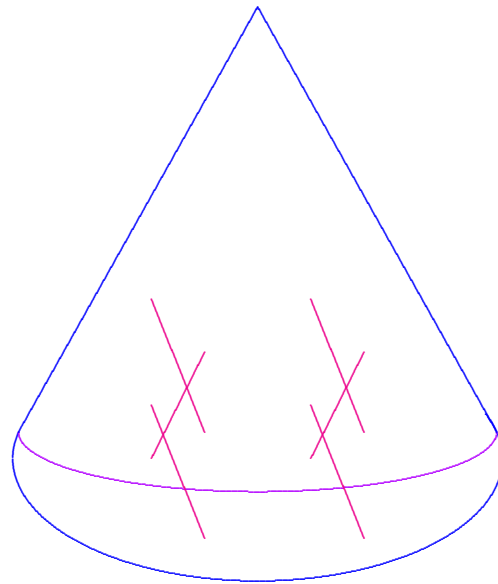
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for some, and hence any, ω -compatible almost-complex structure J .

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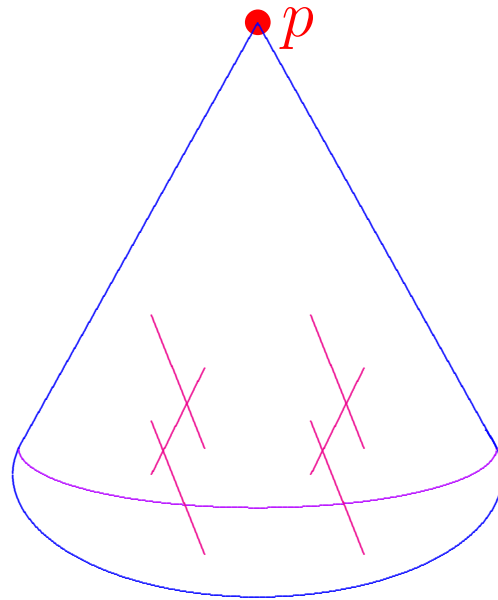
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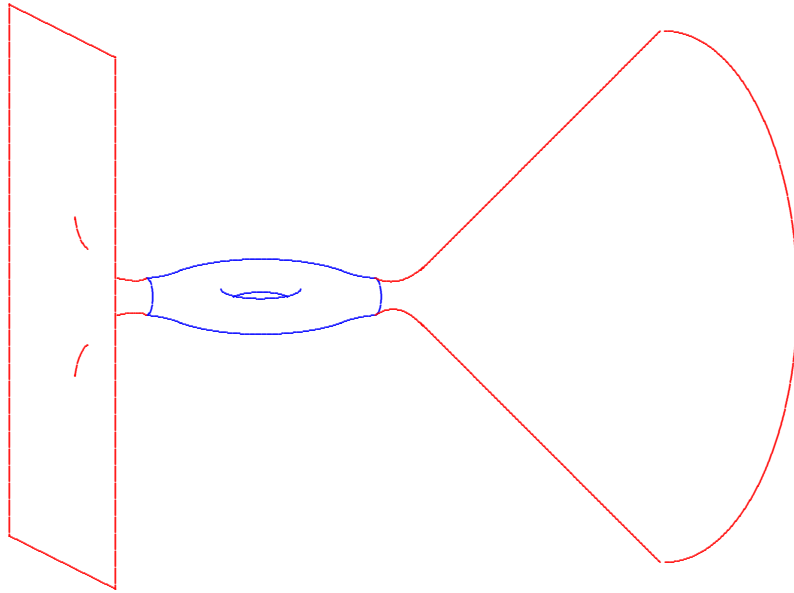
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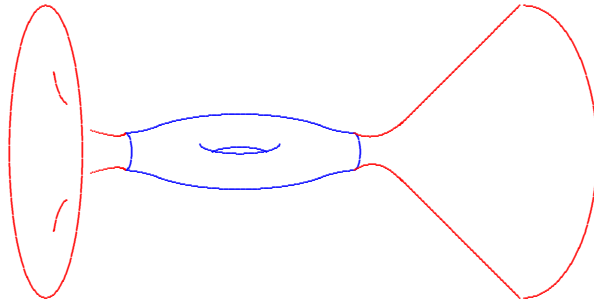
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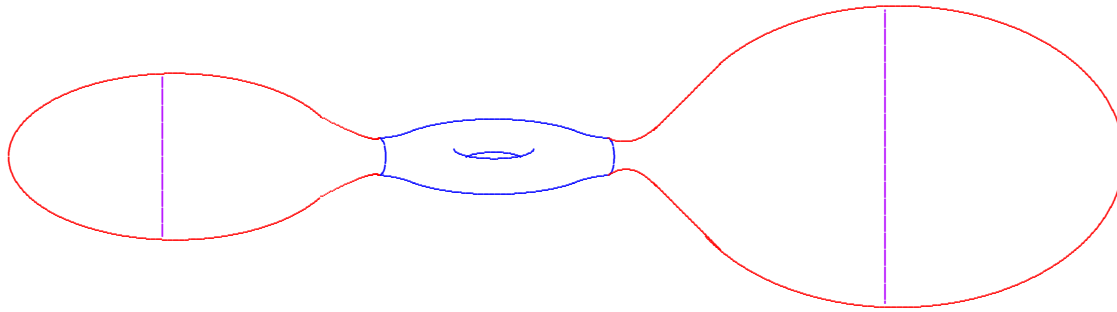
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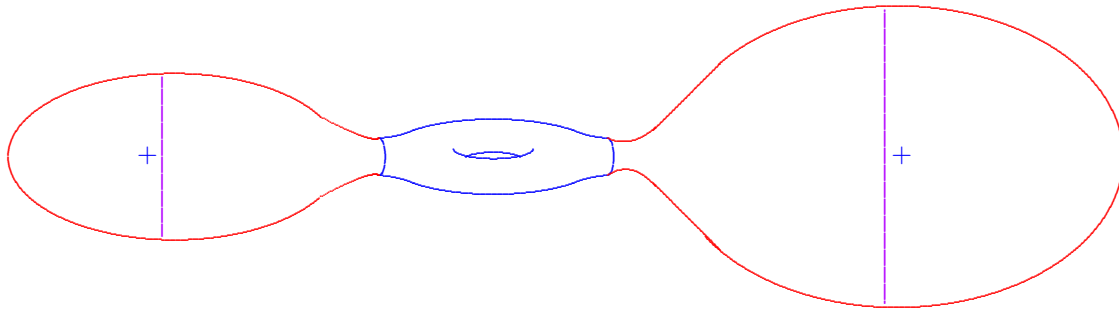
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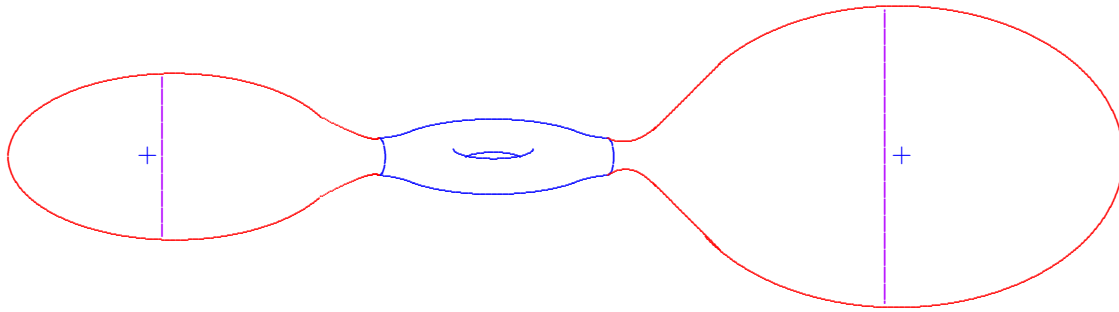


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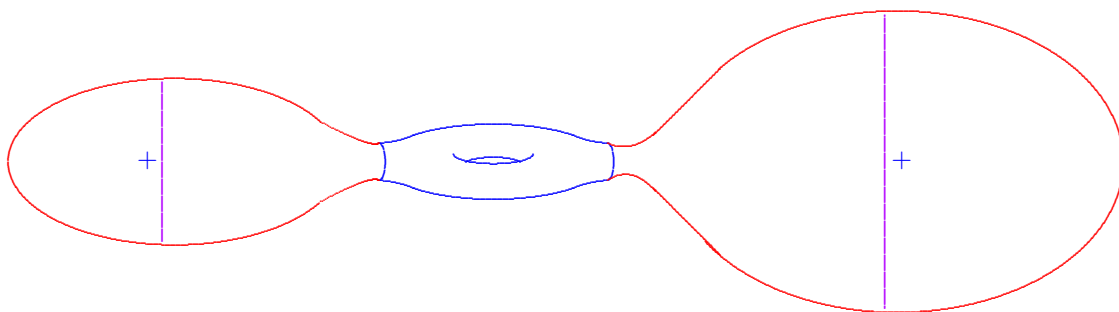
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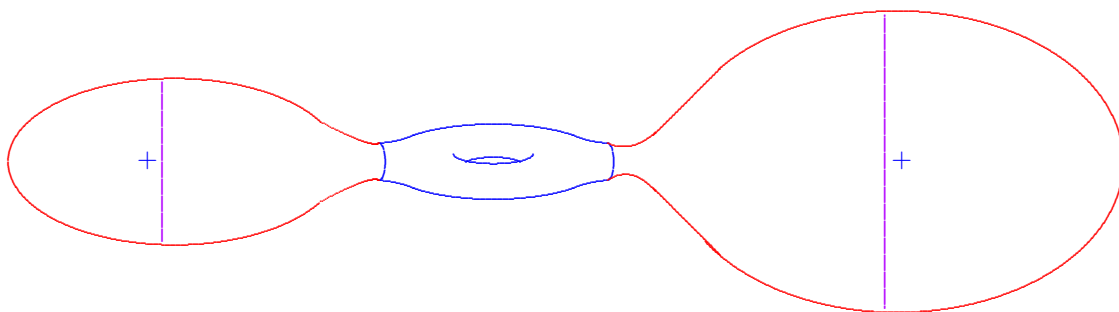
McDuff $\implies \widehat{M} \approx$ rational complex surface.



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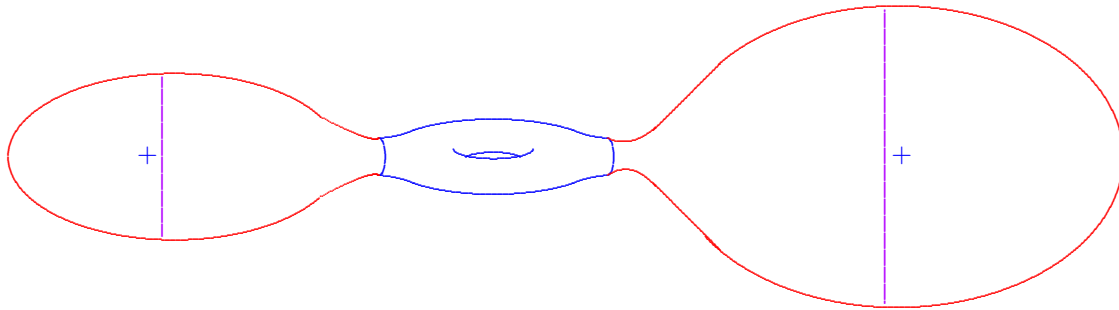
McDuff \implies intersection form $(+- \cdots -)$.



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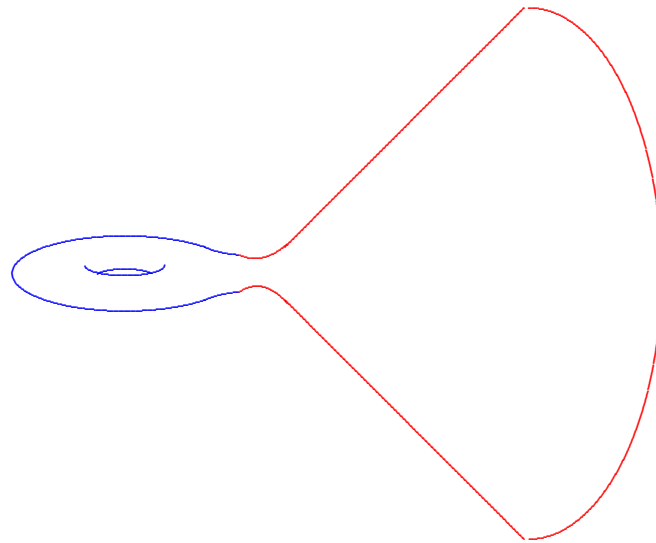
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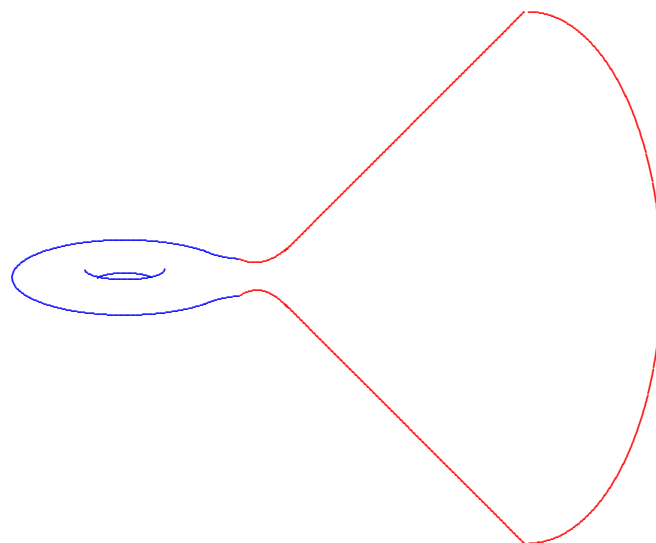
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Since each end contributes positive direction...

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[In higher dimensions, one similarly shows that (M, J) can be compactified as Kähler orbifold. The Hodge theorem on intersection form instead tells one that form on $H^{1,1}(\widehat{M}, \mathbb{R})$ is of type $(+ - \cdots -)$.]

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Proof is heavily computational. Idea:

First rotate coordinates so that $J \rightarrow J_0$ at ∞ ,
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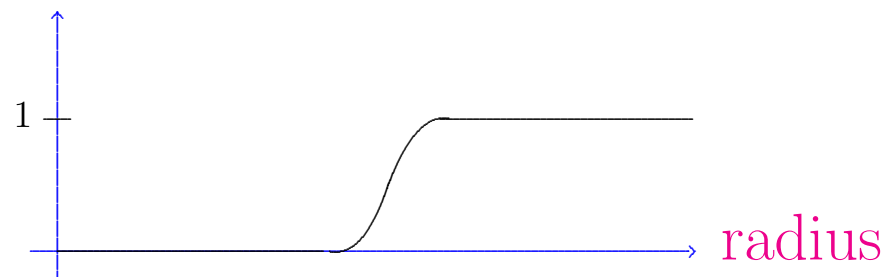
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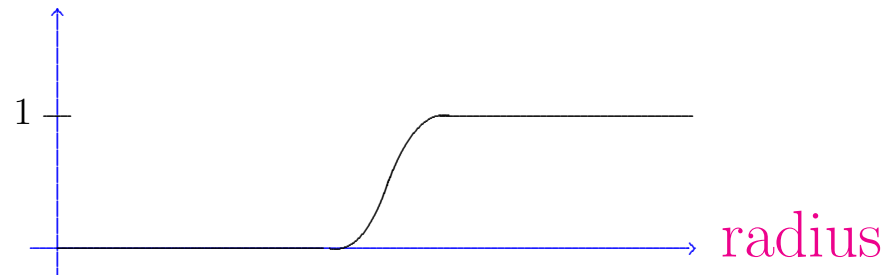
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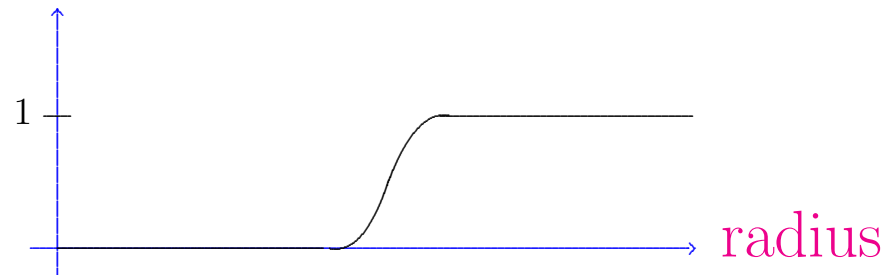
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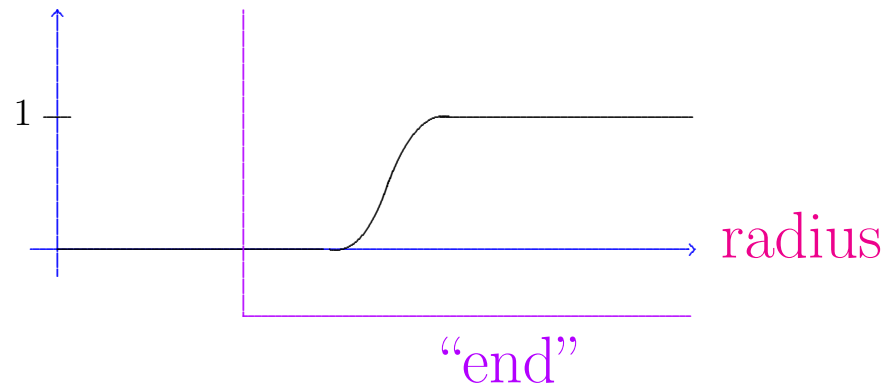
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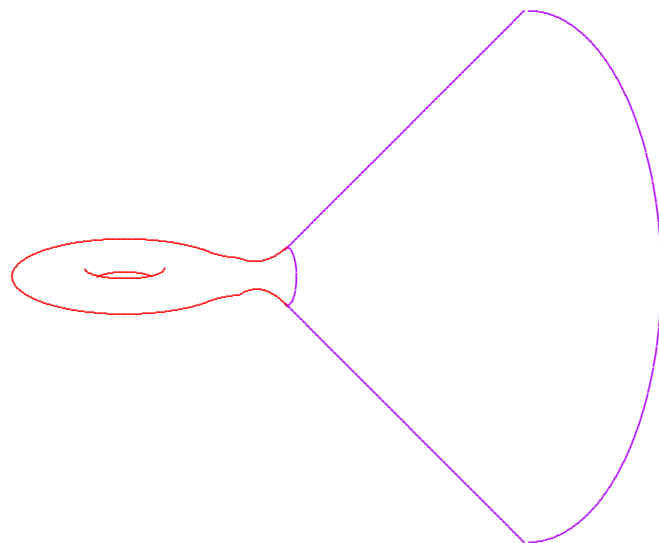
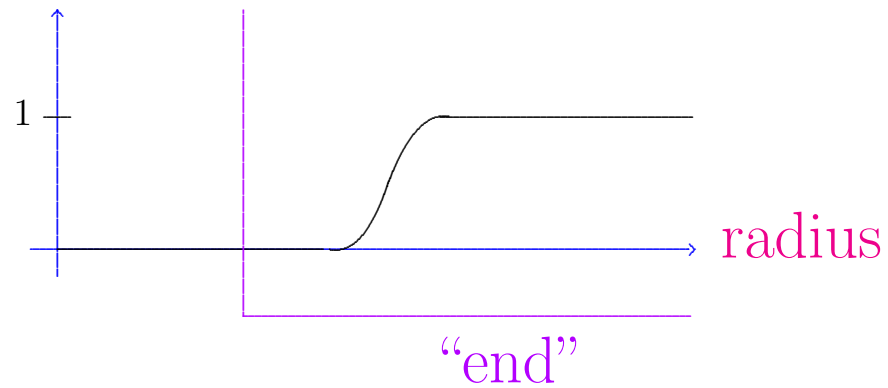
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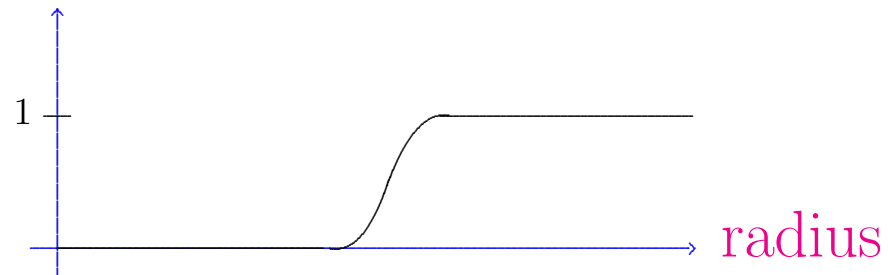
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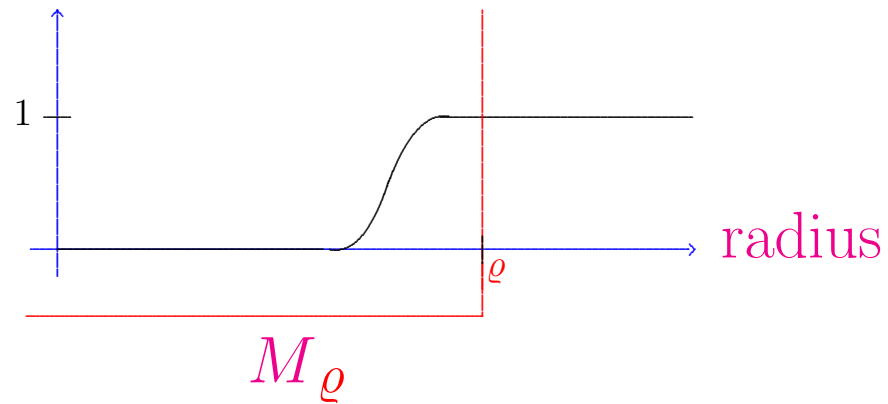
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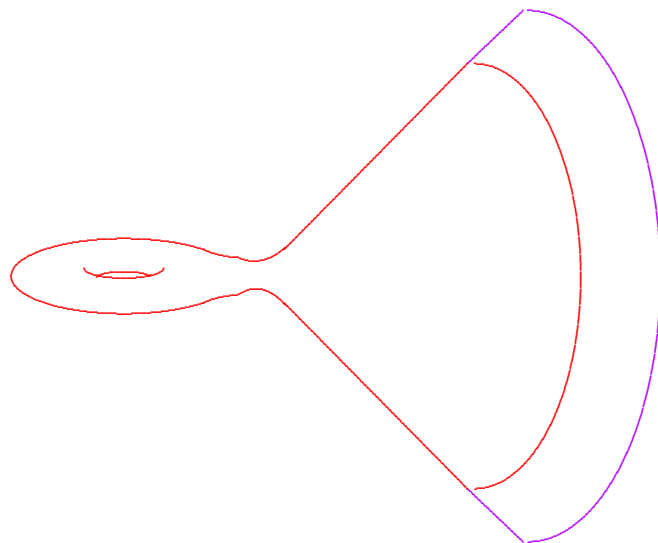
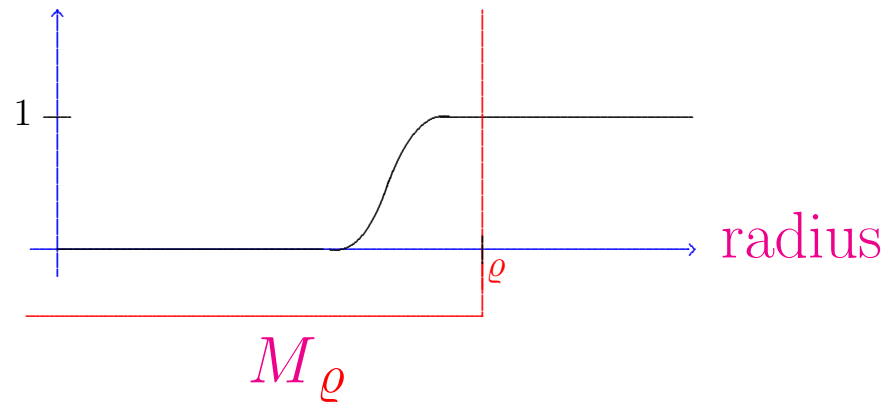
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Compactly supported, because $d\theta = \rho$ near infinity.

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where M_ϱ defined by radius $\leq \varrho$.

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because there is only one end!

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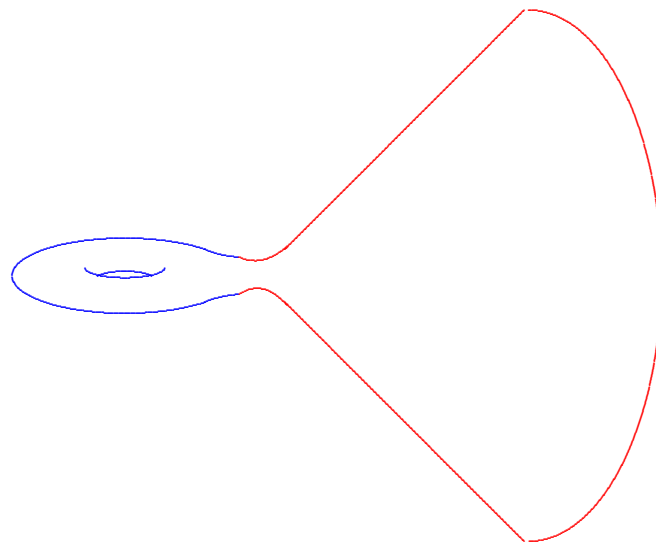
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Limit as $\varrho \rightarrow \infty$ now yields the mass formula.

$$\frac{4\pi^m(2m-1)}{(m-1)!} \mathfrak{m}(M, g) = -\frac{4\pi}{(m-1)!} \langle \clubsuit(c_1), [\omega]^{m-1} \rangle + \int_M s_g d\mu_g$$

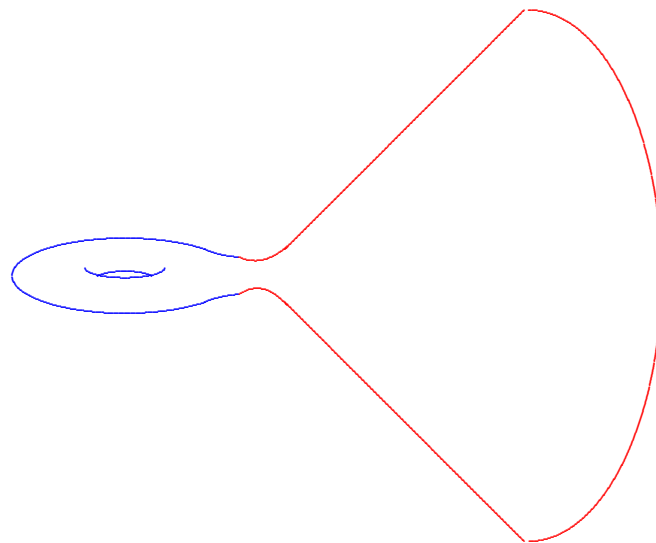
$$m(M, g) = -\frac{\langle \clubsuit(c_1), [\omega]^{m-1} \rangle}{(2m-1)\pi^{m-1}} + \frac{(m-1)!}{4(2m-1)\pi^m} \int_M s_g d\mu_g$$

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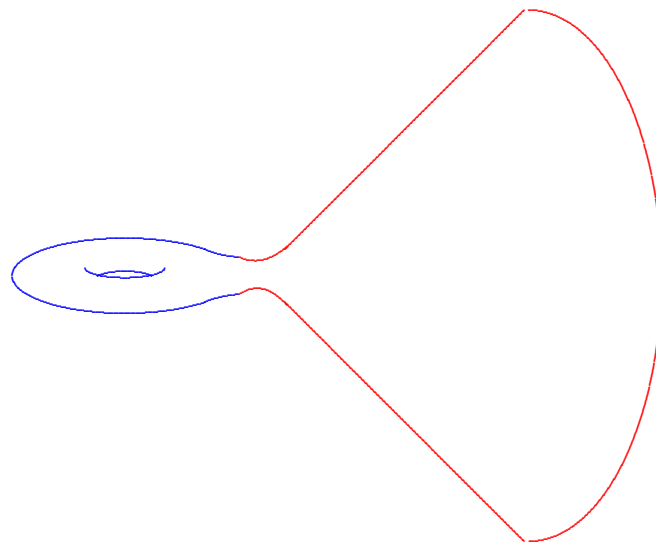
But the Penrose-type inequality is more subtle.

Theorem E (Penrose Inequality). *Let (M^{2m}, g, J) be an **AE** Kähler manifold with scalar curvature $s \geq 0$. Then (M, J) carries a canonical divisor D that is expressed as a sum $\sum_j \mathbf{n}_j D_j$ of compact complex hypersurfaces with positive integer coefficients, with the property that $\bigcup_j D_j \neq \emptyset$ whenever $M \not\cong \mathbb{R}^{2m}$. In terms of this divisor, we then have*

$$m(M, g) \geq \frac{(m-1)!}{(2m-1)\pi^{m-1}} \sum \mathbf{n}_j \text{Vol}(D_j)$$

with $= \iff (M, g, J)$ is scalar-flat Kähler.

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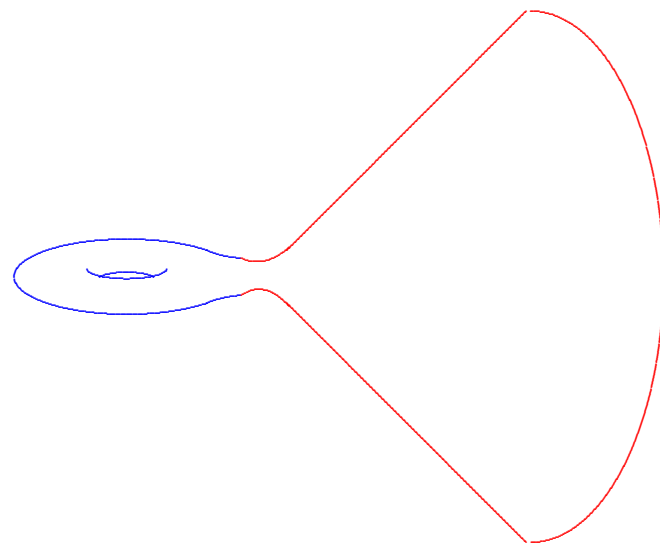
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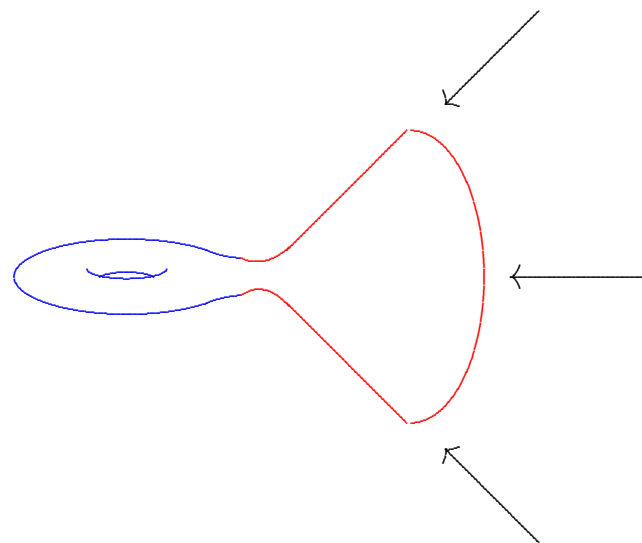
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Technical challenge: Loss of control of derivatives!

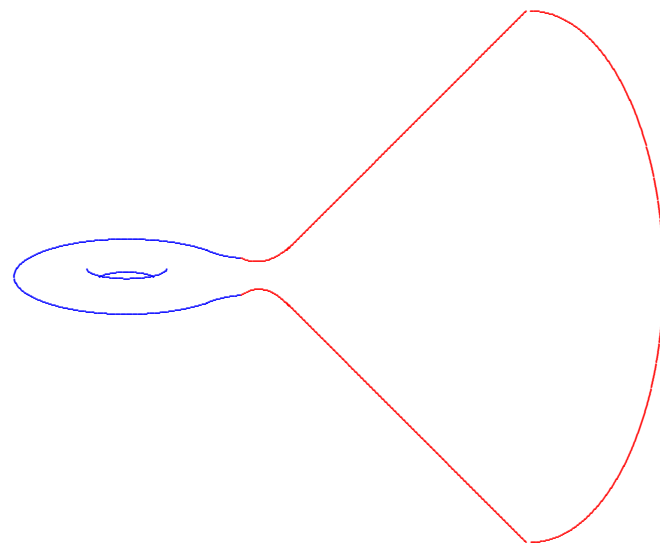
Distance-decreasing map:



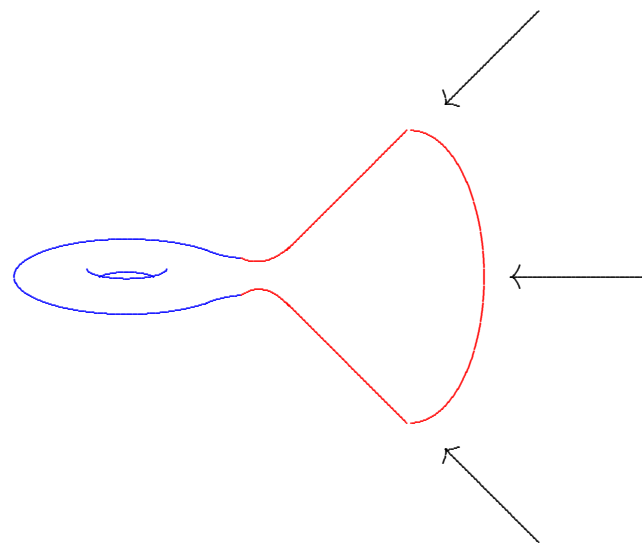
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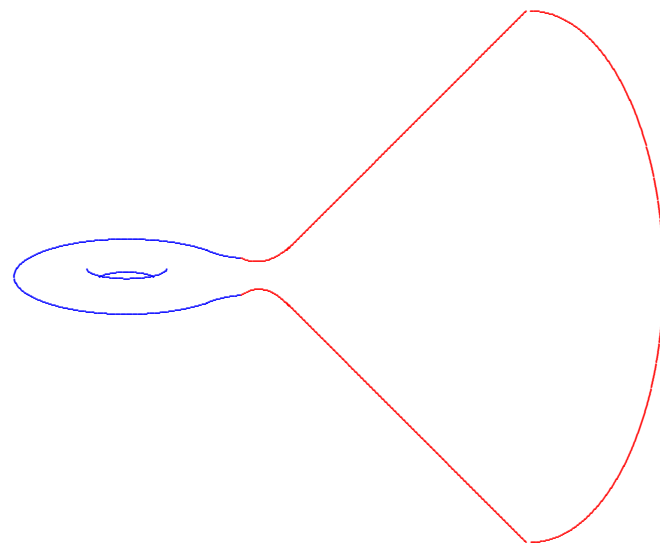
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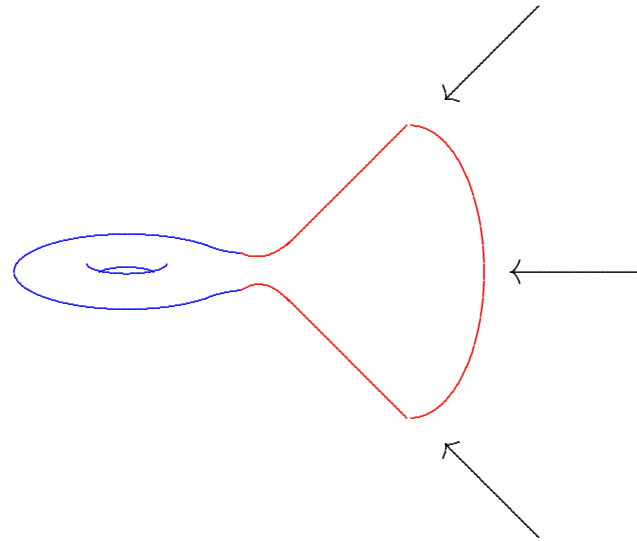
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Robust under distortion of metric in outer region.

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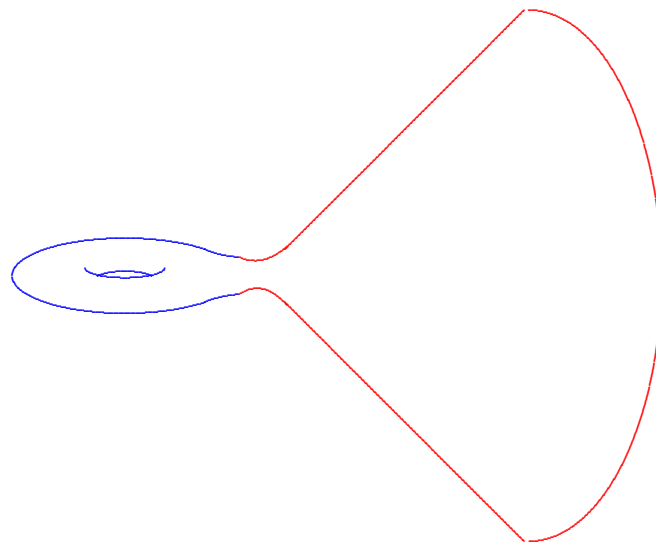
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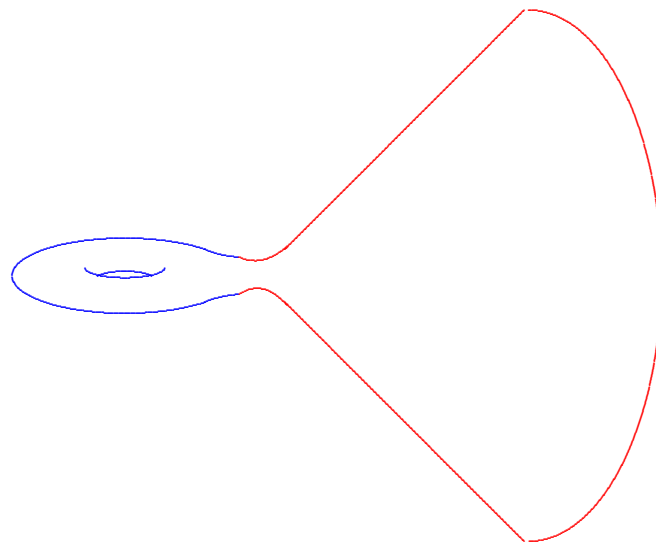
In (M, J) , this gives desired Poincaré dual of $-c_1$.

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**Warmest birthday wishes to my friend
and esteemed colleague Akito Futaki!**

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二木先生, お誕生日おめでとうございます!

非常感谢!

Fēicháng gǎnxiè!

