Einstein Metrics,

Weyl Curvature, and

Conformally Kähler Geometry

Claude LeBrun
Stony Brook University

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Einstein Metrics, Harmonic Forms, and Symplectic Four-Manifolds

Ann. Global An. Geom. 48 (2015) 75–85

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Bach-Flat Kähler Surfaces

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J. Geom. Analysis, *to appear.*

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"...the greatest blunder of my life!"

— A. Einstein, to G. Gamow

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On oriented (M^4, g), $\Longrightarrow$

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^-$$

Λ^+ self-dual 2-forms
Λ^- anti-self-dual 2-forms

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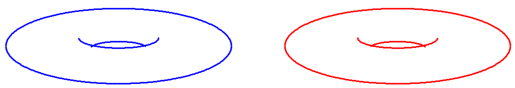

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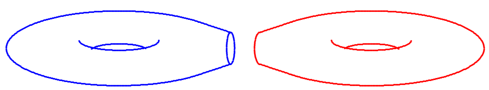

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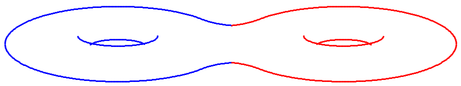

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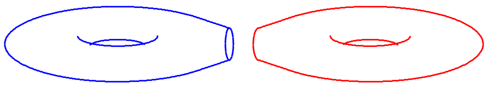

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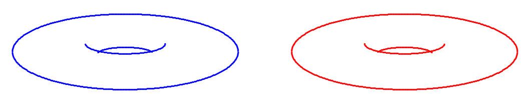

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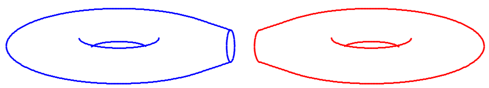

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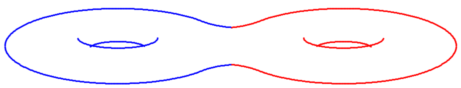

Theorem (L '09). *Suppose that M is a smooth compact oriented 4-manifold which admits a symplectic structure ω. Then M also admits an Einstein metric h with $\lambda \geq 0$ if and only if*

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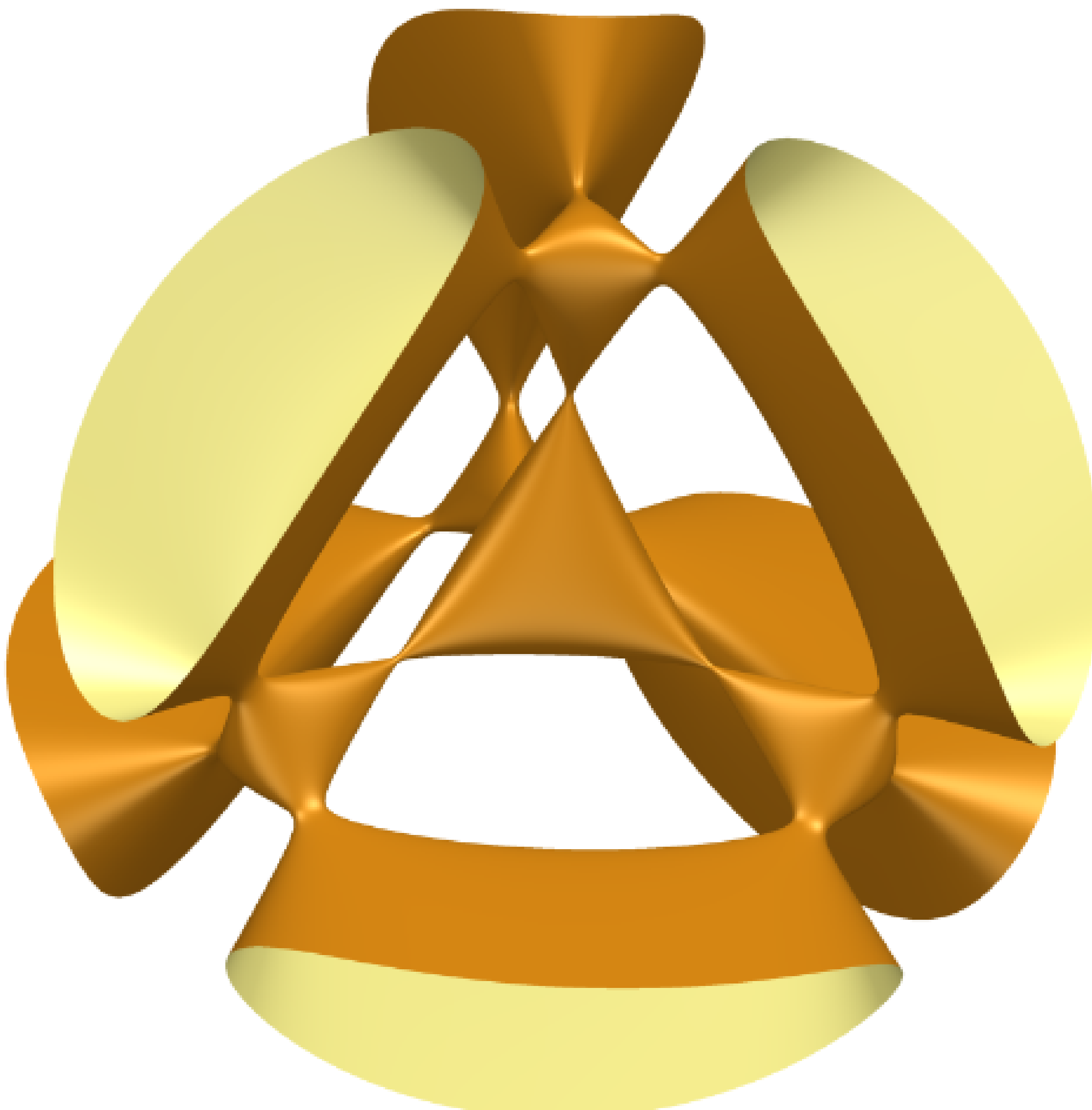

$K3$ = underlying M^4 of a generic quartic in $\mathbb{CP}_3$.

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Del Pezzo surfaces,
K3 surface, Enriques surface,
Abelian surface, Hyper-elliptic surfaces.

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No others: Hitchin-Thorpe $2\chi + 3\tau \geq 0$

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Constructed Einstein metrics all *conformally Kähler:*

$$h = u^2 g, \qquad g \text{ Kähler.}$$

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Key to construction: *Weyl functional*.

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$$\mathcal{R}^{ab}{}_{cd} = W^{ab}{}_{cd} + \frac{4}{n-2}\mathring{r}^{[a}{}_{[c}\delta^{b]}{}_{d]} + \frac{2}{n(n-1)}\, s\, \delta^{a}_{[c}\delta^{b}_{d]}$$

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$W^{a}{}_{bcd}$ unchanged if $g \rightsquigarrow \hat{g} = u^2 g$.

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Proposition. *Assume* $n \geq 4$. *Then* (M^n, g) *locally conformally flat* $\iff W \equiv 0$.

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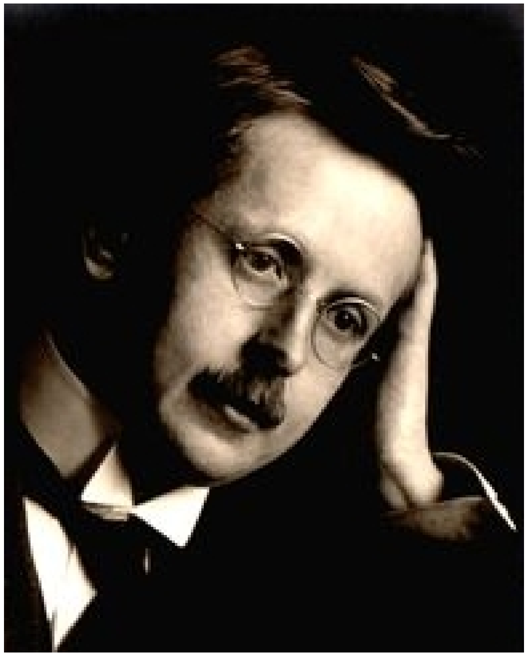

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Basic problems: For given smooth compact M,

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when $\ell > 0$, because $\mathscr{W} \propto \mathrm{Vol}(T^\ell)$!

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$$\tau(M) = \frac{1}{12\pi^2} \int_M \left(|W_+|^2 - |W_-|^2 \right) d\mu$$

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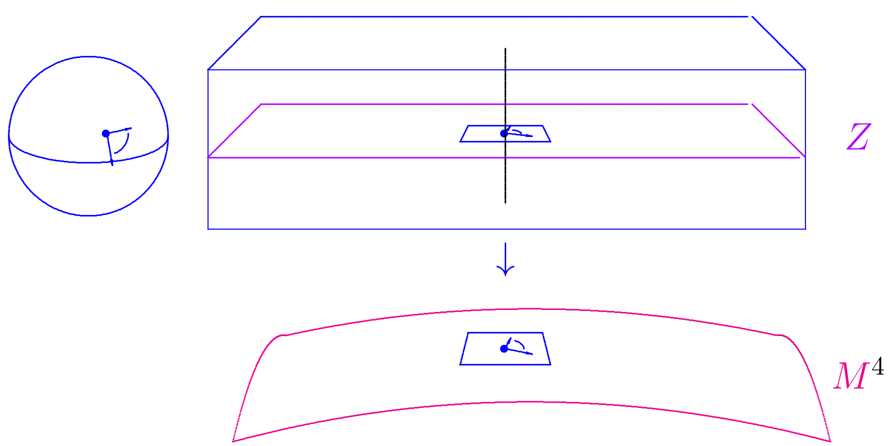

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Violate Hitchin-Thorpe, so $\nexists$ Einstein on such M.

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L-Singer '93, Kim-L-Pontecorovo '97 Any rational/ruled (M, J) has blow-ups admitting SFK.

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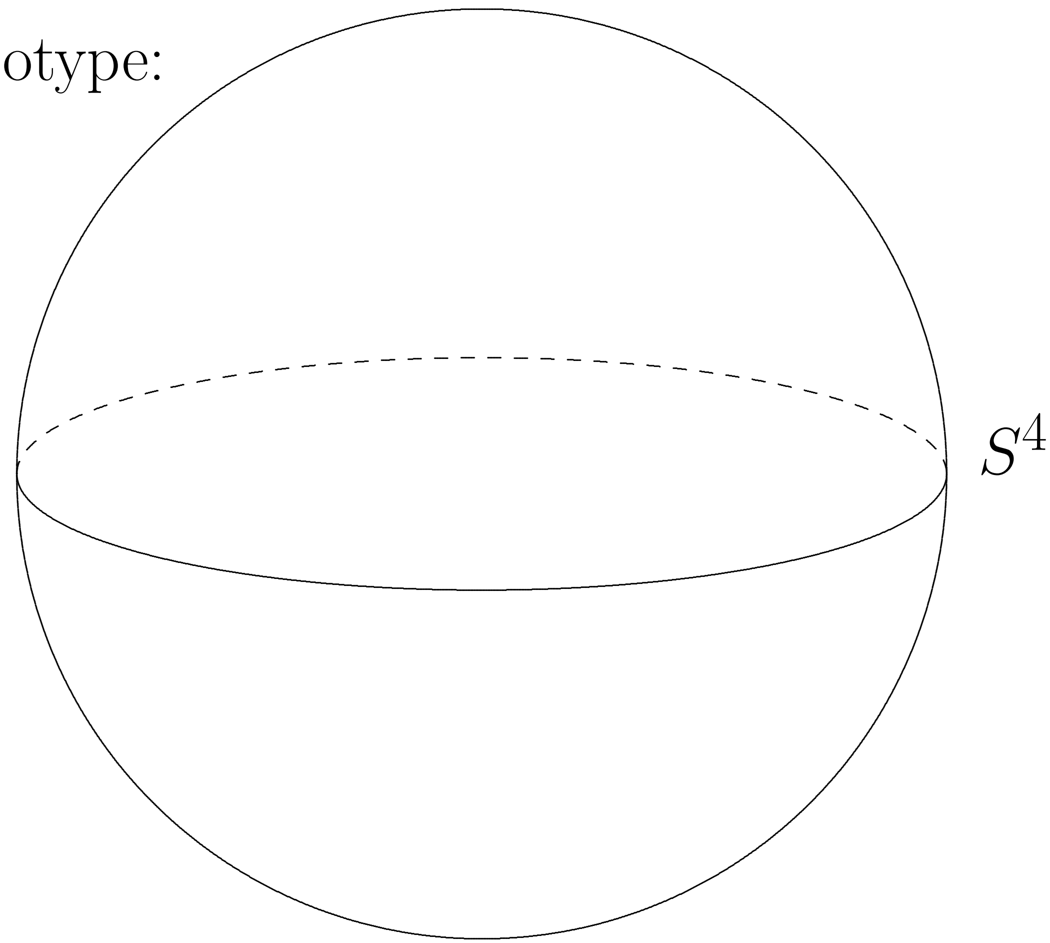

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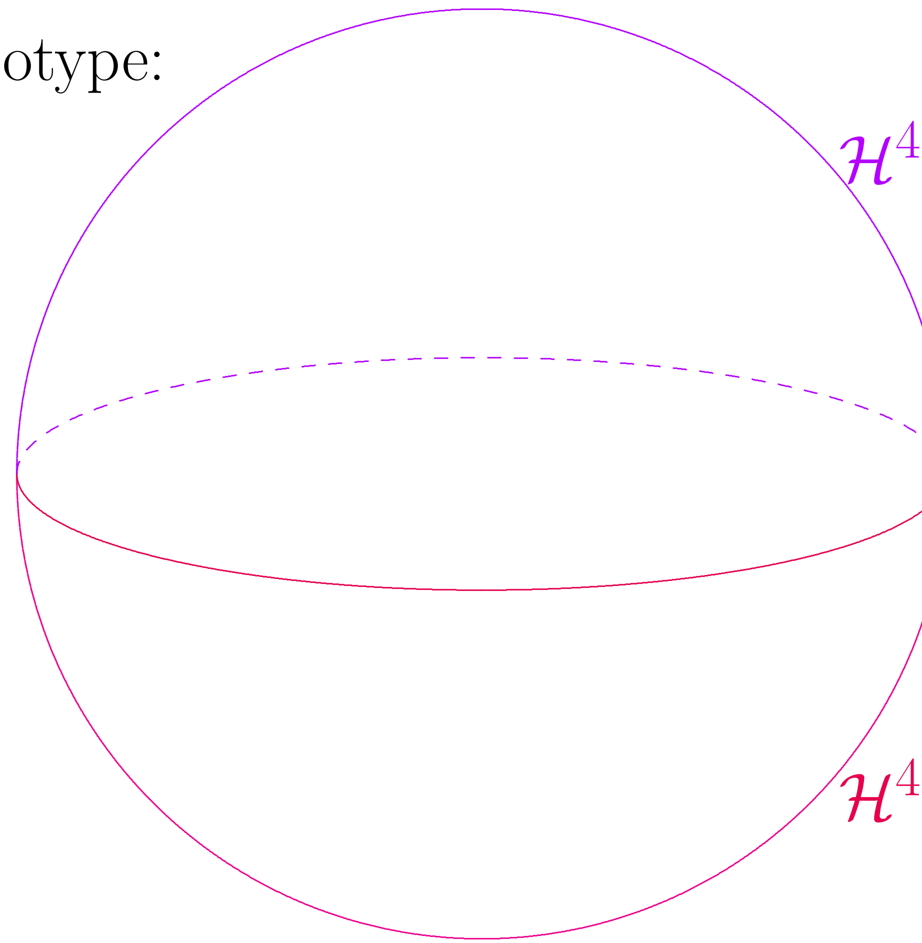

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But $\exists$ genuine examples that aren't.

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But no compact counter-examples are known!

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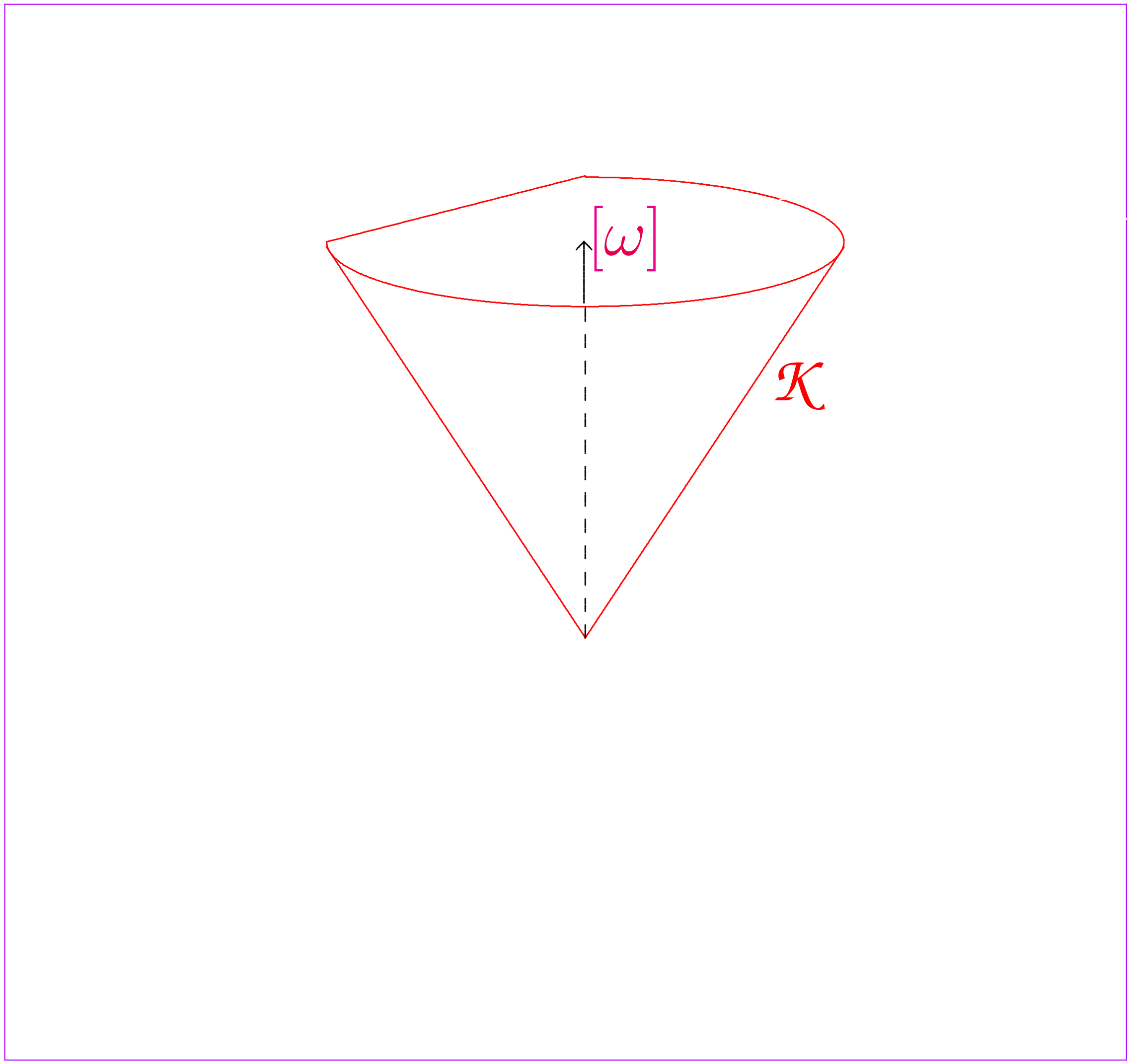
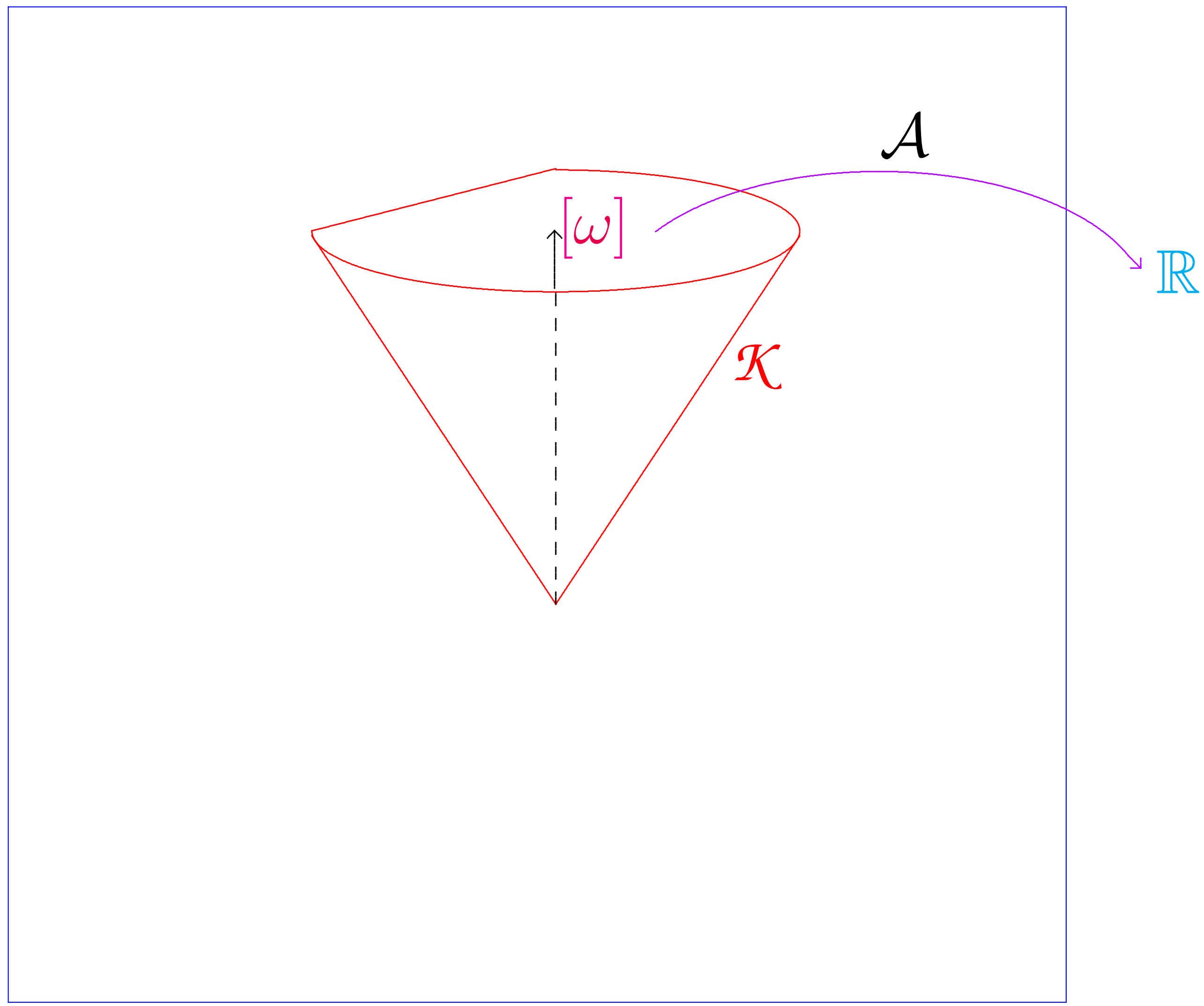
On set where $s \neq 0$, the metric $s^{-2}g$ is Einstein.



$$\mathcal{K} \subset H^{1,1}(M, \mathbb{R}) \subset H^2(M, \mathbb{R})$$



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Action Function on Kähler Cone

For any extremal Kähler (M^4, g, J),

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$$=: \mathcal{A}([\omega])$$

where $\mathcal{F}$ is Futaki invariant.

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$$\frac{1}{32\pi^2}\int s^2 d\mu_g = \frac{(c_1\cdot[\omega])^2}{[\omega]^2} + \frac{1}{32\pi^2}\|\mathcal{F}_{[\omega]}\|^2 =: \mathcal{A}([\omega])$$

where $\mathcal{F}$ is Futaki invariant.

$\mathcal{A}$ is function on Kähler cone $\mathcal{K} \subset H^2(M, \mathbb{R})$.

Proposition. *If g is a Kähler metric on a compact complex surface (M^4, J), with Kähler class $[\omega]$, then g satisfies $B = 0 \iff$*

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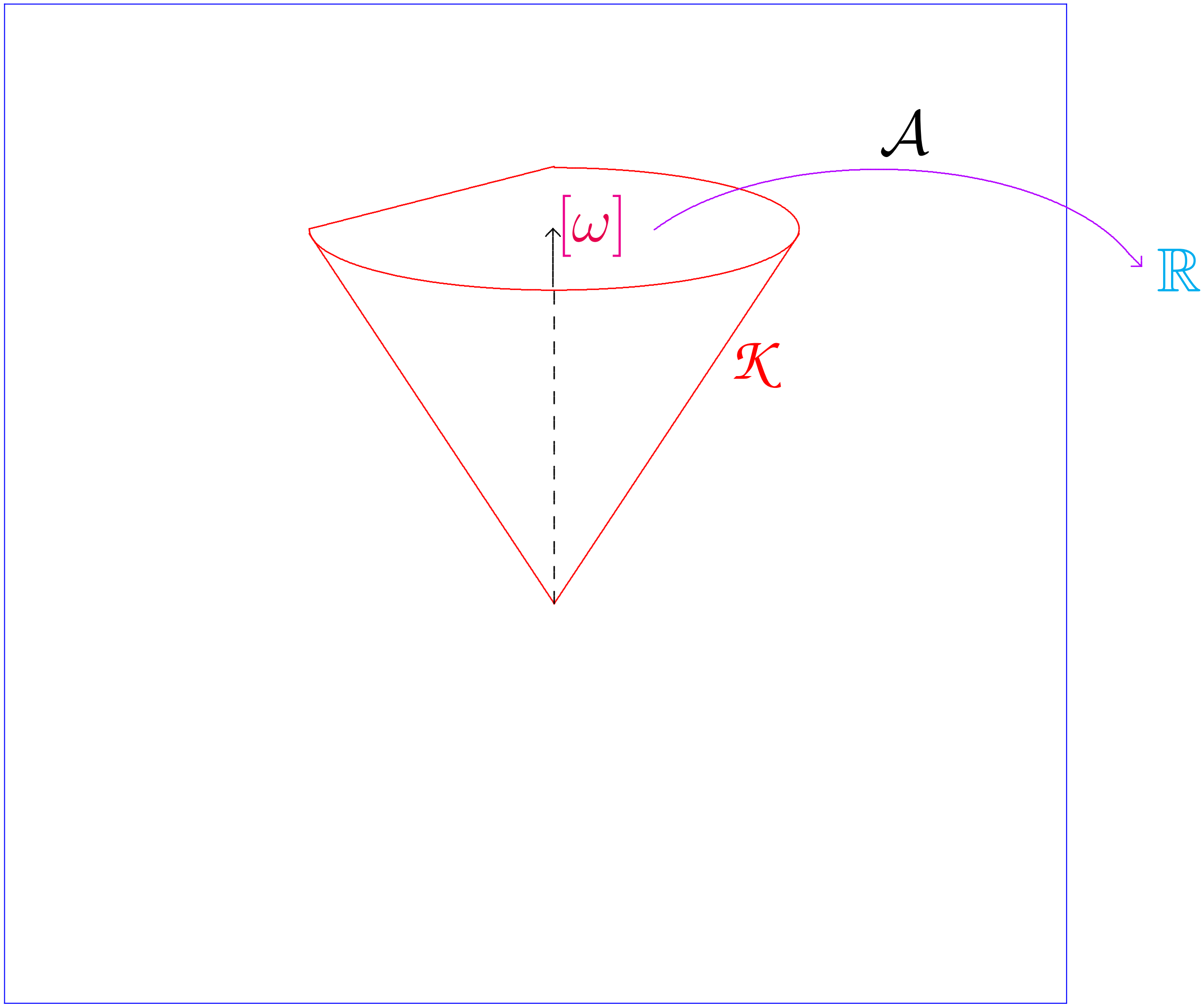

$$\mathcal{K} \subset H^{1,1}(M, \mathbb{R}) \subset H^2(M, \mathbb{R})$$

Henceforth, assume M compact, real dimension 4.

Today:

Bach-flat Kähler $\Longrightarrow$ one of these three types.

Builds on earlier local results of Andrzej Derdziński.

Scalar curvature s plays the starring role.

Kähler surfaces:

$$|W_+|^2 = \frac{s^2}{24}$$

$$\int_M |W_+|^2 d\mu = \frac{1}{24}\int_M s^2 d\mu$$

Bach-flat Kähler $\Longrightarrow$ extremal Kähler

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If not Kähler-Einstein:

I. s *is positive. Then*

$(M, s^{-2}g)$ *Einstein*, $\lambda > 0$, *Hol* $= \mathbf{SO}(4)$.

II. s *is zero. Then*

(M, g, J) *SFK, but not Ricci-flat.*

III. s *changes sign. Then*

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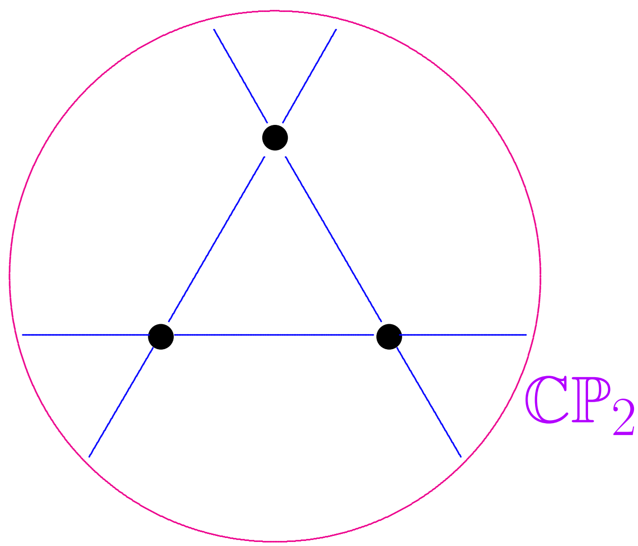

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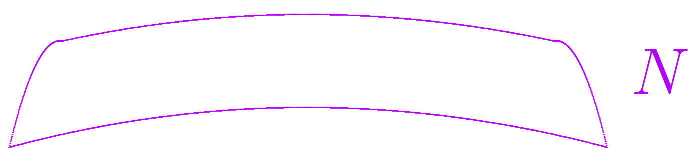

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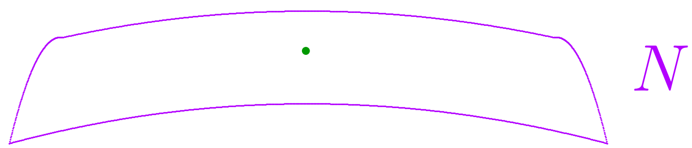

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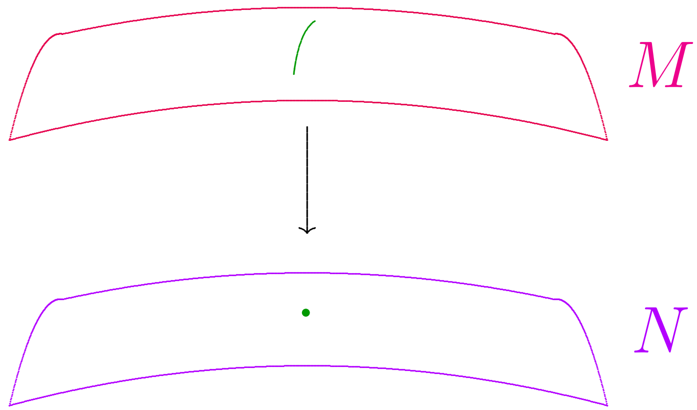
If N is a complex surface, may replace $p \in N$ with $\mathbb{CP}_1$



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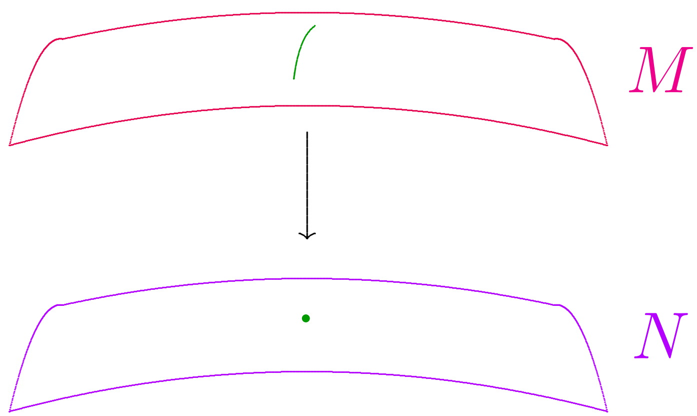
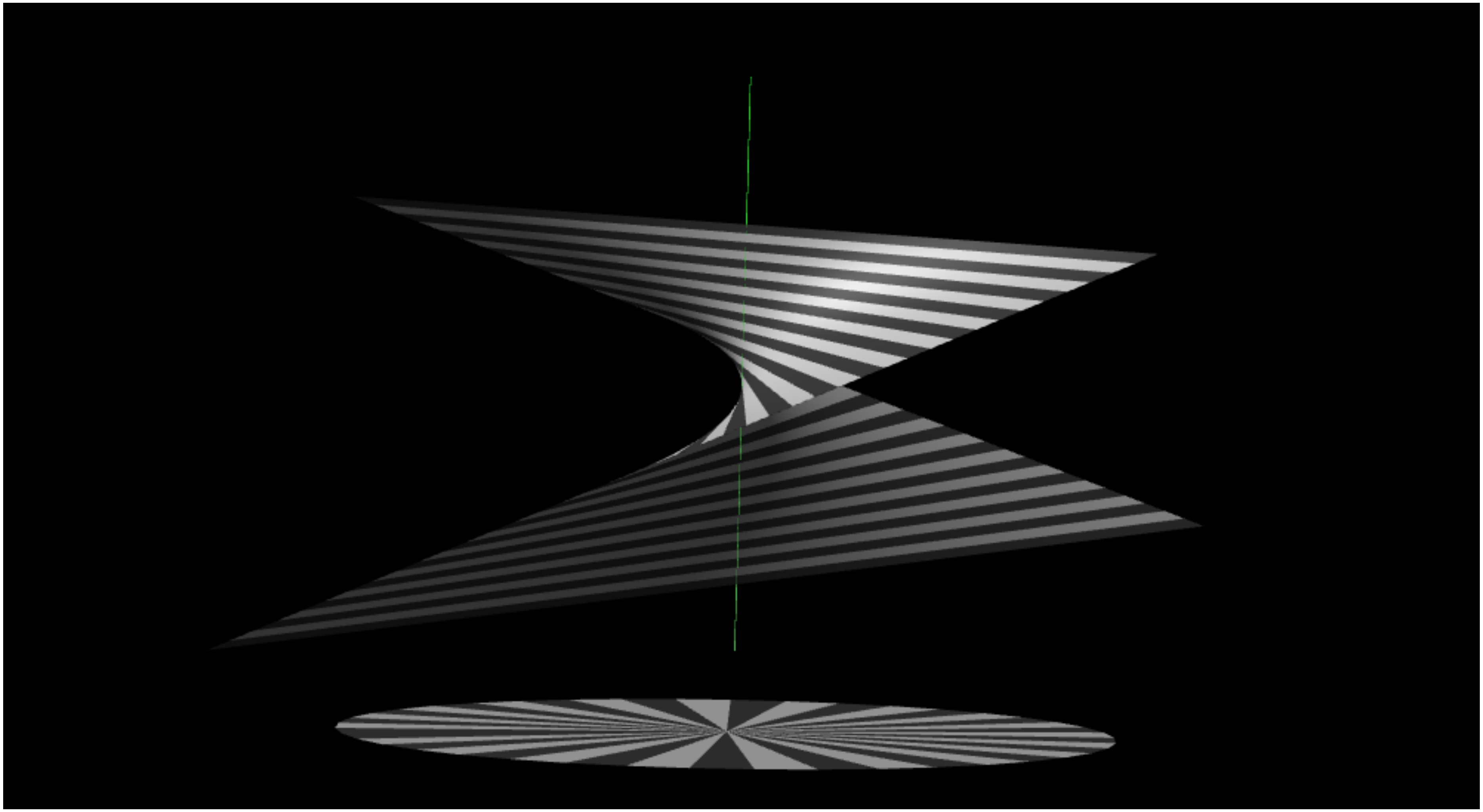
If N is a complex surface, may replace $p \in N$ with $\mathbb{CP}_1$ to obtain blow-up

$$M \approx N \# \overline{\mathbb{CP}}_2$$

in which added $\mathbb{CP}_1$ has normal bundle $\mathcal{O}(-1)$.





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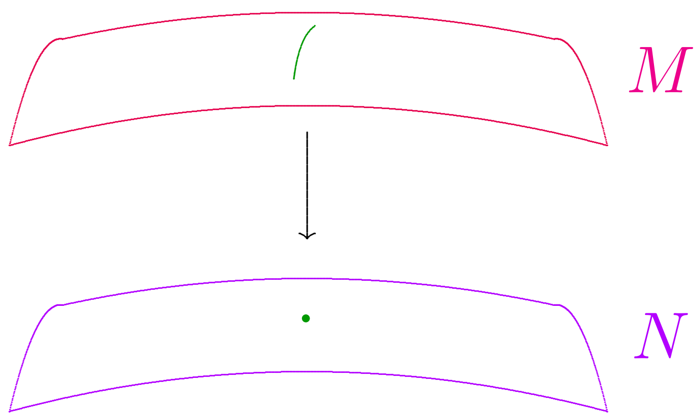

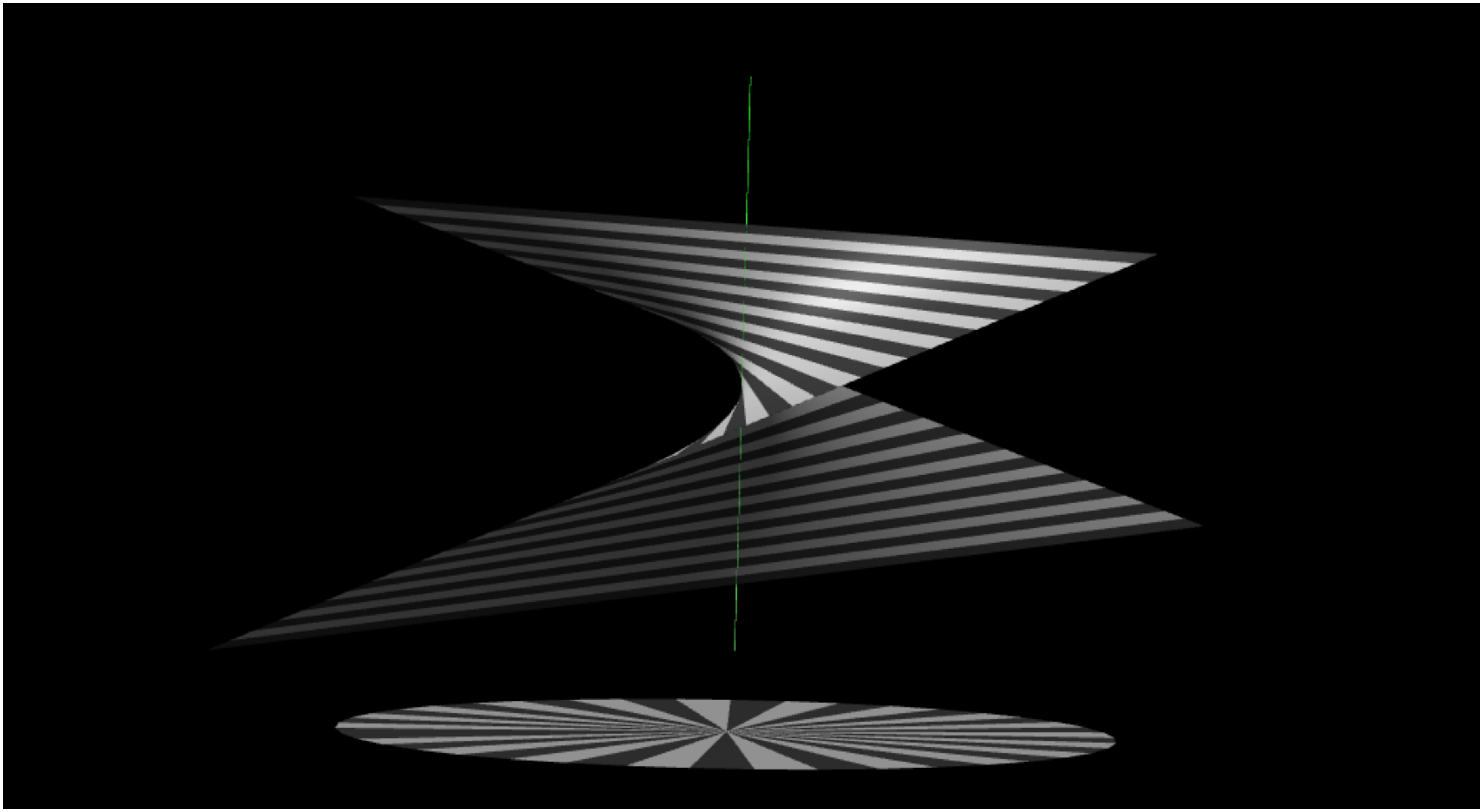

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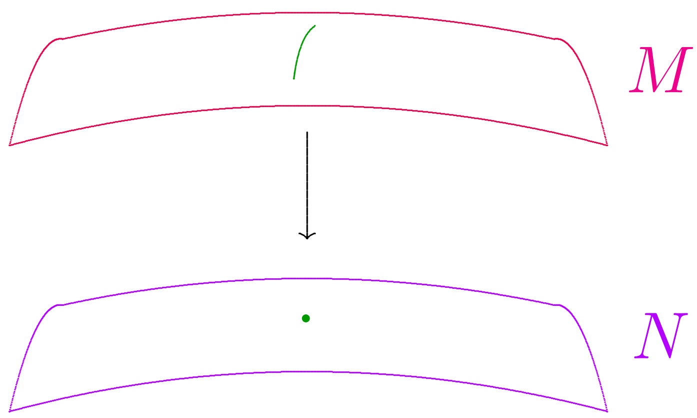

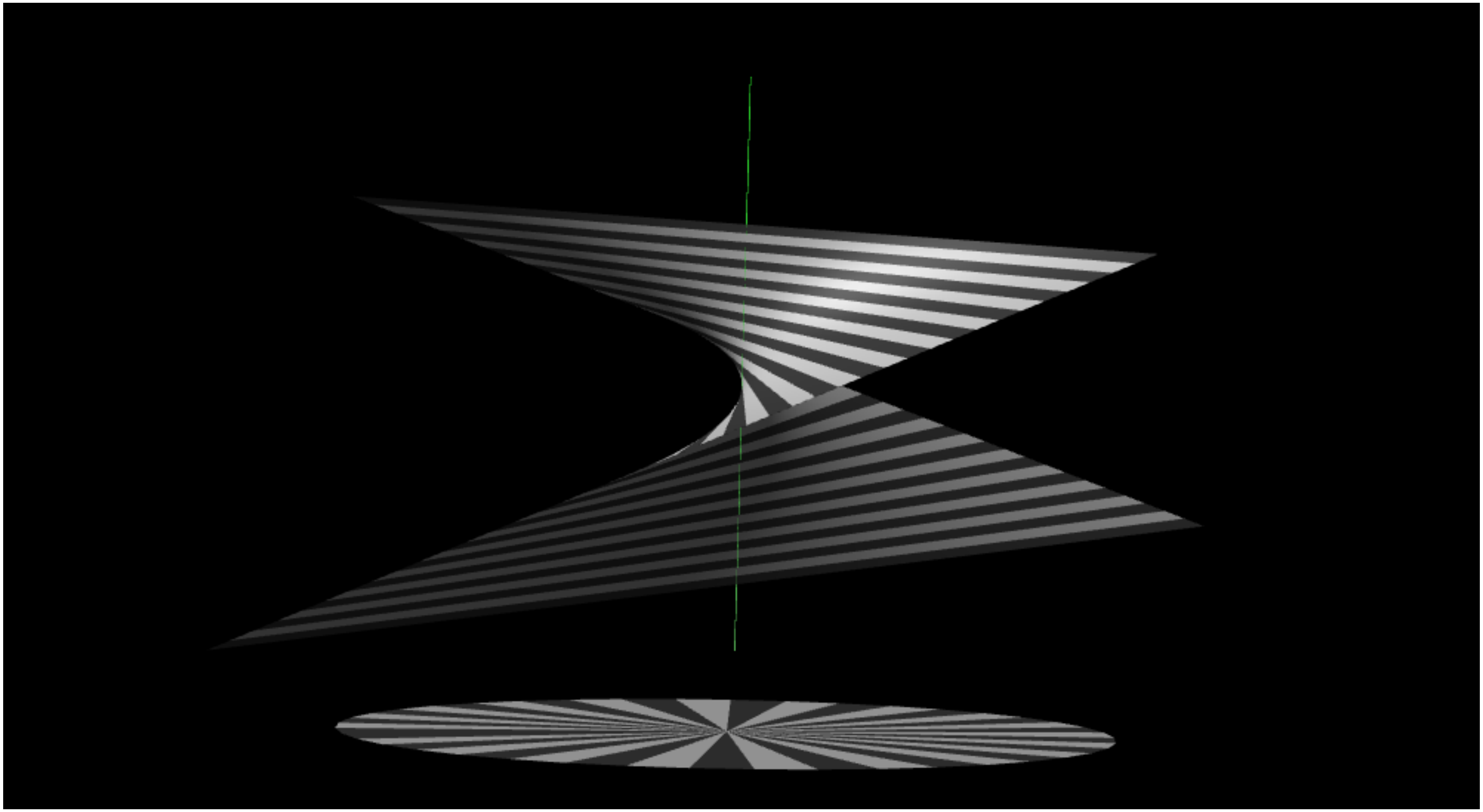

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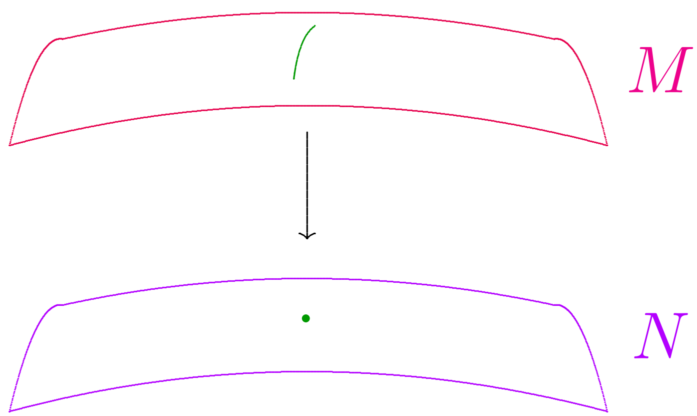

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(M^4, J) for which c_1 is a Kähler class $[\omega]$.
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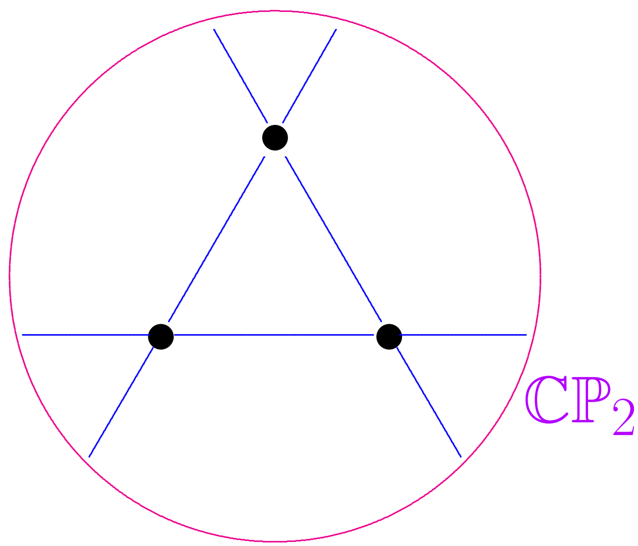
Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.



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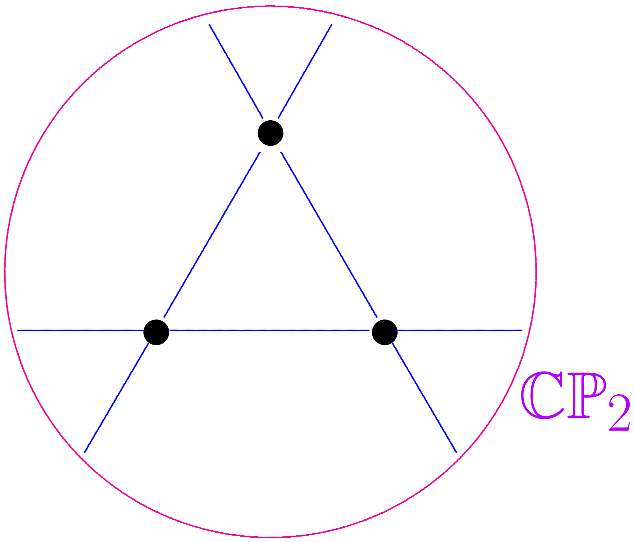

No 3 on a line,

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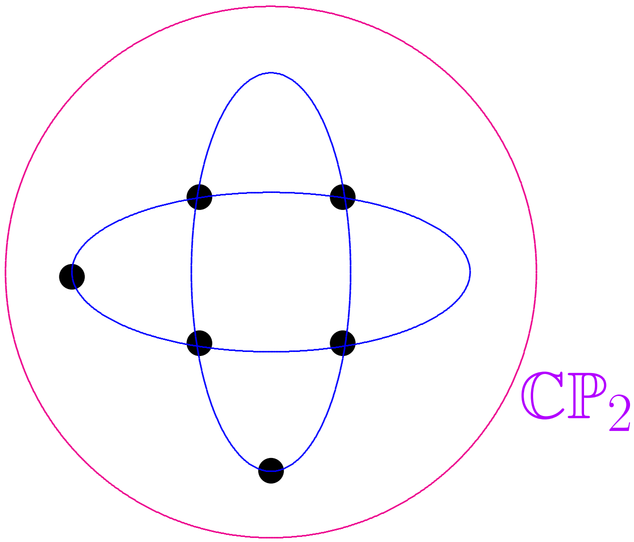

No 3 on a line, no 6 on conic,

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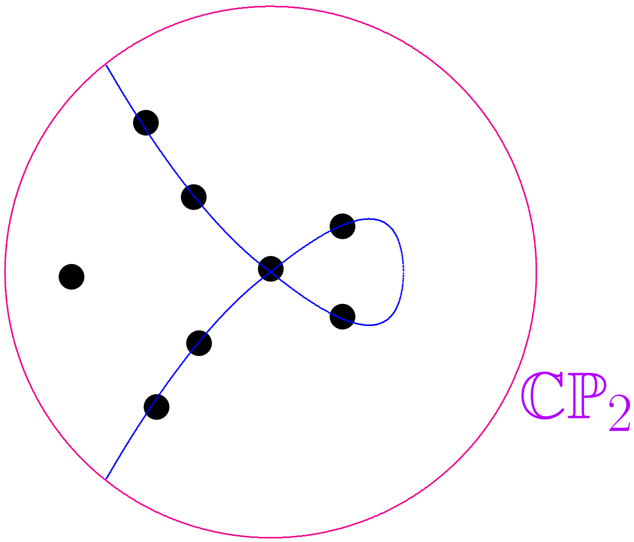

No 3 on a line, no 6 on conic, no 8 on nodal cubic.

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Theorem. *Each Del Pezzo* (M^4, J) *admits a compatible conformally Kähler Einstein metric, and this metric is unique up to automorphisms.*

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Uniqueness: Bando-Mabuchi, L '12…

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Moduli space of such (M^4, J) is connected.

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Just a point if $b_2(M) \leq 5$.

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(b) when $M = \mathbb{CP}_2 \# \overline{\mathbb{CP}}_2$ or $\mathbb{CP}_2 \# 2\overline{\mathbb{CP}}_2$.

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(a) (M, g, J) *Kähler-Einstein*, $\lambda > 0$; *or else*

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II. $s \equiv 0$. *Then*

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(a) $\Longrightarrow$ Kod $(M, J) = 0$.

(b) $\Longrightarrow$ Kod $(M, J) = -\infty$.

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If $\min s < 0$, then s either constant, or changes sign.

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(a) $\Longrightarrow \operatorname{Kod}(M, J) = 2$.

I. $\min s > 0$. *Then*

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II. $s \equiv 0$. *Then*

(a) (M, g, J) *Kähler-Einstein*, $\lambda = 0$; *or else*

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(a) (M, g, J) *Kähler-Einstein*, $\lambda < 0$; *or else*

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Examples of (b): Hwang-Simanca, Tønnesen-Friedman

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Define

$$\kappa := -6s\Delta s - 12|\nabla s|^2 + s^3,$$

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where $\Delta = -\nabla^a \nabla_a$.

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Lemma. *The function* κ *is constant, and has the same sign* $(+, -, 0)$ *as* $\min s$. *On set where* $s \neq 0$, *the metric* $h = s^{-2}g$ *is Einstein, with scalar curvature* κ.

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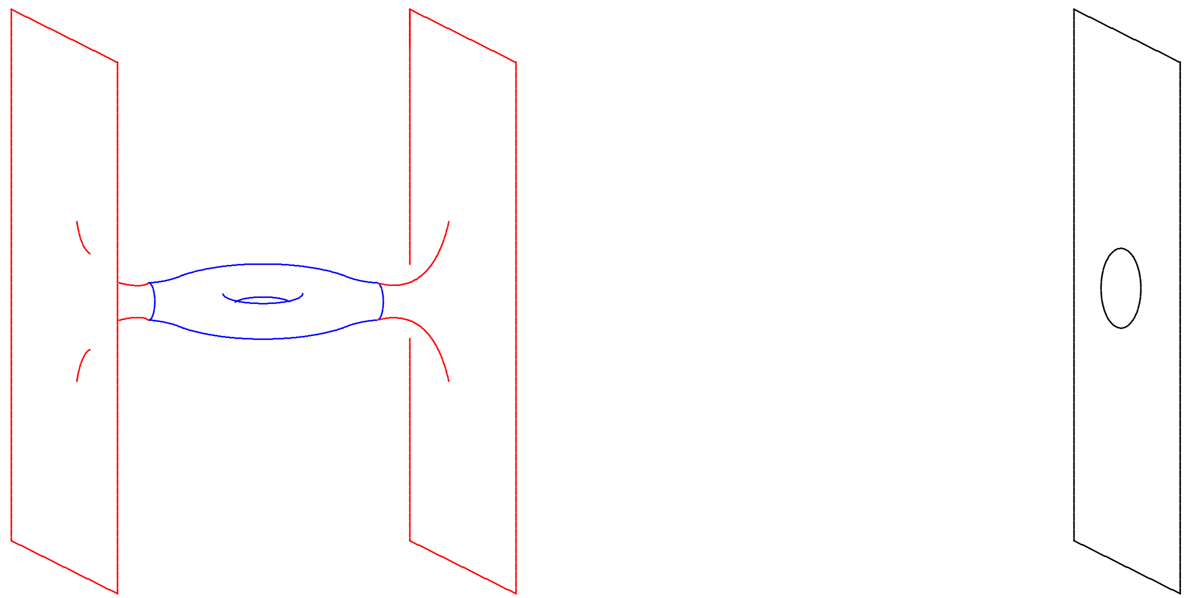
$h = s^{-2}g$ is Ricci-flat, asymptotically Euclidean.

Definition. *A complete, non-compact Riemannian n-manifold (M^n, g) is called* asymptotically Euclidean *(AE) if there is a compact set $K \subset M$ such that each component of $M - K$ is diffeomorphic to $\mathbb{R}^n - D^n$ in such a manner that*



$$g_{jk} = \delta_{jk} + O(|x|^{1-\frac{n}{2}-\varepsilon})$$

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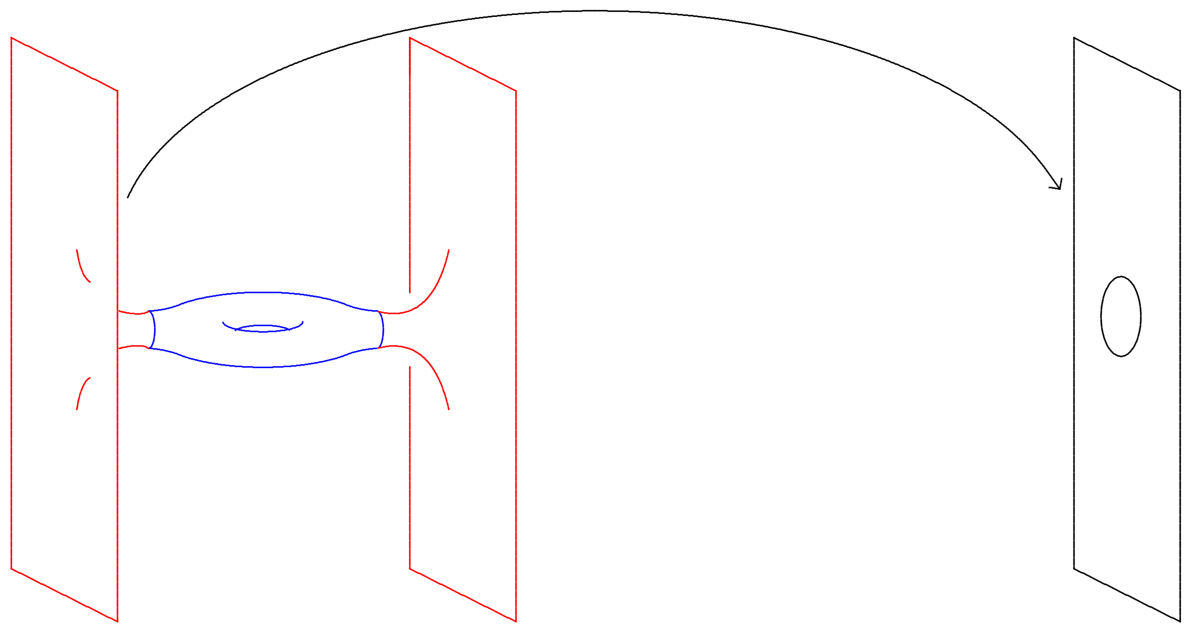

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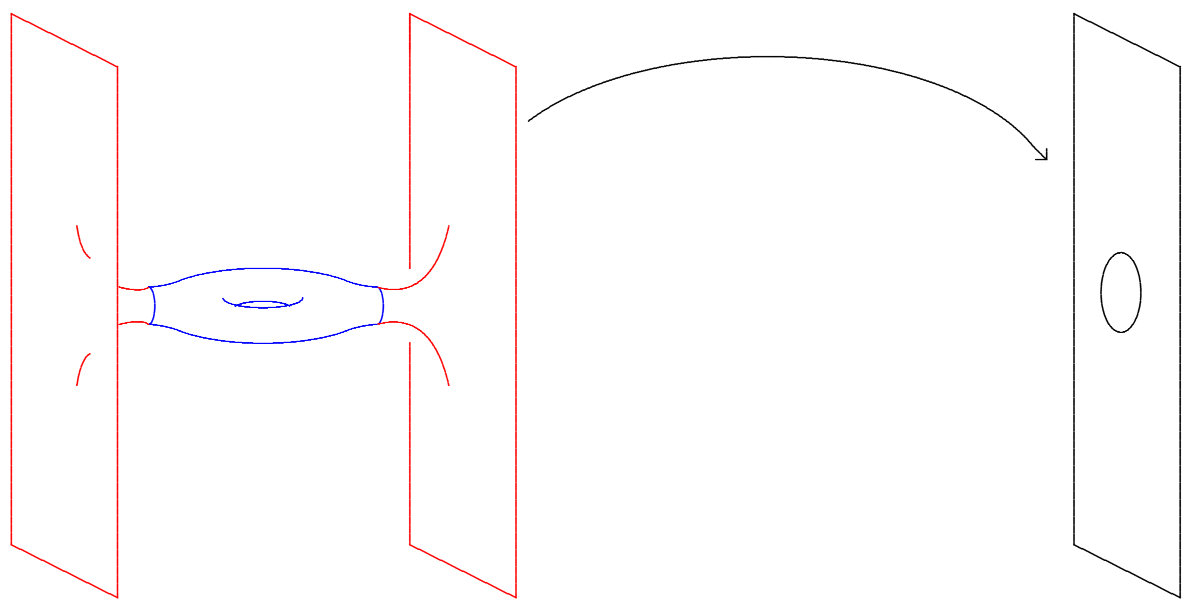

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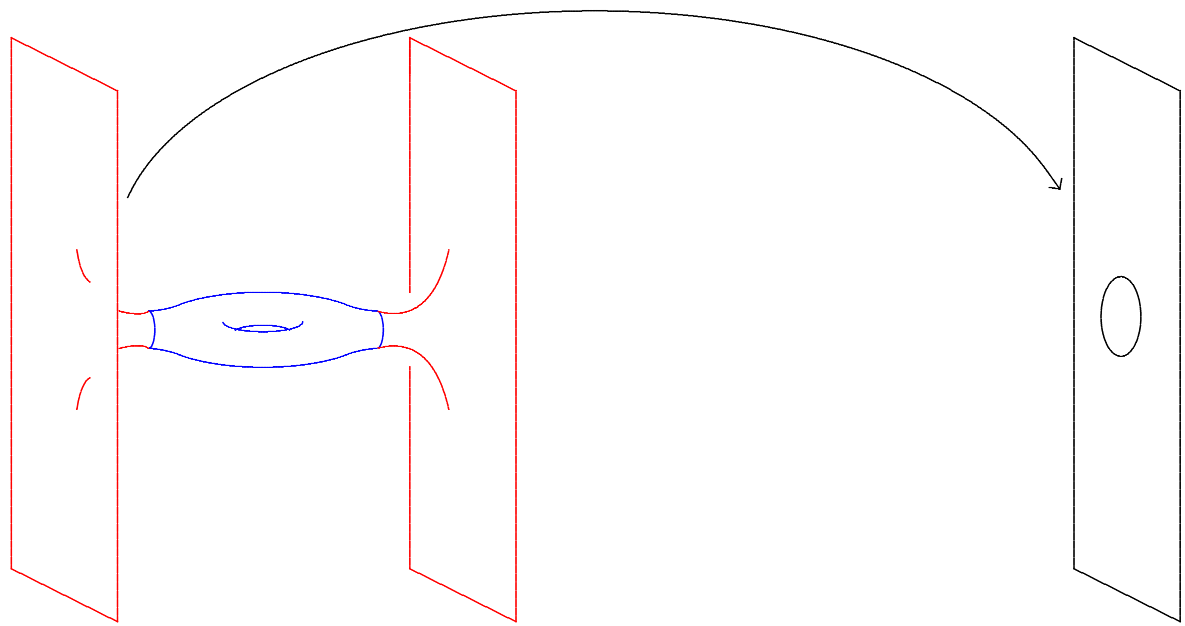

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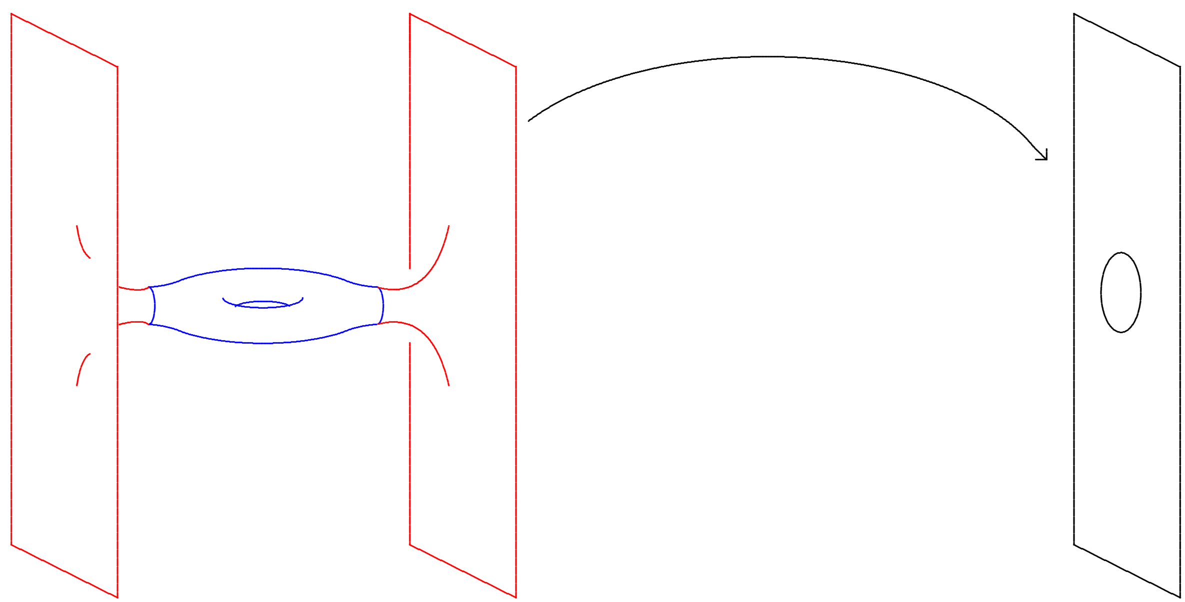

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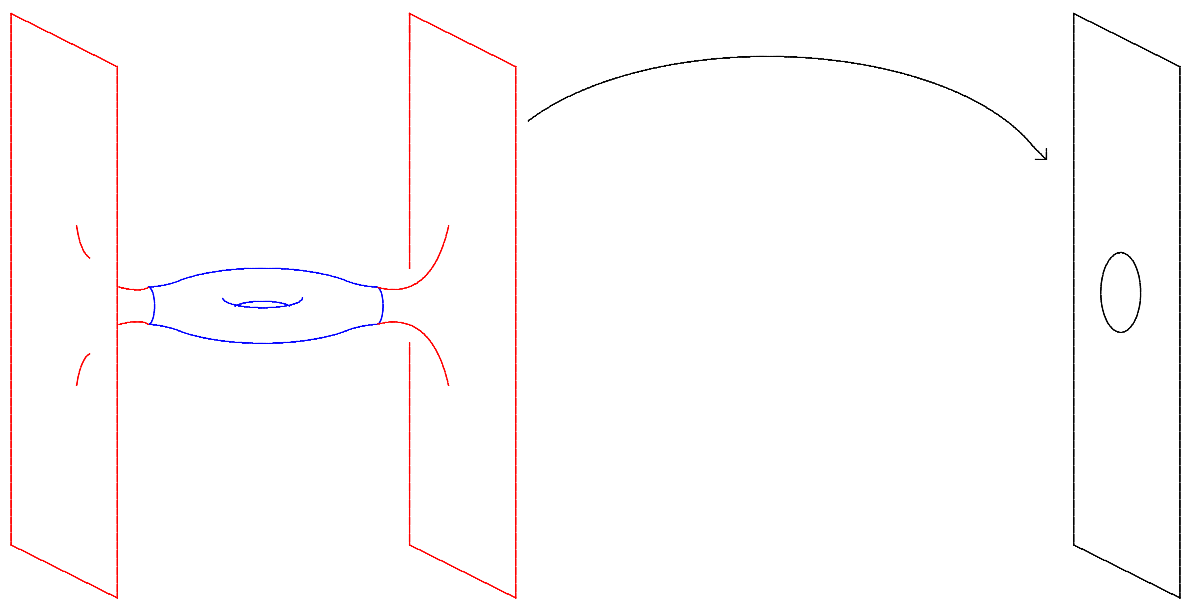

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Same as saying that $\kappa = 0$.

Want to show that s is constant.

If not, $s = 0$ only at finite set.

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Positive mass theorem

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Contradiction! So $s \equiv 0$.

Theorem A. *Let* (M^4, g, J) *be compact connected Bach-flat Kähler surface. Then exactly one holds:*

I. $\min s > 0$. *Then*

(a) (M, g, J) *Kähler-Einstein*, $\lambda > 0$*; or else*

(b) $(M, s^{-2}g)$ *Einstein*, $\lambda > 0$, *Hol* $= \mathbf{SO}(4)$.

II. $s \equiv 0$. *Then*

(a) (M, g, J) *Kähler-Einstein*, $\lambda = 0$*; or else*

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Theorem (L '09). *Suppose that M is a smooth compact oriented 4-manifold which admits a symplectic structure ω. Then M also admits an Einstein metric h with $\lambda \geq 0$ if and only if*

$$M \overset{diff}{\approx} \begin{cases} \mathbb{CP}_2 \# k\overline{\mathbb{CP}}_2, & 0 \leq k \leq 8, \\ S^2 \times S^2, \\ K3, \\ K3/\mathbb{Z}_2, \\ T^4, \\ T^4/\mathbb{Z}_2, T^4/\mathbb{Z}_3, T^4/\mathbb{Z}_4, T^4/\mathbb{Z}_6, \\ T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_2), T^4/(\mathbb{Z}_3 \oplus \mathbb{Z}_3), \text{ or } T^4/(\mathbb{Z}_2 \oplus \mathbb{Z}_4). \end{cases}$$

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Definitive list . . .

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In fact, all known Einstein metrics on Del Pezzo surfaces have these properties. These known metrics are all conformal to Bach-flat Kähler metrics.

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Corollary. *$\mathscr{E}^+_\omega(M)$ is exactly one connected component of $\mathscr{E}(M)$.*

A few words about the proof. . .

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for some g-preserving almost-complex structure J.

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for $fW_+ \in \mathrm{End}(\Lambda^+)$.

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Proposition. *If compact almost-Kähler* (M^4, g, ω) *satisfies* $\delta(fW_+) = 0$ *for some* $f > 0$, *then*

$$0 \geq \int_M W_+(\omega, \omega)|\nabla\omega|^2 f \; d\mu$$

Now take inner product of Weitzenböck formula

$$0 = \nabla^*\nabla(fW_+) + \frac{s}{2}fW_+ - 6fW_+ \circ W_+ + 2f|W_+|^2 I$$

with $2\omega \otimes \omega$ and integrate by parts. One obtains

Proposition. *If compact almost-Kähler* (M^4, g, ω) *satisfies* $\delta(fW_+) = 0$ *for some* $f > 0$, *then*

$$0 \geq \int_M W_+(\omega, \omega)|\nabla\omega|^2 f \, d\mu$$

If $W_+(\omega, \omega) > 0$, we thus conclude that g is Kähler!