Kodaira Dimension

and the

Yamabe Problem,

Revisited

Claude LeBrun
Stony Brook University

Rutgers Geometric Analysis Conference
New Brunswick, NJ, May 17, 2022

Joint work with

Joint work with

Michael Albanese

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Michael Albanese
Université du Québec à Montréal

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e-prints: arXiv:2106.14333 [math.DG]

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e-prints: arXiv:2106.14333 [math.DG]

and arXiv:2105.10785 [math.DG]

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to appear in
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Perspectives on Scalar Curvature,
Gromov and Lawson, editors.

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This talk focuses on the relationship between a complex-analytic invariant called the Kodaira dimension, and a diffeomorphism invariant called the Yamabe invariant (or sigma constant), which encodes information about the scalar curvature.

The new results concern complex surfaces which do not admit Kähler metrics, and thus are far-removed from the original context.

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$$r = \lambda g$$

for some constant $\lambda \in \mathbb{R}$.

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where $V = \mathrm{Vol}(M, g)$ inserted to make scale-invariant.

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Difficulty: $L^2_1 \hookrightarrow L^p$ bounded, but not compact.

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Has $s =$ constant.

Unique up to scale when $s \leq 0$.

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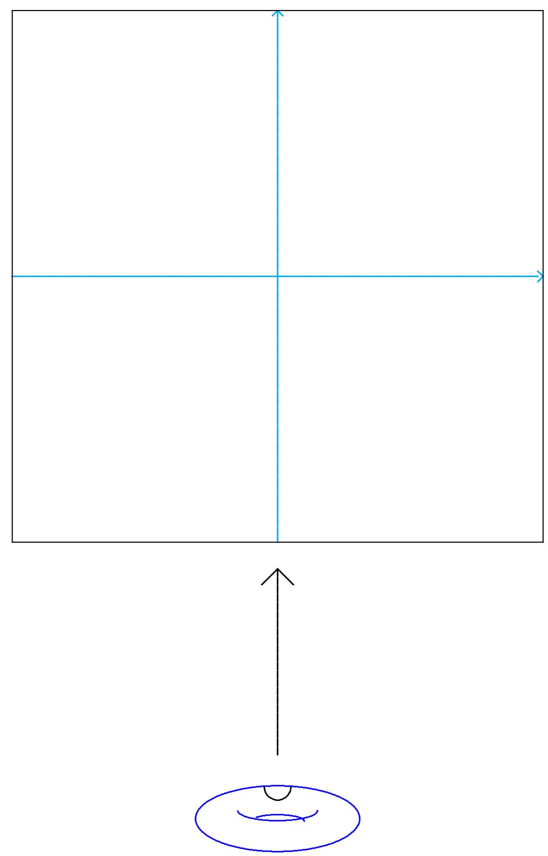
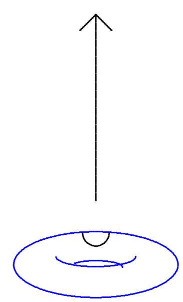
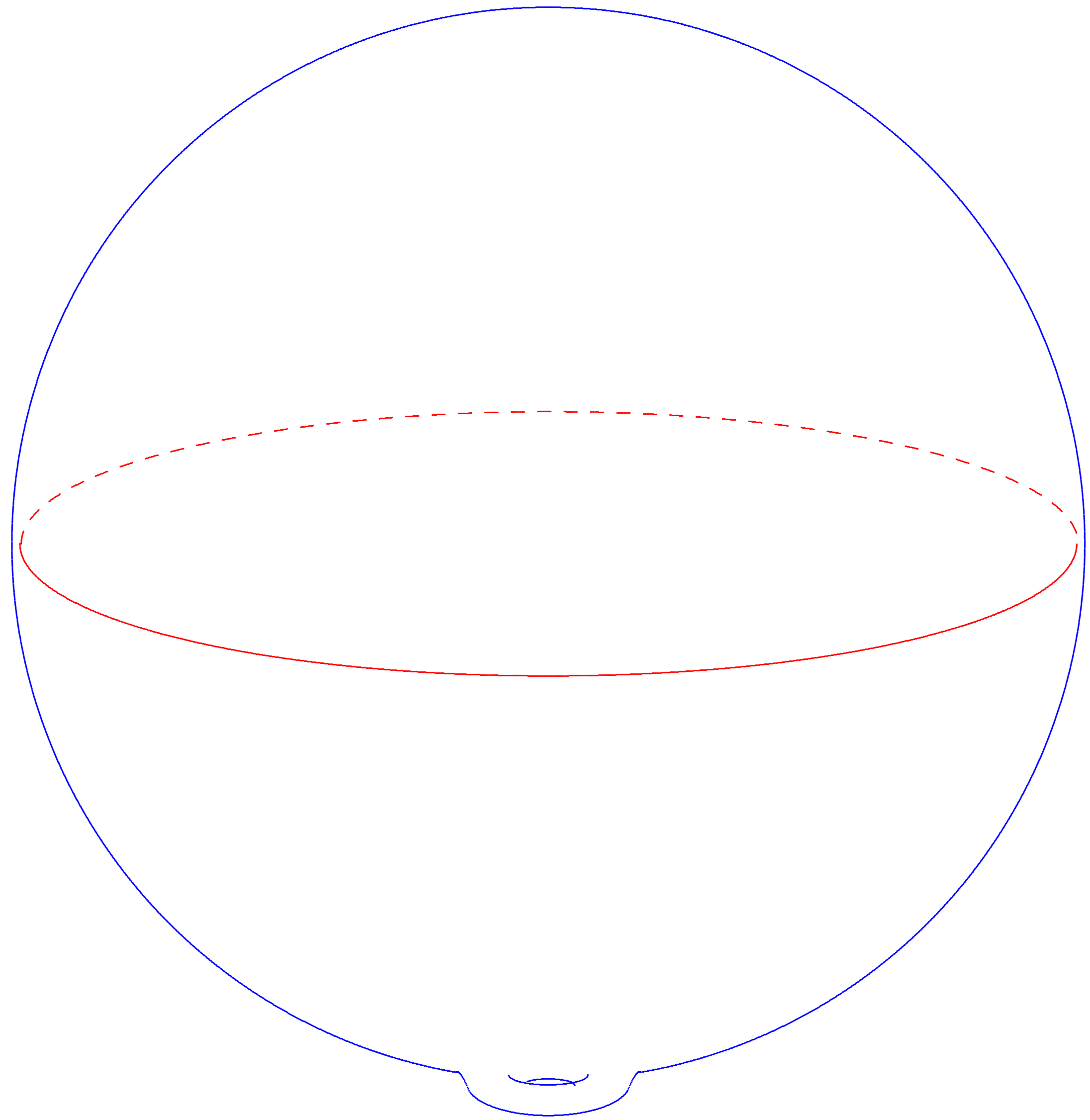


$$g_{jk} = \delta_{jk} + O(|x|^2)$$





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$$= \quad \text{only for round sphere.}$$

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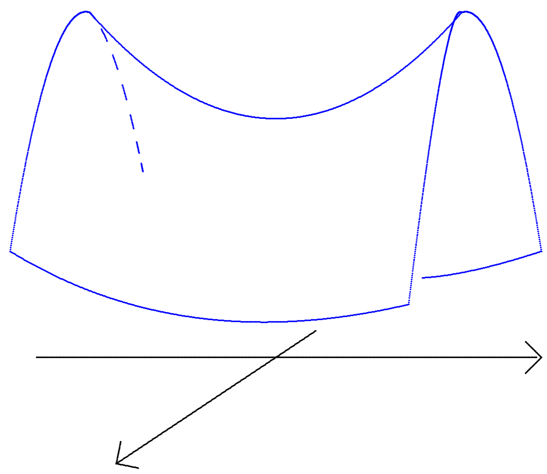

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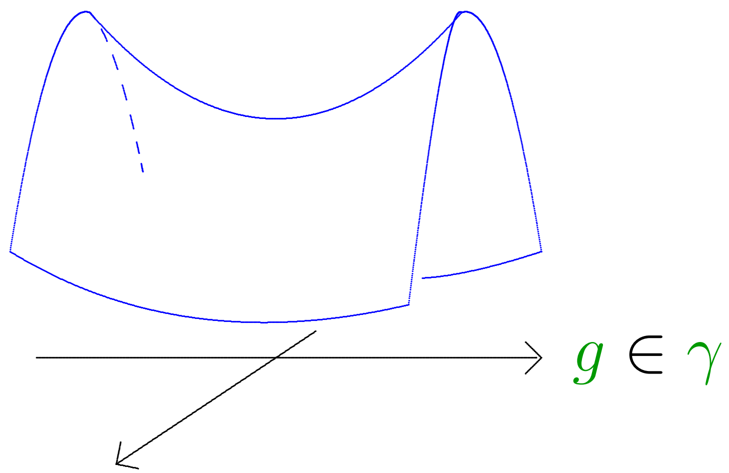

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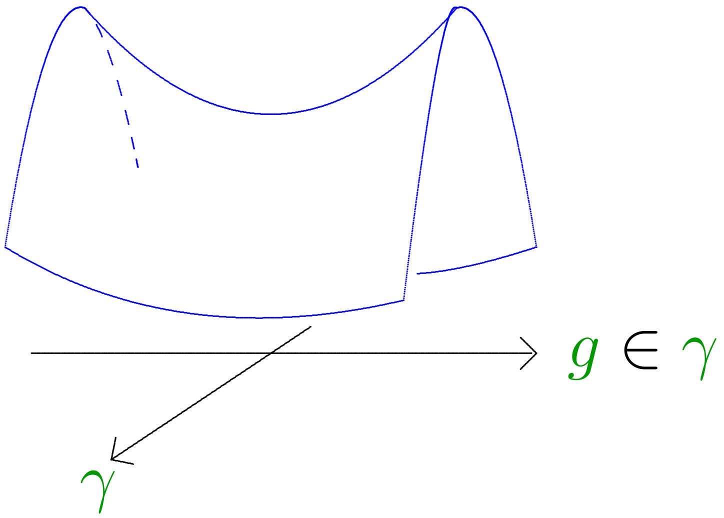

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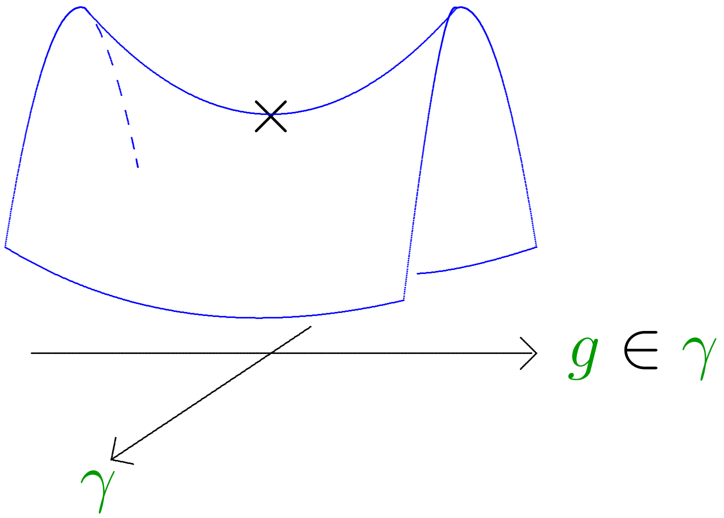

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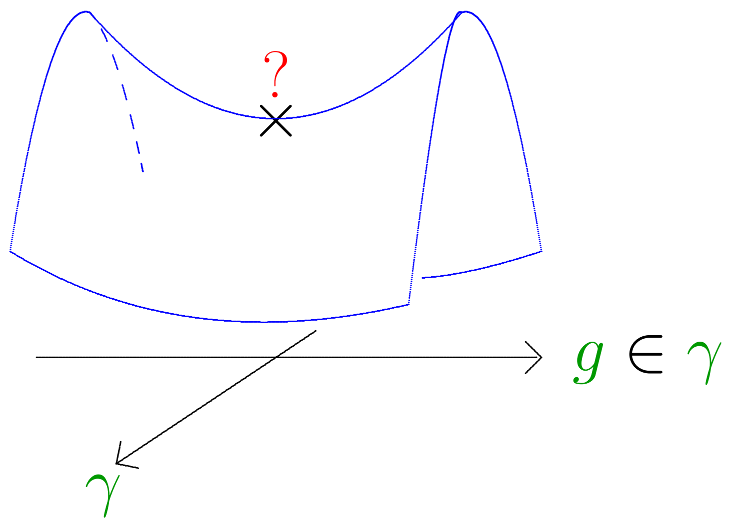

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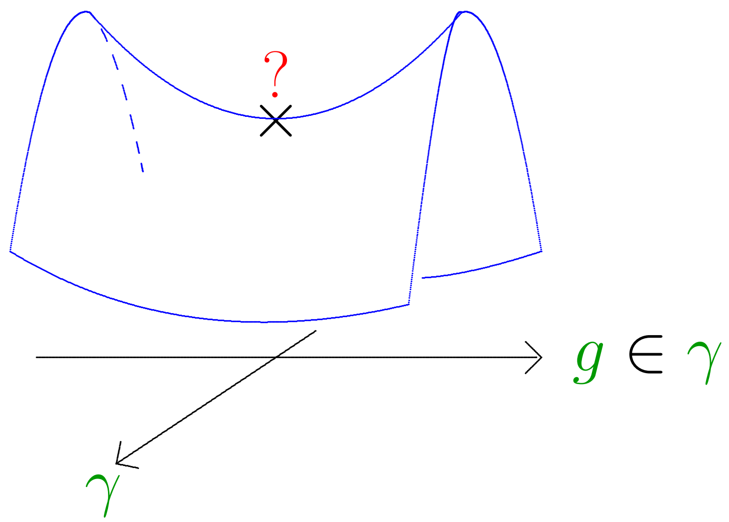

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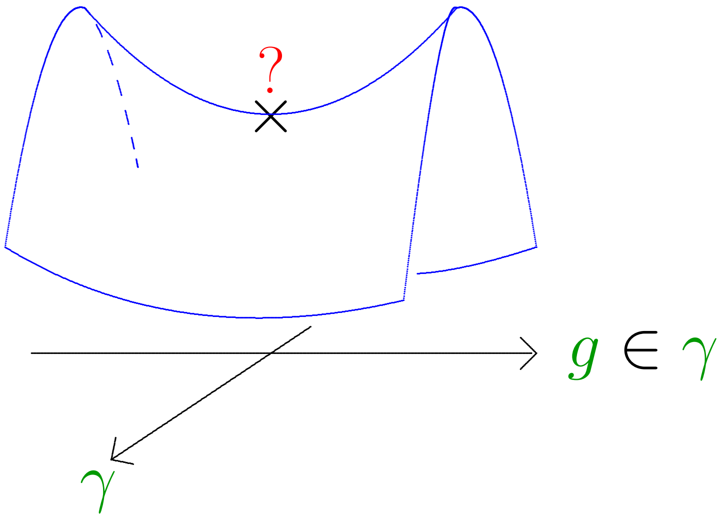

Too good to be true! But . . .

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R. Schoen ('87): "sigma constant"

O. Kobayashi ('87): "mu invariant"

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Problem. *Compute actual value of $\mathscr{Y}(M)$ for concrete, interesting manifolds.*

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Moreover, can choose M_j such that each $\mathscr{Y}(M_j)$ is realized by an Einstein metric g_j.

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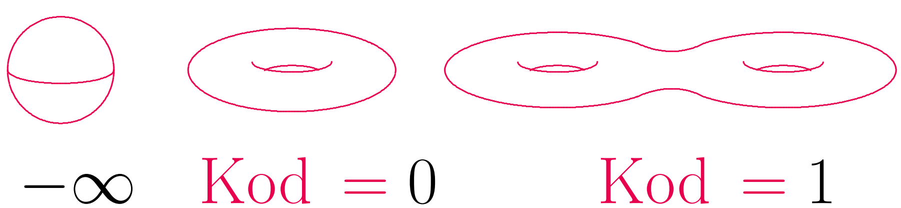

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By contrast, in complex dimension $m \geq 3$, Kod is not a diffeomorphism invariant, and has essentially nothing to do with $\mathscr{Y}(M)$.

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Today: what happens when $b_1(M)$ is odd?

Kodaira Classification

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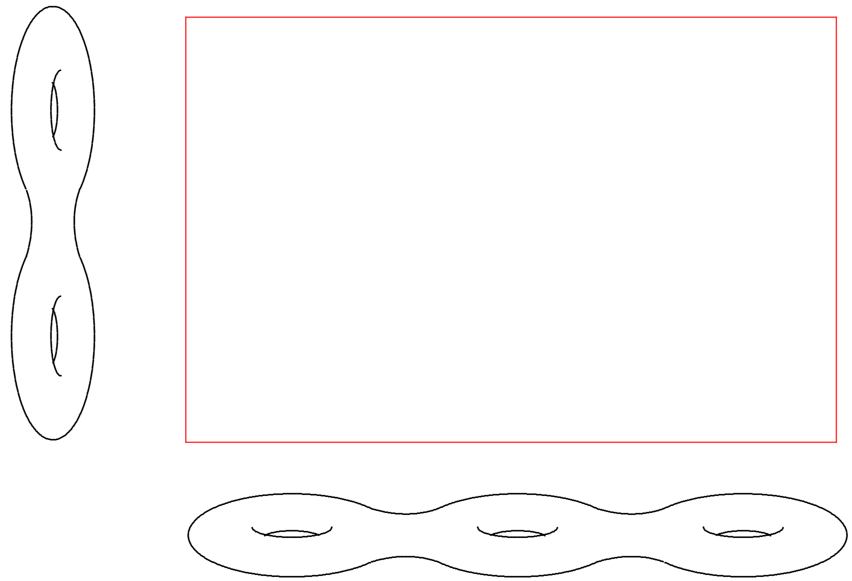

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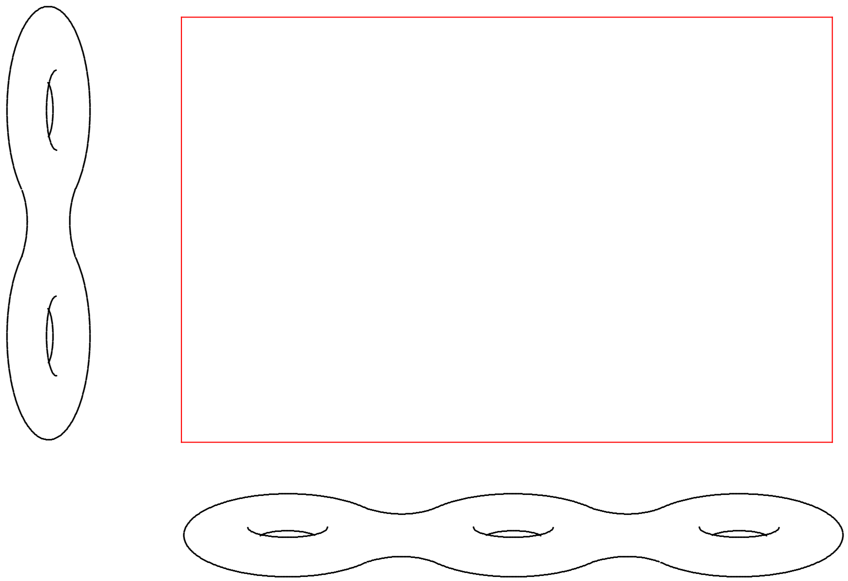

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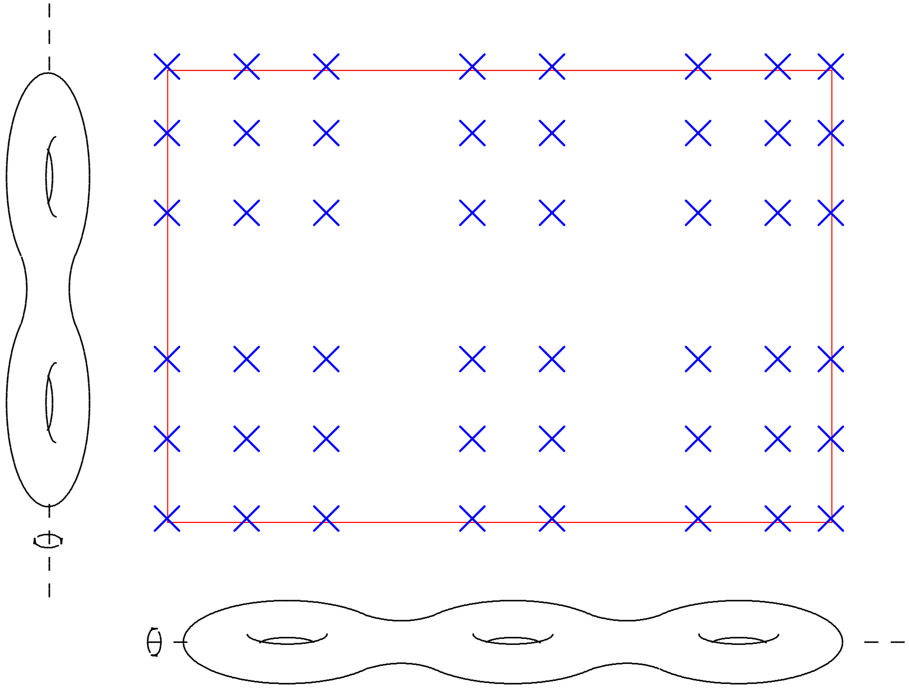

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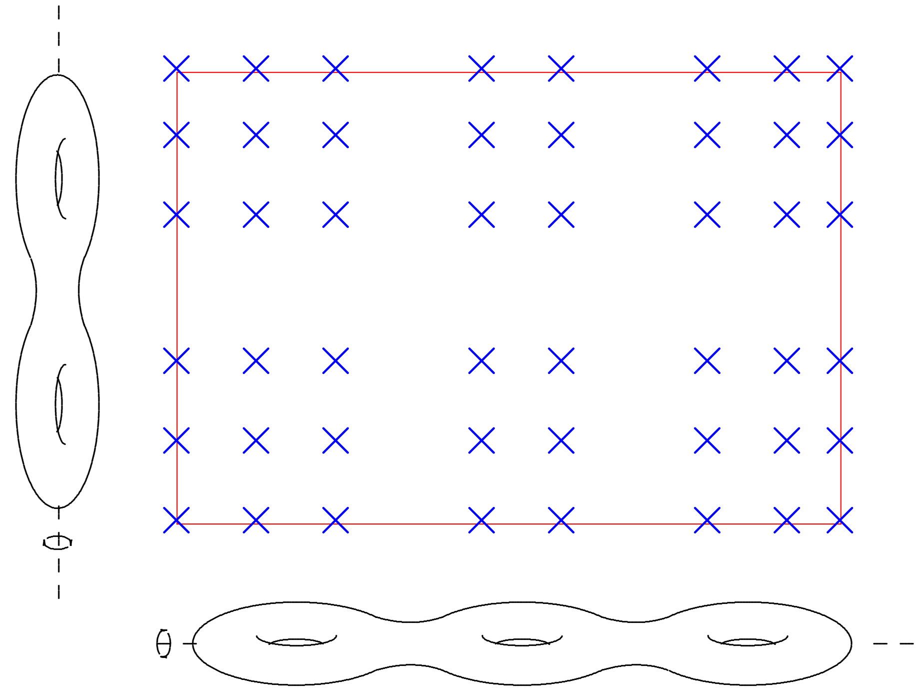

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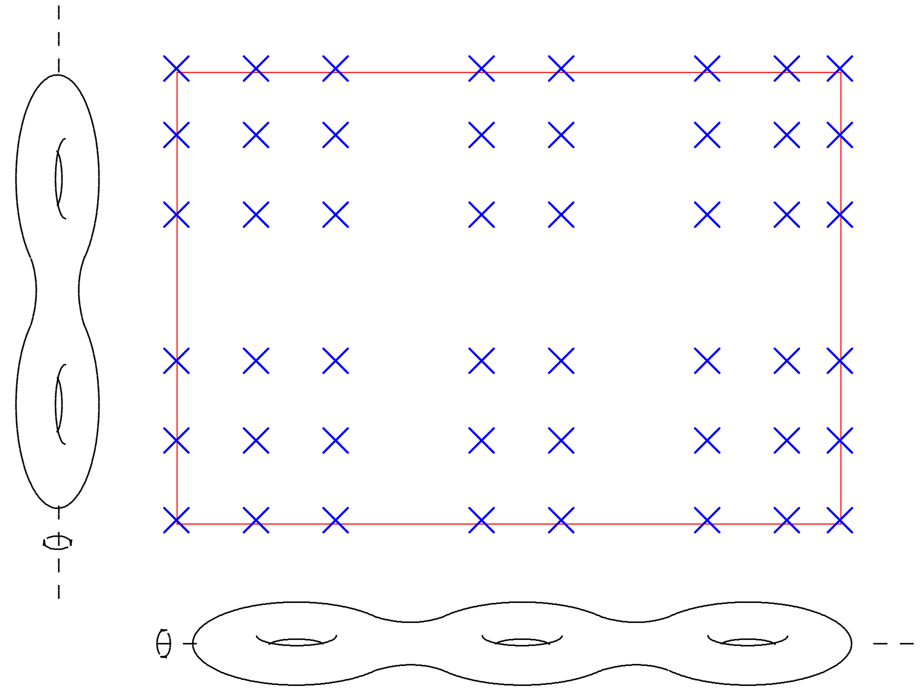

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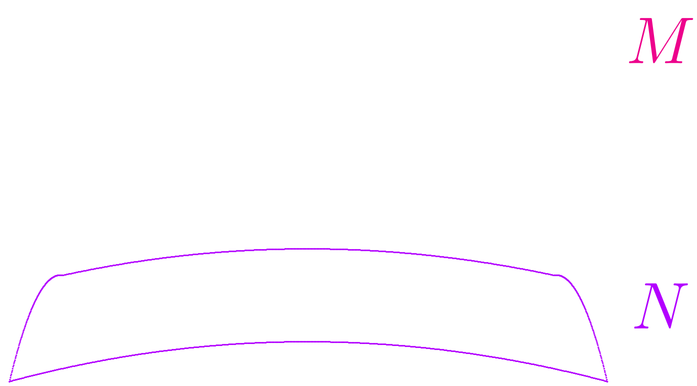

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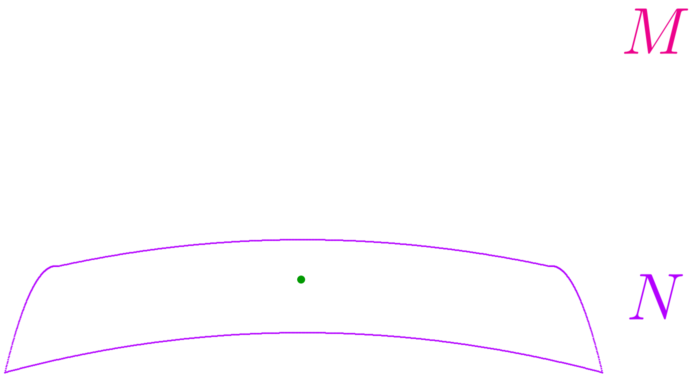

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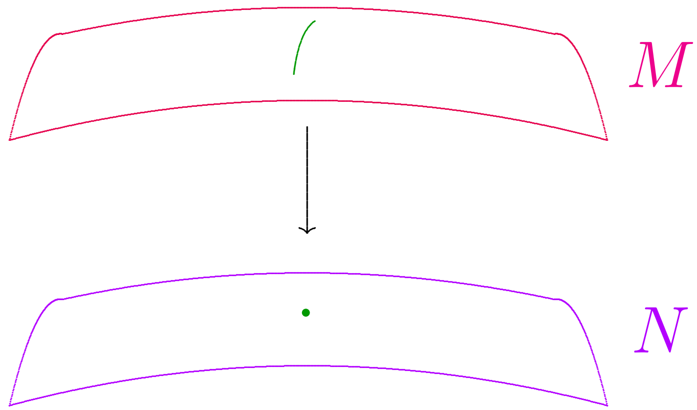

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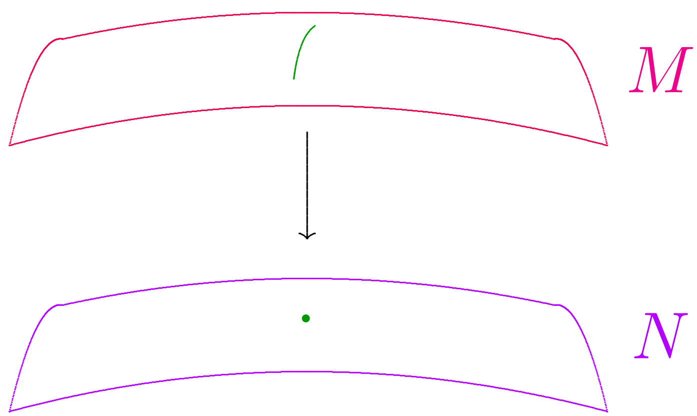

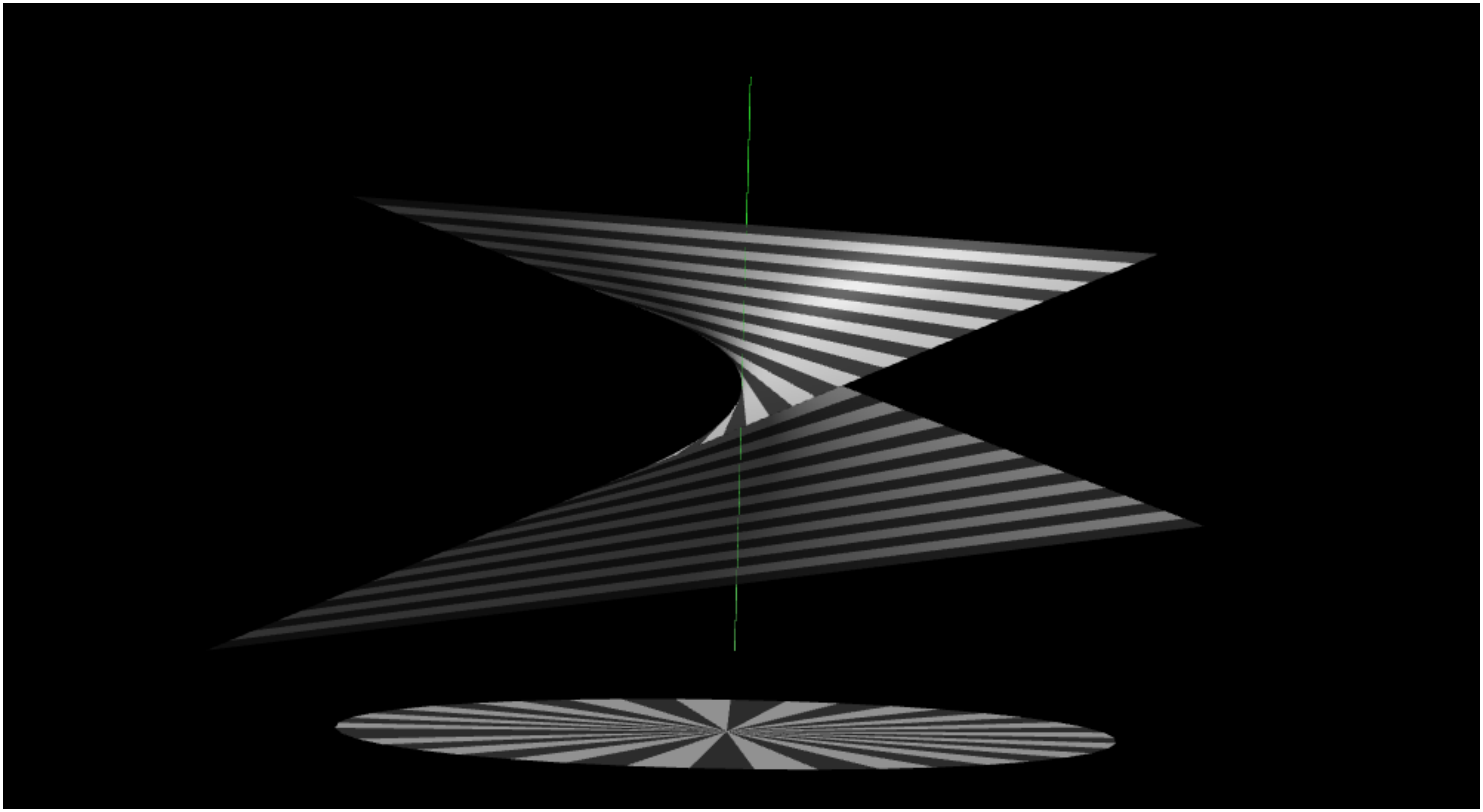

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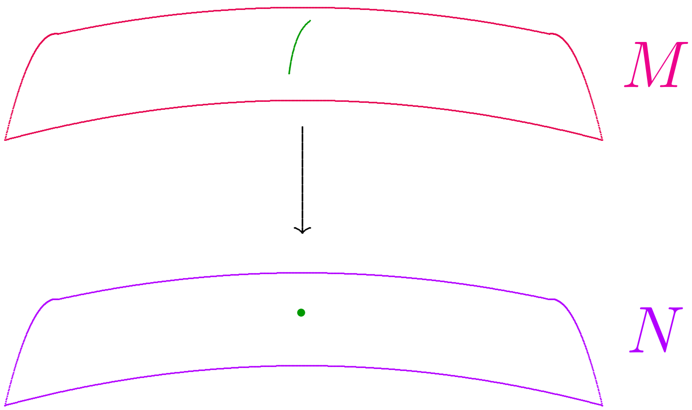

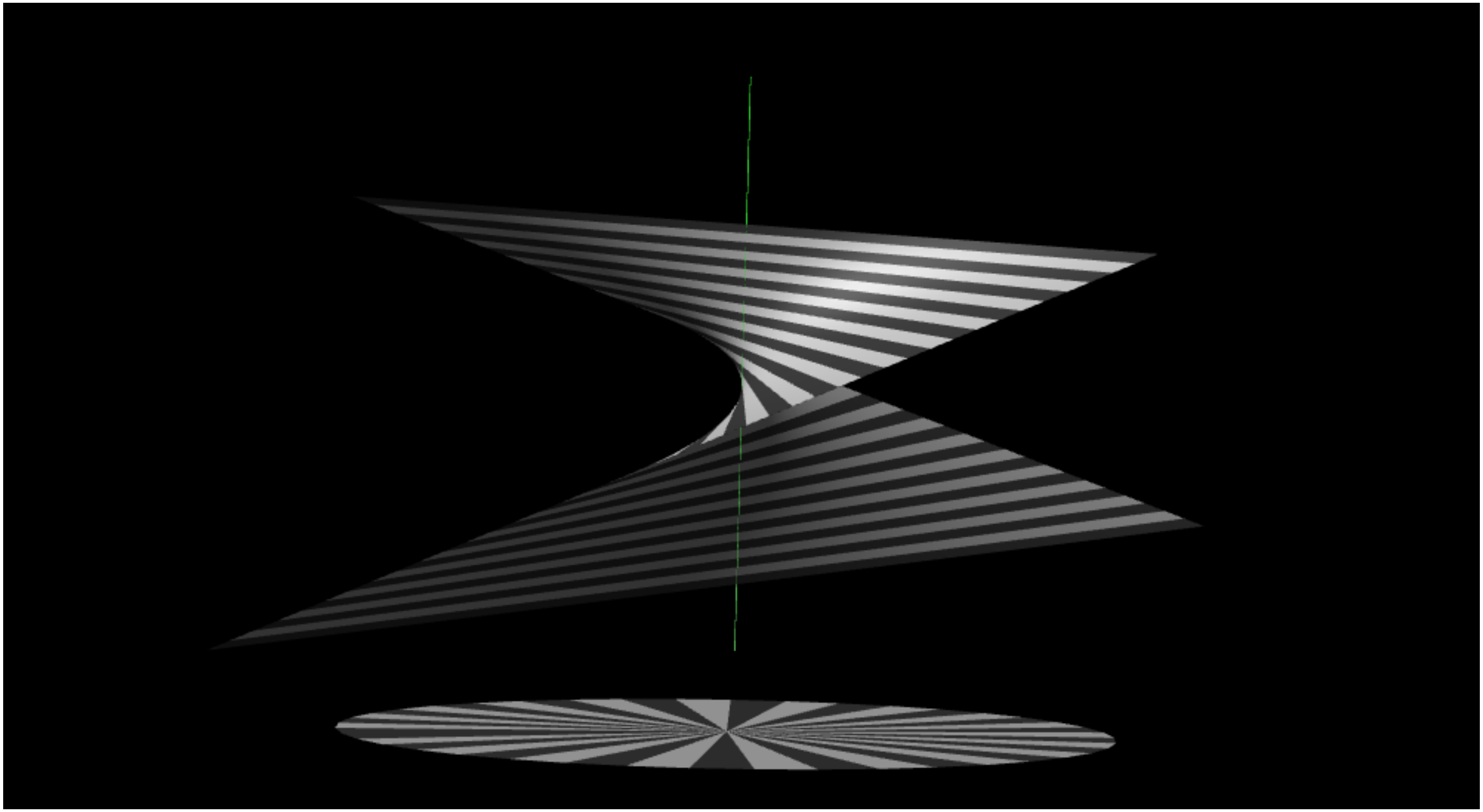

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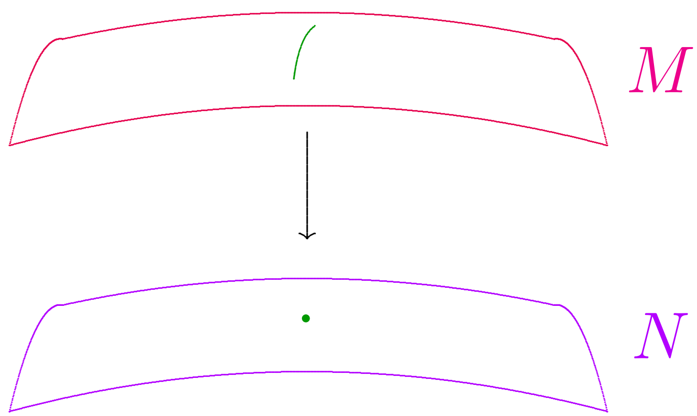

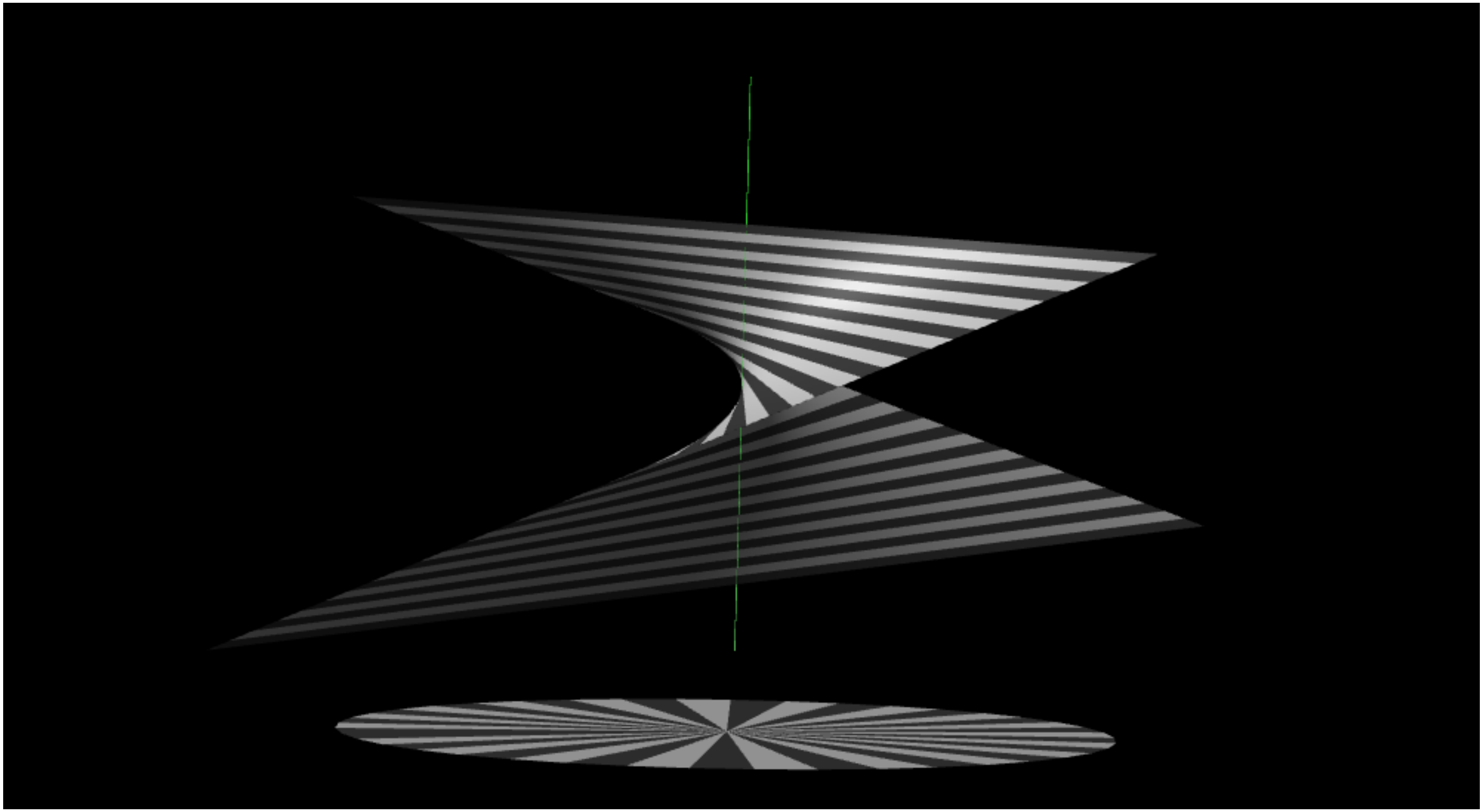

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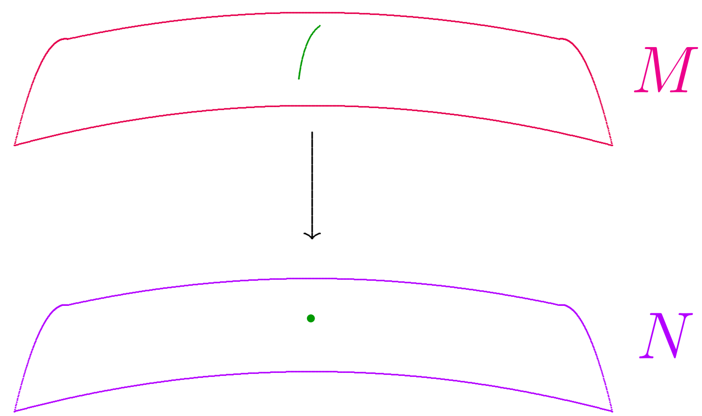

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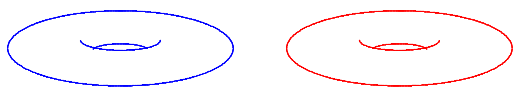

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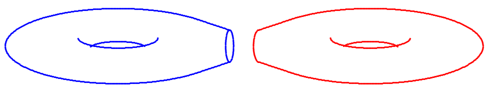

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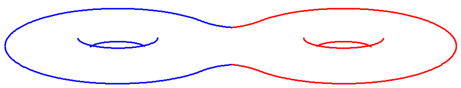

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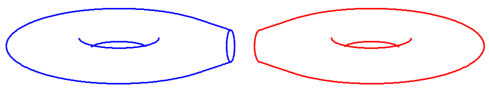

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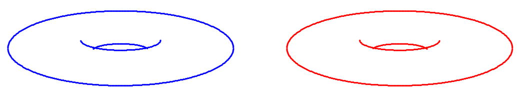

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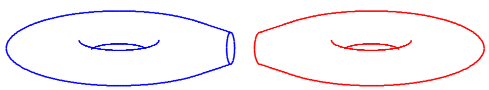

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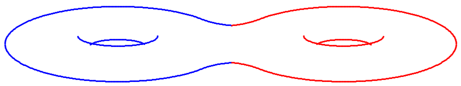

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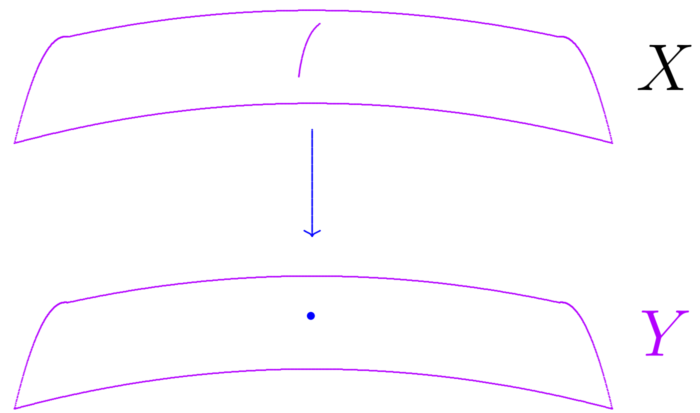

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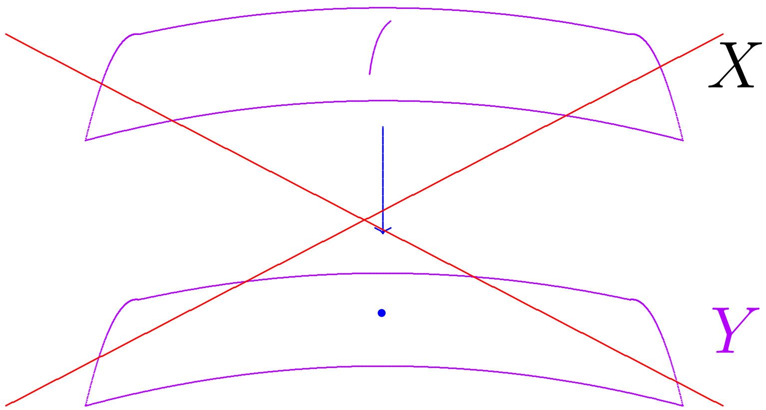

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"Fibration" allows singular fibers, so not fiber-bundle.

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In fact, if X admits K-E metric, achieves $\mathscr{Y}(X)$.

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We'll see that this isn't so when Kod $= -\infty$!

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Missing piece:
Prove $\mathscr{Y}(M) \leq 0$ when Kod $= 1$ and b_1 is odd.

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I will focus on second method in this lecture.

Crash course on Seiberg-Witten Theory. . .

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Any oriented M^4 admits spin^c structures $\mathfrak{c}$.

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where $\mathbb{S}_\pm$ are the (locally defined) left- and right-handed spinor bundles of (M, g).

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where ${F_\theta}^+$ = self-dual part curvature of θ, and $\sigma : \mathbb{V}_+ \to \Lambda^+$ is a natural real-quadratic map,

$$|\sigma(\Phi)| = \frac{1}{2\sqrt{2}}|\Phi|^2.$$

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where $c_1(L)_g^+ =$ self-dual part of harmonic rep.

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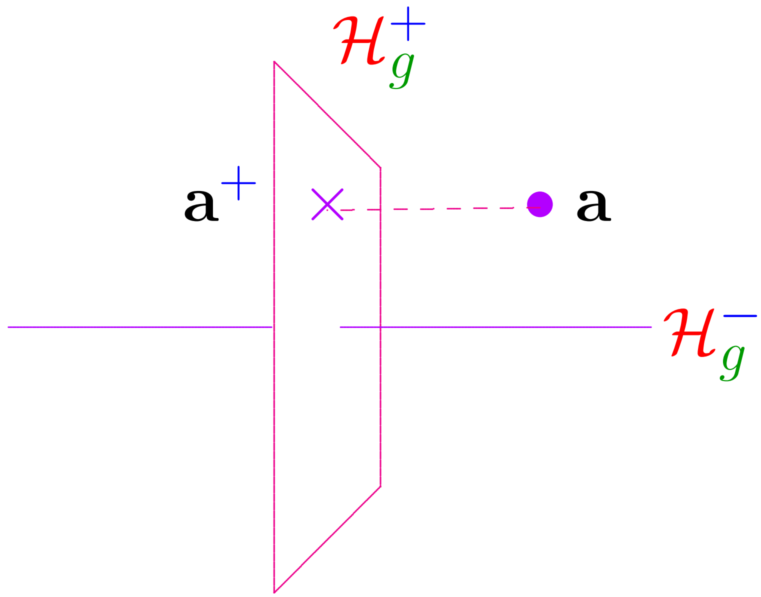

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Instead, with only a modicum of extra work, his method proves the existence of the following. . .

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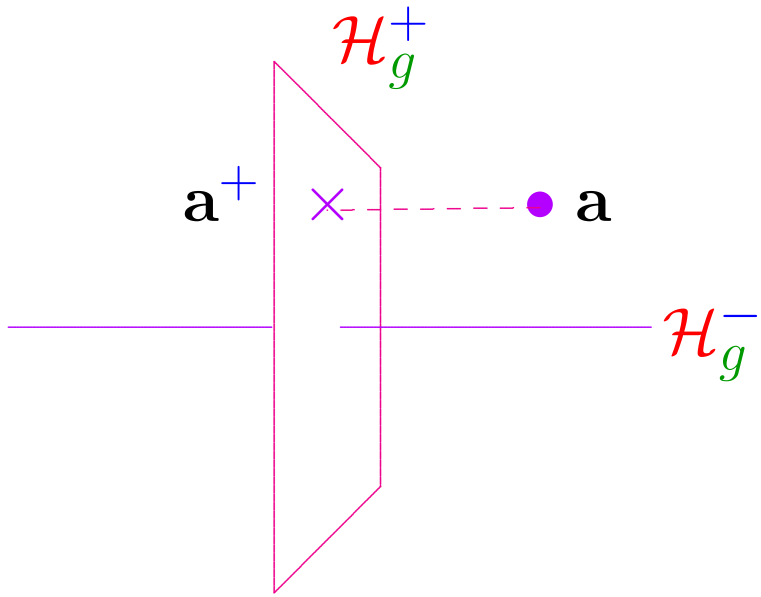

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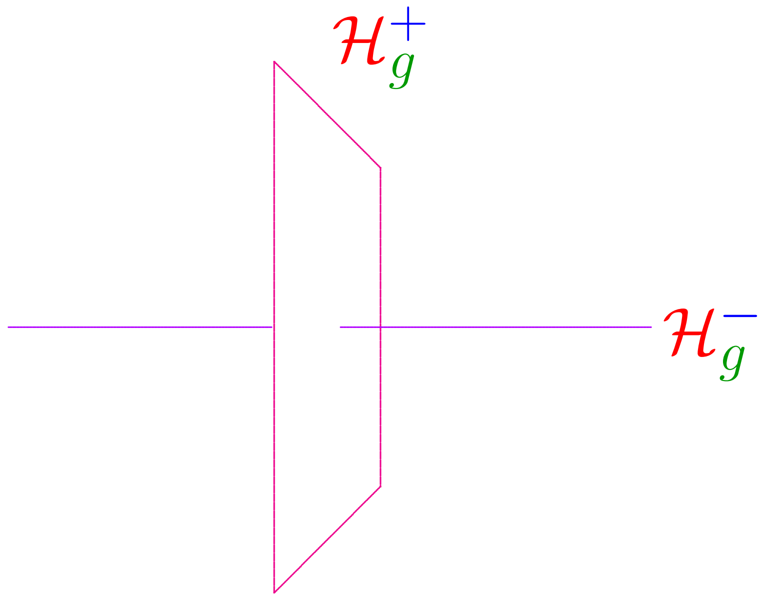

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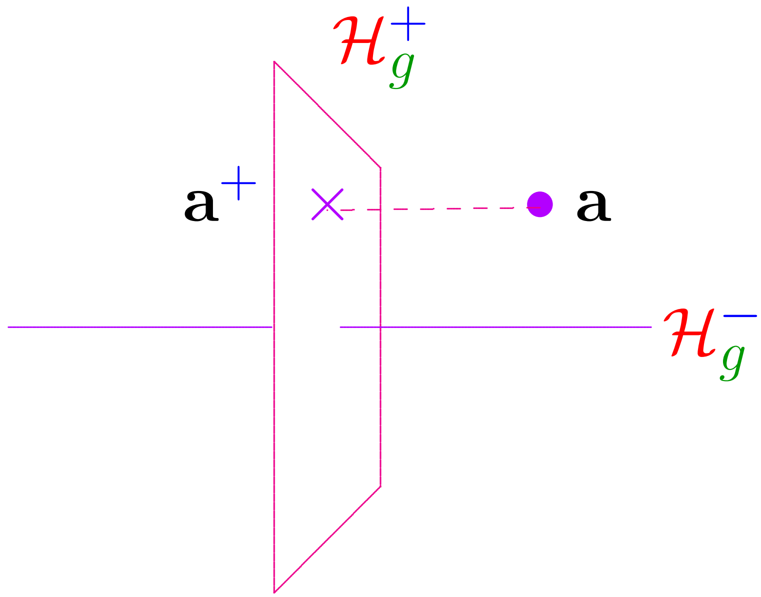

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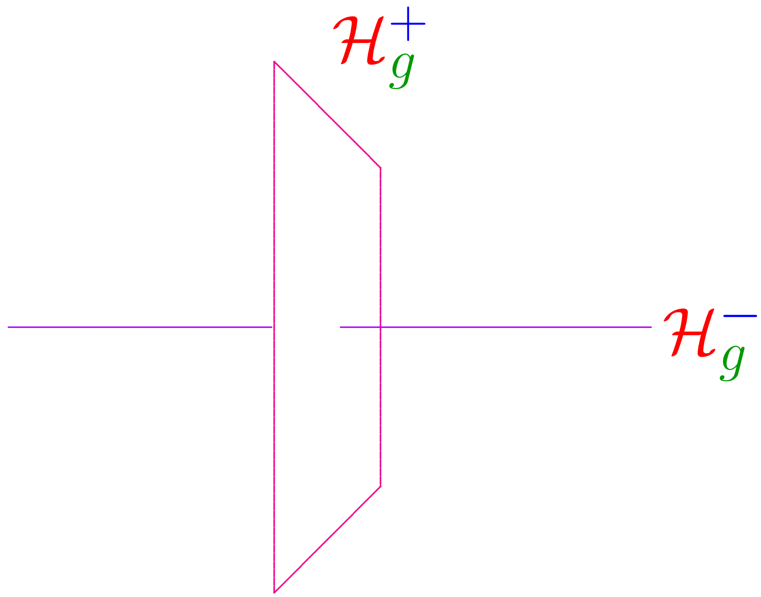

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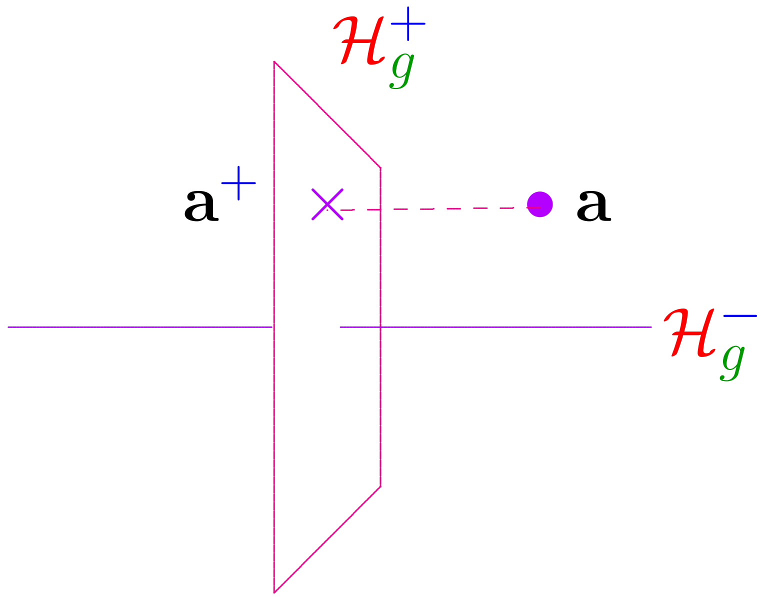

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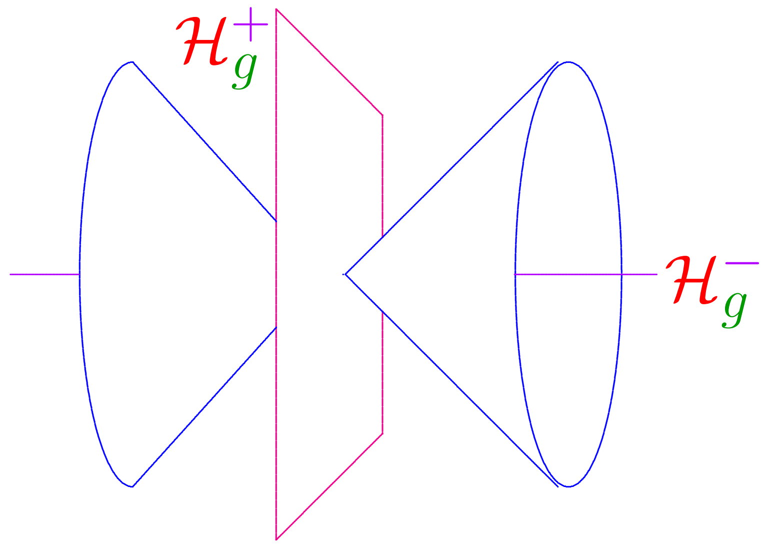

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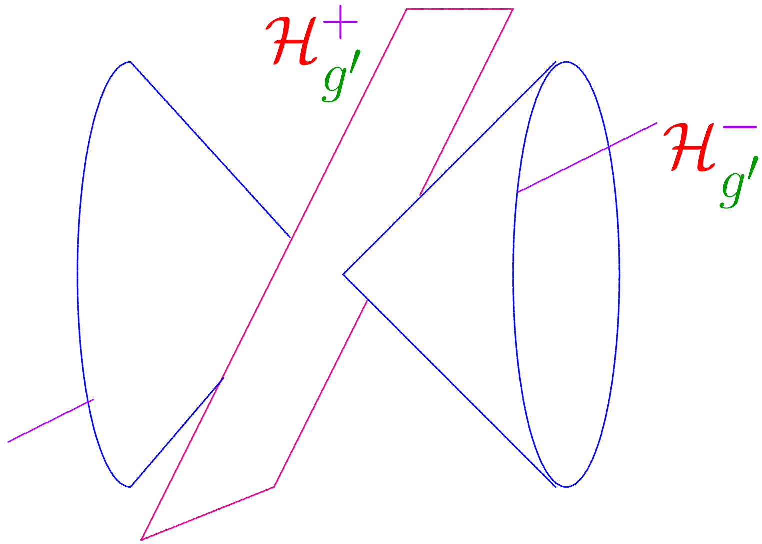

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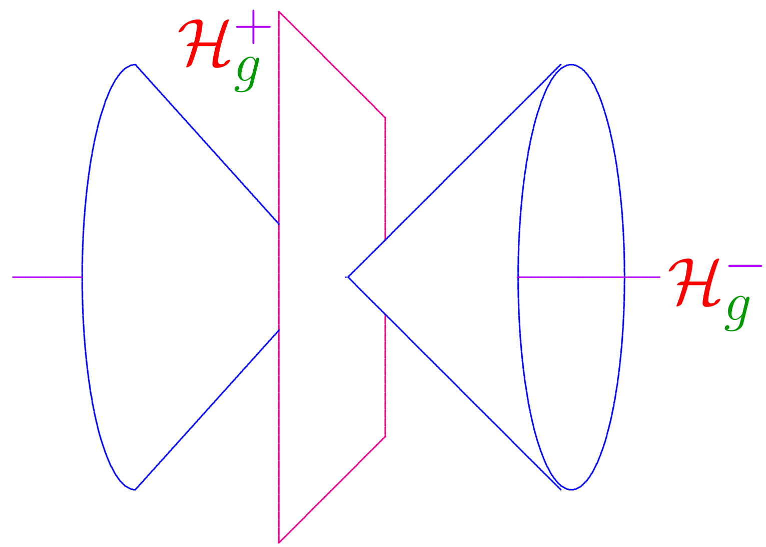

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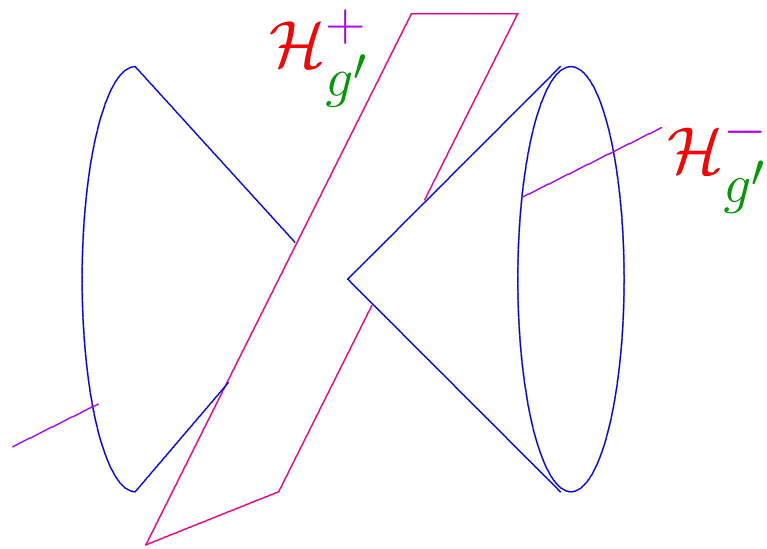

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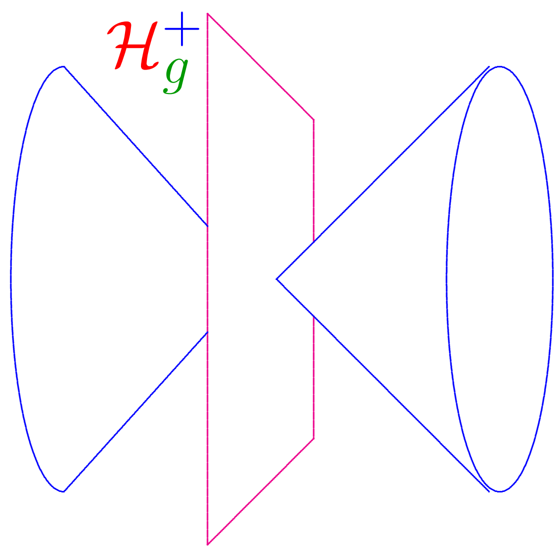

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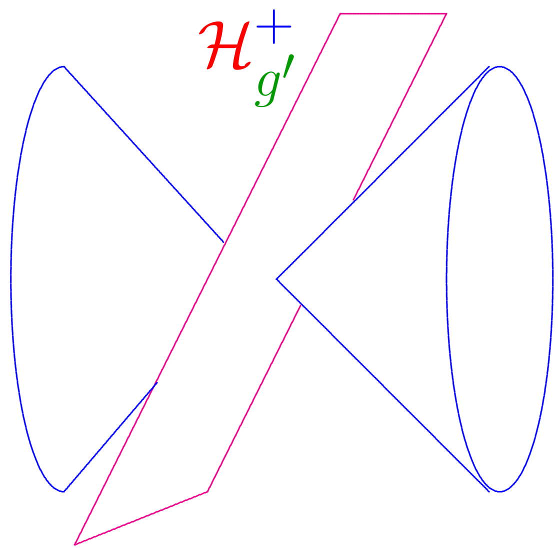

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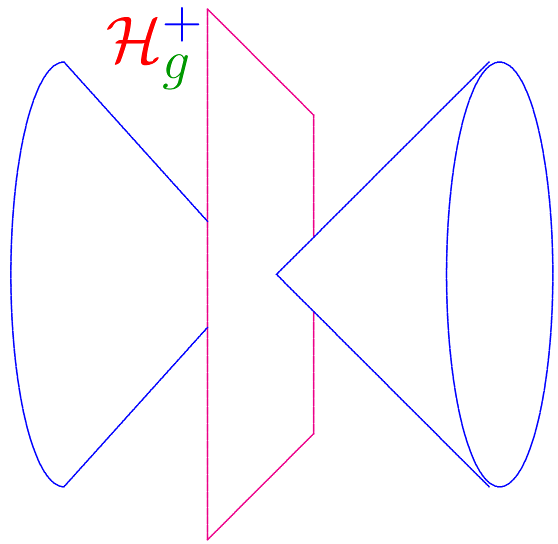

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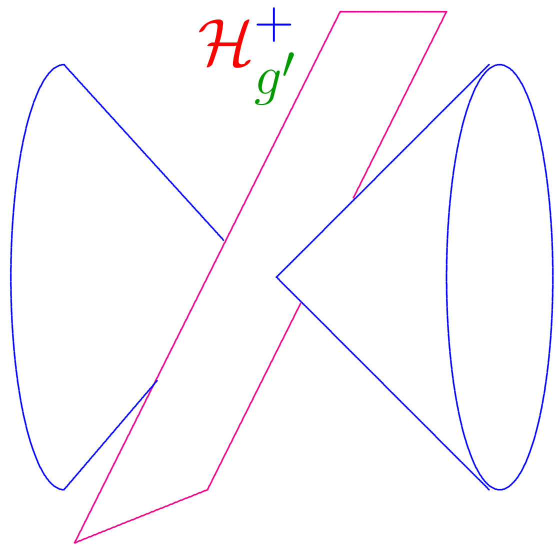

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Characteristic:

$$\mathbf{a} \bullet \mathbf{b} \equiv \mathbf{b} \bullet \mathbf{b} \bmod 2 \qquad \forall \mathbf{b} \in H^2(M, \mathbb{Z})/\text{torsion}$$

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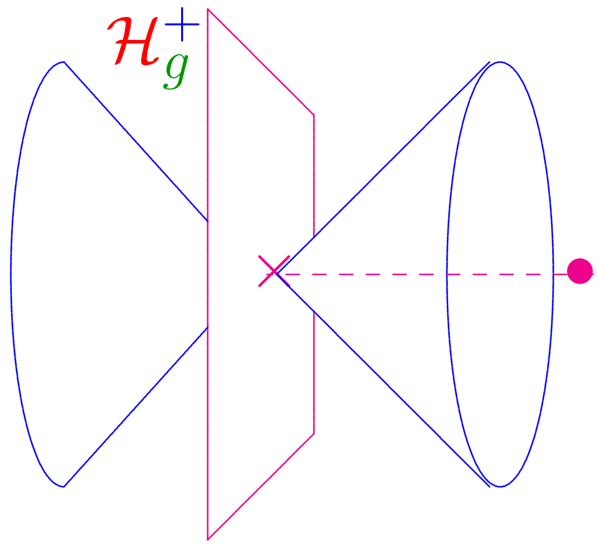

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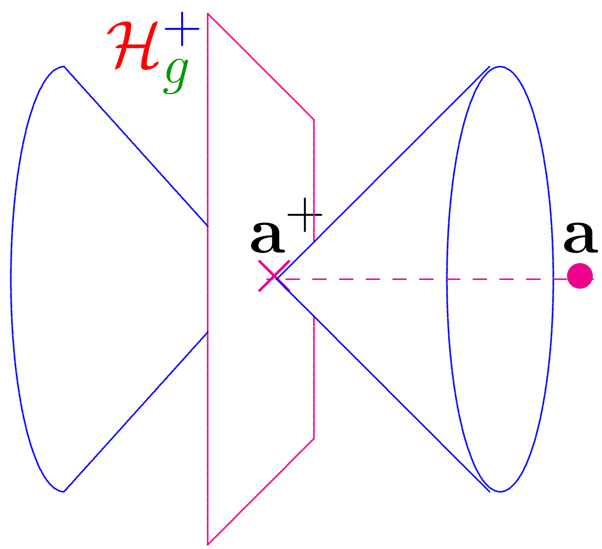

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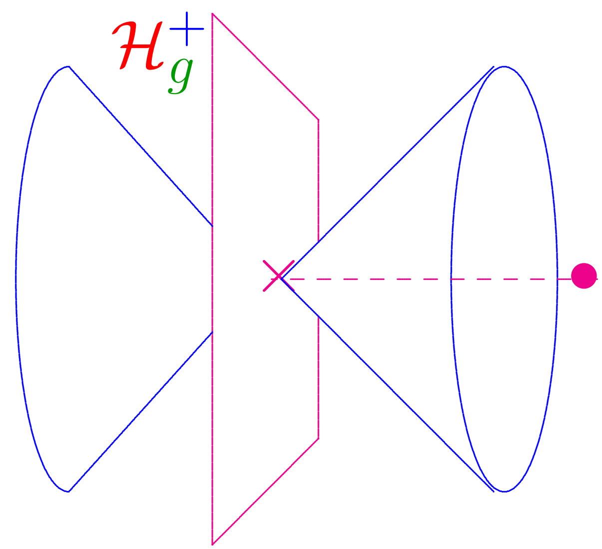

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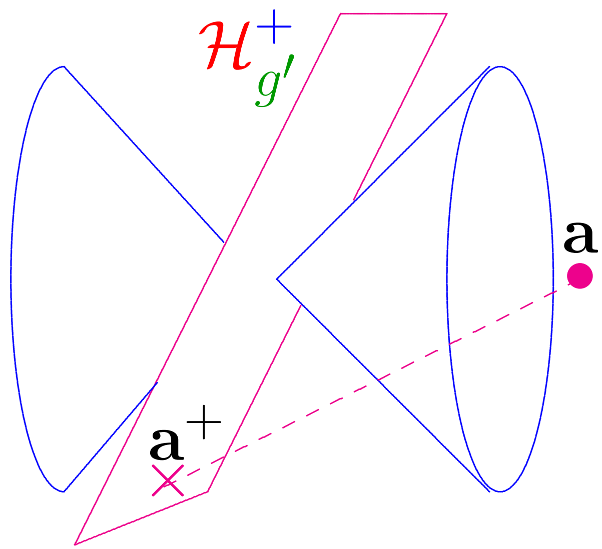

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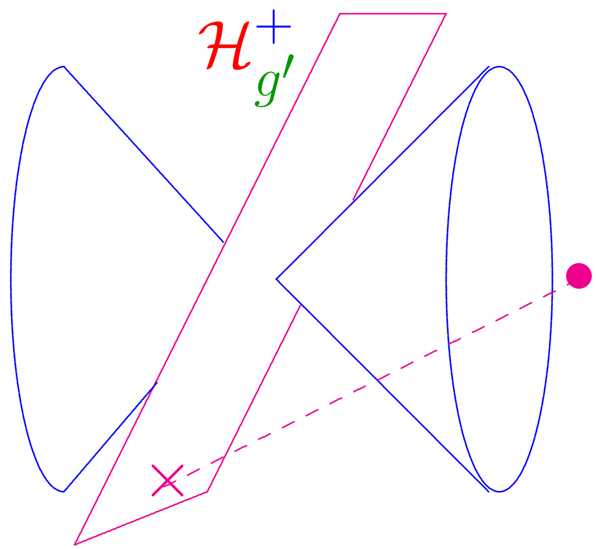

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Characteristic:

$$\mathbf{a} \bullet \mathbf{b} \equiv \mathbf{b} \bullet \mathbf{b} \bmod 2 \qquad \forall \mathbf{b} \in H^2(M, \mathbb{Z})/\text{torsion}$$

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On $M = X\#k\overline{\mathbb{CP}}_2$, mock-monopole $\mathbf{a} \in H^2(M, \mathbb{Z})/\text{torsion}$ must be non-zero, because pairing with Poincaré dual of the generator of $H_2(\overline{\mathbb{CP}}_2, \mathbb{Z})$ must be odd.

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Schoen-Yau, Gromov-Lawson:

$\mathscr{Y} > 0$ preserved under connected sums ($n \geq 3$).

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$\mathscr{Y}(X) > 0 \Longrightarrow \mathscr{Y}(M) > 0.$

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Key Point: Brinzănescu '94 $\Longrightarrow$ minimal model X has unbranched covers diffeomorphic to $N \times S^1$, where $N \to \Sigma$ Chern-class-1 circle bundle over Σ of genus ≥ 2.

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Idea of the proof hidden in **Kronheimer '99**, which did not define the concept or quite prove the needed estimate. Objective was instead to estimate

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Kronheimer's method is to construct approximate solutions of the SW equations on a sequence of high-degree covers $\widetilde{M} \to M$, with error term uniformly bounded as the degree of the cover $\to +\infty$.

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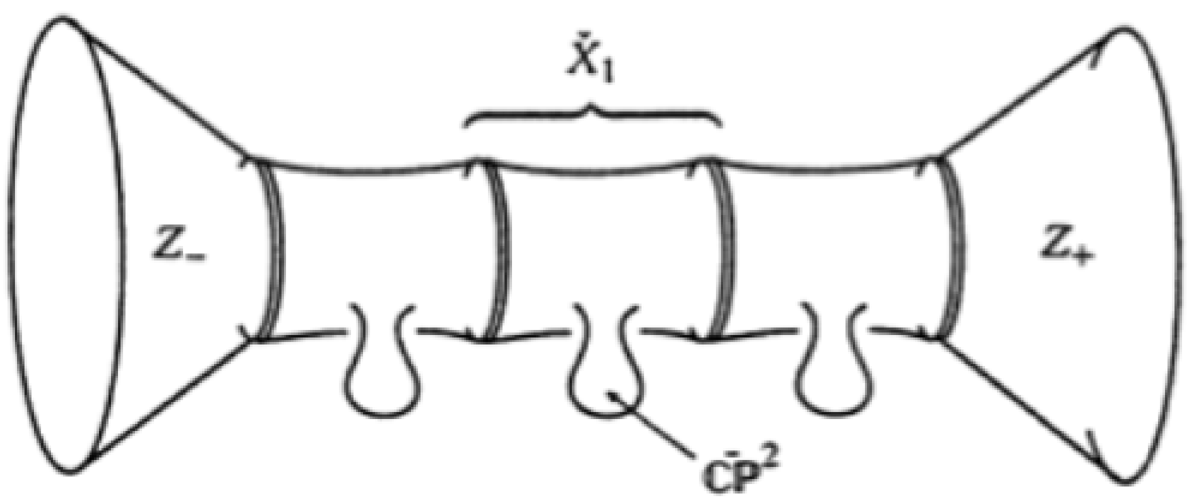

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In limit, one obtains desired inequality

$$\int_M (s_-)^2 \, d\mu_g \geq 32\pi^2 [\mathbf{a}^+]^2$$

for any Riemannian metric g on M.

Lemma C. *Let (M, J) be a compact complex surface with b_1 odd and $Kod(M) = 1$. Then M does not admit a Riemannian metric of positive scalar curvature.*

Theorem A. *Let M be the smooth 4-manifold underlying any compact complex surface (M^4, J) of Kodaira dimension $\neq -\infty$. Then*

$$\mathscr{Y}(M) = 0 \iff Kod(M, J) = 0 \text{ or } 1,$$
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For known classes of examples, sign of $\mathscr{Y}(M)$ is left unchanged by blowing up.

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Global Spherical Shell Conjecture claims that all possible diffeotypes are already known. This would mean $\mathscr{Y}(M) \geq 0$ for any class-VII surface.

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However, this **Conjecture** is very difficult, and has only been proved with $b_2(M) \leq 3$.

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Thanks for the invitation!