Twistors,

Self-Duality,

and

Spin^c Structures

Claude LeBrun
Stony Brook University

RP90: Twistors from Geometry to Physics.
University of Oxford, July 23, 2021.

Revised version.

For my friend and teacher

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Sir Roger Penrose,

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In celebration of his 90th birthday

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$\{$infinitesimal rotations$\}$ = $\{$skew matrices$\}$ = $\{$2-forms$\}$.

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Λ^+ self-dual 2-forms

Λ^- anti-self-dual 2-forms

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which is almost-complex structure compatible with metric and determining given orientation.

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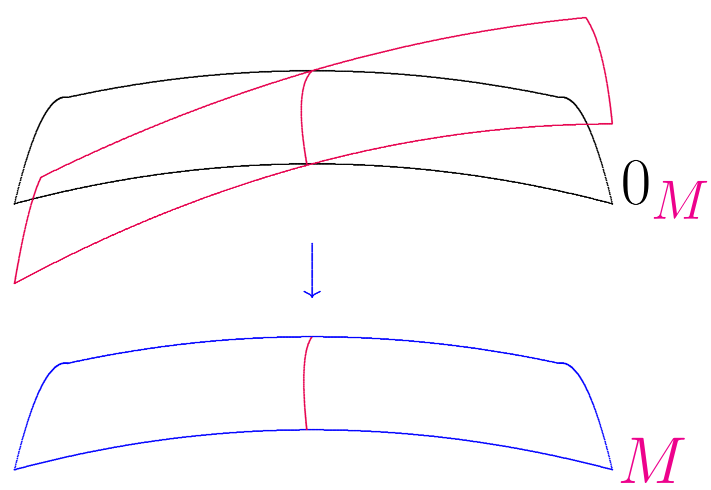

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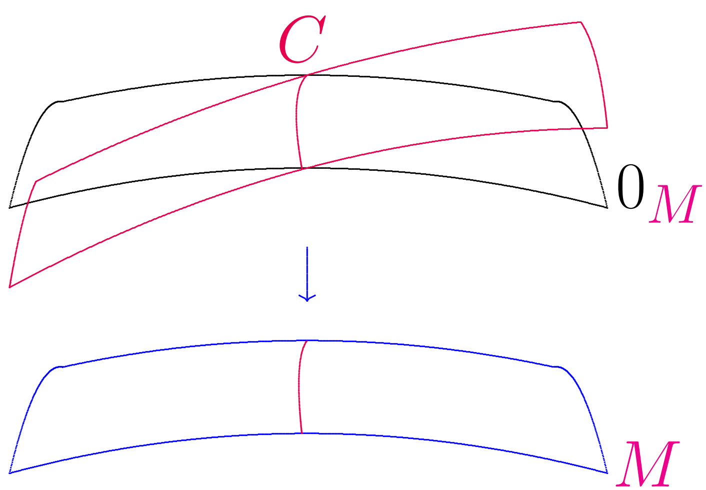

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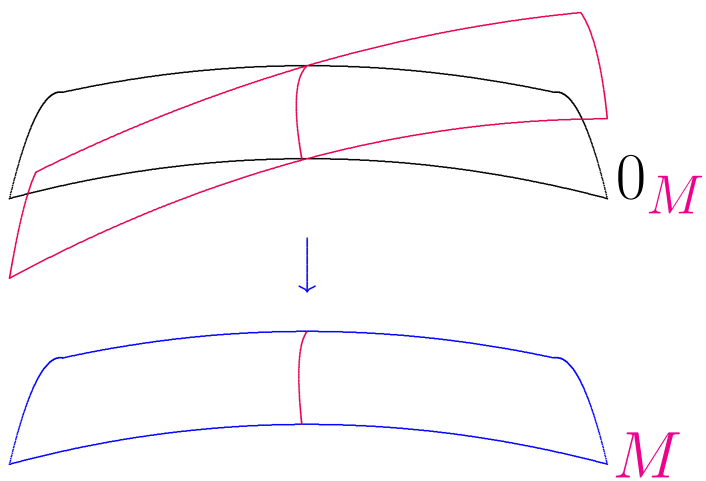

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Can't just cancel zeroes, as in case of $TM \to M$.

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But $\nexists$ non-zero section of $\Lambda^+ \to S^4$ because S^4 not almost-complex!
For example, $Td(S^4) = \frac{(\chi+\tau)}{4}(S^4) = \frac{1}{2} \not\in \mathbb{Z}$.

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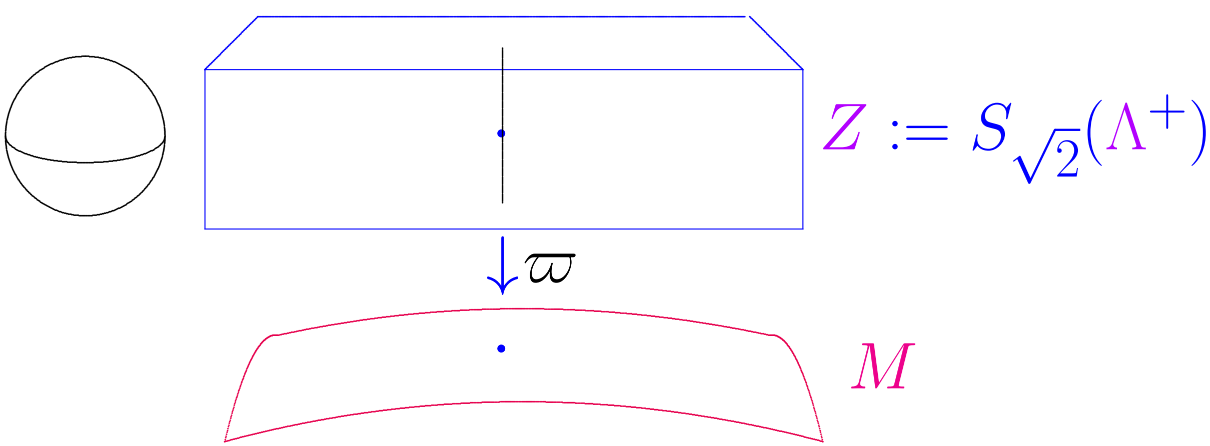

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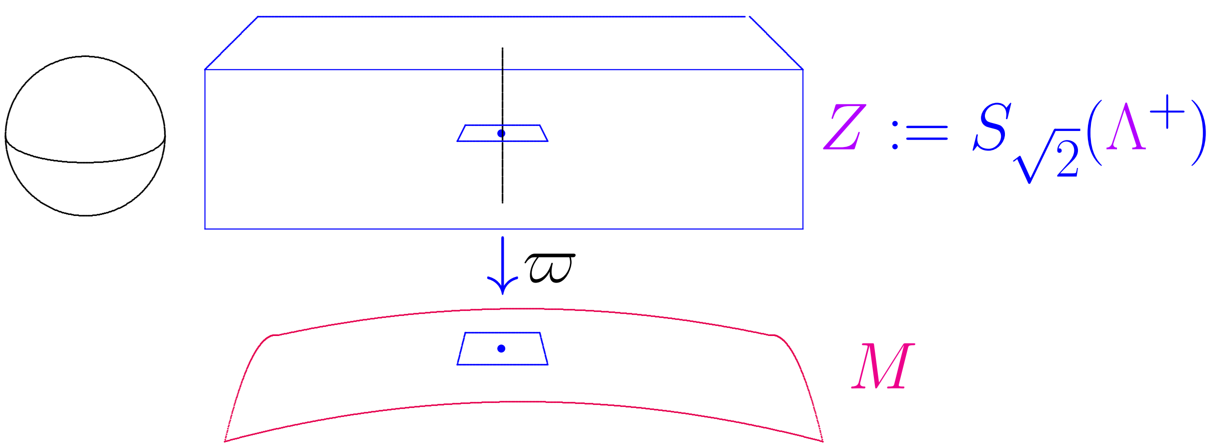

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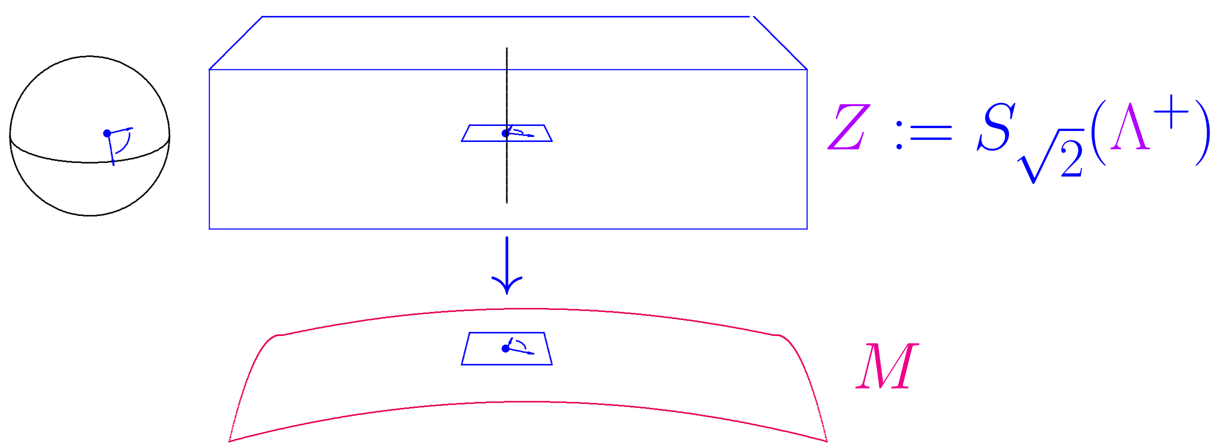

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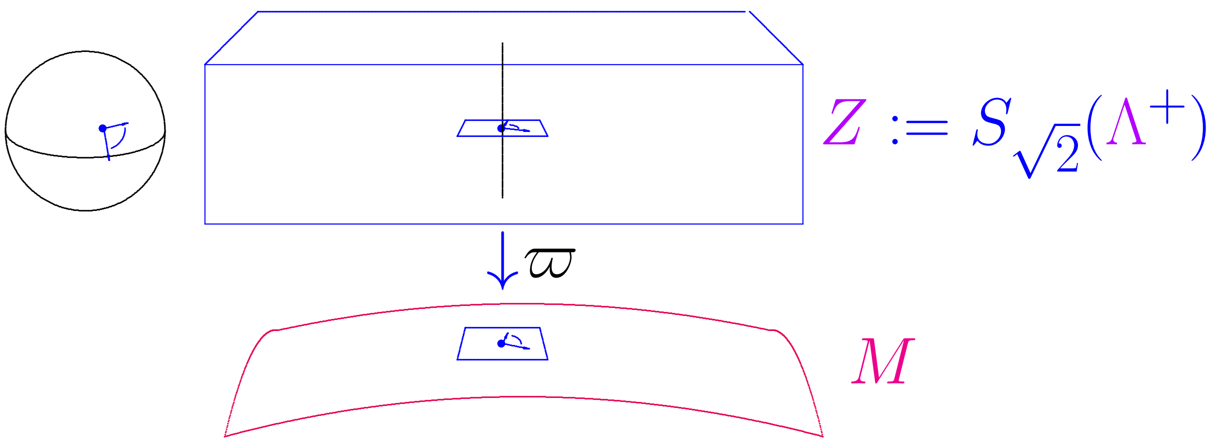

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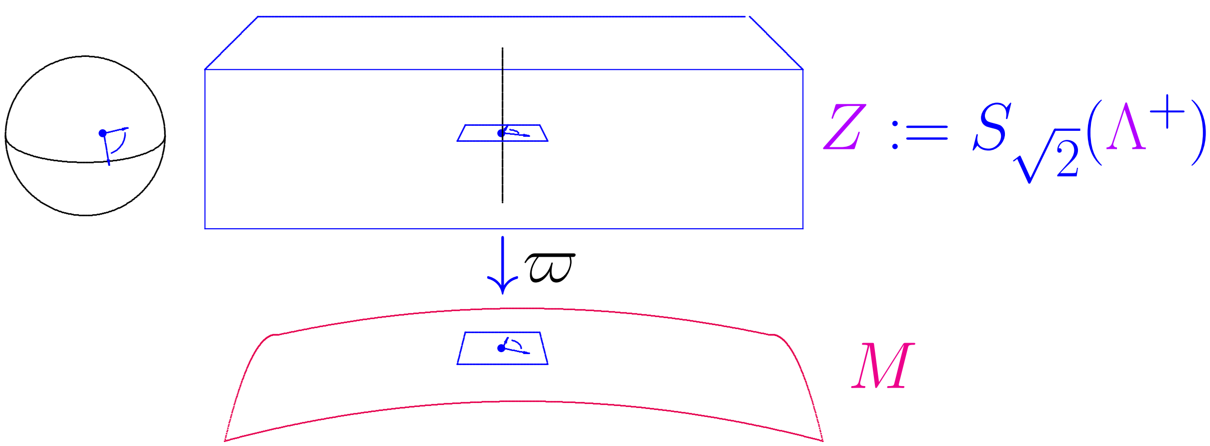

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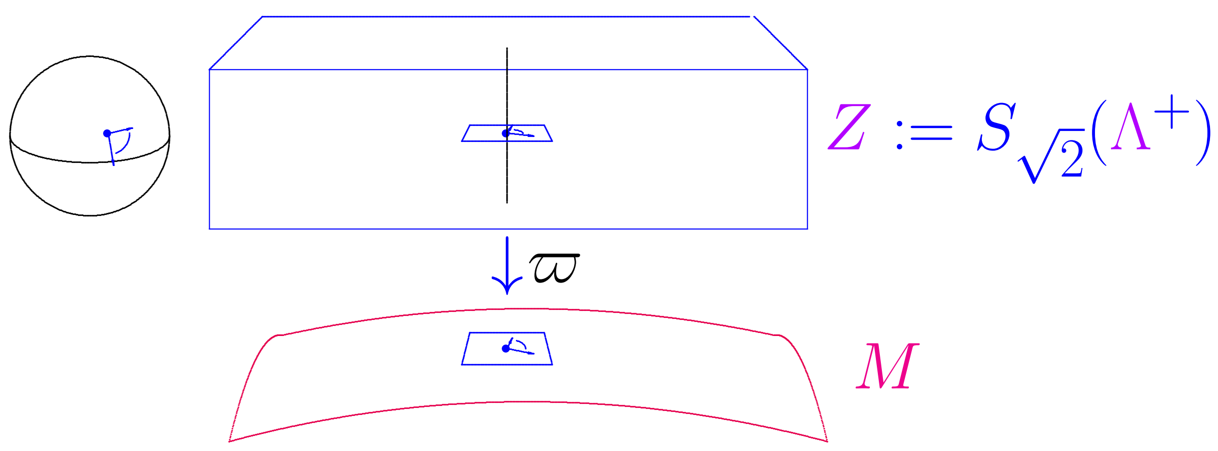

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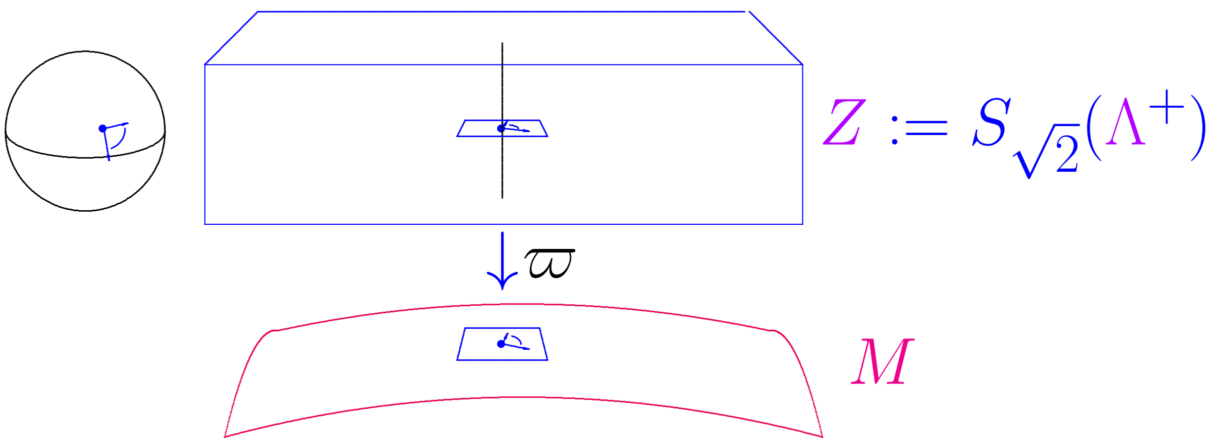

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Non-integrability only involves $[\text{normal}, \text{normal}]$.

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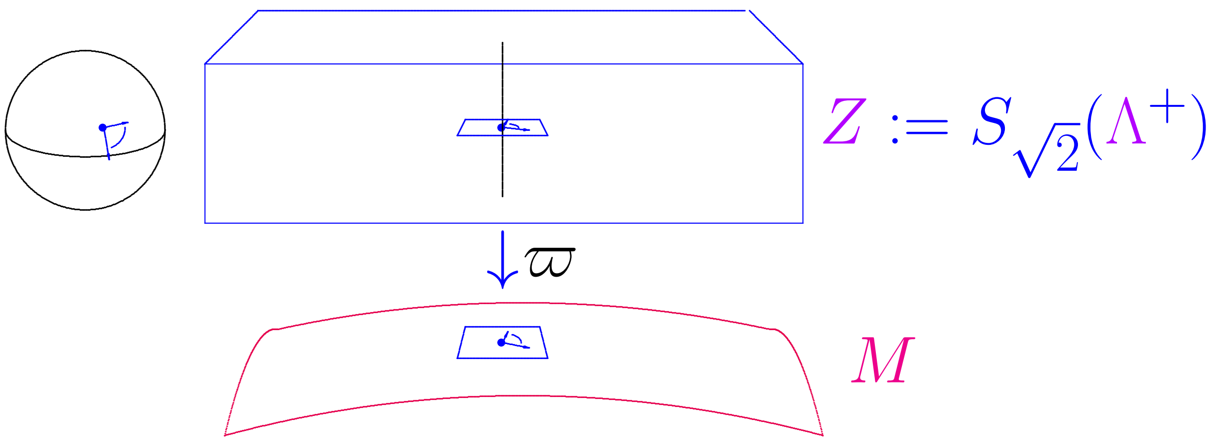
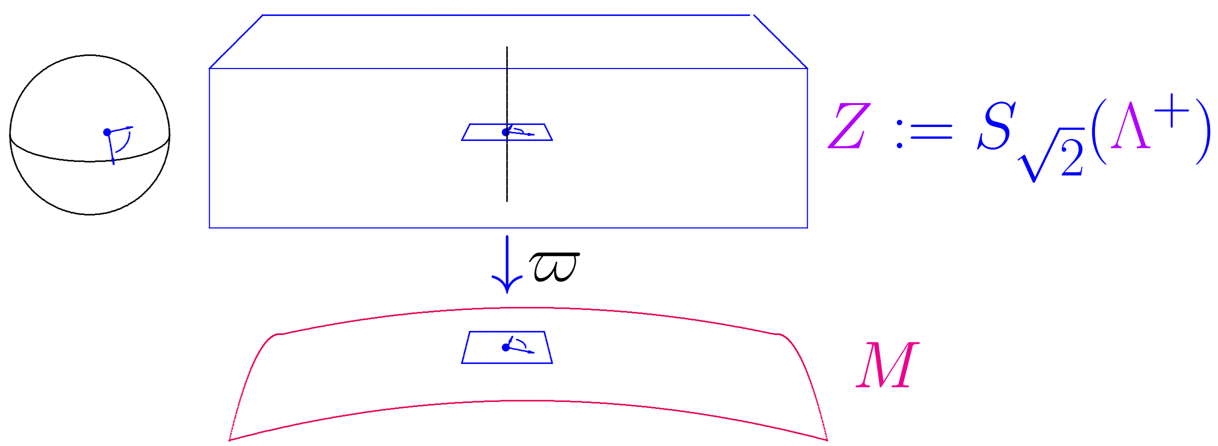
$\Longleftrightarrow \exists$ spin^c structures!

Geometric Definition.



Geometric Definition. A spin^c structure



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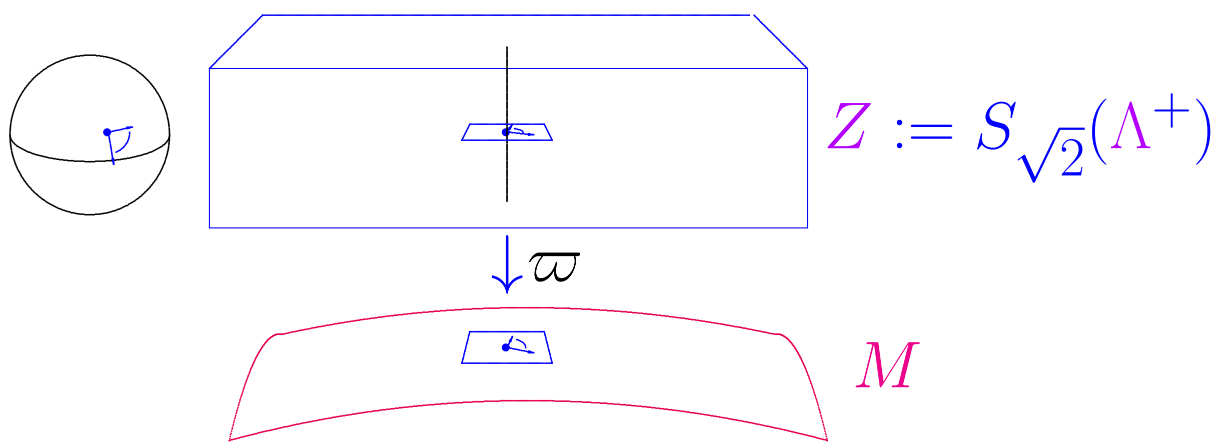

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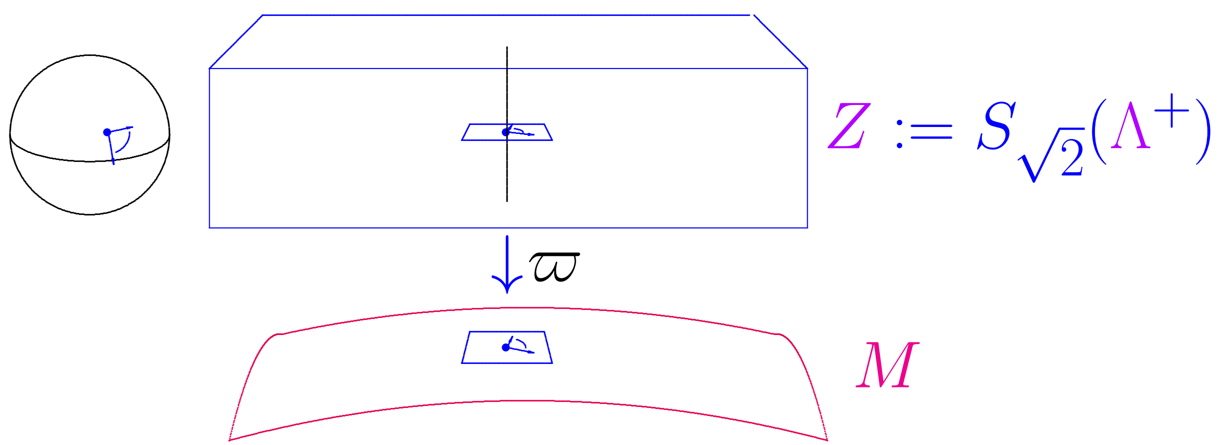

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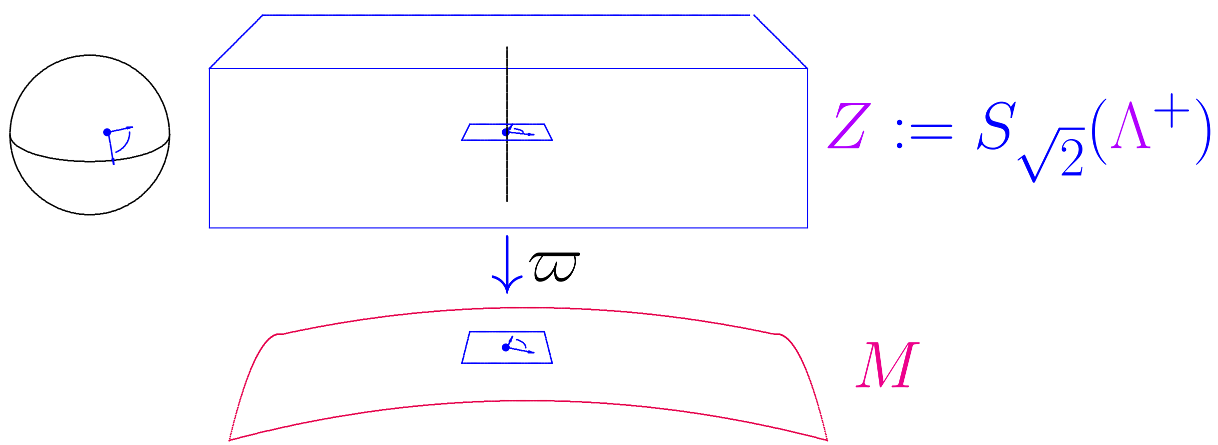
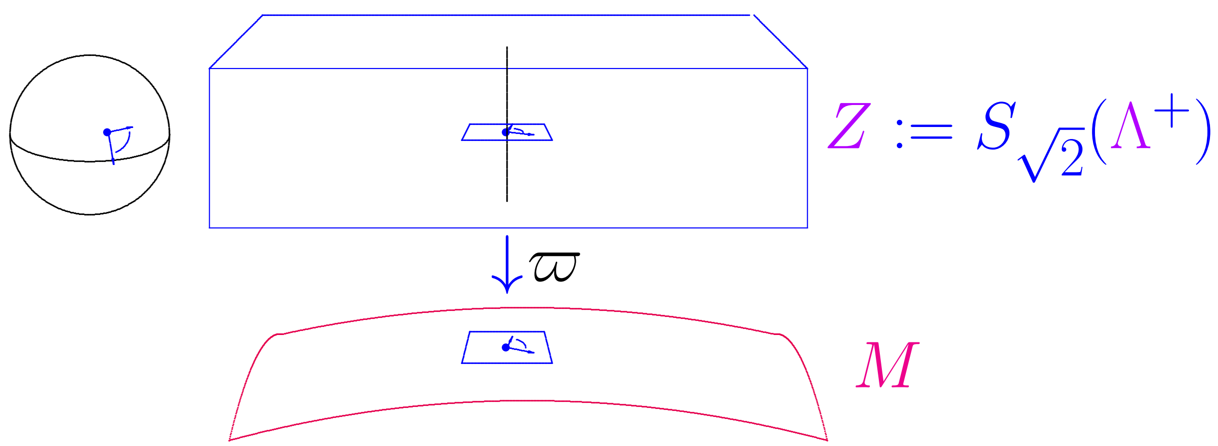
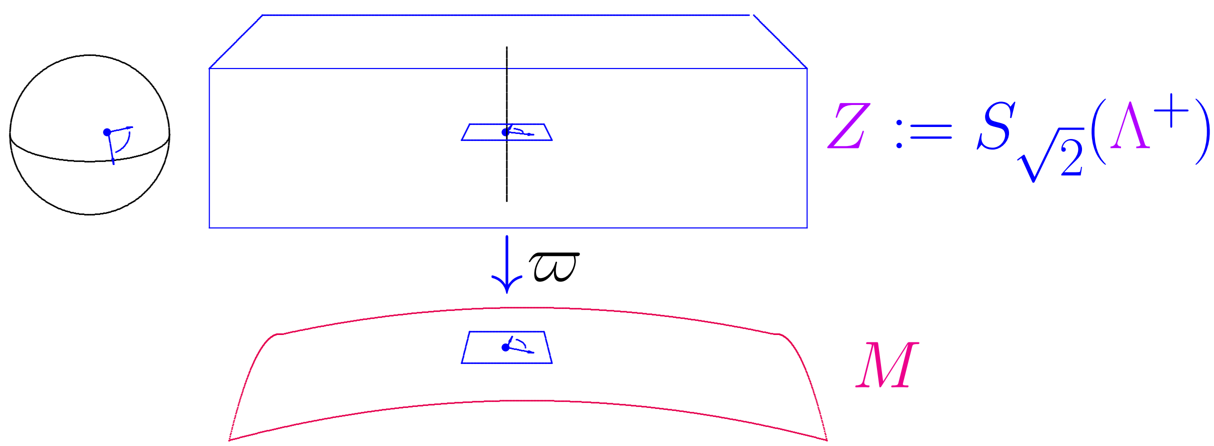


Geometric Definition. *A* spin^c structure *on a smooth oriented* 4*-manifold* M *is a complex line bundle* $\mathscr{L} \to Z$ *on the twistor space that has degree* 1



Geometric Definition. *A* spin^c structure *on a smooth oriented* 4*-manifold* M *is a complex line bundle* $\mathscr{L} \to Z$ *on the twistor space that has degree* 1 *on any* S^2 *fiber of* $Z \to M$*.*



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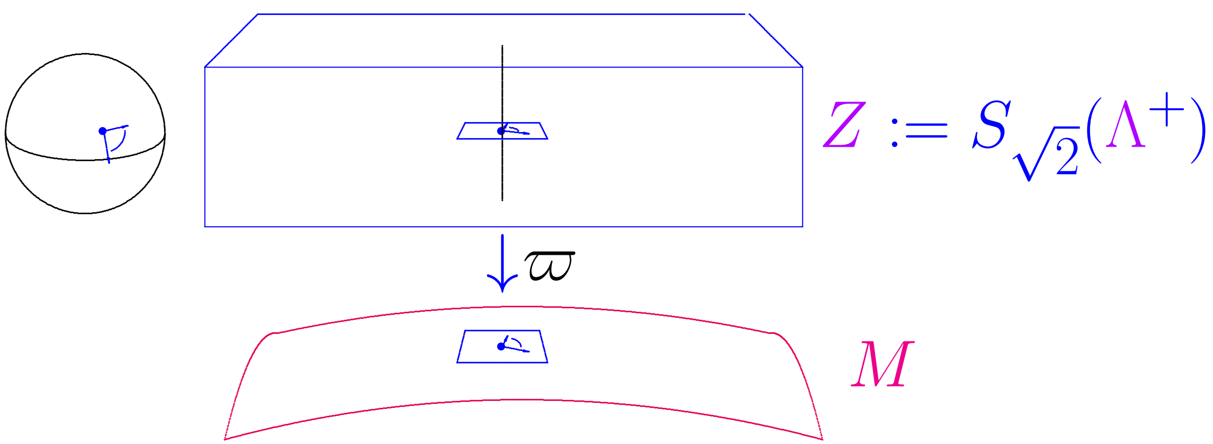

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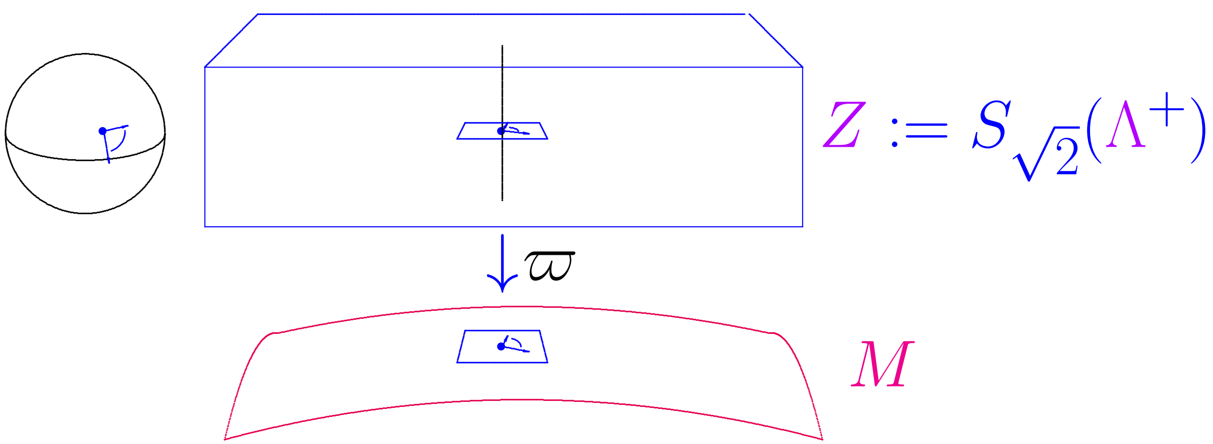

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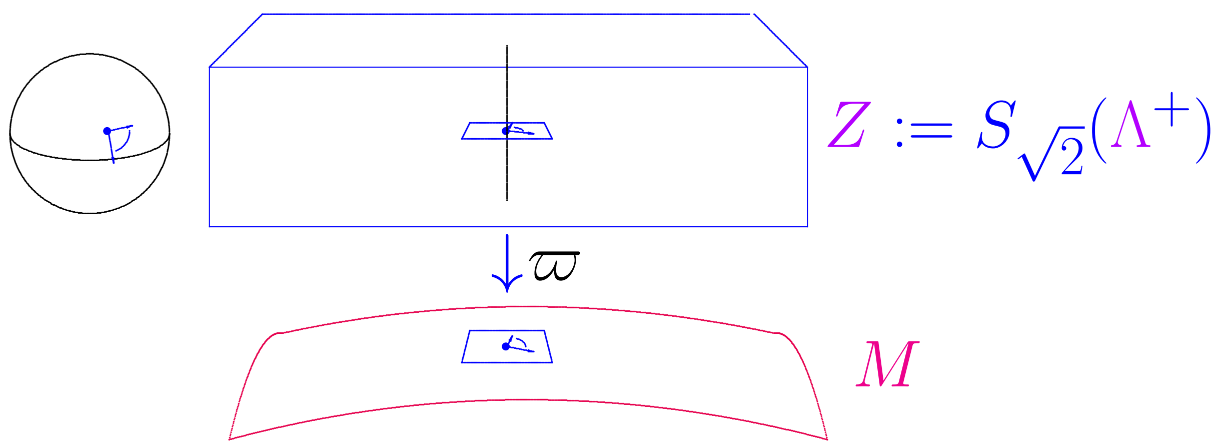

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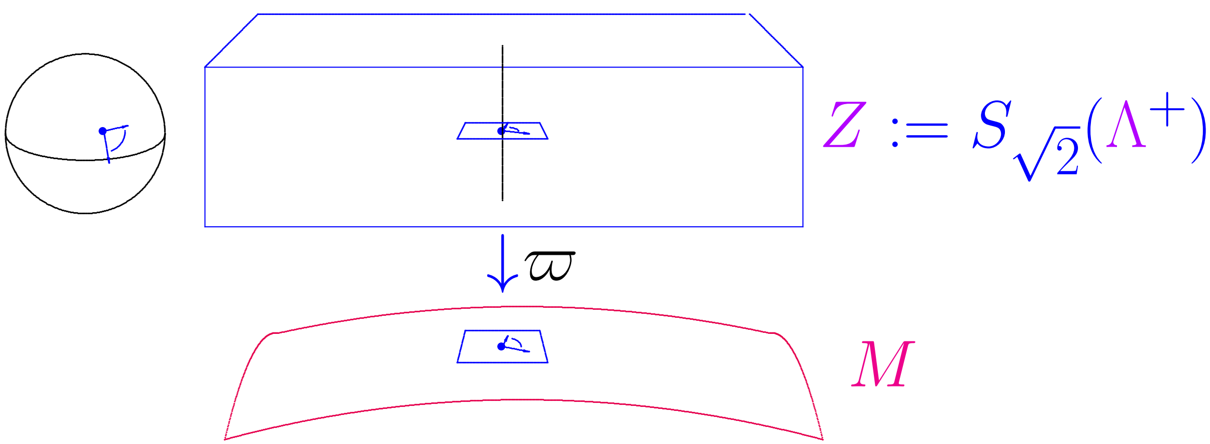

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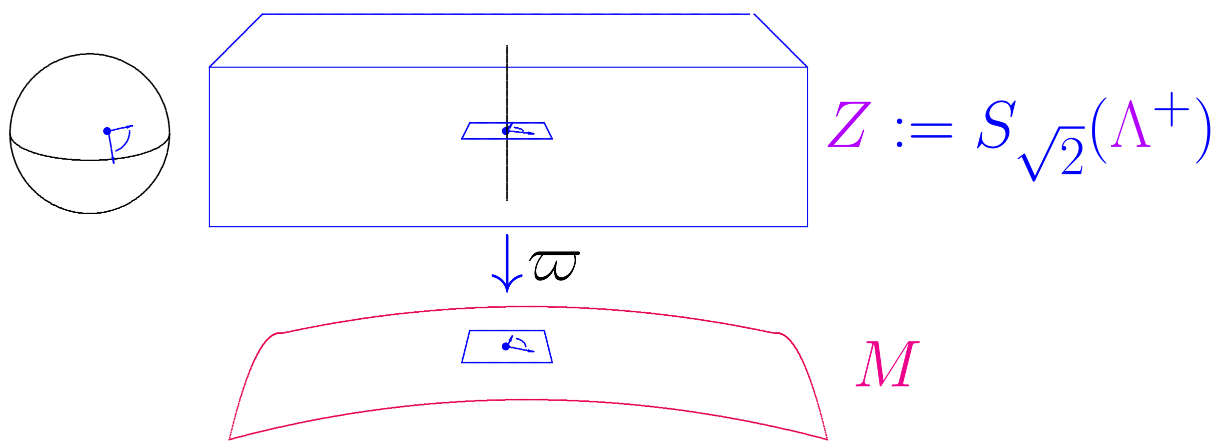

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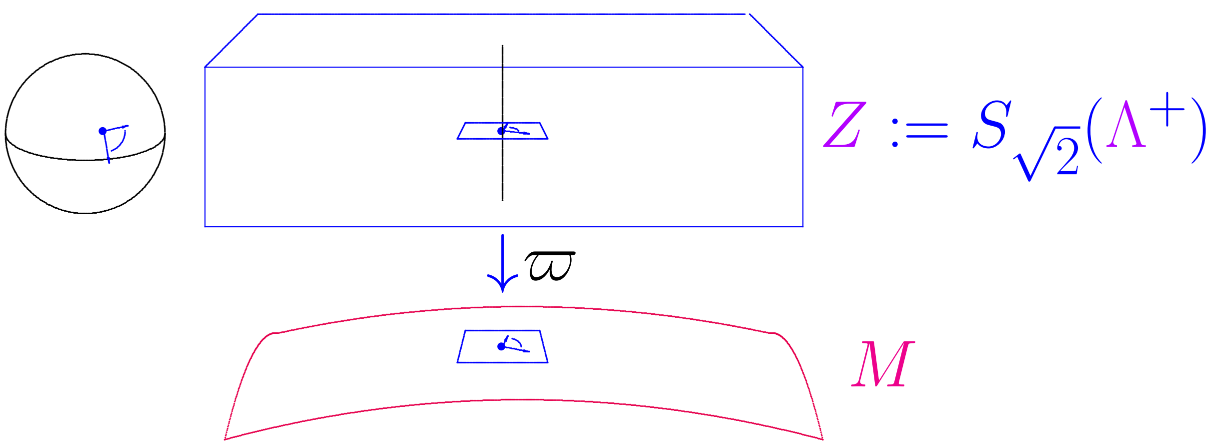

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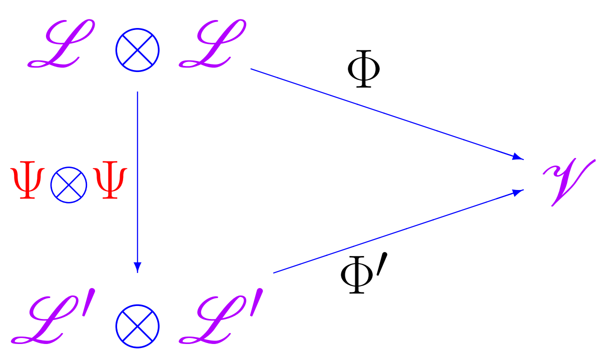

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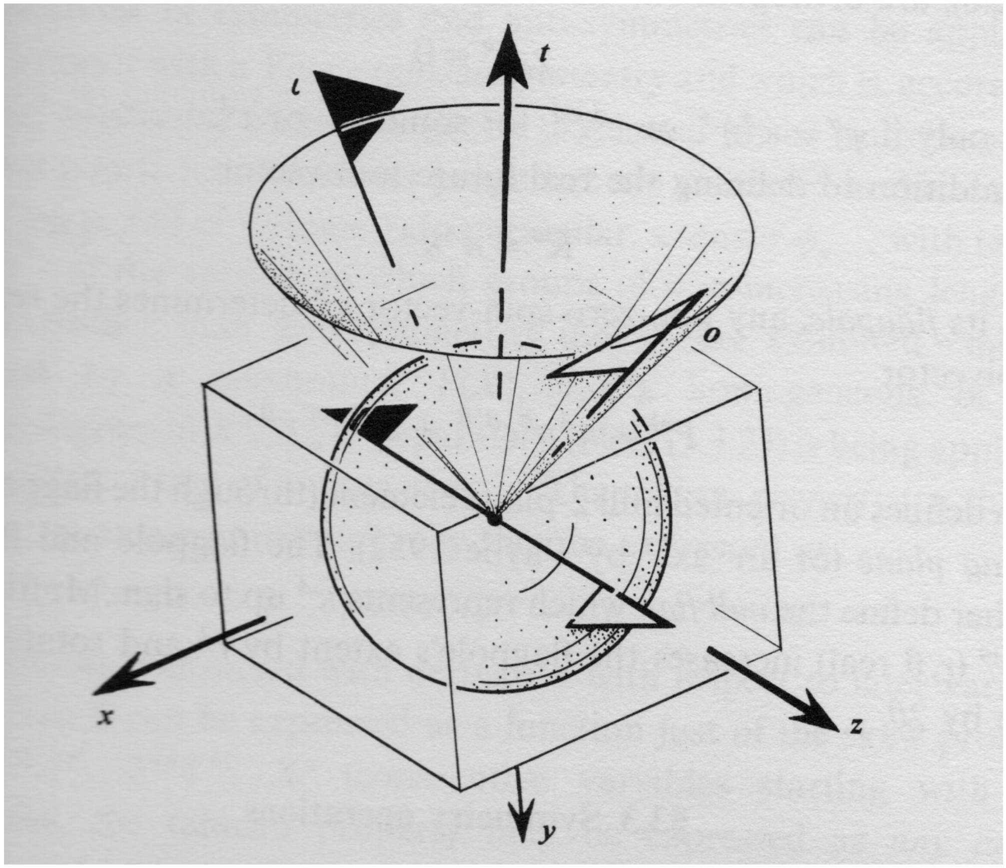

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Key tool:

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$$
\begin{array}{l}
0 \longrightarrow H^2(M,\mathbb{Z}) \longrightarrow H^2(\mathcal{Z},\mathbb{Z}) \longrightarrow H^0(M,\mathbb{Z}) \xrightarrow{e\cup} \\
H^3(M,\mathbb{Z}) \longrightarrow H^3(\mathcal{Z},\mathbb{Z}) \longrightarrow H^1(M,\mathbb{Z}) \xrightarrow{e\cup} \\
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Exact sequence of pair:

$$\to H^2(\mathsf{E}) \to H^2(\mathsf{E}-0) \to$$
$$H^3(\mathsf{E}, \mathsf{E}-0) \to H^3(\mathsf{E}) \to H^3(\mathsf{E}-0) \to$$
$$H^4(\mathsf{E}, \mathsf{E}-0) \to H^4(\mathsf{E}) \to H^4(\mathsf{E}-0) \to$$
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Thom isomorphism:

$$H^{k-3}(M) \xrightarrow{\cong} H^k(\mathsf{E}, \mathsf{E} - 0)$$

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More obvious implications:

$$\text{(i)} \Longrightarrow \text{(ii)} \quad \text{and} \quad \text{(iii)} \Longrightarrow \text{(iv)}.$$

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$$
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What about

$$\text{(ii)} \Longrightarrow \text{(iii)?}$$

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$$
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\to\ & H^3(M,\mathbb{Z}) \xrightarrow{\varpi^*} H^3(\mathcal{Z},\mathbb{Z}) \xrightarrow{\varpi_*} H^1(M,\mathbb{Z}) \xrightarrow{\cup \mathsf{e}} \\
\to\ & H^4(M,\mathbb{Z}) \xrightarrow{\varpi^*} H^4(\mathcal{Z},\mathbb{Z}) \xrightarrow{\varpi_*} H^2(M,\mathbb{Z}) \xrightarrow{\cup \mathsf{e}} 0
\end{aligned}
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$H^4(M,\mathbb{Z}) = \mathbb{Z}$ is free,

$$
\begin{aligned}
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\end{aligned}
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$H^4(M,\mathbb{Z}) = \mathbb{Z}$ is free,

$\mathsf{e}(\mathsf{E})$ is torsion.

$$0 \to H^2(M,\mathbb{Z}) \xrightarrow{\varpi^*} H^2(\mathcal{Z},\mathbb{Z}) \xrightarrow{\varpi_*} H^0(M,\mathbb{Z}) \xrightarrow{\cup \mathsf{e}}$$
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$$0 \to H^4(M,\mathbb{Z}) \xrightarrow{\varpi^*} H^4(\mathcal{Z},\mathbb{Z}) \xrightarrow{\varpi_*} H^2(M,\mathbb{Z}) \to 0$$

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$$0 \to H^4(M,\mathbb{Z}) \xrightarrow{\varpi^*} H^4(\mathcal{Z},\mathbb{Z}) \xrightarrow{\varpi_*} H^2(M,\mathbb{Z}) \to 0$$

Poincaré duality:

$$0 \to H^2(M,\mathbb{Z}) \xrightarrow{\varpi^*} H^2(\mathcal{Z},\mathbb{Z}) \xrightarrow{\varpi_*} H^0(M,\mathbb{Z}) \xrightarrow{\cup \mathsf{e}}$$
$$\to H^3(M,\mathbb{Z}) \xrightarrow{\varpi^*} H^3(\mathcal{Z},\mathbb{Z}) \xrightarrow{\varpi_*} H^1(M,\mathbb{Z}) \xrightarrow{\cup \mathsf{e}}$$
$$\to H^4(M,\mathbb{Z}) \xrightarrow{\varpi^*} H^4(\mathcal{Z},\mathbb{Z}) \xrightarrow{\varpi_*} H^2(M,\mathbb{Z}) \xrightarrow{\cup \mathsf{e}} 0$$

$H^4(M,\mathbb{Z}) = \mathbb{Z}$ is free,

$\mathsf{e}(\mathsf{E})$ is torsion.

$$0 \to H^4(M,\mathbb{Z}) \xrightarrow{\varpi^*} H^4(\mathcal{Z},\mathbb{Z}) \xrightarrow{\varpi_*} H^2(M,\mathbb{Z}) \to 0$$

Poincaré duality:

$$0 \to \mathbb{Z} \xrightarrow{\cdot F} H_2(\mathcal{Z},\mathbb{Z}) \to H_2(M,\mathbb{Z}) \to 0$$

$$0 \to H^2(M,\mathbb{Z}) \xrightarrow{\varpi^*} H^2(\mathcal{Z},\mathbb{Z}) \xrightarrow{\varpi_*} H^0(M,\mathbb{Z}) \xrightarrow{\cup \mathsf{e}}$$
$$\to H^3(M,\mathbb{Z}) \xrightarrow{\varpi^*} H^3(\mathcal{Z},\mathbb{Z}) \xrightarrow{\varpi_*} H^1(M,\mathbb{Z}) \xrightarrow{\cup \mathsf{e}}$$
$$\to H^4(M,\mathbb{Z}) \xrightarrow{\varpi^*} H^4(\mathcal{Z},\mathbb{Z}) \xrightarrow{\varpi_*} H^2(M,\mathbb{Z}) \xrightarrow{\cup \mathsf{e}} 0$$

$H^4(M,\mathbb{Z}) = \mathbb{Z}$ is free,

$\mathsf{e}(\mathsf{E})$ is torsion.

$$0 \to H^4(M,\mathbb{Z}) \xrightarrow{\varpi^*} H^4(\mathcal{Z},\mathbb{Z}) \xrightarrow{\varpi_*} H^2(M,\mathbb{Z}) \to 0$$

Poincaré duality:

$$0 \to \mathbb{Z} \xrightarrow{\cdot F} H_2(\mathcal{Z},\mathbb{Z}) \to H_2(M,\mathbb{Z}) \to 0$$

If $\exists$ $\mathbf{a} \in H^2(\mathcal{Z},\mathbb{Z})$ with $\langle \mathbf{a}, F\rangle = 1$, gives splitting

$$0 \to \mathbb{Z} \longrightarrow H_2(\mathcal{Z},\mathbb{Z}) \to H_2(M,\mathbb{Z}) \to 0$$

$$0 \to H^2(M,\mathbb{Z}) \xrightarrow{\varpi^*} H^2(\mathcal{Z},\mathbb{Z}) \xrightarrow{\varpi_*} H^0(M,\mathbb{Z}) \xrightarrow{\cup \mathsf{e}}$$
$$\to H^3(M,\mathbb{Z}) \xrightarrow{\varpi^*} H^3(\mathcal{Z},\mathbb{Z}) \xrightarrow{\varpi_*} H^1(M,\mathbb{Z}) \xrightarrow{\cup \mathsf{e}}$$
$$\to H^4(M,\mathbb{Z}) \xrightarrow{\varpi^*} H^4(\mathcal{Z},\mathbb{Z}) \xrightarrow{\varpi_*} H^2(M,\mathbb{Z}) \xrightarrow{\cup \mathsf{e}} 0$$

$H^4(M,\mathbb{Z}) = \mathbb{Z}$ is free,

$\mathsf{e}(\mathsf{E})$ is torsion.

$$0 \to H^4(M,\mathbb{Z}) \xrightarrow{\varpi^*} H^4(\mathcal{Z},\mathbb{Z}) \xrightarrow{\varpi_*} H^2(M,\mathbb{Z}) \to 0$$

Poincaré duality:

$$0 \to \mathbb{Z} \xrightarrow{\cdot F} H_2(\mathcal{Z},\mathbb{Z}) \to H_2(M,\mathbb{Z}) \to 0$$

If $\exists$ $\mathbf{a} \in H^2(\mathcal{Z},\mathbb{Z})$ with $\langle \mathbf{a}, F\rangle = 1$, gives splitting

$$H_2(\mathcal{Z},\mathbb{Z}) \cong \mathbb{Z} \oplus H_2(M,\mathbb{Z}).$$

So we now have

$$\text{(i)} \Longrightarrow \text{(ii)} \Longrightarrow \text{(iii)} \Longrightarrow \text{(iv)}.$$

Proposition. *The following are equivalent:*

(i) The Euler class $\mathsf{e}(\mathsf{E}) \in H^3(M, \mathbb{Z})$ vanishes;

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(iv) $|\mathfrak{T}_2(\mathcal{Z})| = |\mathfrak{T}_2(M)|$, where $\mathfrak{T}_2$ is the torsion subgroup of $H_2(\underline{}, \mathbb{Z})$.

Finally, we'll show

$$\text{(iv)} \Longrightarrow \text{(i)}$$

Proposition. *The following are equivalent:*

(i) The Euler class $\mathsf{e}(\mathsf{E}) \in H^3(M, \mathbb{Z})$ vanishes;

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$$
\begin{aligned}
0 \to\ & H^2(M,\mathbb{Z}) \xrightarrow{\varpi^*} H^2(\mathcal{Z},\mathbb{Z}) \xrightarrow{\varpi_*} H^0(M,\mathbb{Z}) \xrightarrow{\cup \mathsf{e}} \\
\to\ & H^3(M,\mathbb{Z}) \xrightarrow{\varpi^*} H^3(\mathcal{Z},\mathbb{Z}) \xrightarrow{\varpi_*} H^1(M,\mathbb{Z}) \xrightarrow{\cup \mathsf{e}} \\
\to\ & H^4(M,\mathbb{Z}) \xrightarrow{\varpi^*} H^4(\mathcal{Z},\mathbb{Z}) \xrightarrow{\varpi_*} H^2(M,\mathbb{Z}) \xrightarrow{\cup \mathsf{e}} 0
\end{aligned}
$$

$$
\mathfrak{T}_2(M) \cong \mathfrak{T}^3(M), \qquad \mathfrak{T}_2(\mathcal{Z}) \cong \mathfrak{T}^3(\mathcal{Z})
$$

$$0 \to H^2(M,\mathbb{Z}) \xrightarrow{\varpi^*} H^2(\mathcal{Z},\mathbb{Z}) \xrightarrow{\varpi_*} H^0(M,\mathbb{Z}) \xrightarrow{\cup \mathsf{e}}$$
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$$|\mathfrak{T}_2(\mathcal{Z})| = |\mathfrak{T}_2(M)| \quad \Longrightarrow \quad |\mathfrak{T}^3(\mathcal{Z})| = |\mathfrak{T}^3(M)|$$

$$0 \to H^2(M,\mathbb{Z}) \xrightarrow{\varpi^*} H^2(\mathcal{Z},\mathbb{Z}) \xrightarrow{\varpi_*} H^0(M,\mathbb{Z}) \xrightarrow{\cup \mathsf{e}}$$

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$$|\mathfrak{T}_2(\mathcal{Z})| = |\mathfrak{T}_2(M)| \quad \Longrightarrow \quad |\mathfrak{T}^3(\mathcal{Z})| = |\mathfrak{T}^3(M)|$$

But

$$H^0(M,\mathbb{Z}) \xrightarrow{\cup \mathsf{e}} H^3(M,\mathbb{Z}) \xrightarrow{\varpi^*} H^3(\mathcal{Z},\mathbb{Z}) \xrightarrow{\varpi_*} H^1(M,\mathbb{Z})$$

$$0 \to H^2(M,\mathbb{Z}) \xrightarrow{\varpi^*} H^2(\mathcal{Z},\mathbb{Z}) \xrightarrow{\varpi_*} H^0(M,\mathbb{Z}) \xrightarrow{\cup \mathsf{e}}$$
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So if $[\Sigma] \in H_2(M, \mathbb{Z})$ is a torsion class, then

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Why torsion?

$$H^2(Z, \mathbb{R}) = \mathbb{R}c_1(H^{1,0}) \oplus \varpi^* H^2(M, \mathbb{R})$$

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which e.g. appears in Seiberg-Witten equations

$$\begin{aligned} \not{D}_\theta \Phi &= 0 \\ F_\theta^+ &= i\sigma(\Phi). \end{aligned}$$

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But since $\mathrm{rank}_{\mathbb{R}}(\mathbb{V}_+) = 4 = \dim M$, always have sections of $\mathbb{V}_+$ that only vanish at one point!

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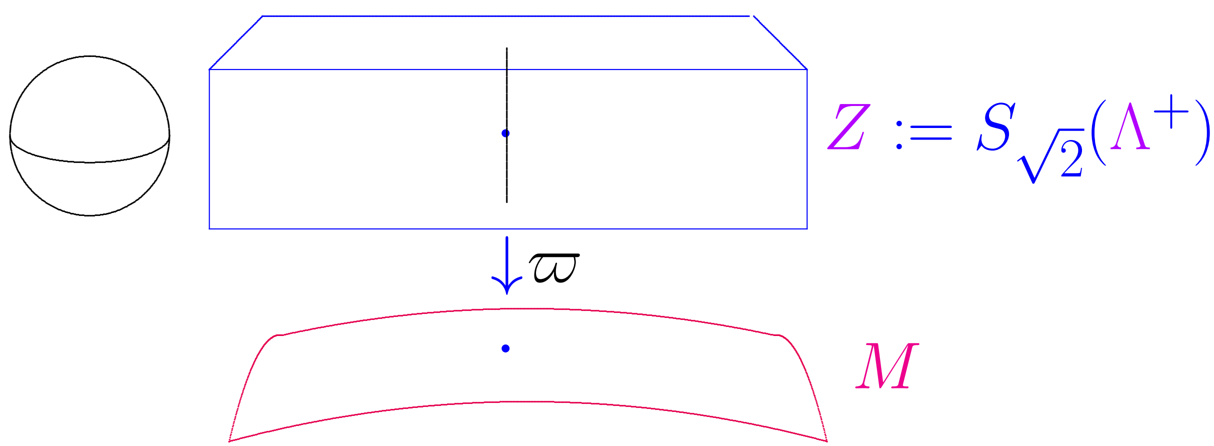

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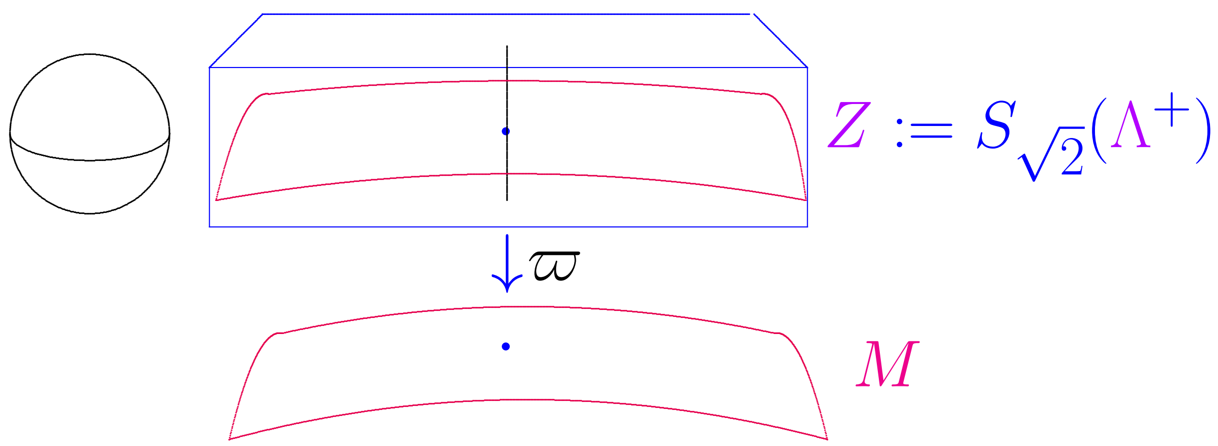

Happy Birthday, Roger!

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And Many Happy Returns!