Gravitational Instantons,

Weyl Curvature, &

Conformally Kähler Geometry

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Differential Geometry and
Geometric Analysis Seminar
Princeton University
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Joint work with

Joint work with

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e-print:
arXiv:2310.14387 [math.DG]

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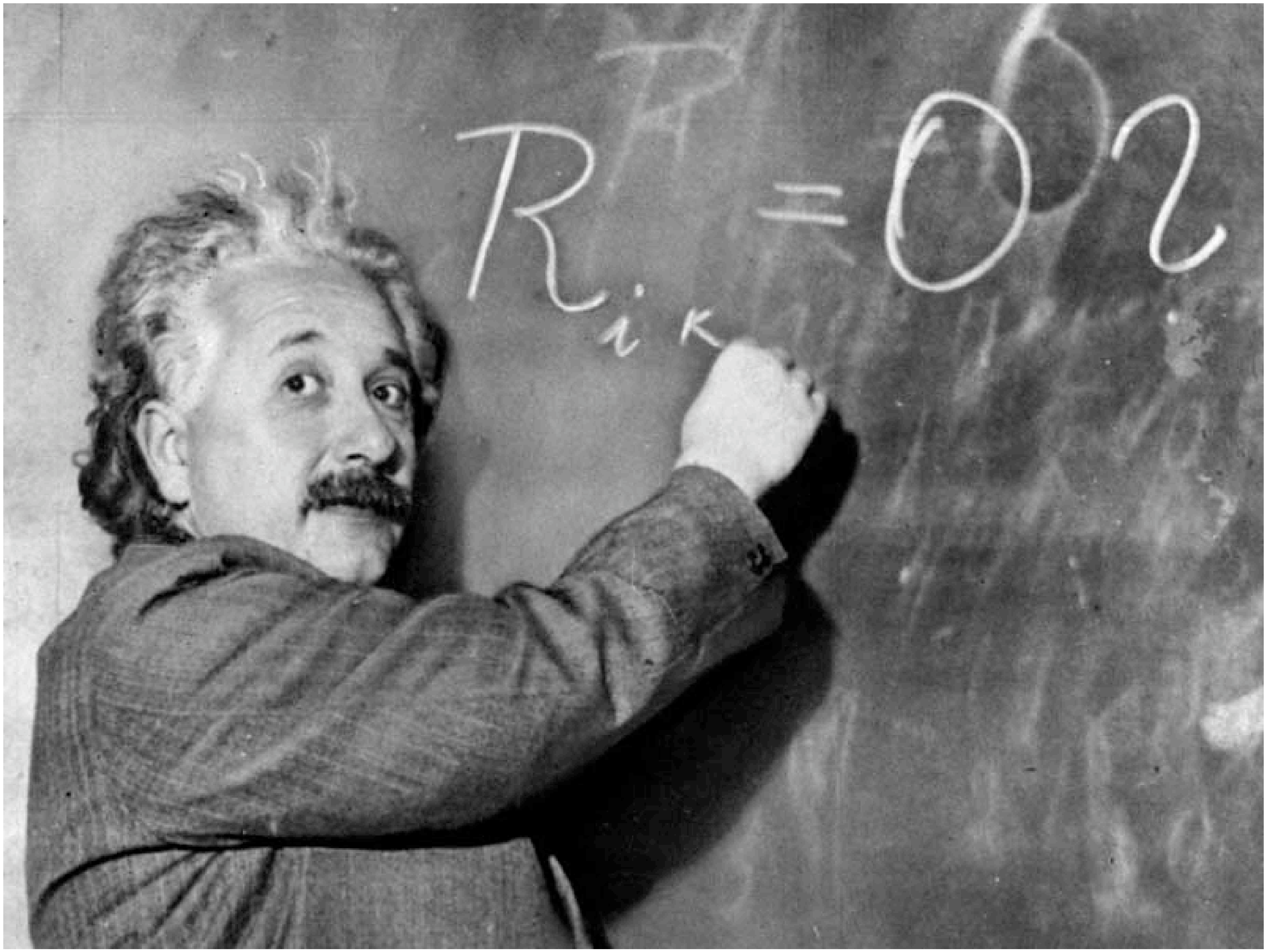

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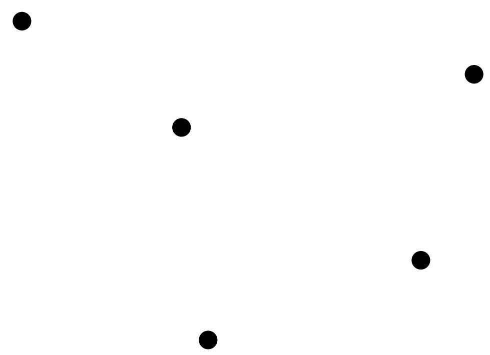

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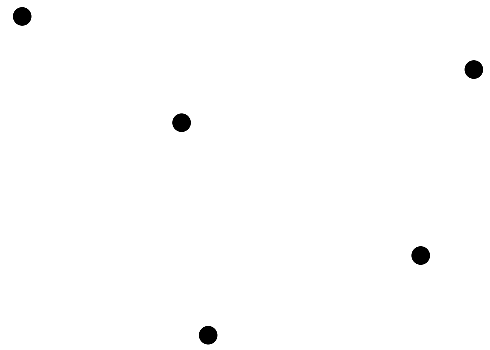

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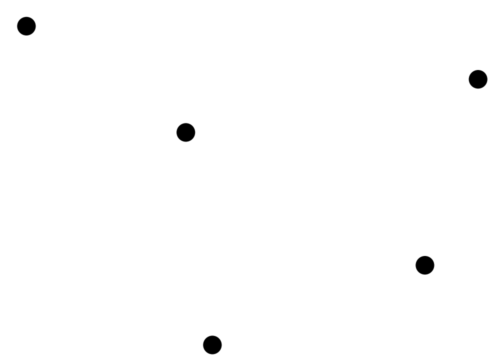

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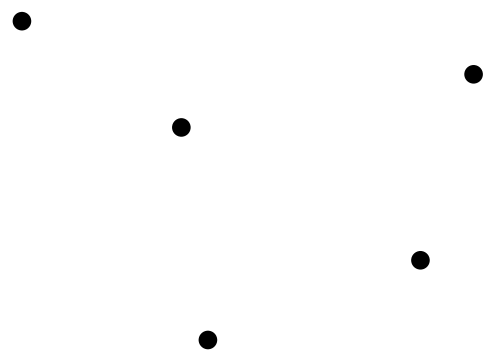

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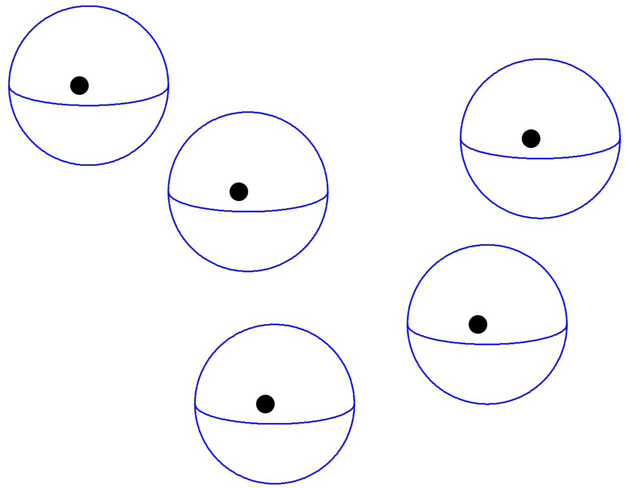

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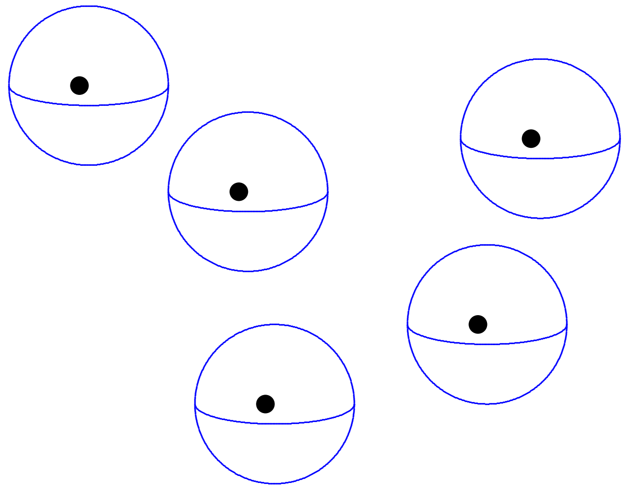

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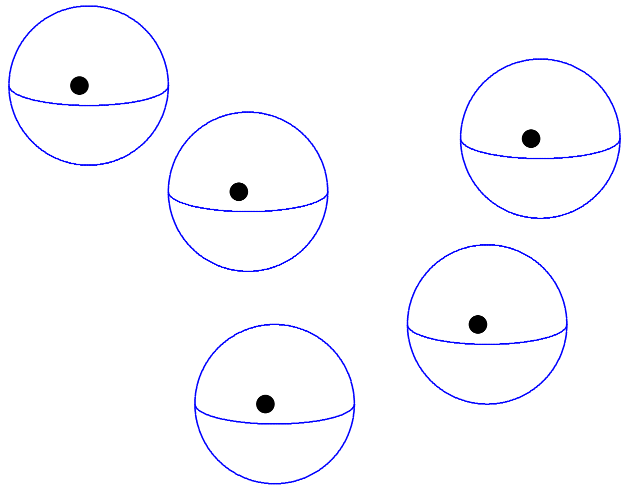

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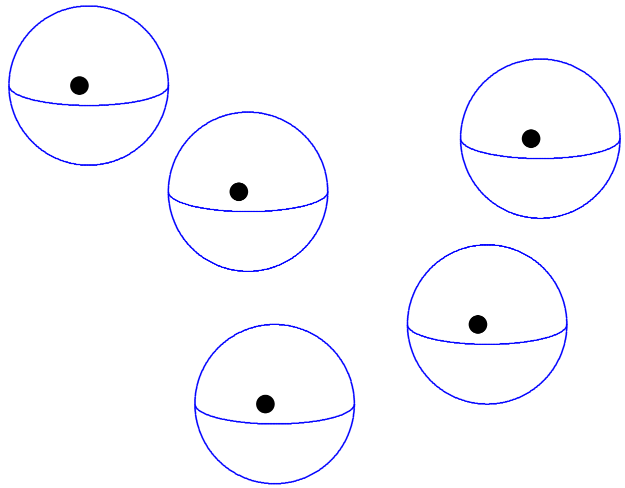

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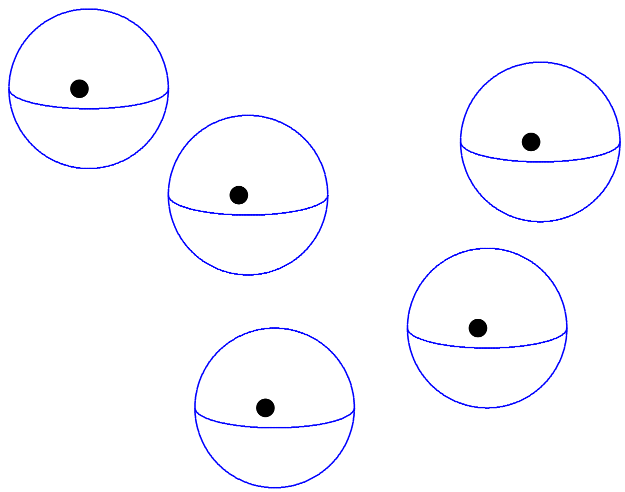

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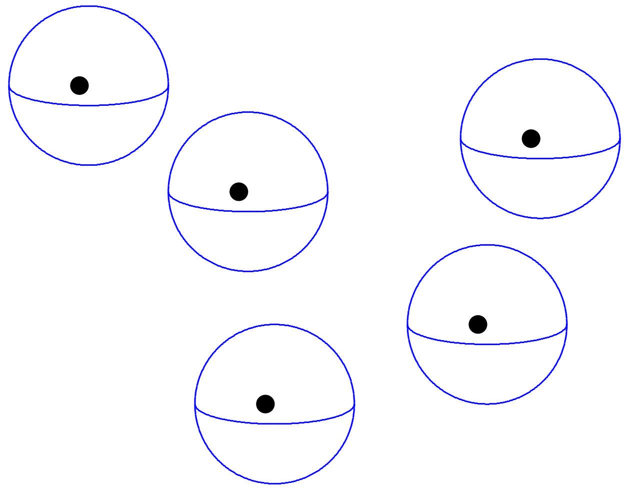

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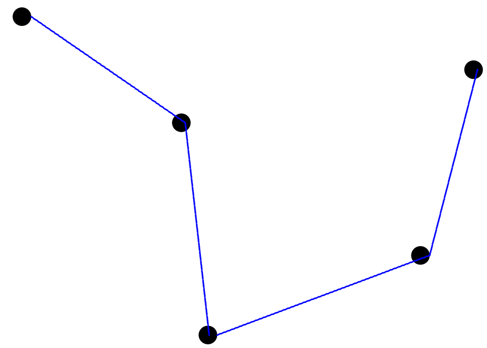

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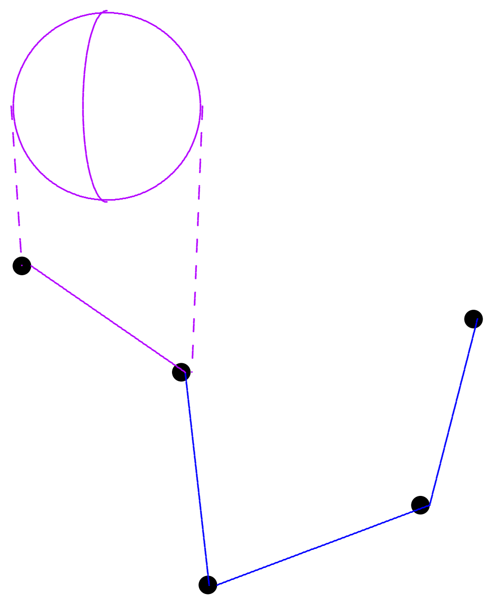

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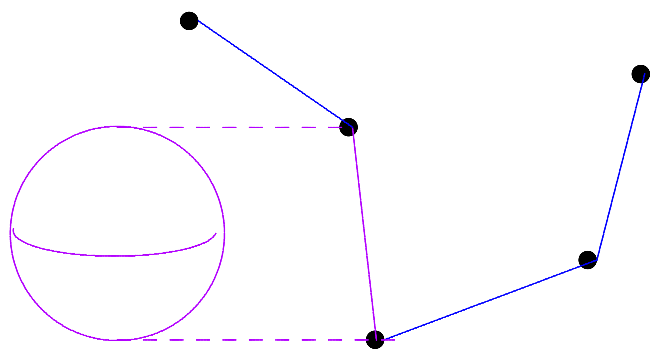

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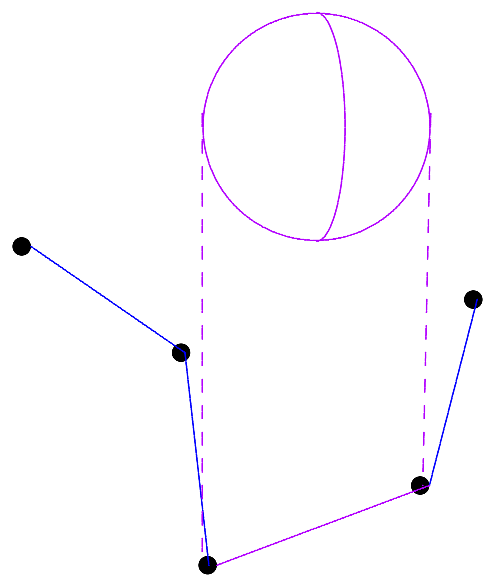

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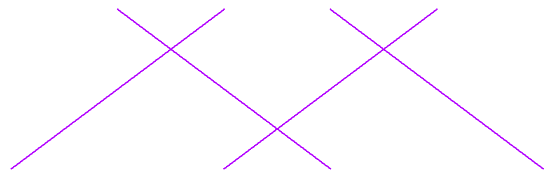

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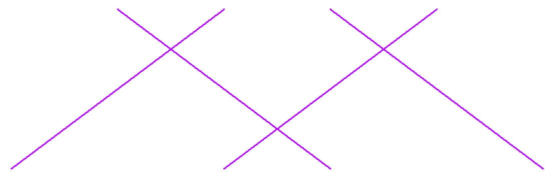

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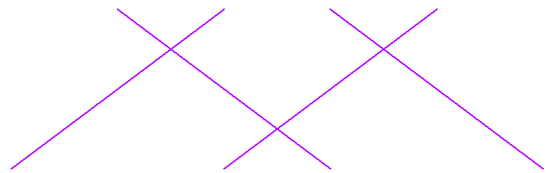

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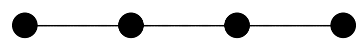

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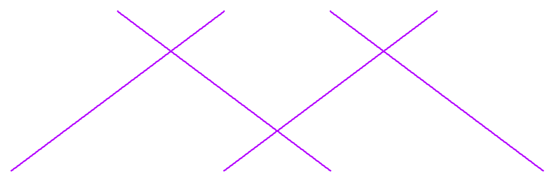

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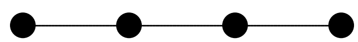

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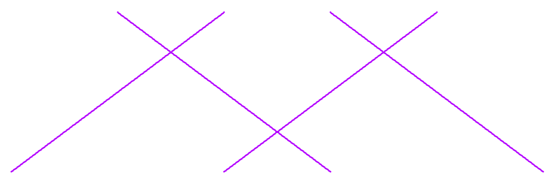

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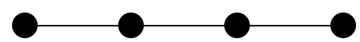

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Plumb together k copies of T^*S^2
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Gibbons and Hawking were unaware of all this!

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cf. Bishop-Gromov inequality!

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This last property distinguishes the ALF spaces from other classes of gravitational instantons:

ALG, ALH, ALG*, ALH*, ...

Example.

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Taub-NUT becomes Hermitian metric on $\mathbb{C}^2$.

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for left-invariant coframe $\{\sigma_j\}$ on $S^3 = \mathbf{SU}(2)$.

Taub-NUT becomes Hermitian metric on $\mathbb{C}^2$.

$$dr \mapsto \frac{2r}{1+r}\sigma_3, \quad \sigma_1 \mapsto \sigma_2$$

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This J determines opposite orientation from the hyper-Kähler complex structures.

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Non-Kähler, but conformally Kähler!

Hawking also explored non-hyper-Kähler examples. . .

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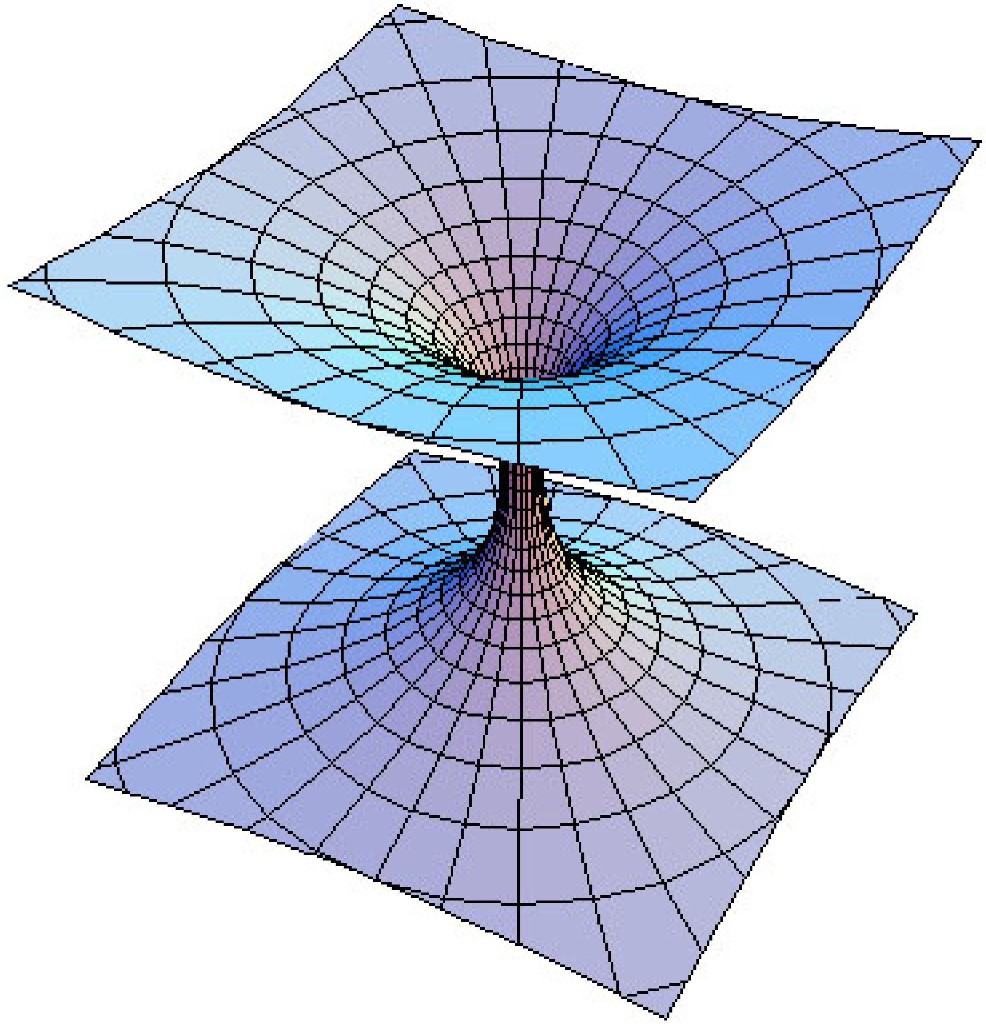
Makes h into extremal Kähler metric on $\mathbb{C} \times \mathbb{CP}_1$.



$$\mathbb{R} \times S^2 \subset \mathbb{R}^2 \times S^2$$

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This might lend some credence to the aphorism…

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Y. Chen & E. Teo, 2011

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By the Riemannian Goldberg-Sachs Theorem, the Hermitian assumption is equivalent to assuming that the Ricci-flat g is conformally Kähler.

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It is false in all higher dimensions!

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Theorem (Biquard-Gauduchon '23). *Let* (M^4, g) *be a smooth, complete, non-flat, Ricci-flat* 4*-manifold that is* toric*, ALF, and Hermitian with respect to some integrable complex structure* J*. Also assume that* (M, g, J) *is not Kähler. Then* (M, g) *is one of the following explicit examples:*

- *the (reverse-oriented) Taub-NUT* metric*;*
- *the Taub-bolt* metric*;*
- *a metric of the Kerr family; or*
- *a metric in the Chen-Teo family.*

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Notational warning:
Here, g and h interchanged relative to our e-print!

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Let $\alpha \geq \beta \geq \gamma$ be eigenvalues of W^+:

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$$2\sqrt{6}|W^+|_h + |s_h| \geq |\omega \wedge \star d\omega|^2$$

everywhere on (M, h).

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Theorem A.

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However, if g_0 is Kerr or Taub-bolt, we can prove a more definitive rigidity result, because g_0 then has both $\det(W^+) > 0$ and $\det(W^-) > 0$.

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Ambi-Kähler Einstein metrics are ambi-toric!

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Conjecture. *Let (M, g_0) be any toric Hermitian ALF gravitational instanton. Then any Ricci-flat metric g on M which is sufficiently C^3_1 close to g_0 must be another one of the gravitational instantons classified by Biquard-Gauduchon.*


GREAT BRITISH
DRAWINGS

Thank you for inviting me!

It's a pleasure to be here!