

Anti-Self-Dual 4-Manifolds,

Quasi-Fuchsian Groups, &

Almost-Kähler Geometry

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Stony Brook University

Differential Geometry & Analysis Seminar

Princeton University, April 23, 2018

Most recent results joint with

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Christopher J. Bishop
Stony Brook University

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e-print: [arXiv:1708.03824](https://arxiv.org/abs/1708.03824) [math.DG]

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To appear in **Comm. An. Geom.**

Facets of Kähler Geometry:

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Complex Geometry

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Symplectic Geometry

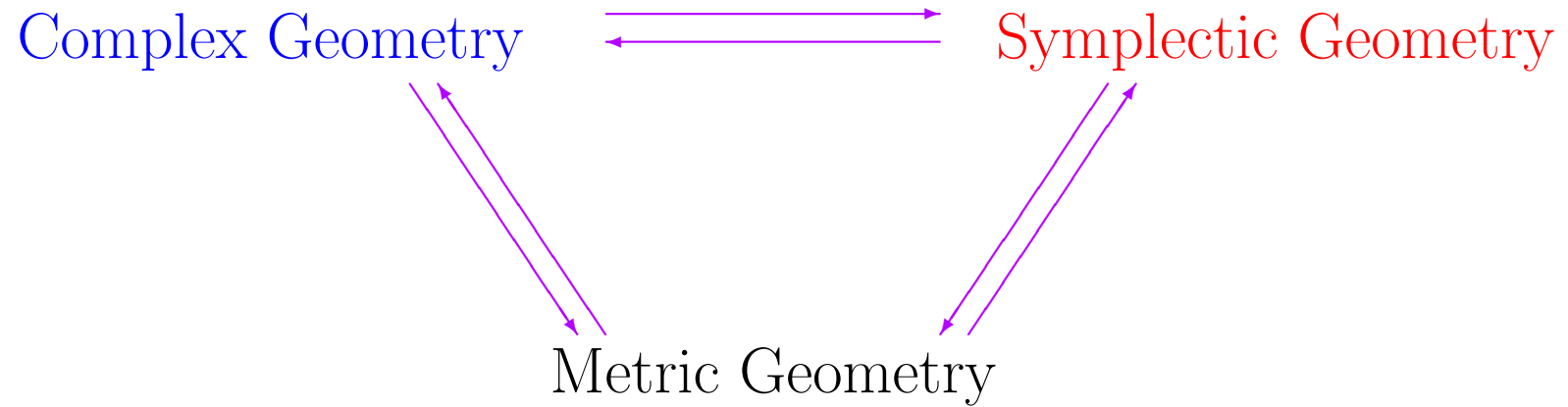
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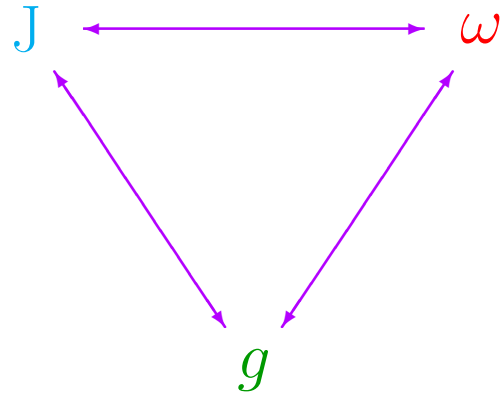
Symplectic Geometry

Metric Geometry

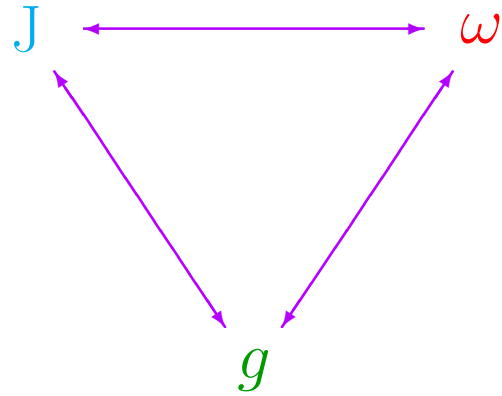
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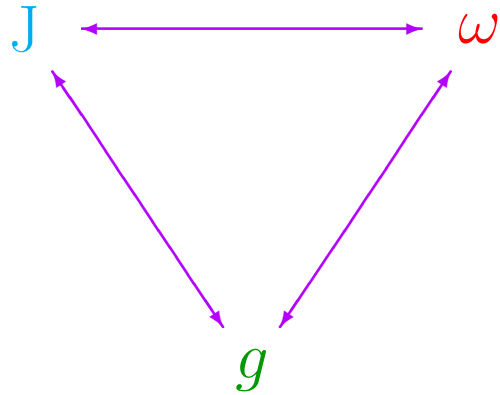
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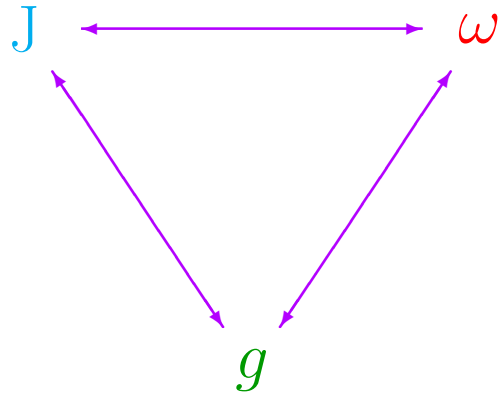


Almost-Kähler Geometry:



Drop demand that J be integrable.

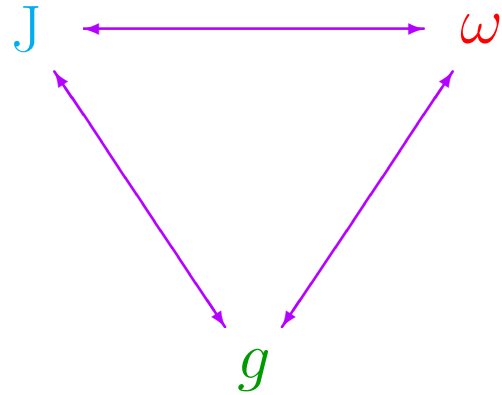
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Linked to conformal geometry in dimension 4.

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Higher dimensions are demonstrably different.

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Imitates Kähler geometry in a non-Kähler setting.

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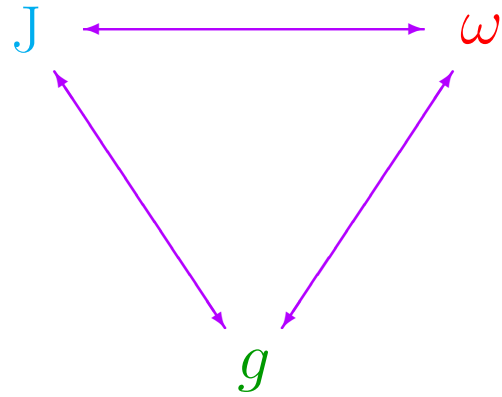
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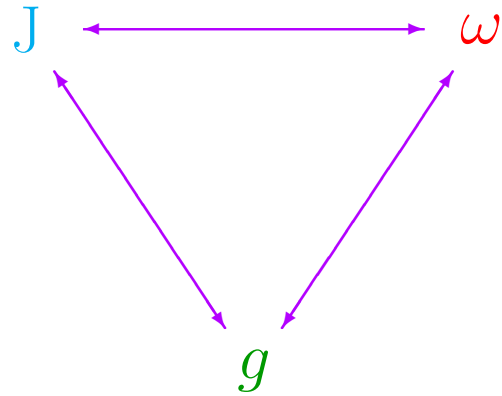
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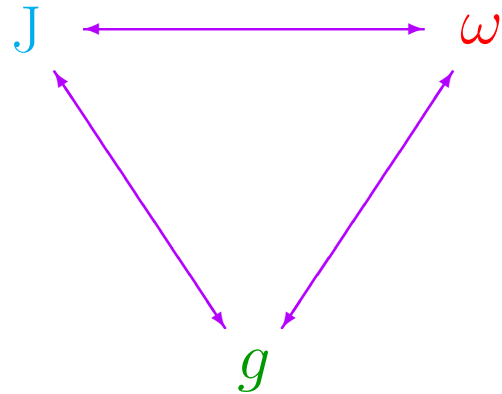


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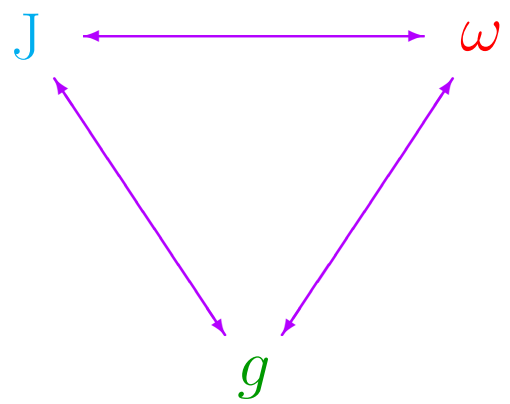
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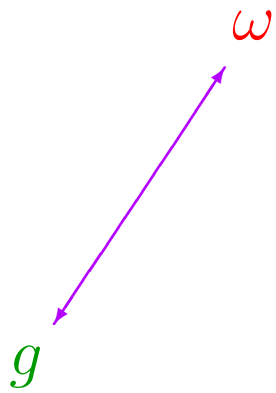
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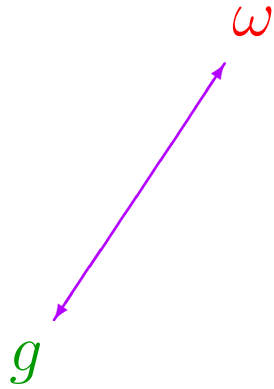


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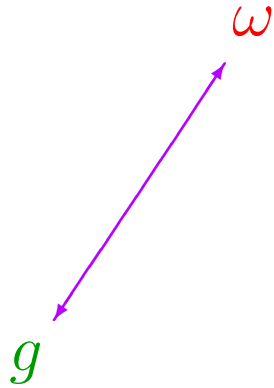
For example, can avoid explicitly mentioning J .



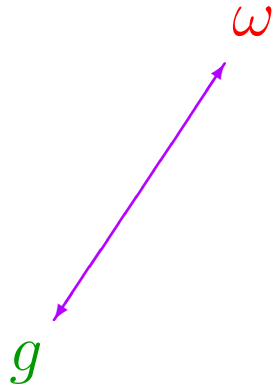




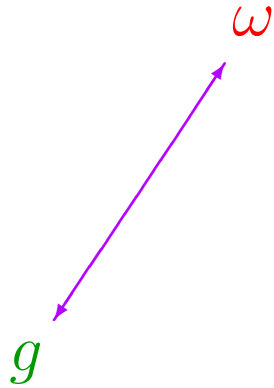
Lemma. *An oriented Riemannian manifold (M^{2m}, g)*



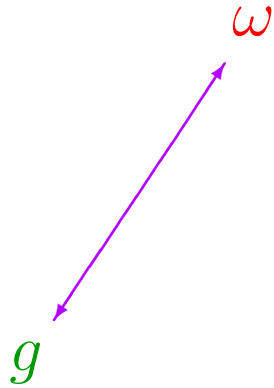
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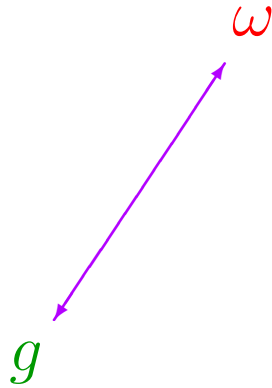


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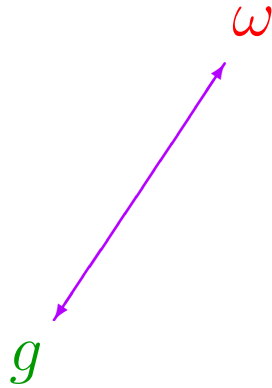
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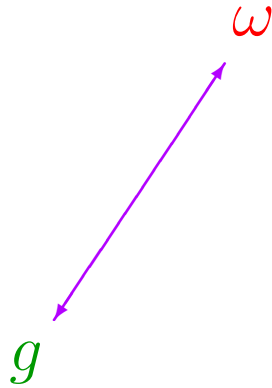
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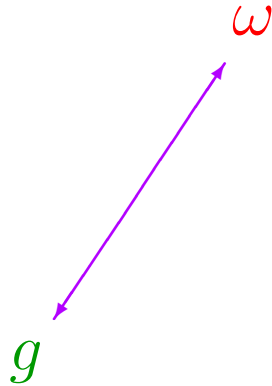
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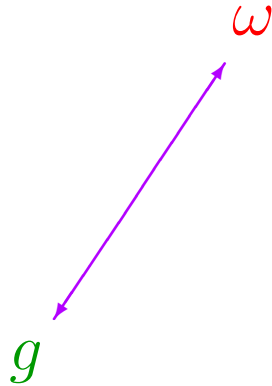
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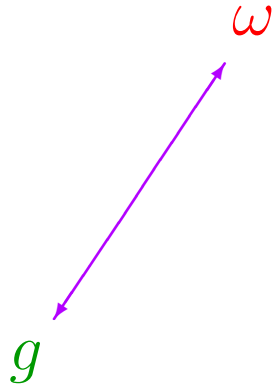
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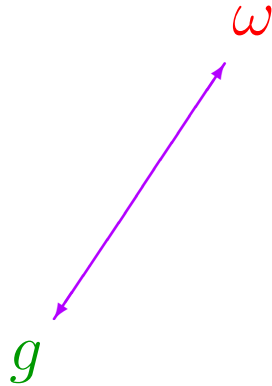
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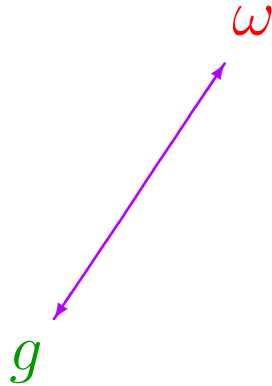
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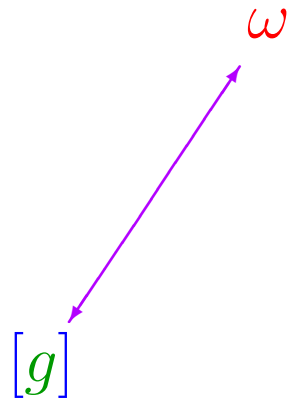
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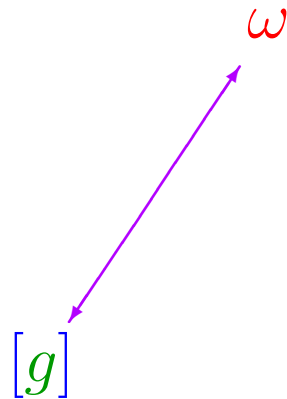


Lemma. *An oriented Riemannian 4-manifold (M, g) is almost-Kähler w/ respect to the 2-form $\omega \iff$*

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Moreover, the set of conformal classes $[g]$ on M that carry such a harmonic form ω is open in the C^2 topology.

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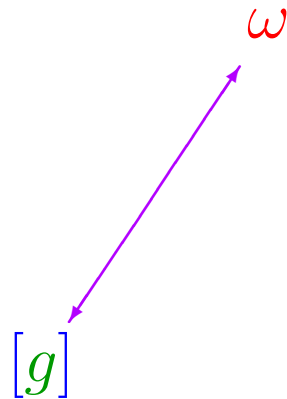
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In particular, the numbers

$$b_\pm(M) = \dim \mathcal{H}_g^\pm$$

are independent of g , and so are invariants of M .

$b_{\pm}(M)?$

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Diagonalize:

$$\begin{bmatrix} +1 & & & & & & & & \\ & \dots & & & & & & & \\ & & +1 & & & & & & \\ & & & & & & & & \\ & & & & -1 & & & & \\ & & & & & \dots & & & \\ & & & & & & & & \\ & & & & & & & -1 & \\ & & & & & & & & \end{bmatrix} \cdot$$

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 +1 & & \\
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 & & +1 \\
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 & & -1 \\
 & \left. \begin{array}{c} \\ \\ \\ \end{array} \right\} b_-(M) & \dots \\
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$$b_2(M) = b_+(M) + b_-(M)$$

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$$\tau(M) = b_+(M) - b_-(M)$$

“Signature” of M .

Signature defined in terms of intersection pairing

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$$\tau(M) = b_+(M) - b_-(M)$$

Signature of M .

Signature defined in terms of intersection pairing

$$\begin{aligned} H^2(M, \mathbb{R}) \times H^2(M, \mathbb{R}) &\longrightarrow \mathbb{R} \\ ([\varphi], [\psi]) &\longmapsto \int_M \varphi \wedge \psi \end{aligned}$$

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(Thom-Hirzebruch Signature Formula)

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Has major consequences in conformal geometry.

On oriented (M^4, g) ,

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^-$$

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Basic problems: For given smooth compact M^4 ,

- What is $\inf \mathcal{W}$?
- Do there exist minimizers?

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So Weyl functional is essentially equivalent to

$$[g] \longmapsto \int_M |W_+|^2 d\mu_g$$

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(ASD)

Twistor picture of anti-self-duality condition:

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Oriented $(M^4, g) \longleftrightarrow (Z, J)$.

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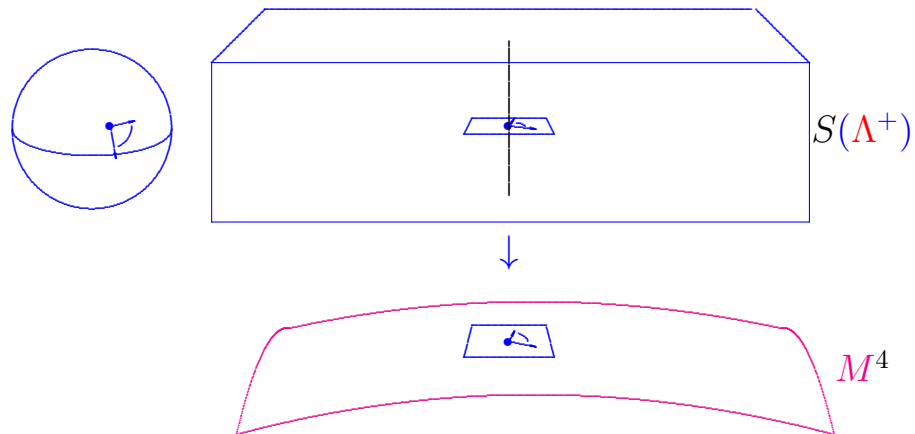
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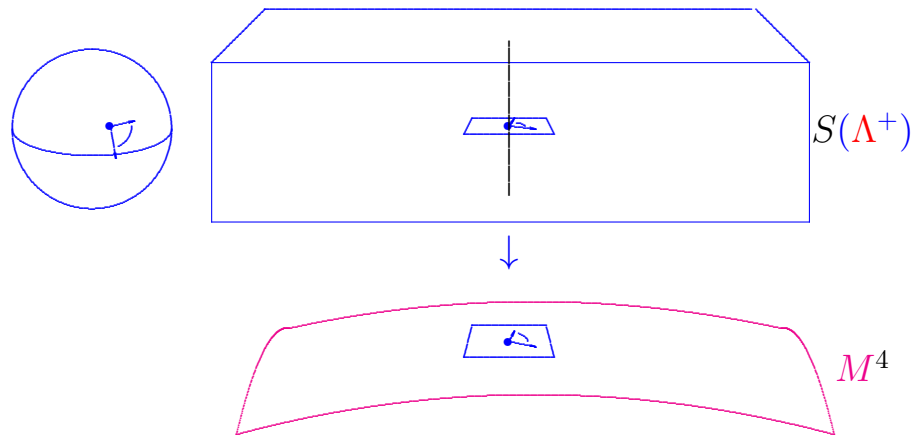
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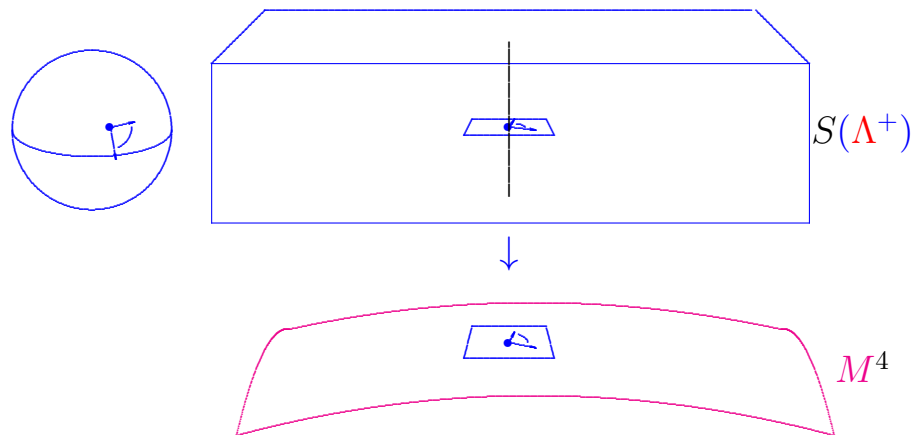


Theorem (Atiyah-Hitchin-Singer). (Z, J) is a complex 3-manifold iff $W_+ = 0$.

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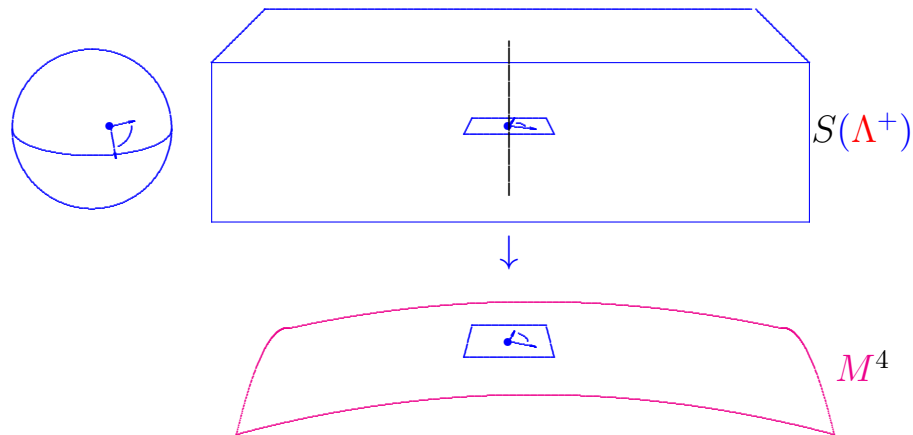
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Reconceptualizes earlier work by Penrose.

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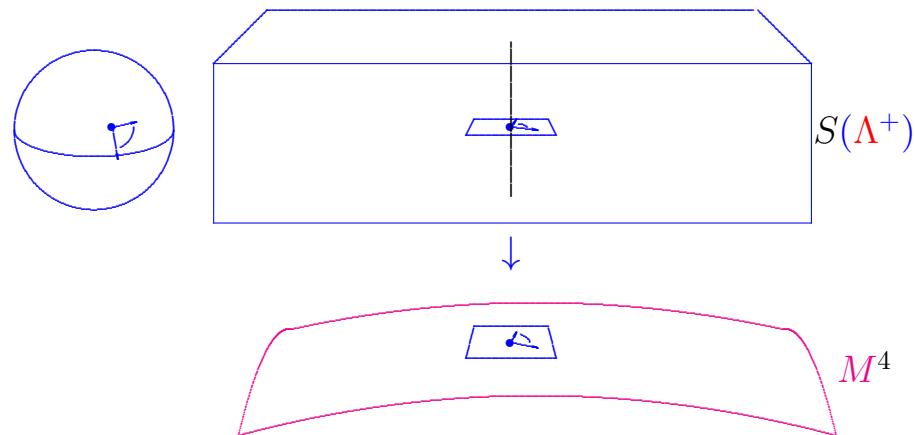


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Motivates study of ASD metrics,
and yields methods for constructing them.

So ASD metrics are linked to complex geometry. . .

A different link with complex geometry:

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If (M^4, g, J) is a Kähler surface,

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Results proved about SFK in '90s foreshadowed
many more recent results about general case.

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Scalar-flat Kähler surfaces:

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 - eight specific finite quotients of these
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Scalar-flat Kähler surfaces:

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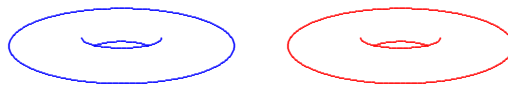
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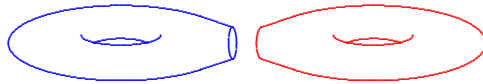
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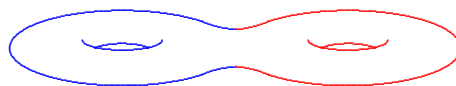
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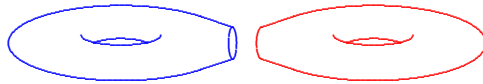
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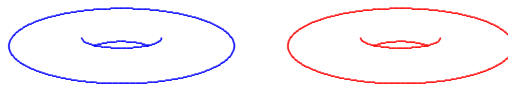
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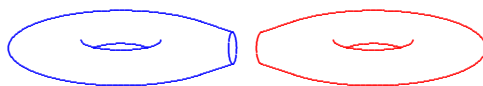
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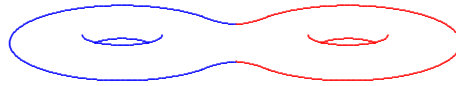
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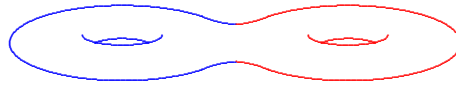
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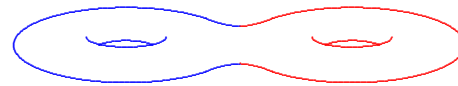
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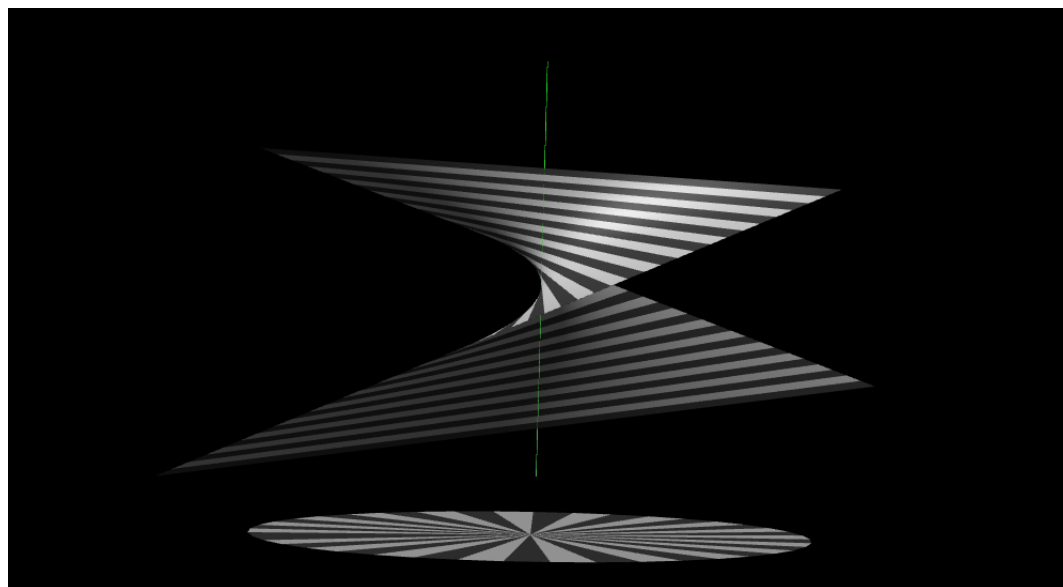
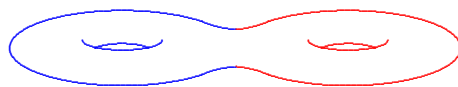


Blowing Up: $M \rightsquigarrow M \# \overline{\mathbb{C}P}_2$

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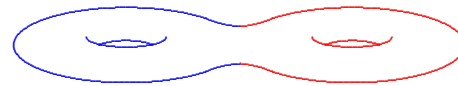
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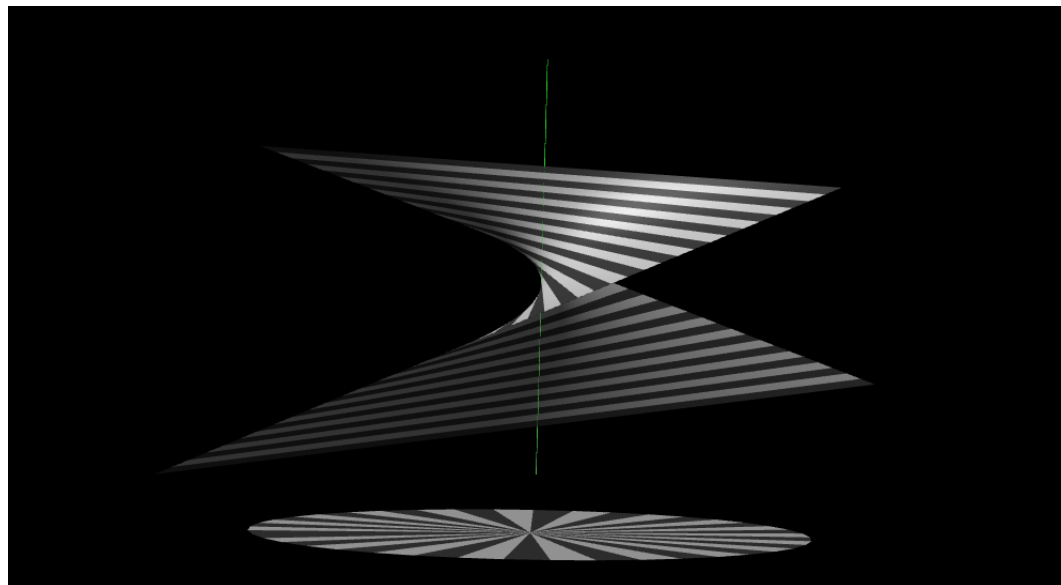
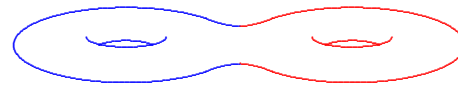


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If (M^4, g, J) is a Kähler surface, then $[g]$ is ASD
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Scalar-flat Kähler surfaces:

Classification up to diffeomorphism:

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these manifolds don't admit any ASD metrics.

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Stronger conjecture:

any metric on one of these manifolds satisfies

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Notice that the 4-manifolds $\mathbb{C}P_2 \#^k \overline{\mathbb{C}P_2}$
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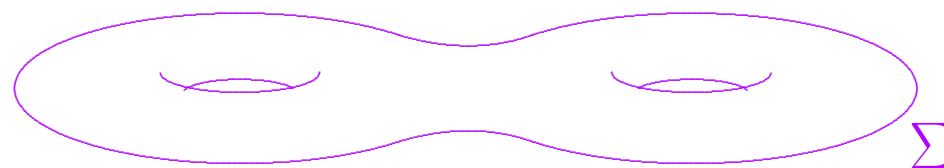
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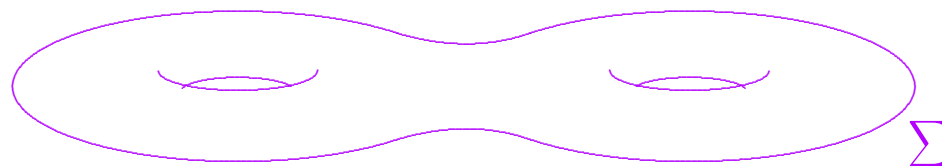
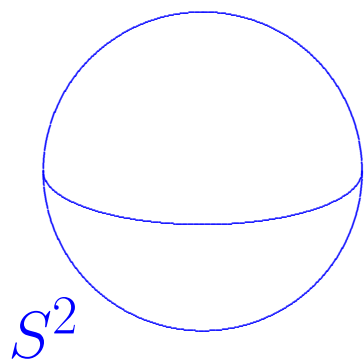
Does this say anything about general ASD metrics?

Almost-Kähler ASD metrics sweep out an open set in the ASD moduli space.

Example.

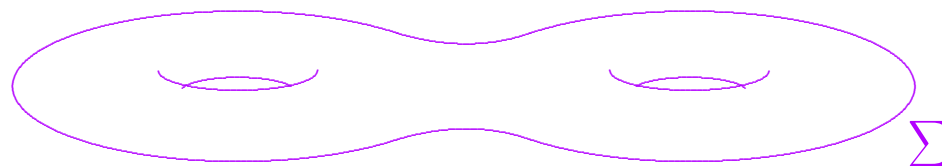
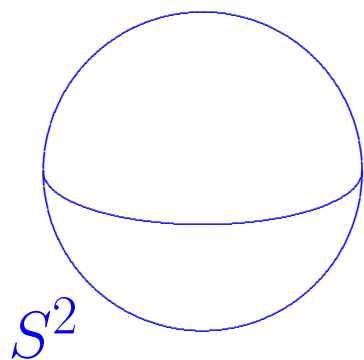


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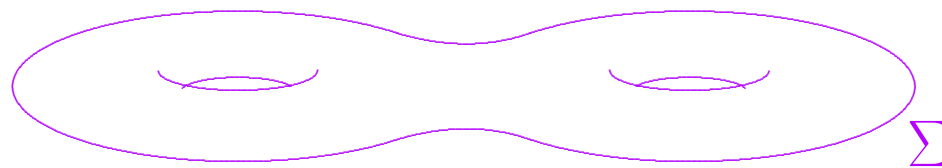
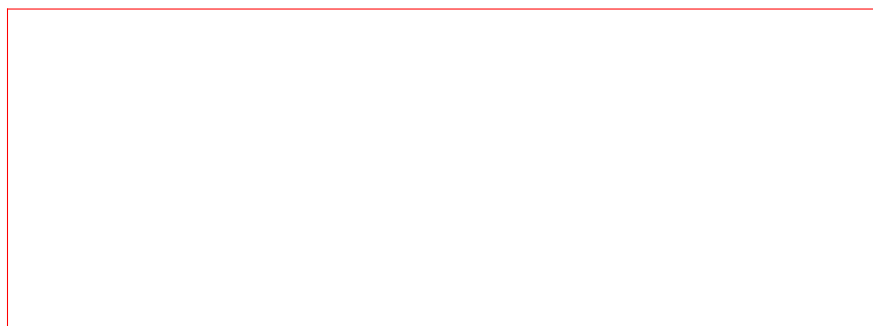
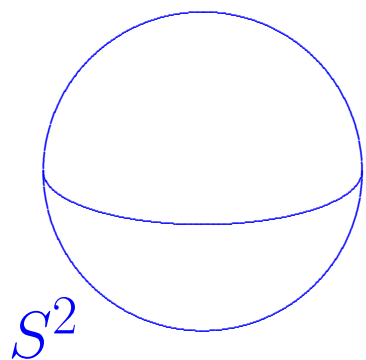
Example.

$$M = \Sigma \times S^2$$



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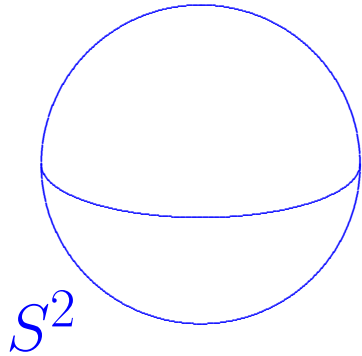
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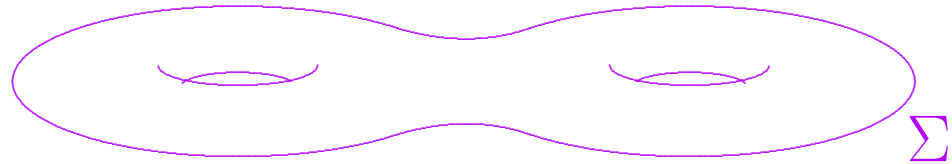
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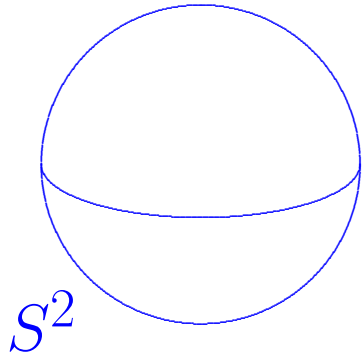


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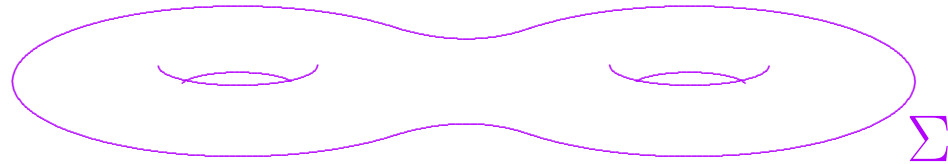
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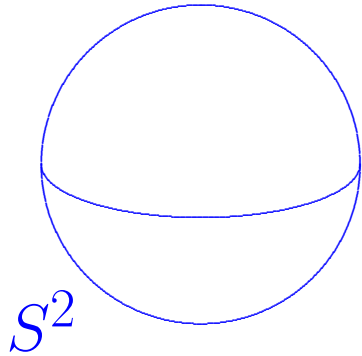
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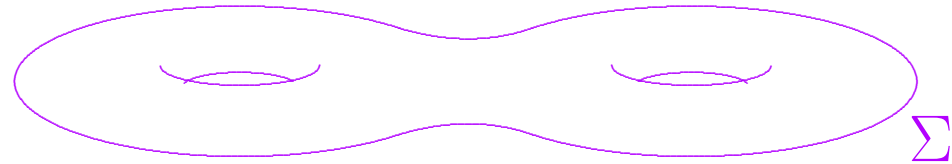
Product is scalar-flat

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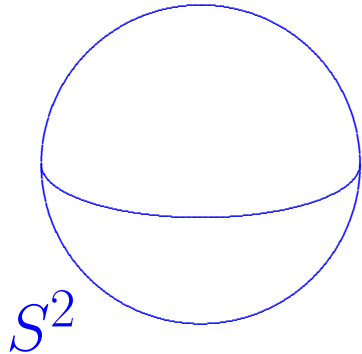


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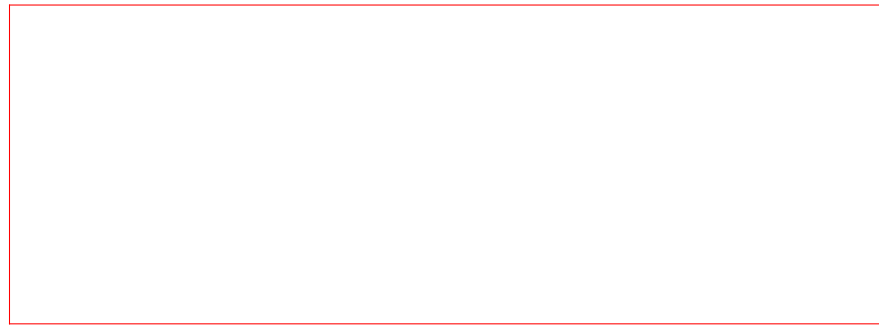
Product is scalar-flat Kähler.

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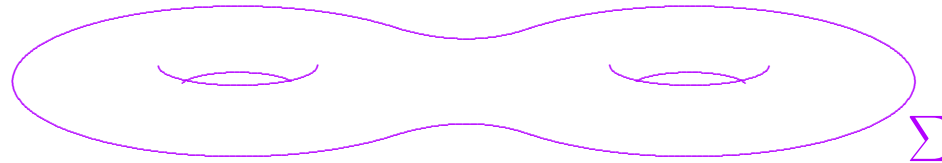
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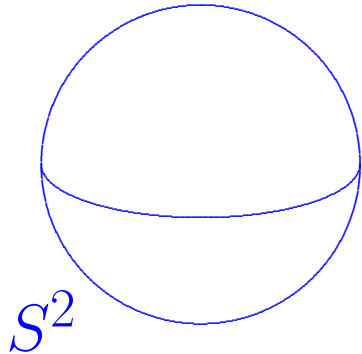
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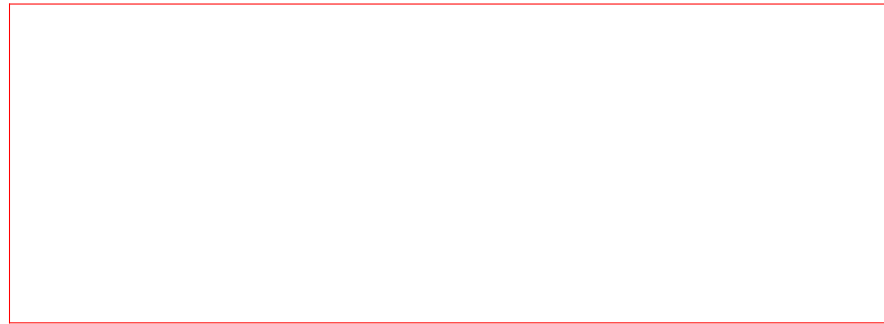
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For both orientations!

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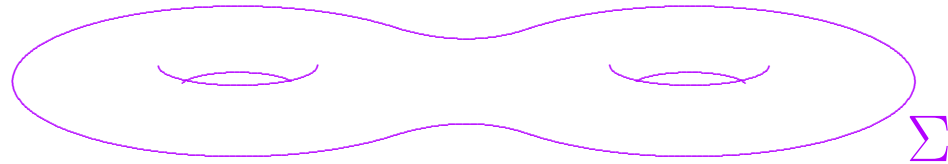
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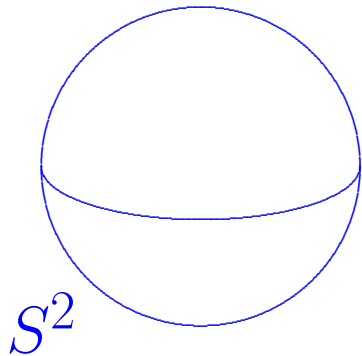
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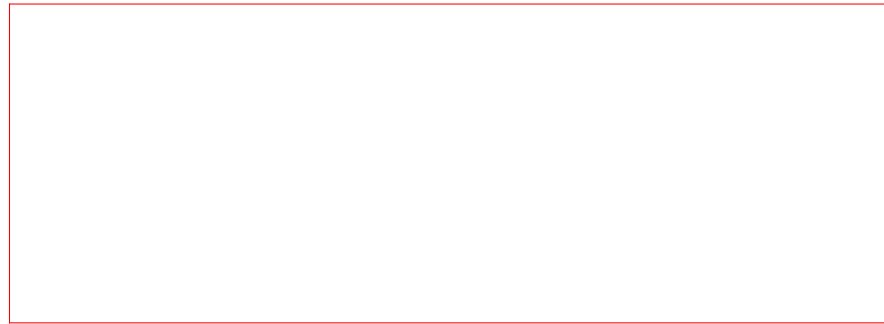
$$W_+ = 0.$$

Example.

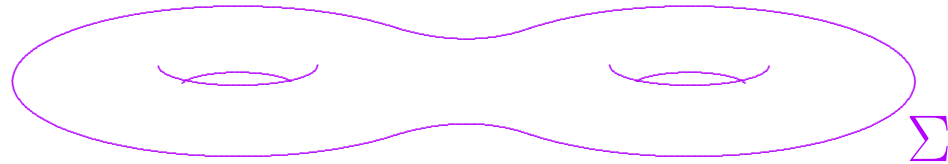
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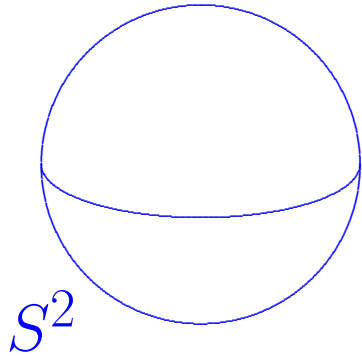
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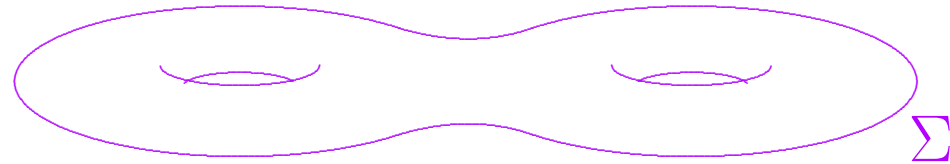
$$W_{\pm} = 0.$$

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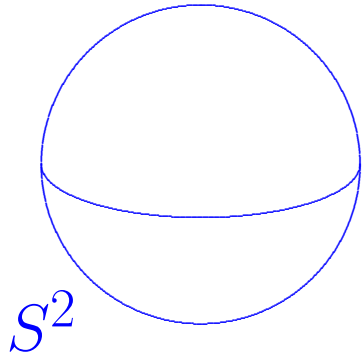
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Locally conformally flat!

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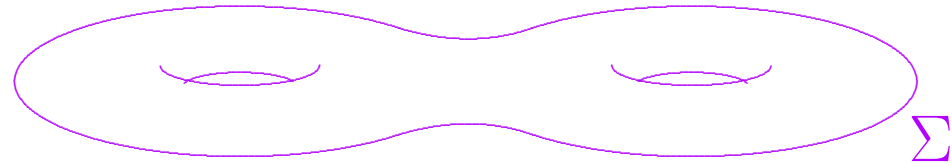
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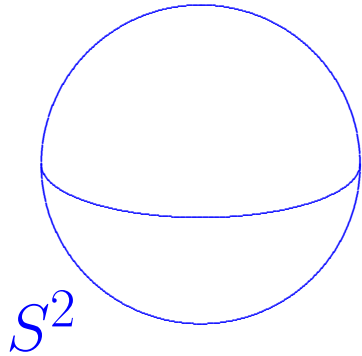
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$$\widetilde{M} = \mathcal{H}^2 \times S^2$$

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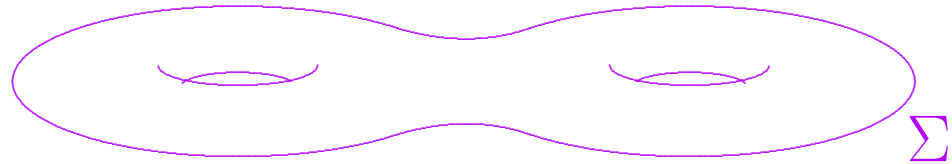
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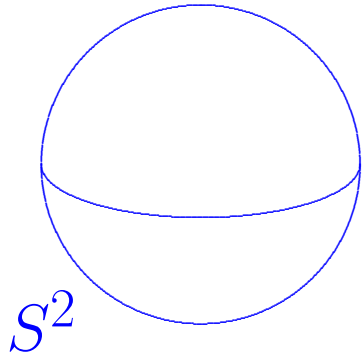
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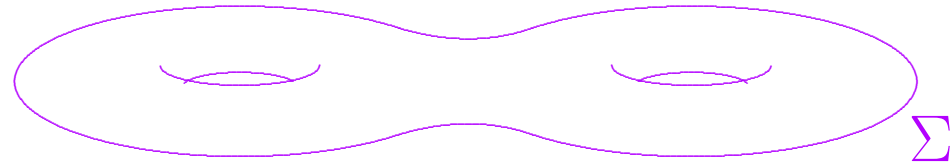
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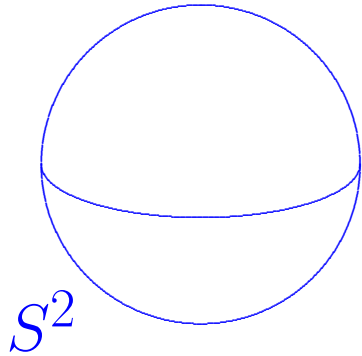


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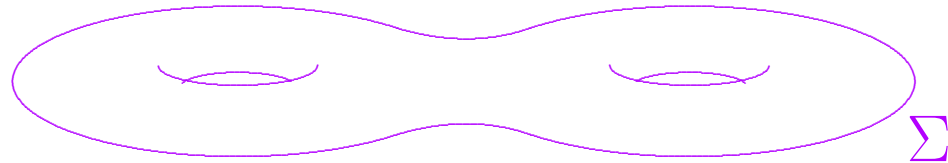
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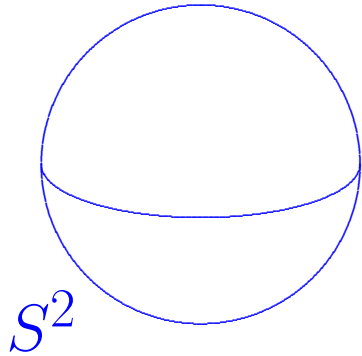


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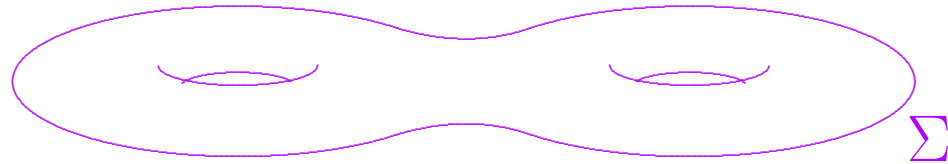
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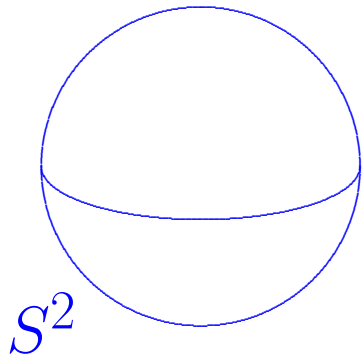


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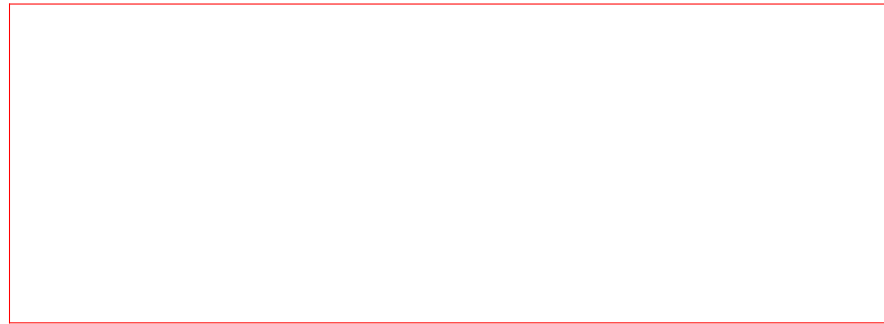
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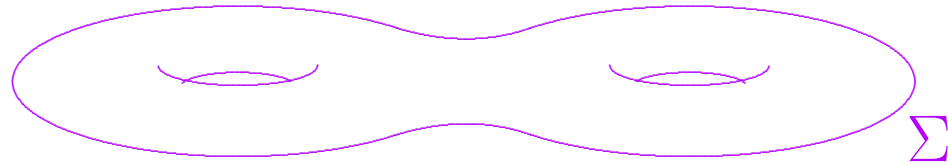
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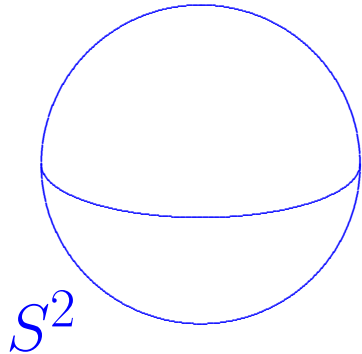
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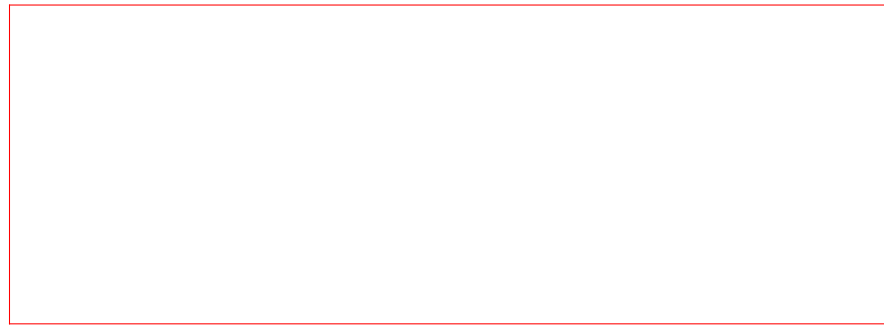
Scalar-flat Kähler deformations: $12(g - 1)$ moduli

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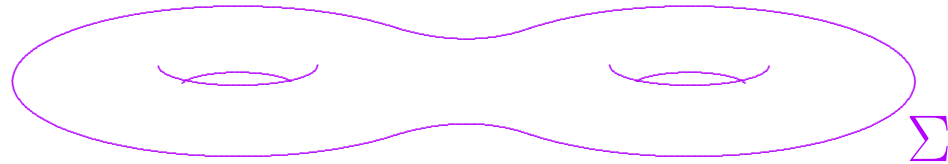
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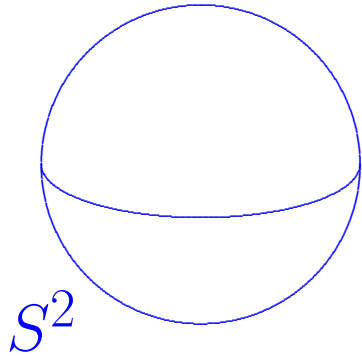
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Scalar-flat Kähler deformations: $12(g - 1)$ moduli
Locally conformally flat def'ms: $30(g - 1)$ moduli

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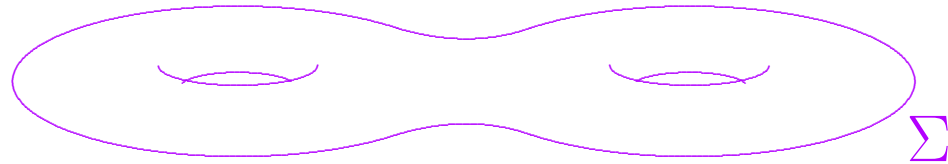
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Scalar-flat Kähler deformations: $12(g-1)$ moduli
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Inyoung Kim '16: classification of almost-Kähler ASD roughly the same as in scalar-flat Kähler case.

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Same method simultaneously proves...

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Proof hinges on a construction of hyperbolic 3-manifolds.

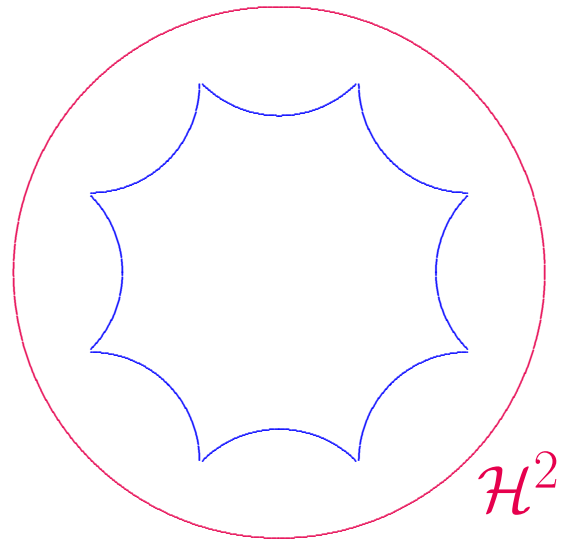
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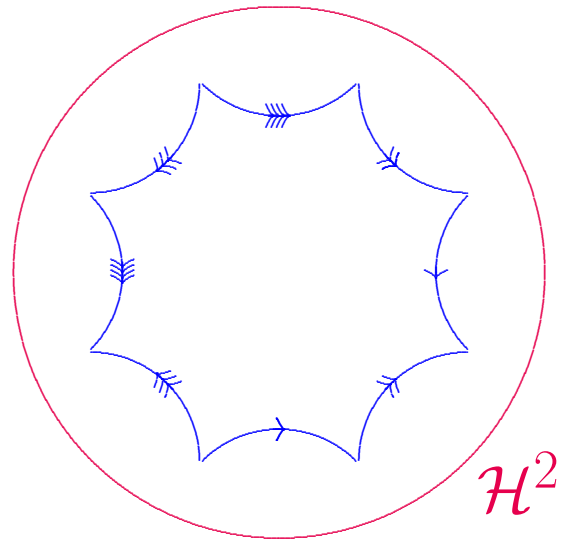
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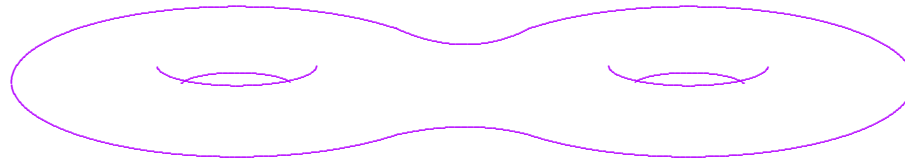
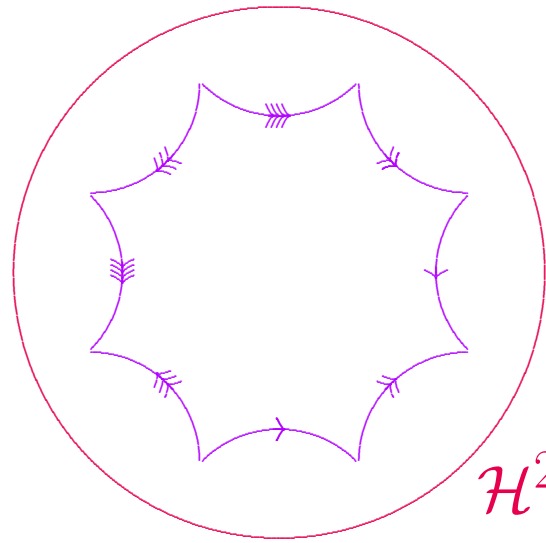
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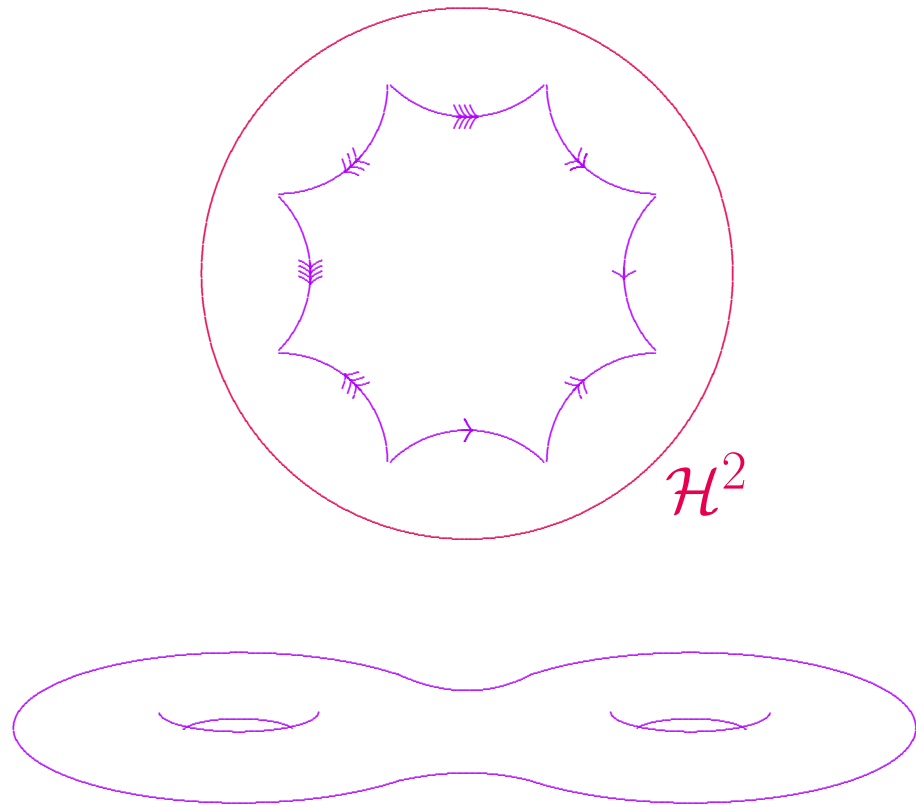
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We begin by revisiting hyperbolic metrics on Σ .

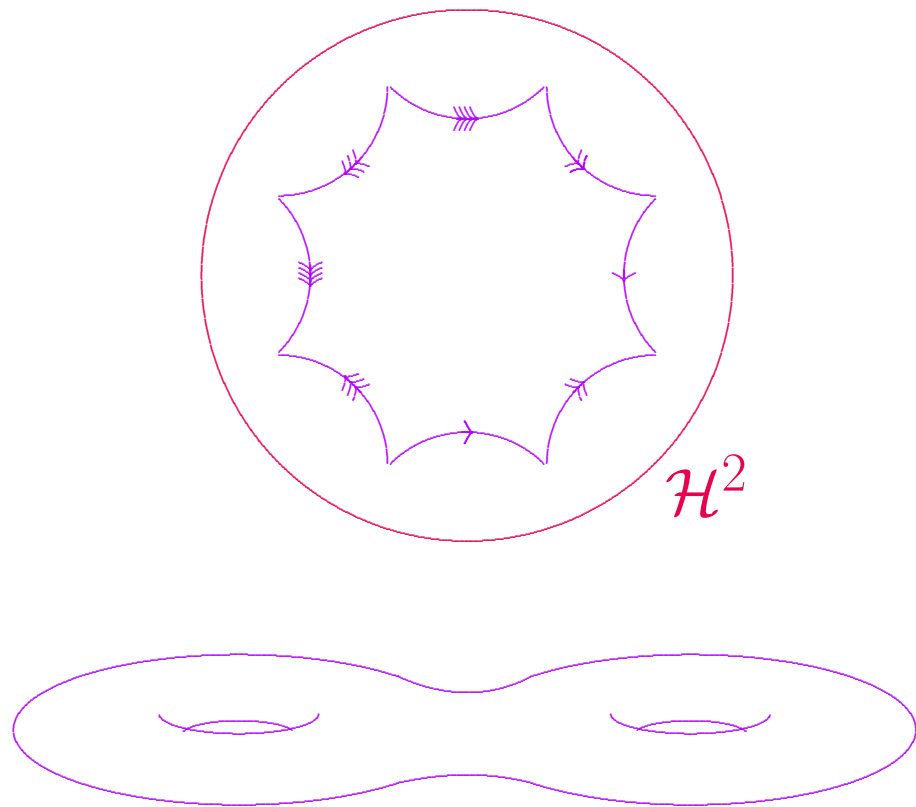




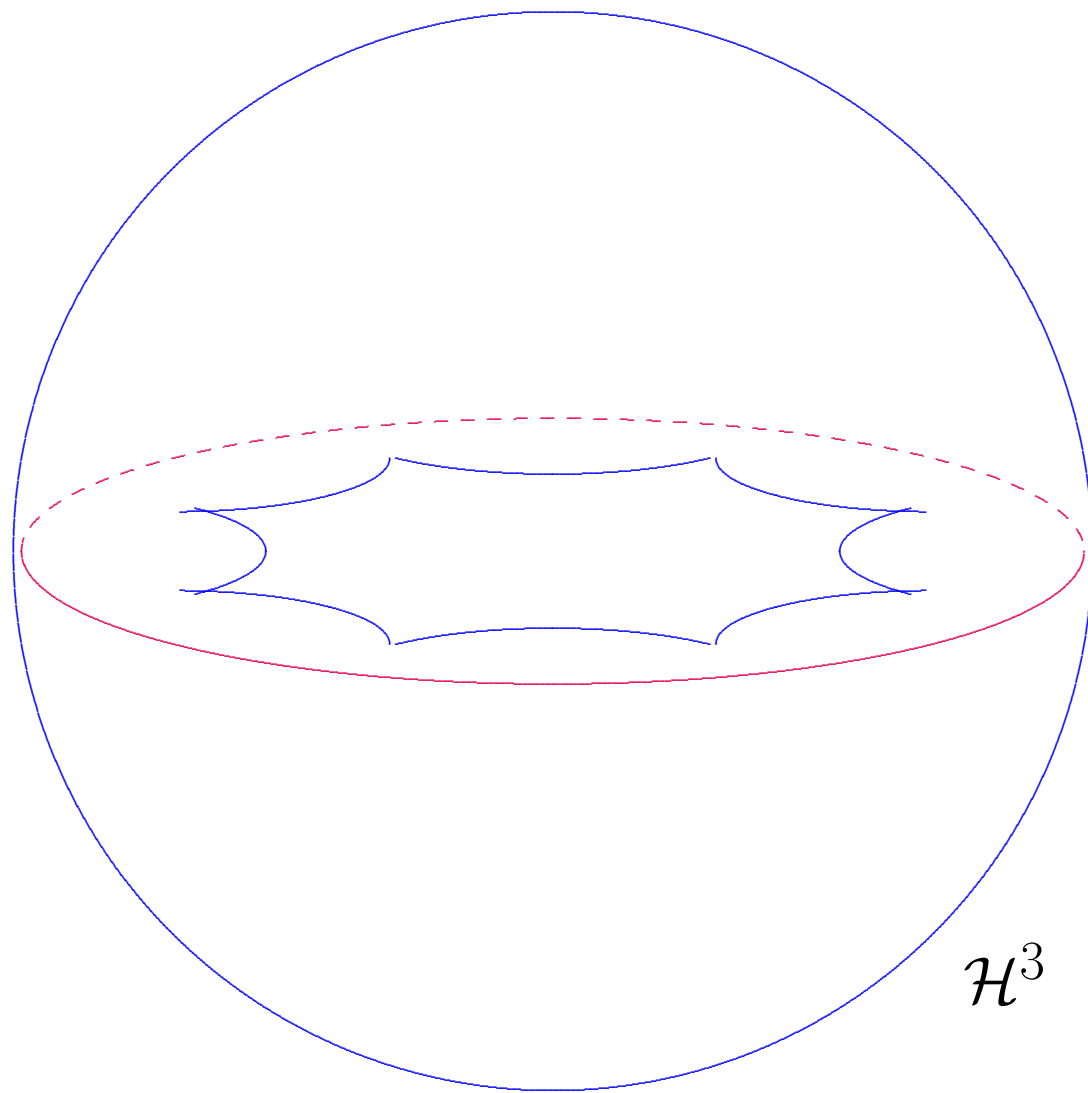




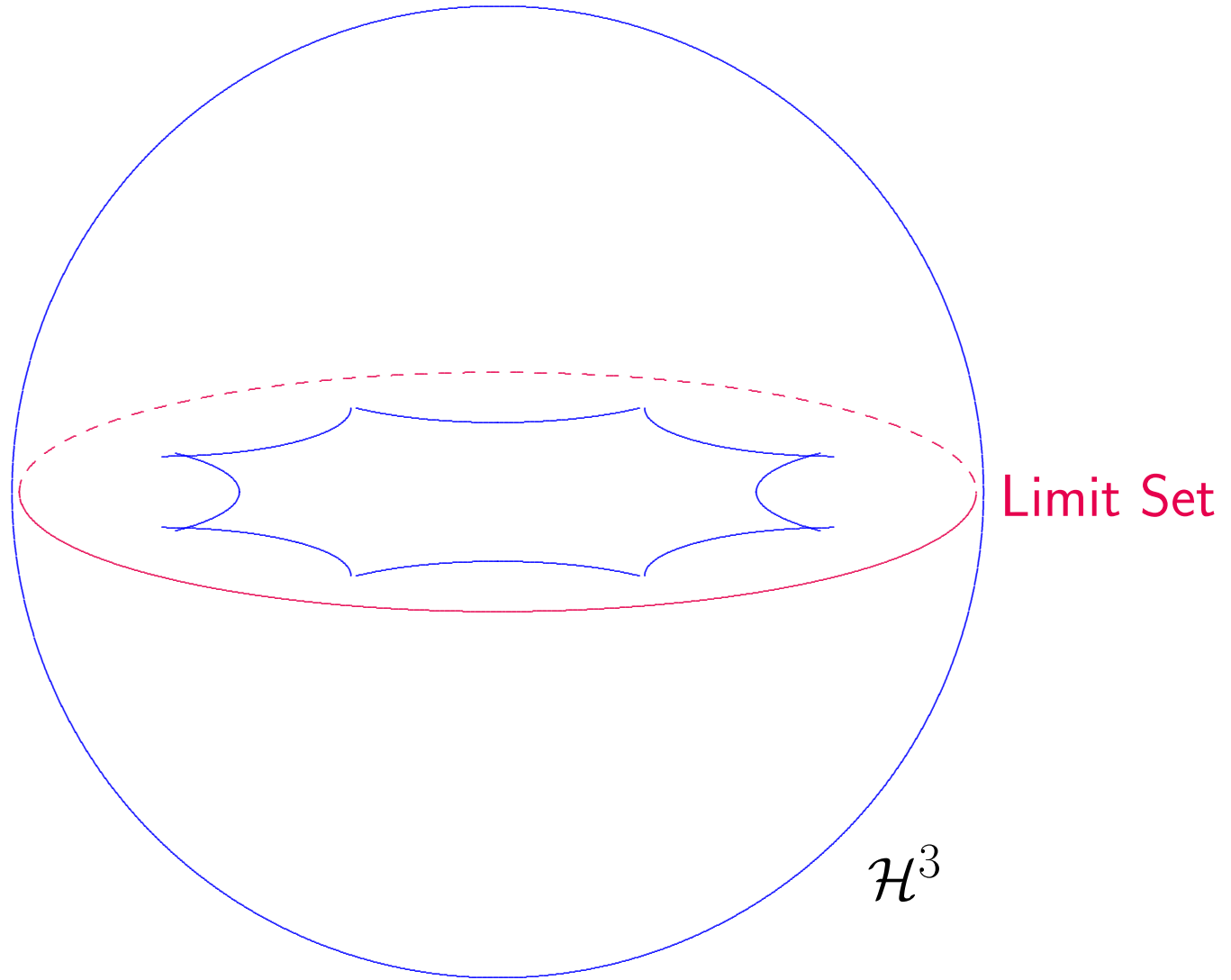
$$\pi_1(\Sigma) \hookrightarrow \mathbf{SO}_+(1, 2) = \mathbf{PSL}(2, \mathbb{R})$$

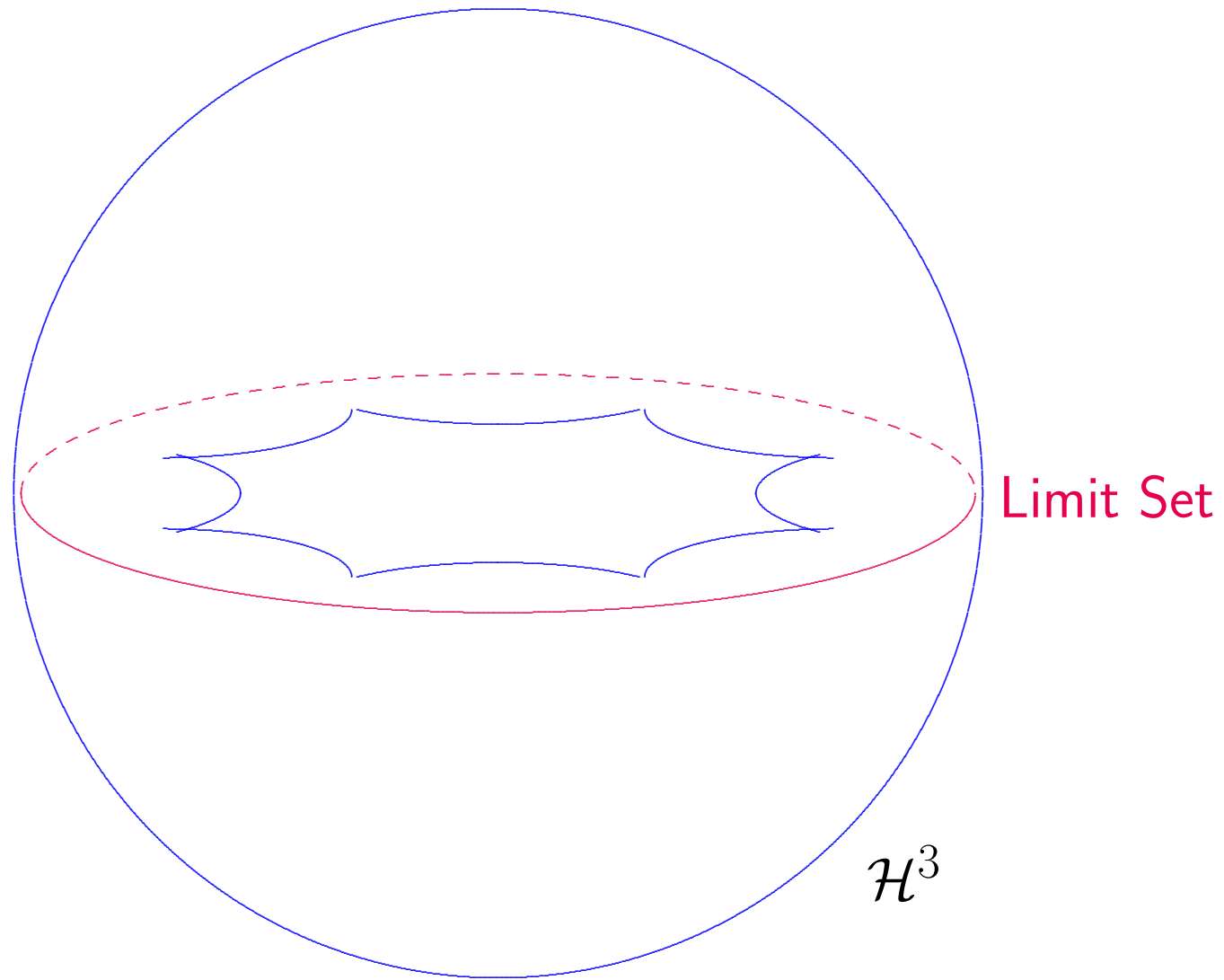


$$\begin{array}{ccc}
 \pi_1(\Sigma) & \hookrightarrow & \mathbf{SO}_+(1, 2) = \mathbf{PSL}(2, \mathbb{R}) \\
 & & \cap \qquad \qquad \cap \\
 & & \mathbf{SO}_+(1, 3) = \mathbf{PSL}(2, \mathbb{C})
 \end{array}$$

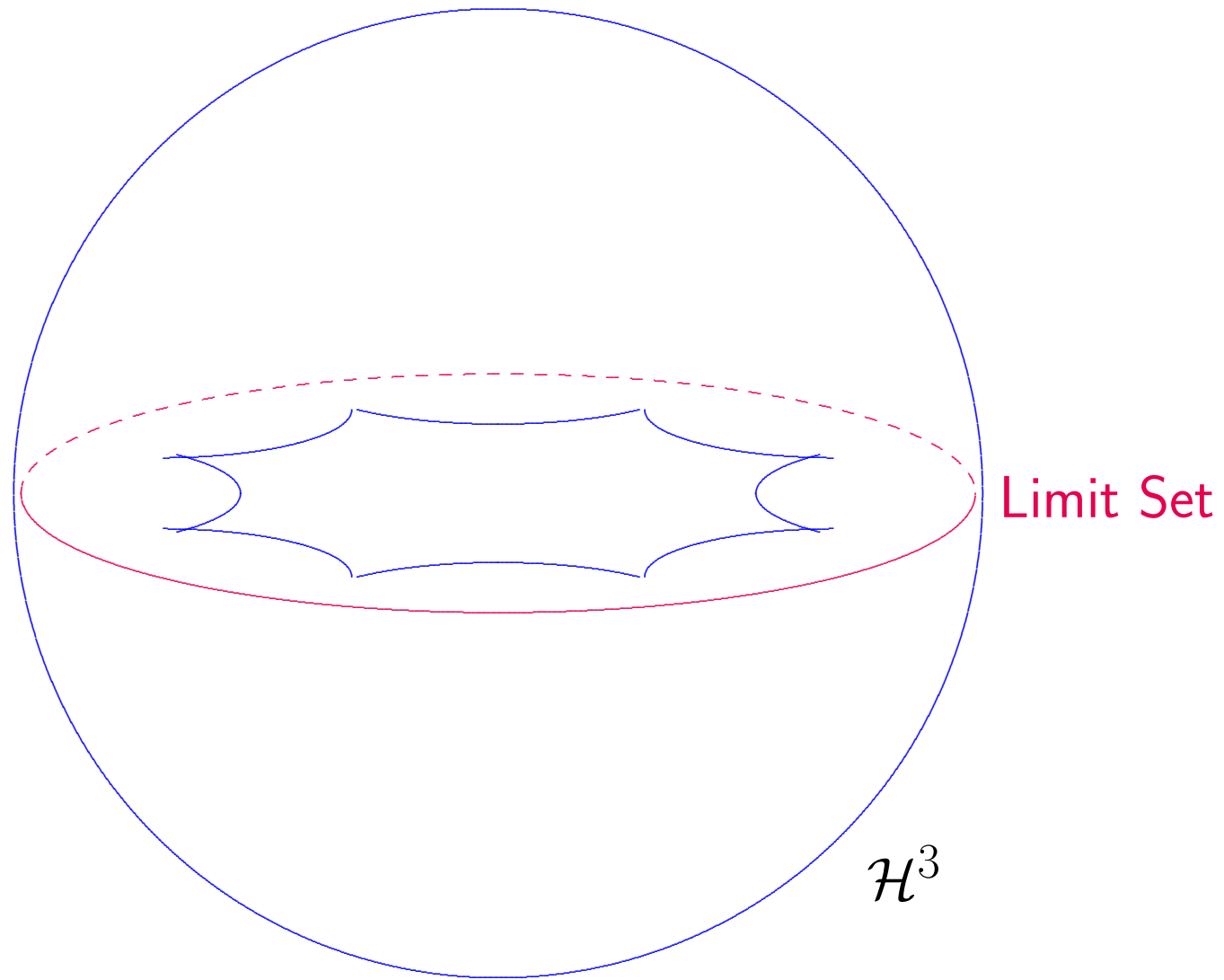


\mathcal{H}^3

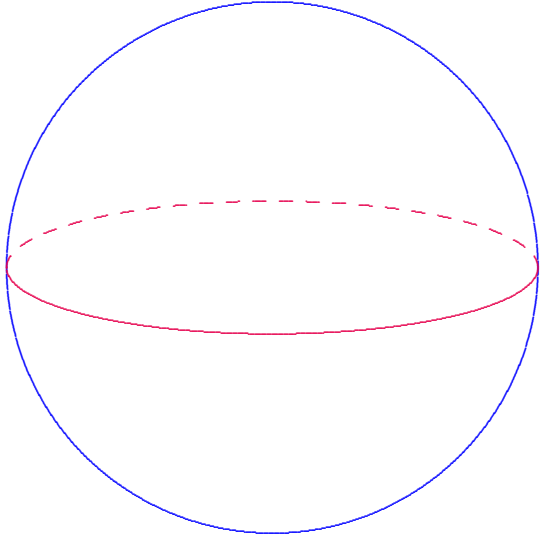


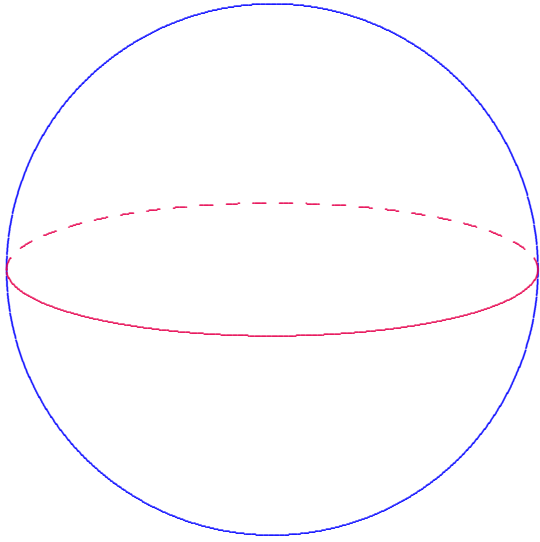


$$\pi_1(\Sigma) \xrightarrow{\cong} \Gamma \subset \mathbf{PSL}(2, \mathbb{R}) \text{ Fuchsian group}$$

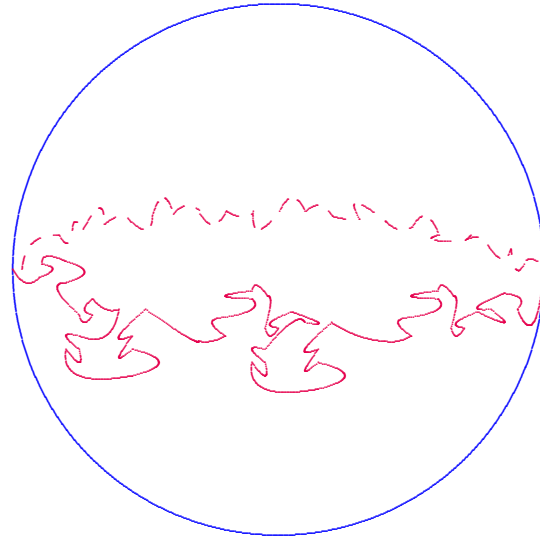
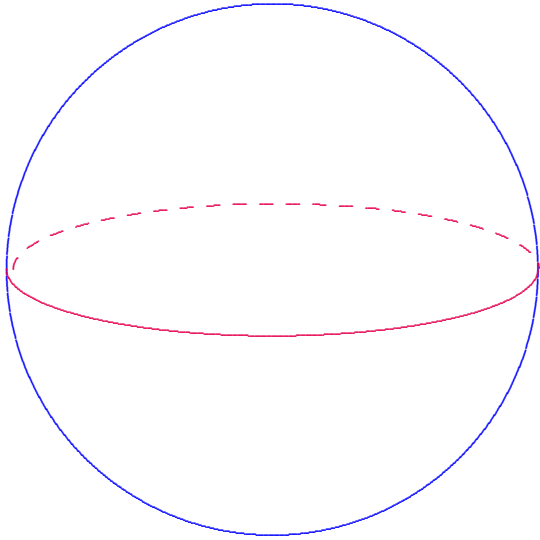


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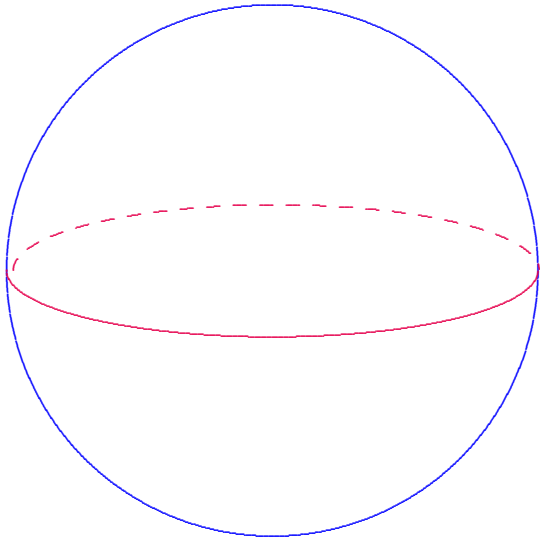




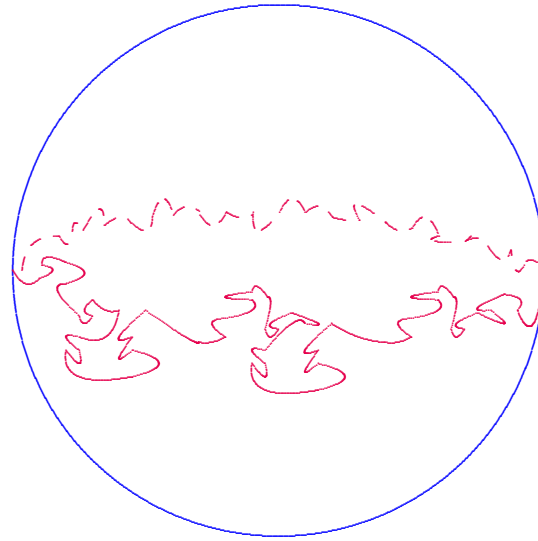
Fuchsian



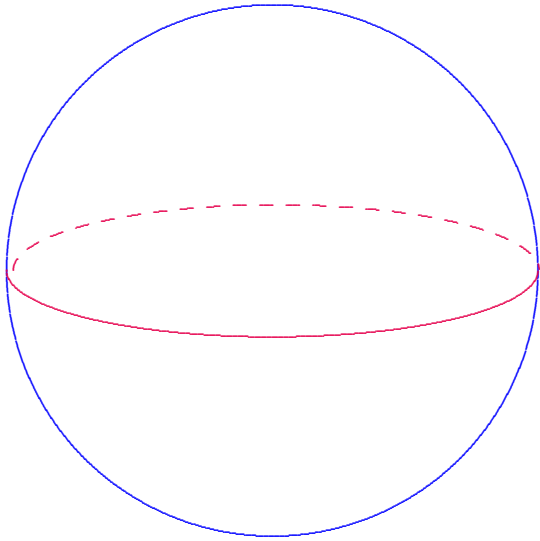
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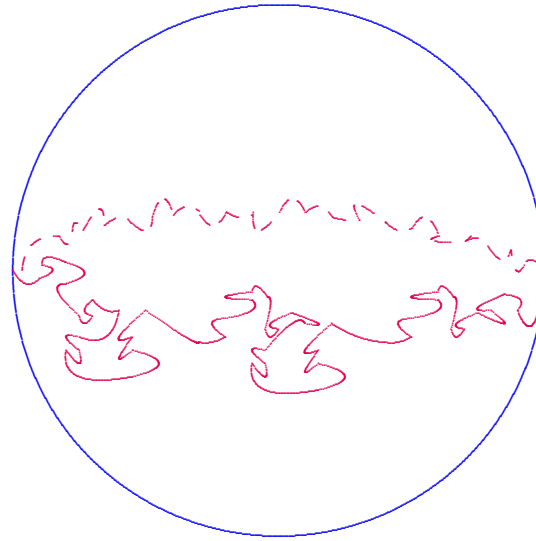
Fuchsian



quasi-Fuchsian

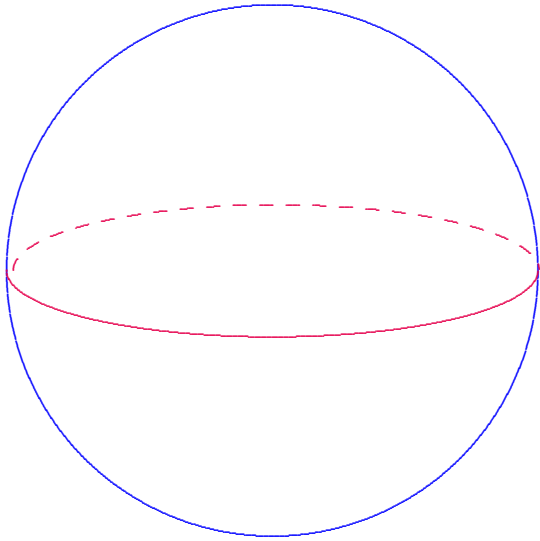


Fuchsian

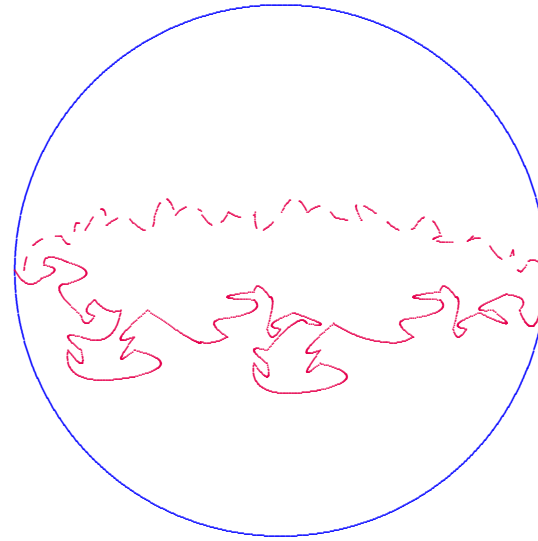


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$$\pi_1(\Sigma) \xrightarrow{\cong} \Gamma \subset \mathbf{PSL}(2, \mathbb{C}) \text{ quasi-Fuchsian group}$$

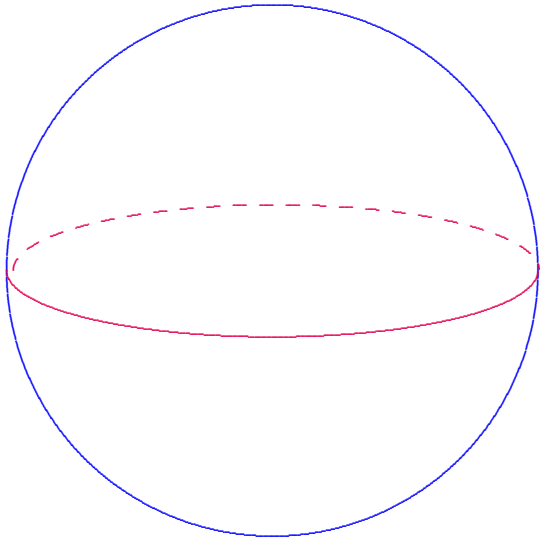


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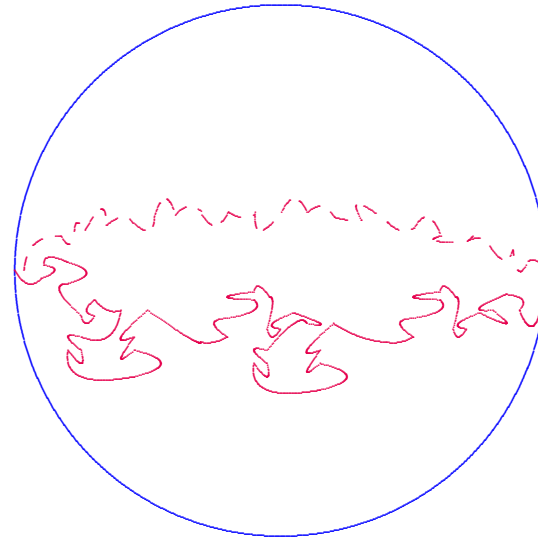


quasi-Fuchsian

$$\pi_1(\Sigma) \xrightarrow{\cong} \Gamma \subset \mathbf{PSL}(2, \mathbb{C}) \text{ quasi-Fuchsian group of Bers type}$$



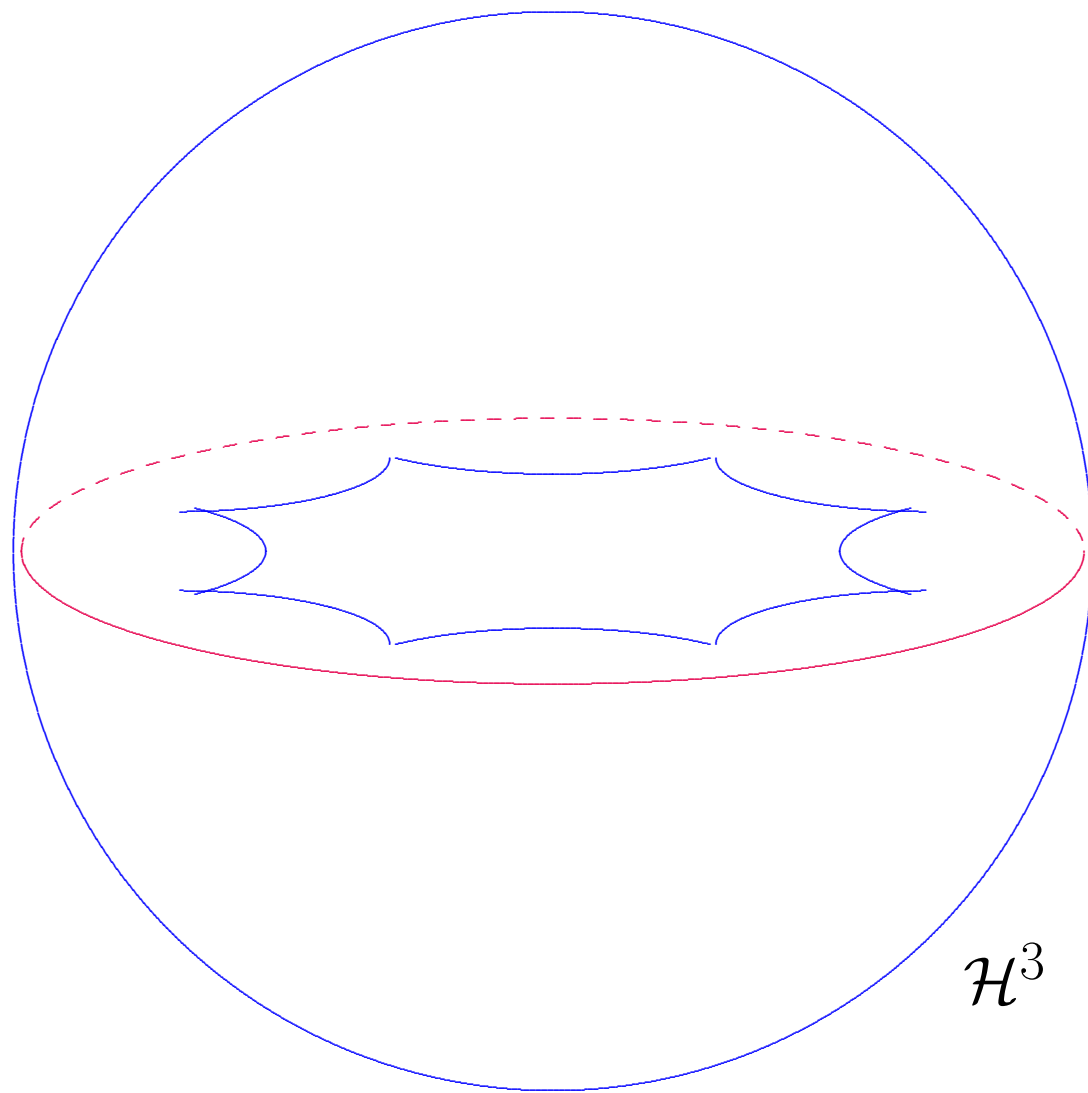
Fuchsian



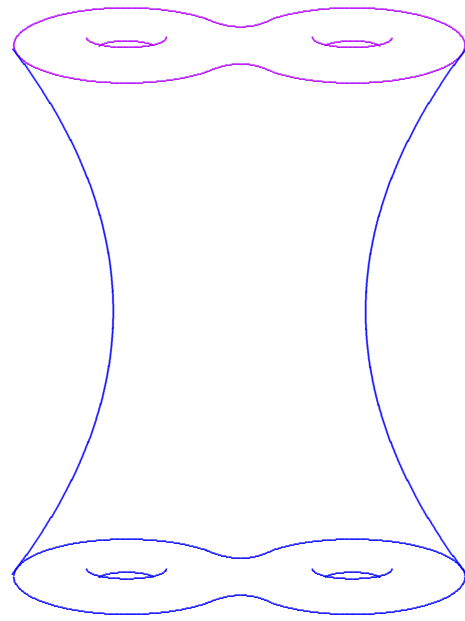
quasi-Fuchsian

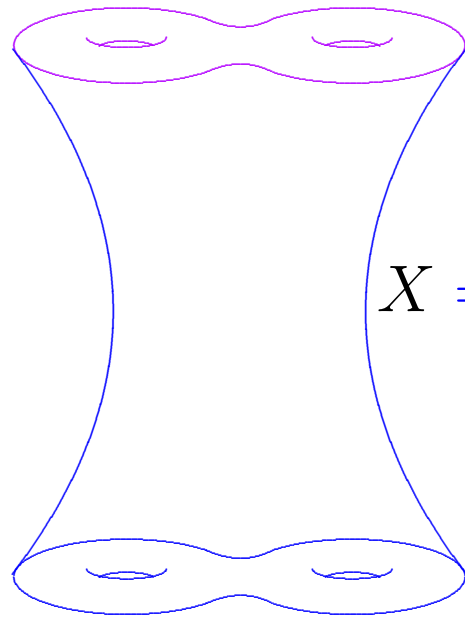
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Quasi-conformally conjugate to Fuchsian.

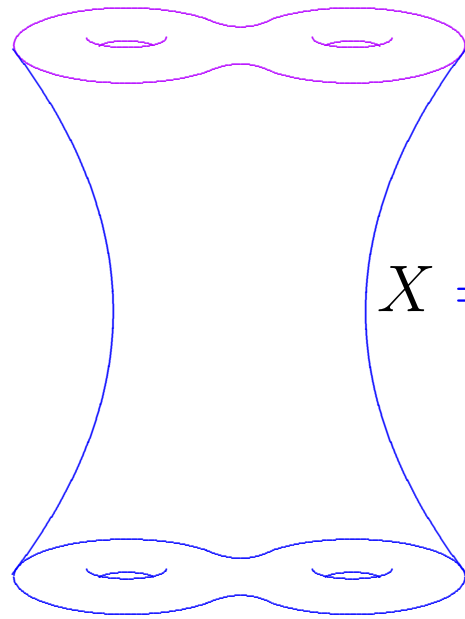


\mathcal{H}^3



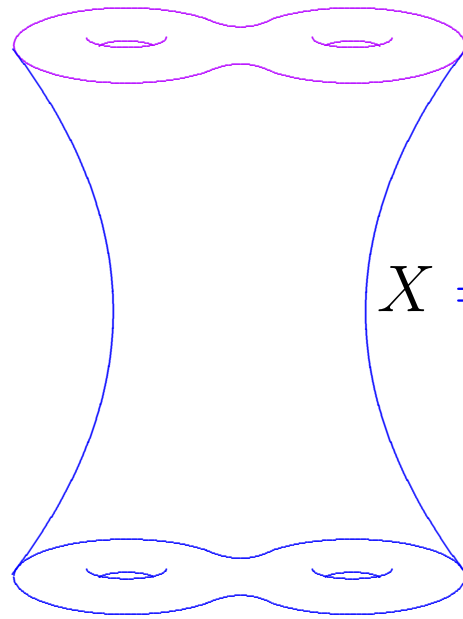


$$X = \mathcal{H}^3 / \Gamma$$



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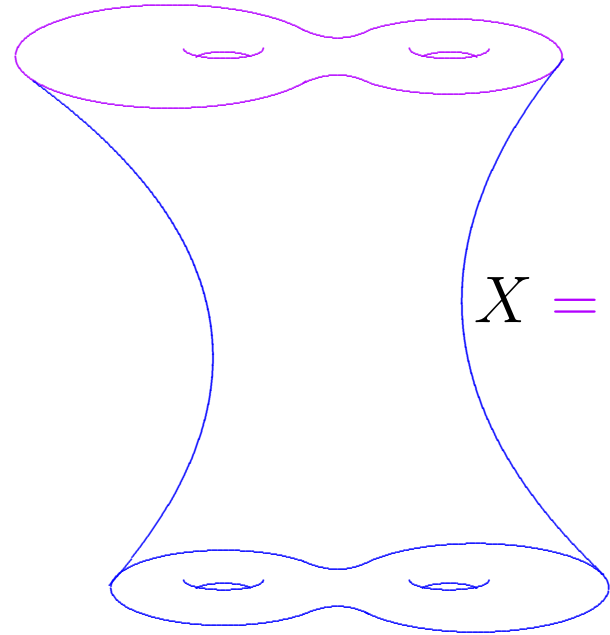
Γ Fuchsian



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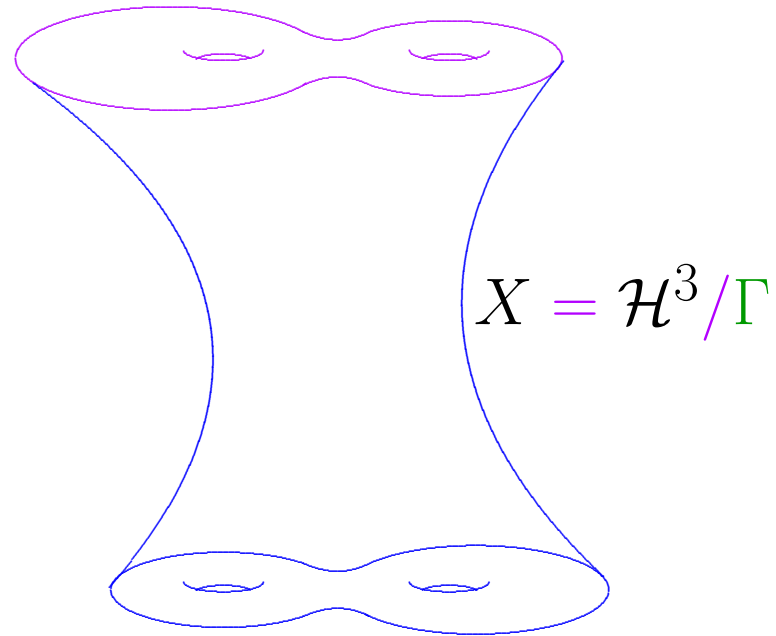
$$X \approx \Sigma \times \mathbb{R}$$



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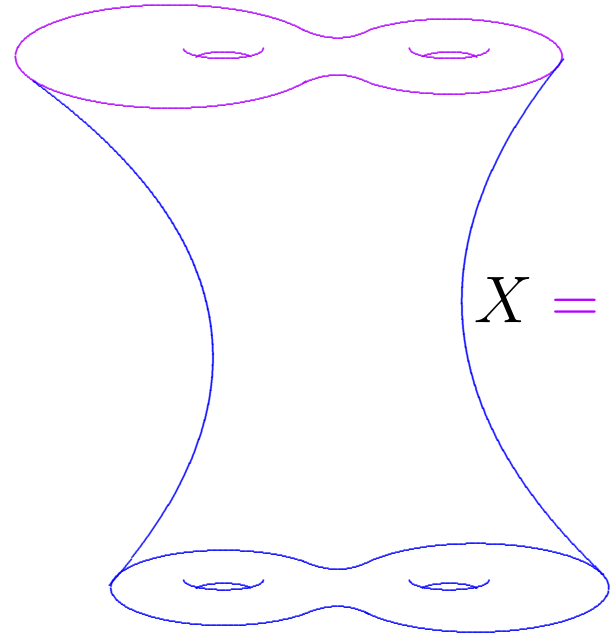
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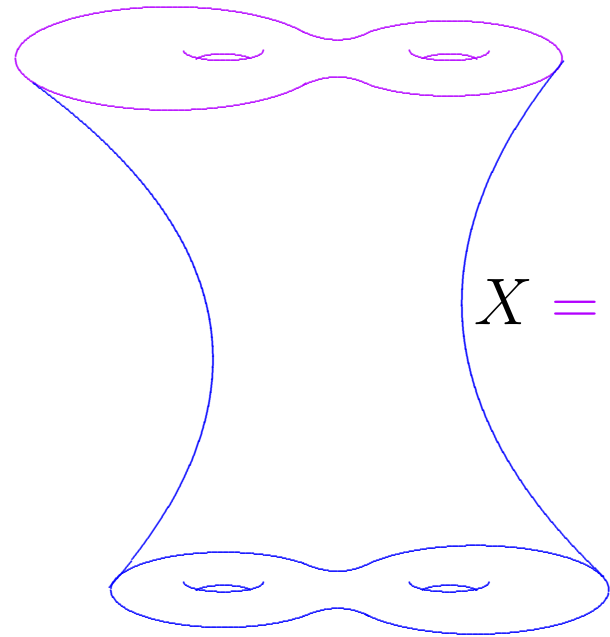
Freedom: two points in Teichmüller space.



$$X = \mathcal{H}^3 / \Gamma$$

Γ quasi-Fuchsian

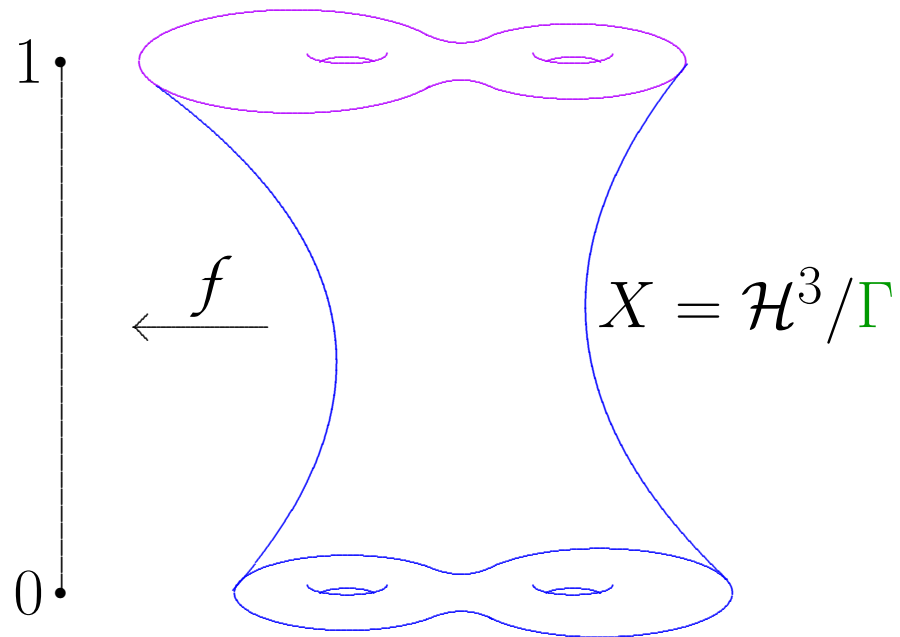
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Γ quasi-Fuchsian

$$\overline{X} \approx \Sigma \times [0, 1]$$

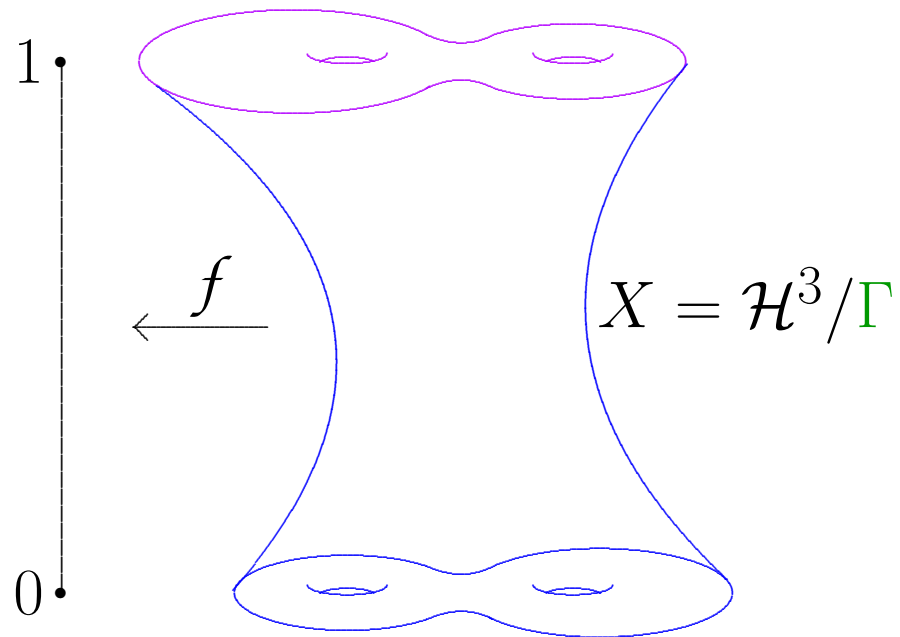


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Tunnel-Vision function:

$$f : \overline{X} \rightarrow [0, 1]$$



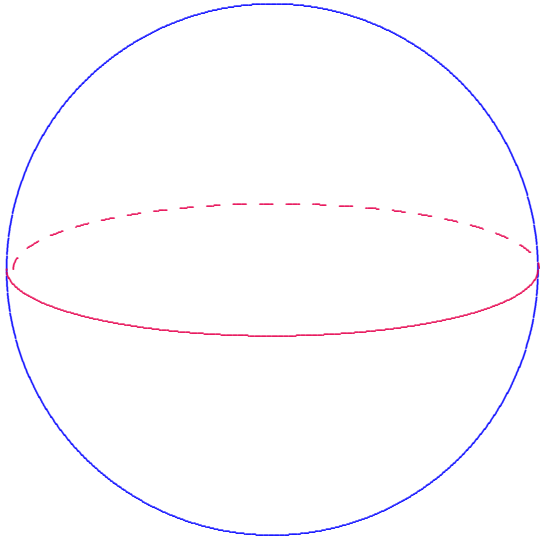
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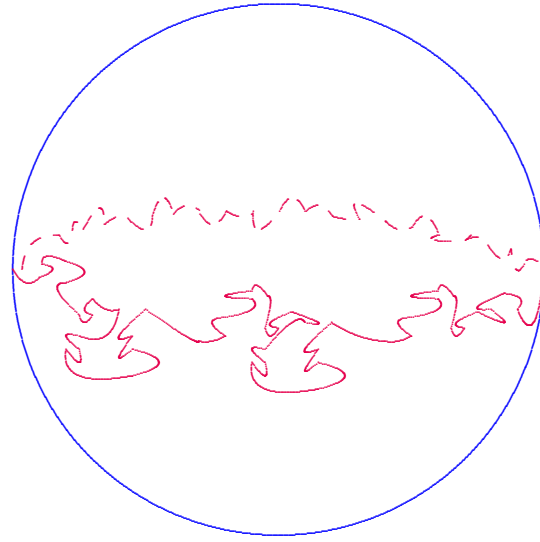
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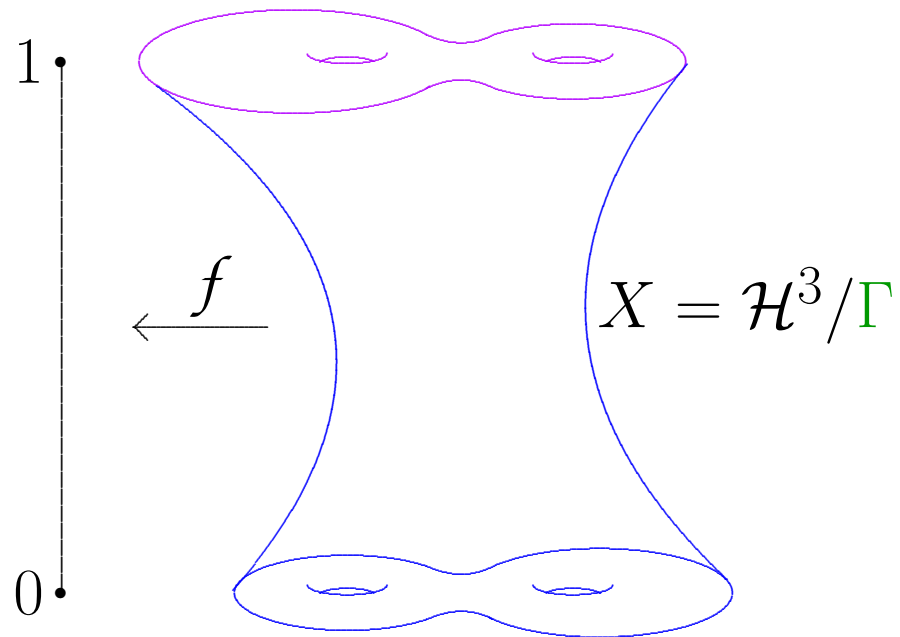
$$\Delta f = 0$$



Fuchsian



quasi-Fuchsian



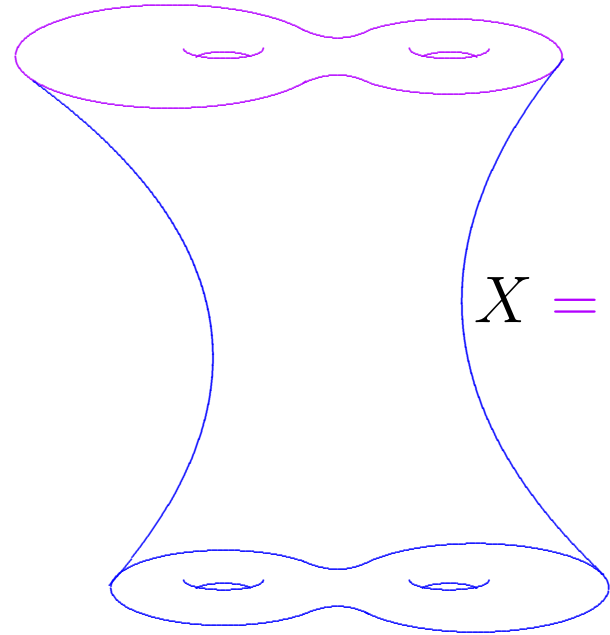
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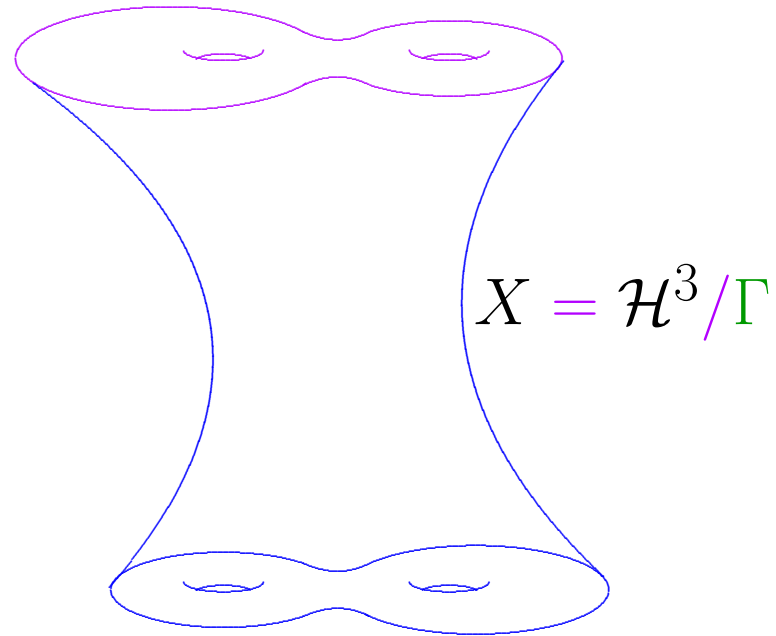
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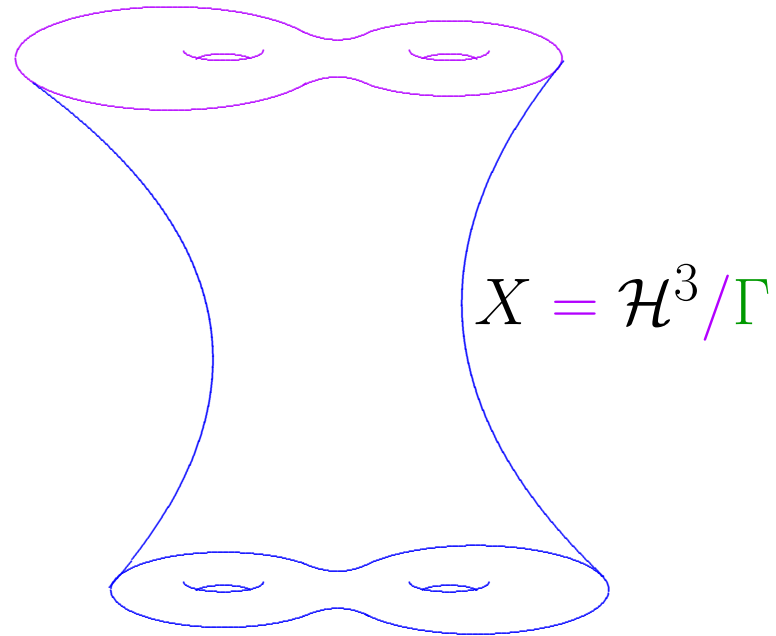
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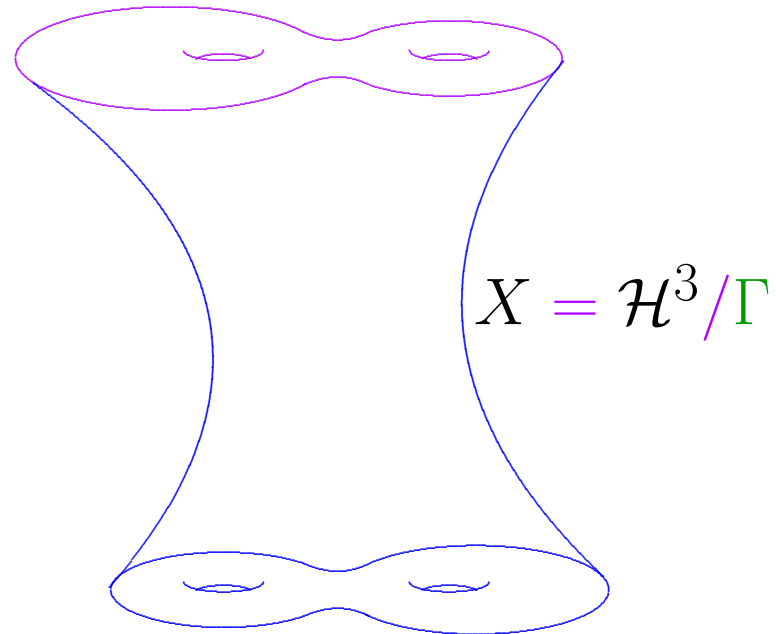


Construction of conformally flat 4-manifolds:



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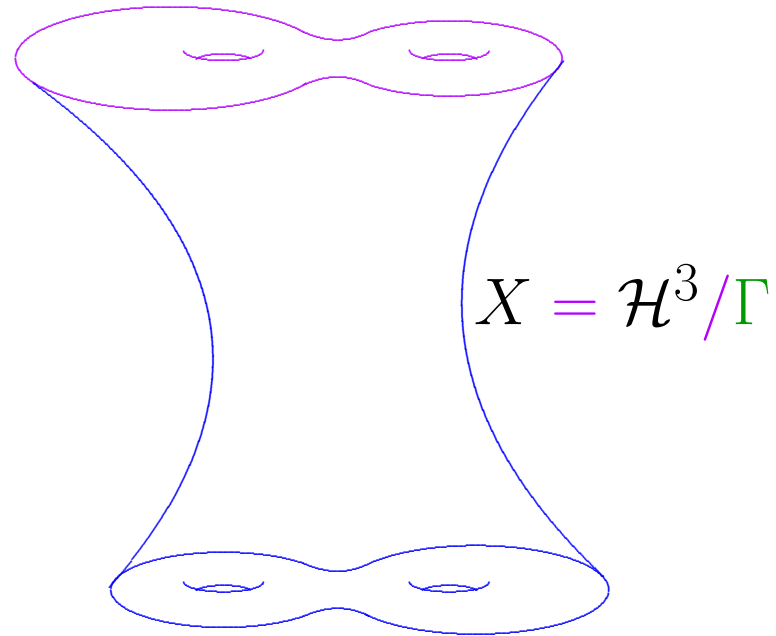
$$M = [\bar{X} \times S^1] / \sim$$



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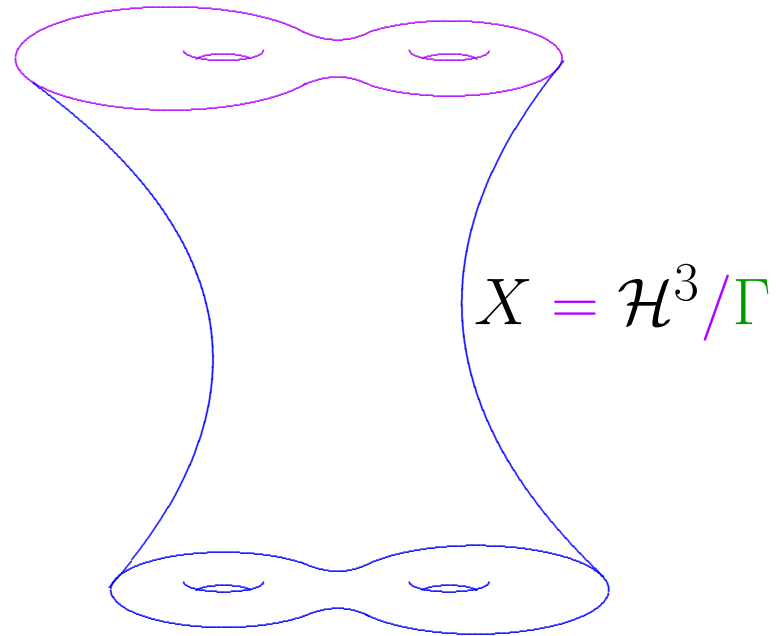
$$M = [\bar{X} \times S^1] / \sim$$

\sim : crush $\partial\bar{X} \times S^1$ to $\partial\bar{X}$.



Construction of conformally flat 4-manifolds:

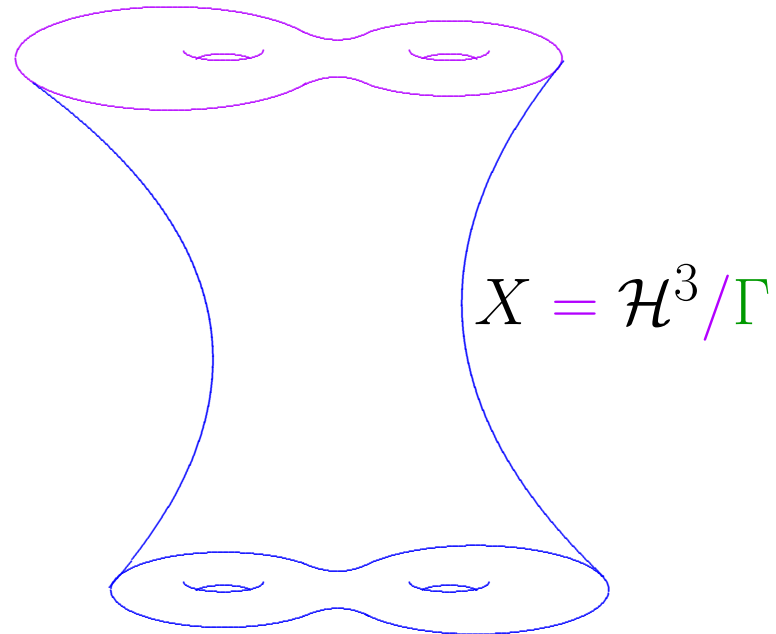
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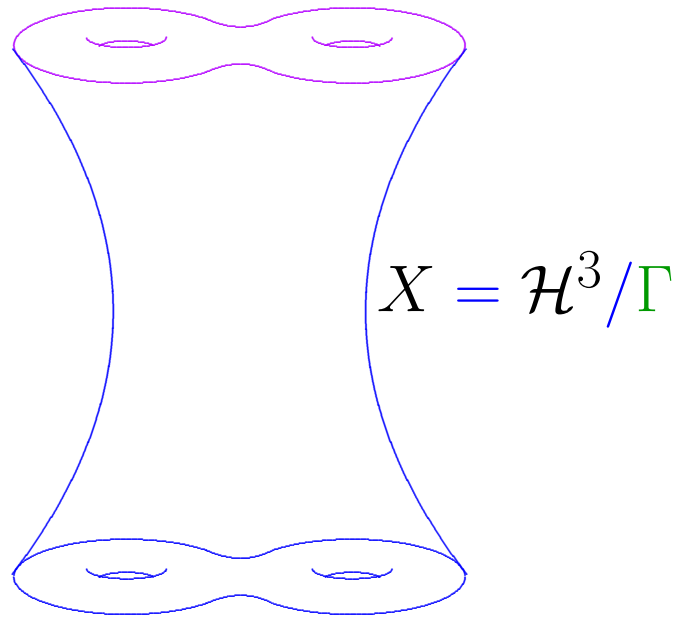
$$g = h + dt^2$$



Construction of conformally flat 4-manifolds:

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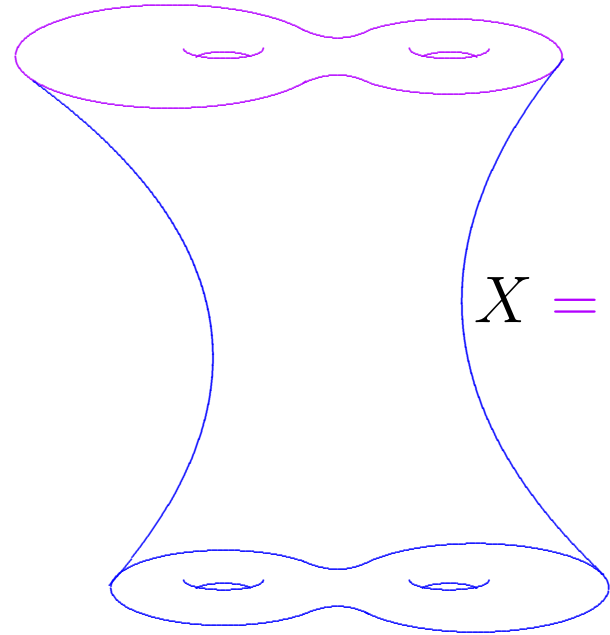


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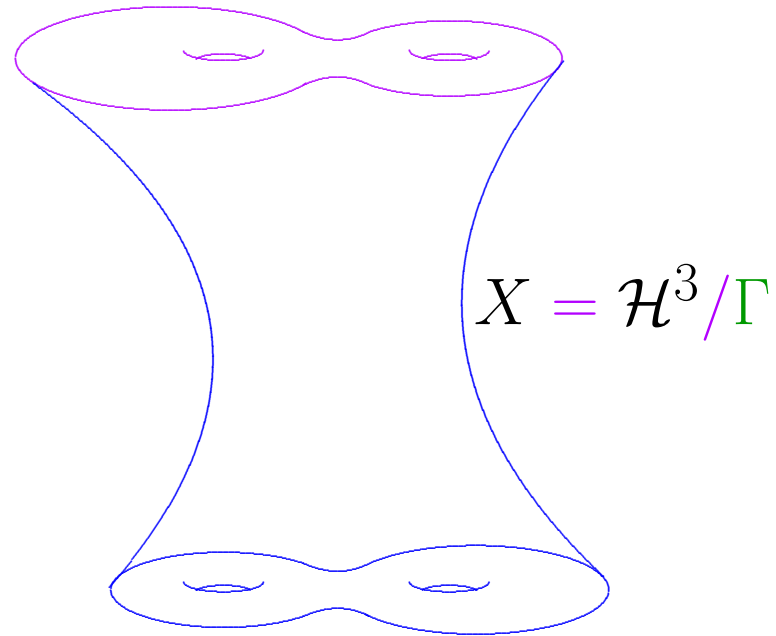
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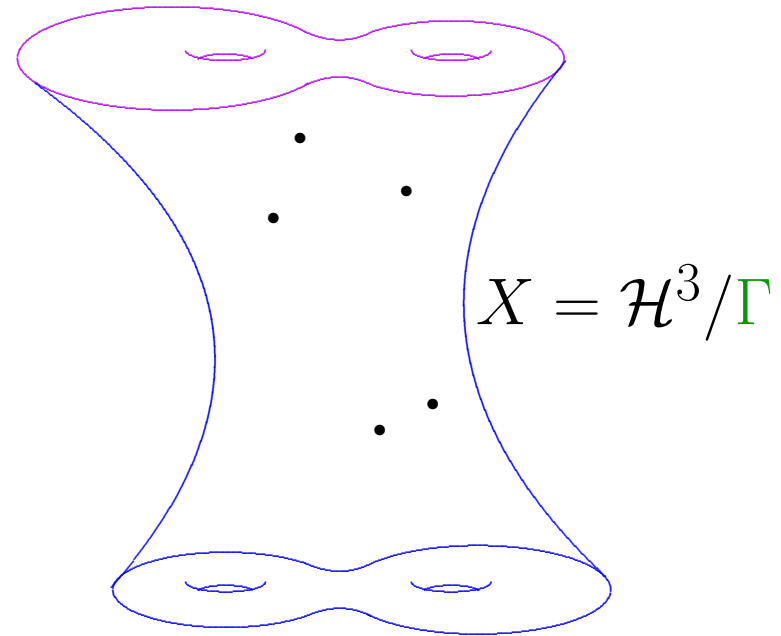
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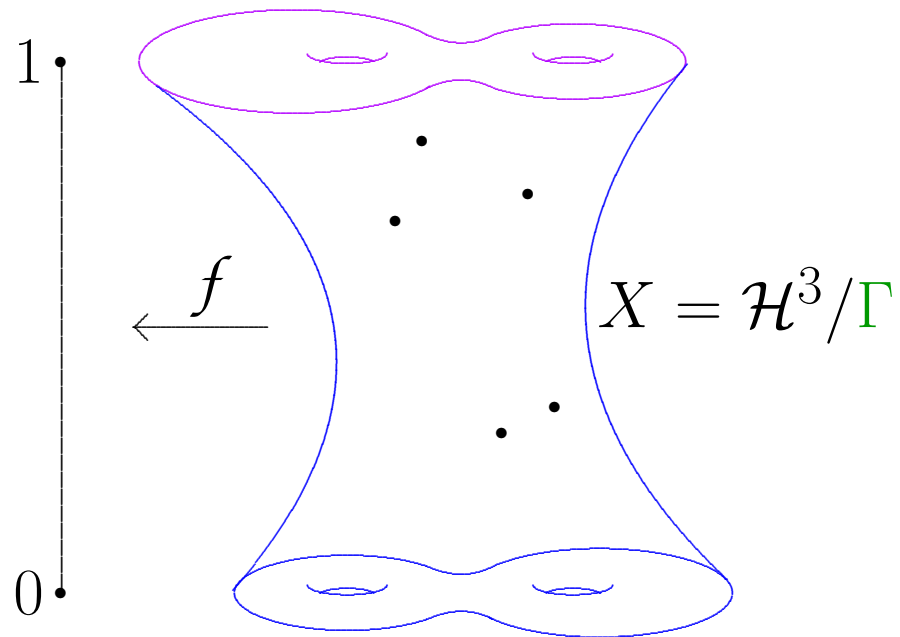


Construction of ASD 4-manifolds:



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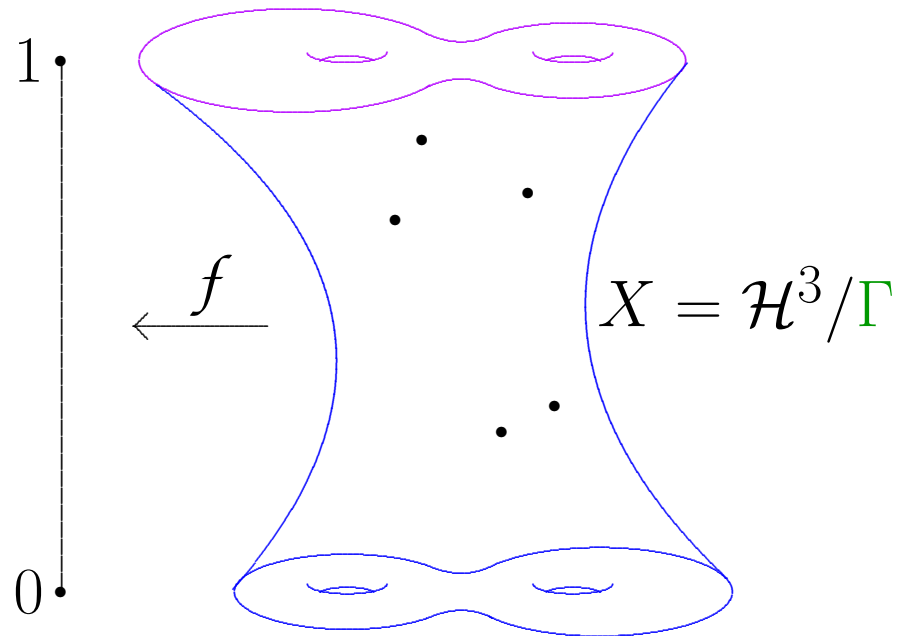
Choose k points $p_1, \dots, p_k \in X$



Construction of ASD 4-manifolds:

Choose k points $p_1, \dots, p_k \in X$

satisfying $\sum_{j=1}^k f(p_j) \in \mathbb{Z}$.

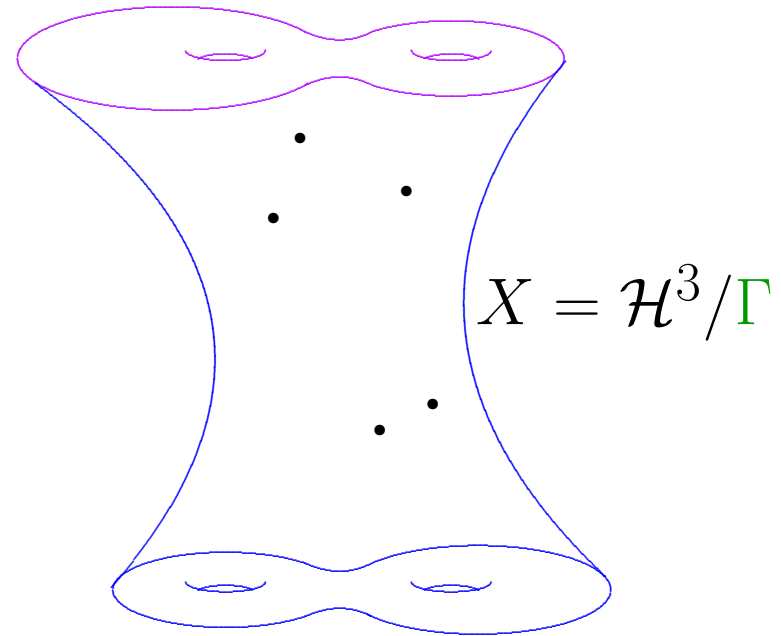


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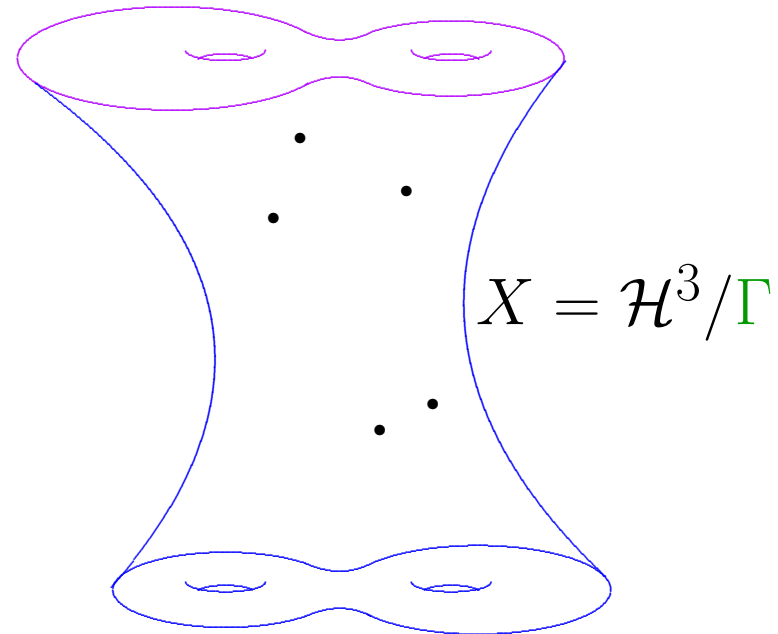
satisfying $\sum_{j=1}^k f(p_j) \in \mathbb{Z}$.

Can do if $k \neq 1$.



Construction of ASD 4-manifolds:

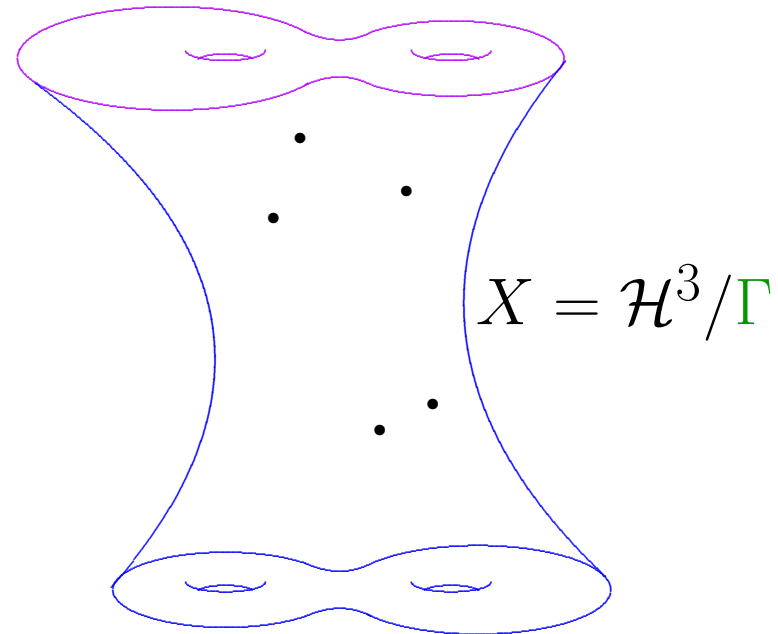
Let G_j be the Green's function of p_j :



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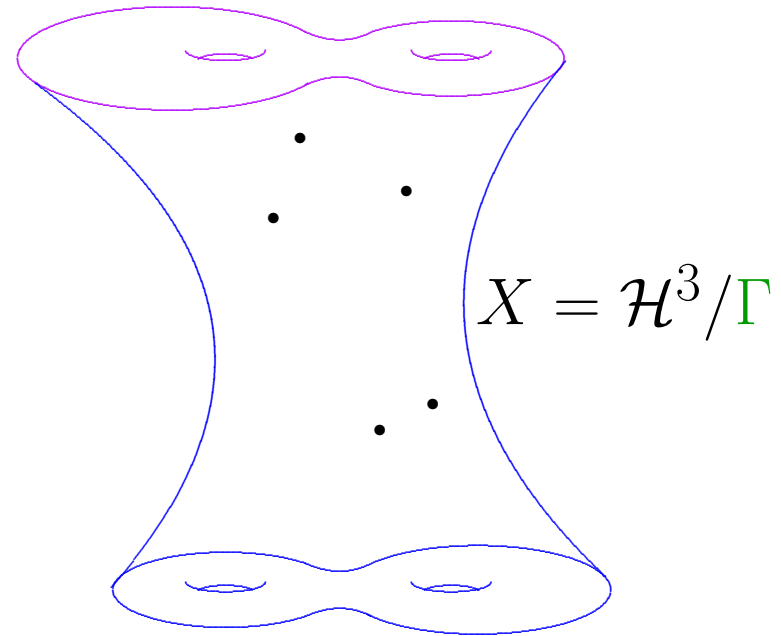
$$\Delta G_j = 2\pi\delta_{p_j}, \quad G_j \rightarrow 0 \text{ at } \partial\bar{X}$$



Construction of ASD 4-manifolds:

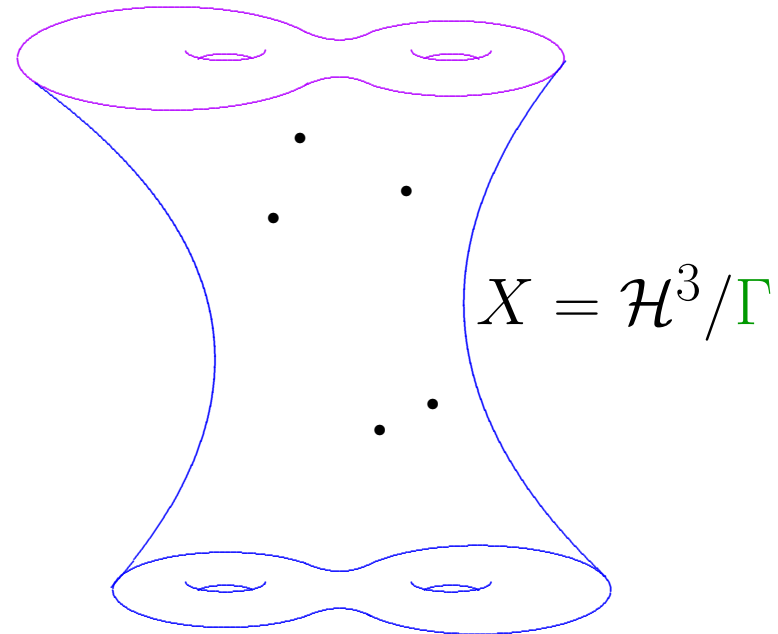
Let G_j be the Green's function of p_j , and set

$$V = 1 + \sum_{j=1}^k G_j.$$



Construction of ASD 4-manifolds:

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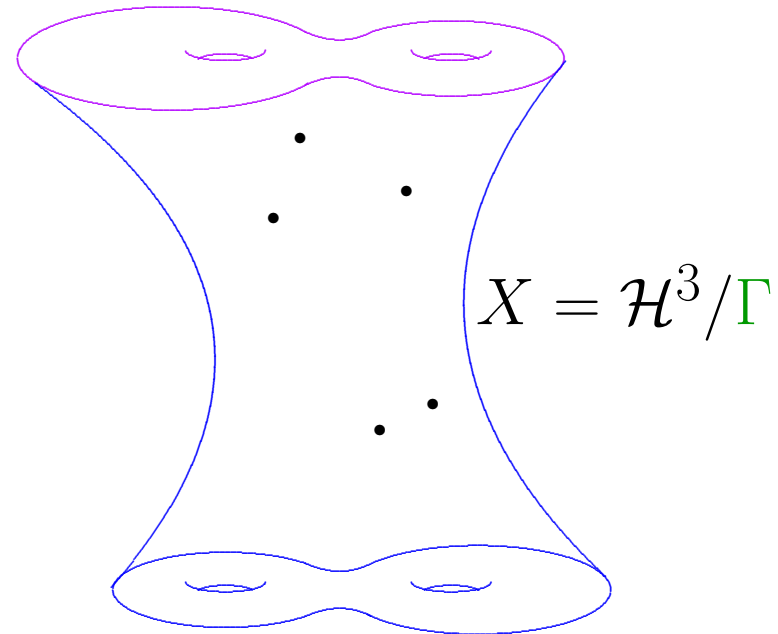


Construction of ASD 4-manifolds:

$$V = 1 + \sum_{j=1}^k G_j.$$

Choose $P \rightarrow (X - \{p_1, \dots, p_k\})$ circle bundle with connection form θ such that

$$d\theta = \star dV.$$

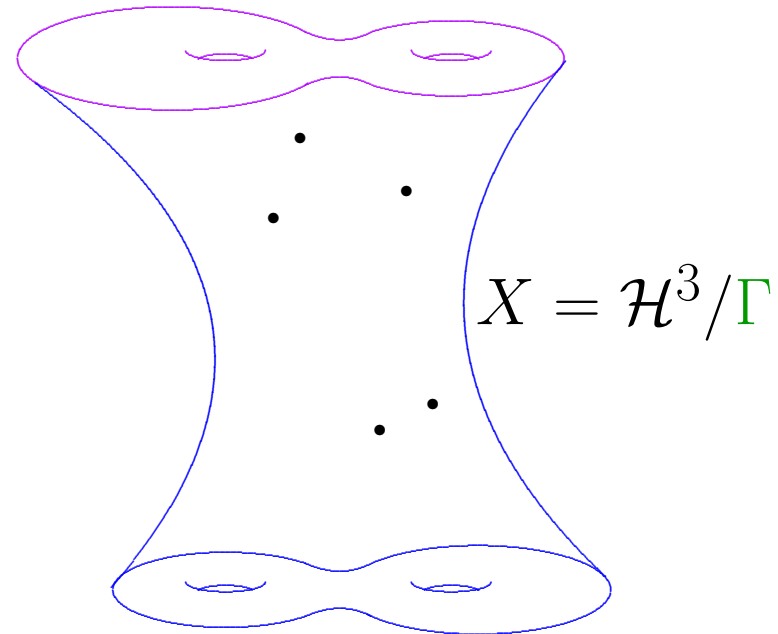


Construction of ASD 4-manifolds:

$$g = Vh + V^{-1}\theta^2$$

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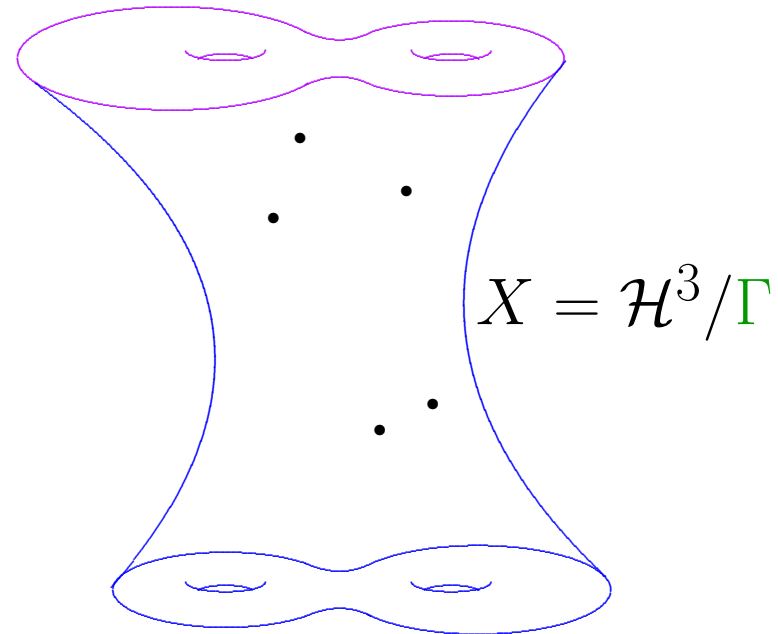


Construction of ASD 4-manifolds:

$$g = f(1 - f)[Vh + V^{-1}\theta^2]$$

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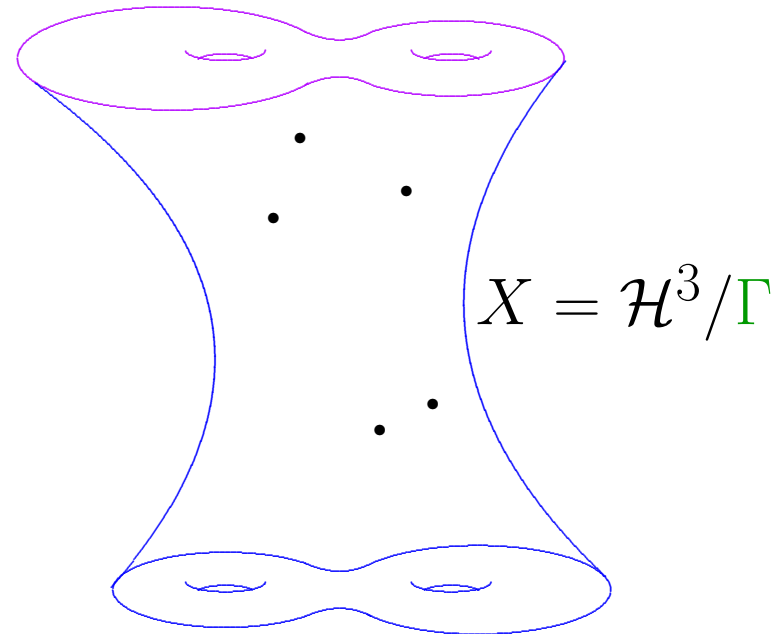
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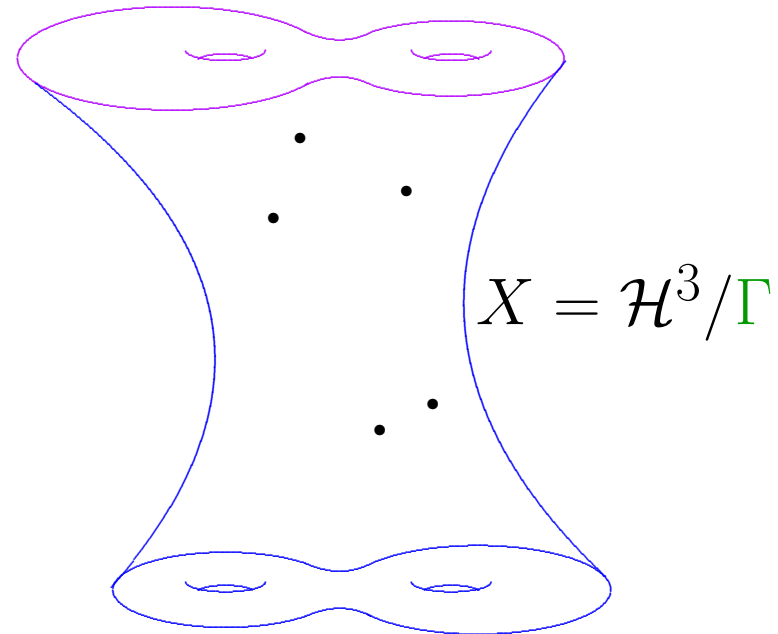
$$M = P \cup \{\hat{p}_1, \dots, \hat{p}_k\} \cup \partial\bar{X}$$



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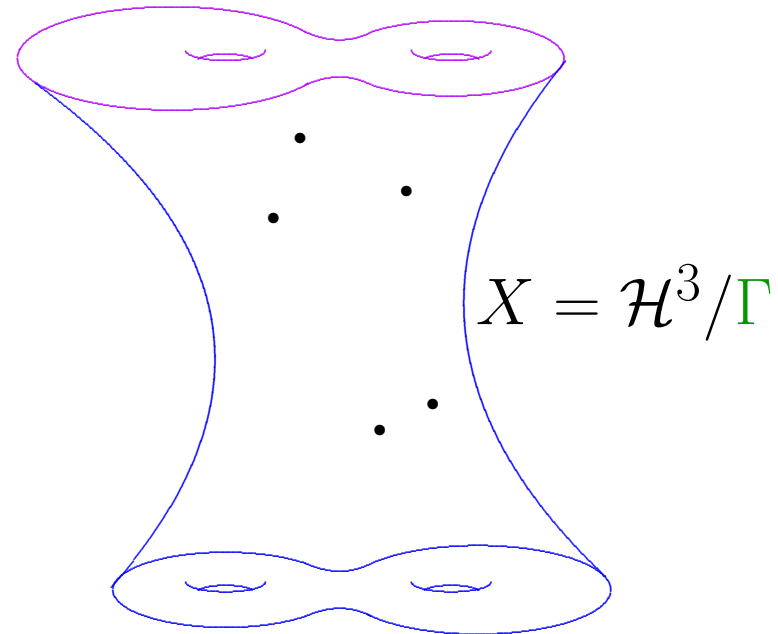
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 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 \bar{X} & = & X - \{p_1, \dots, p_k\} & \cup & \{p_1, \dots, p_k\} & \cup & \partial \bar{X}
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Construction of ASD 4-manifolds:

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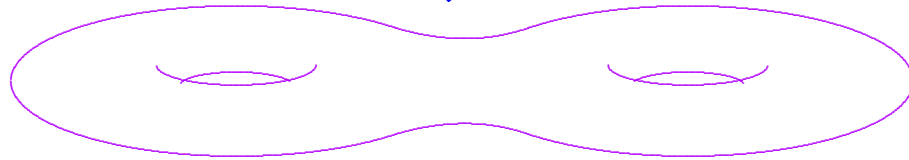
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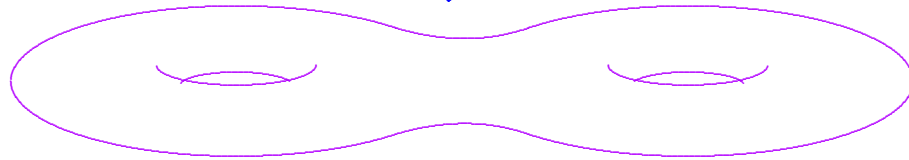
$$\approx (\Sigma \times S^2) \# k \bar{\mathbb{C}P}_2$$

$$\overline{X} = \Sigma \times [0, 1]$$



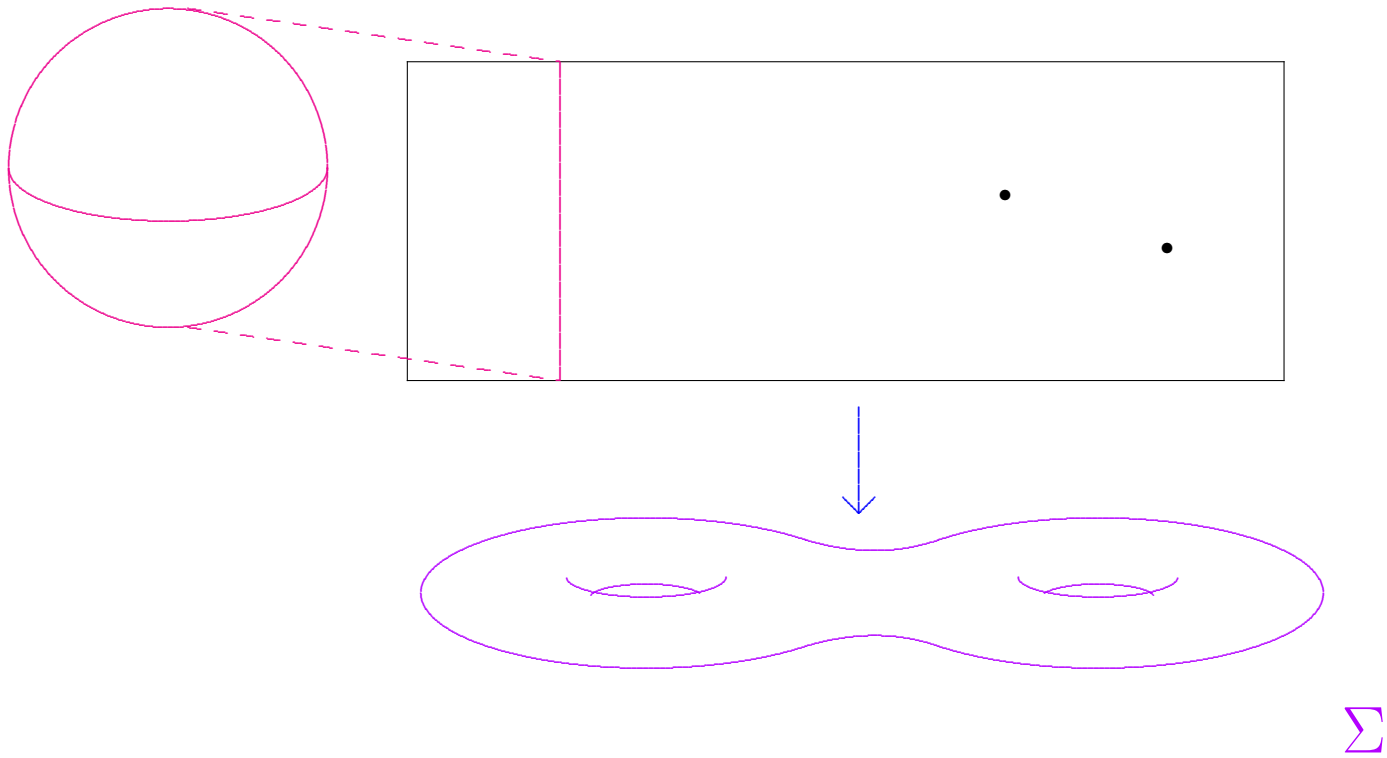
Σ

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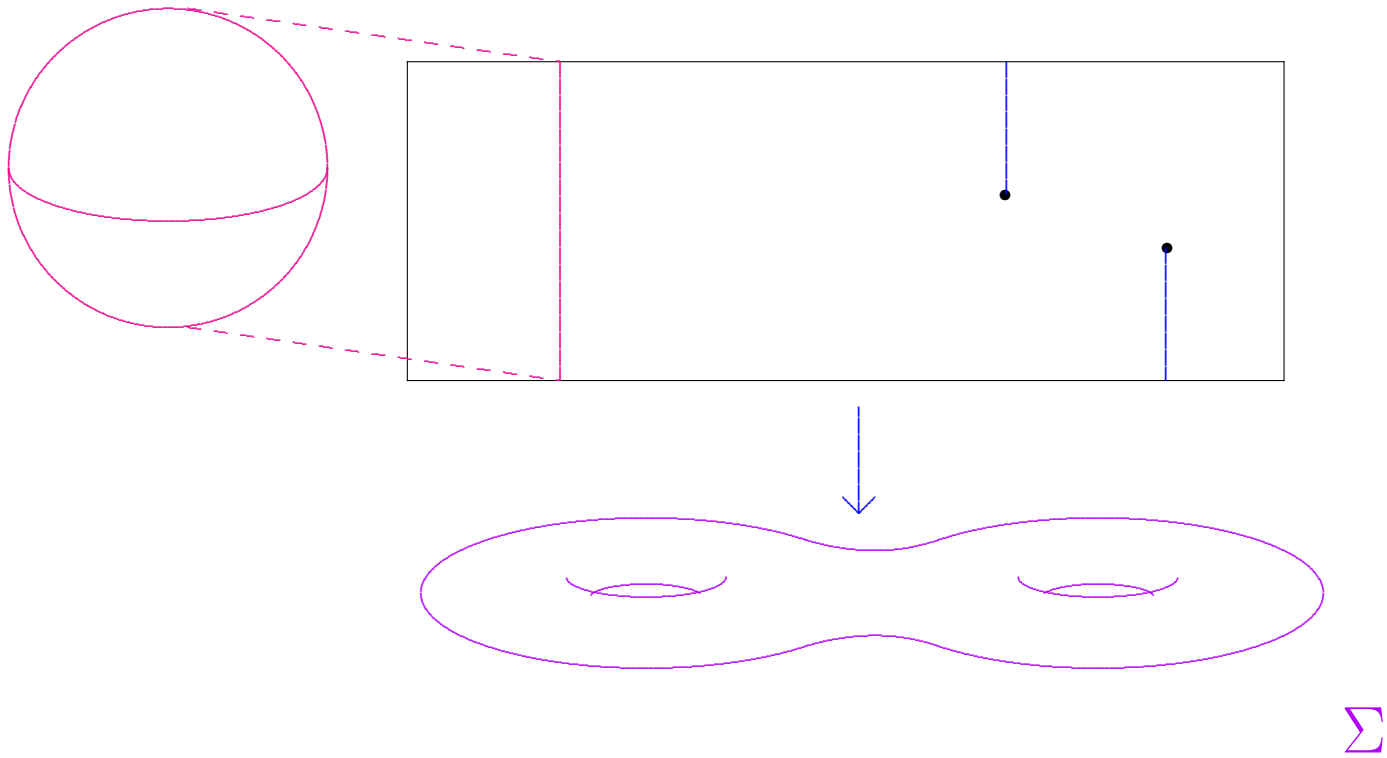


Σ

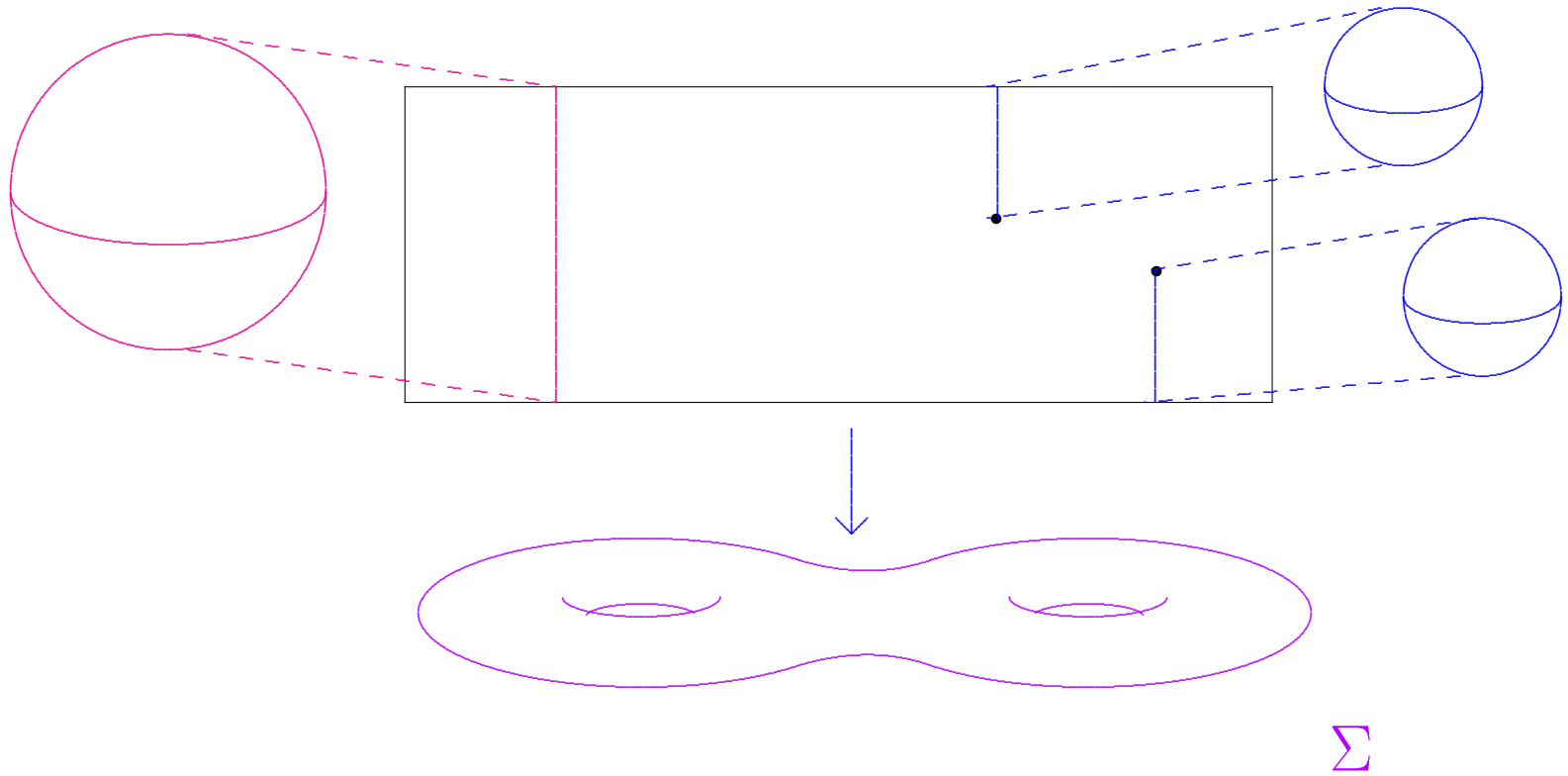
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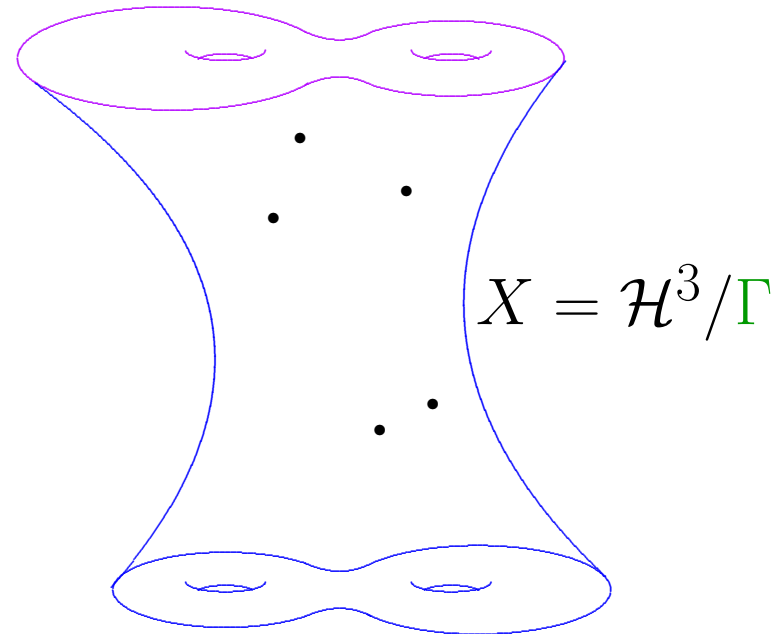


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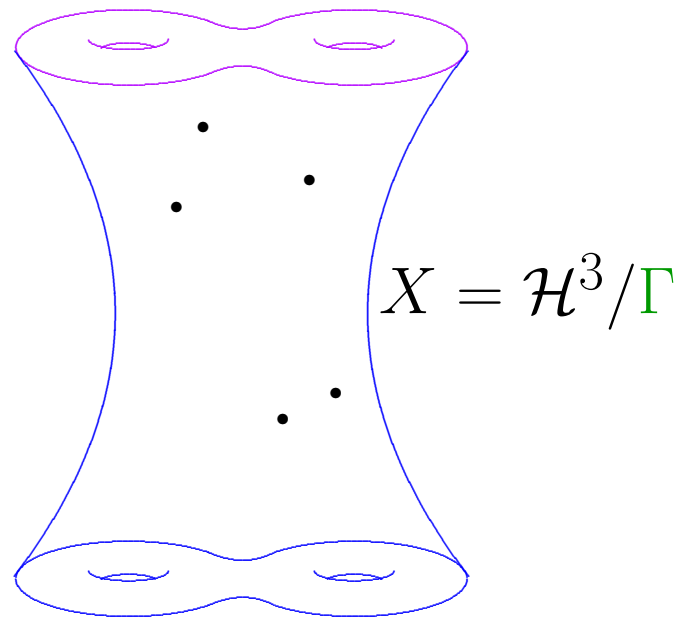


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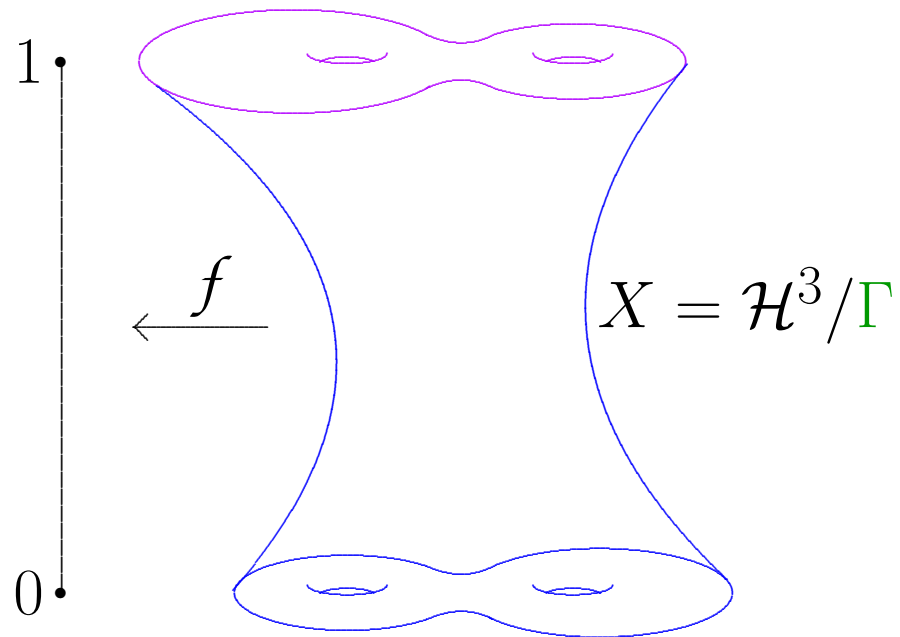


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Fuchsian case: $(\Sigma \times S^2) \# k\overline{\mathbb{C}\mathbb{P}}_2$ scalar-flat Kähler



Γ quasi-Fuchsian

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$$\Delta f = 0$$

Theorem.

Theorem. *Let $(M, [g])$ be ASD manifold*

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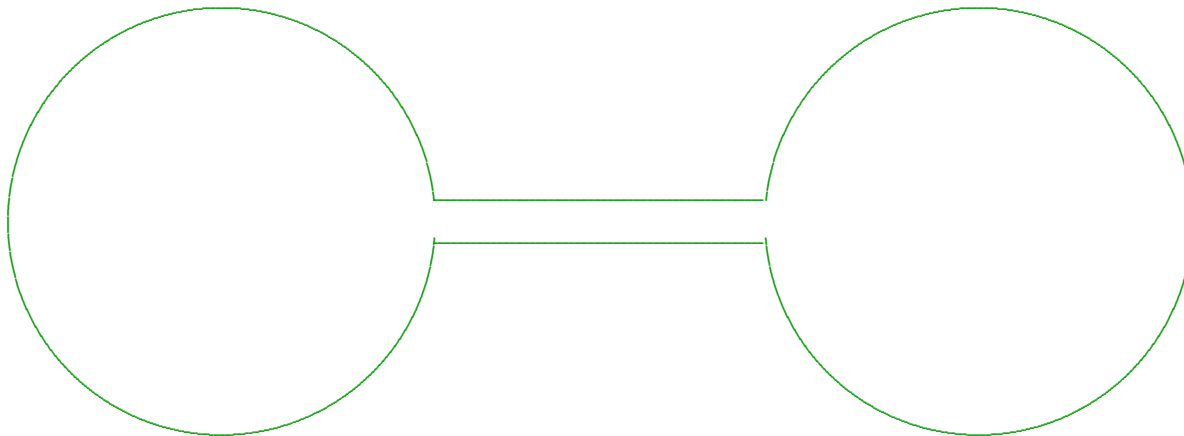
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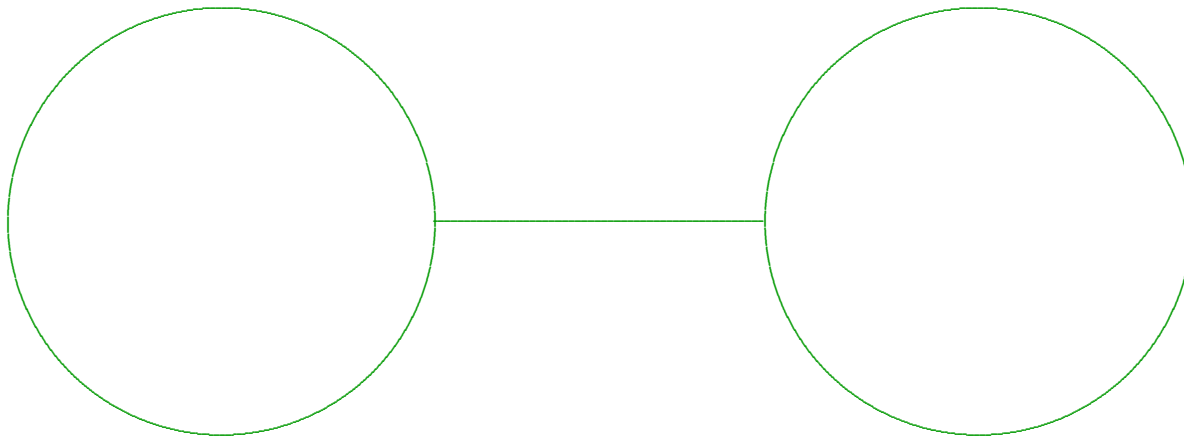
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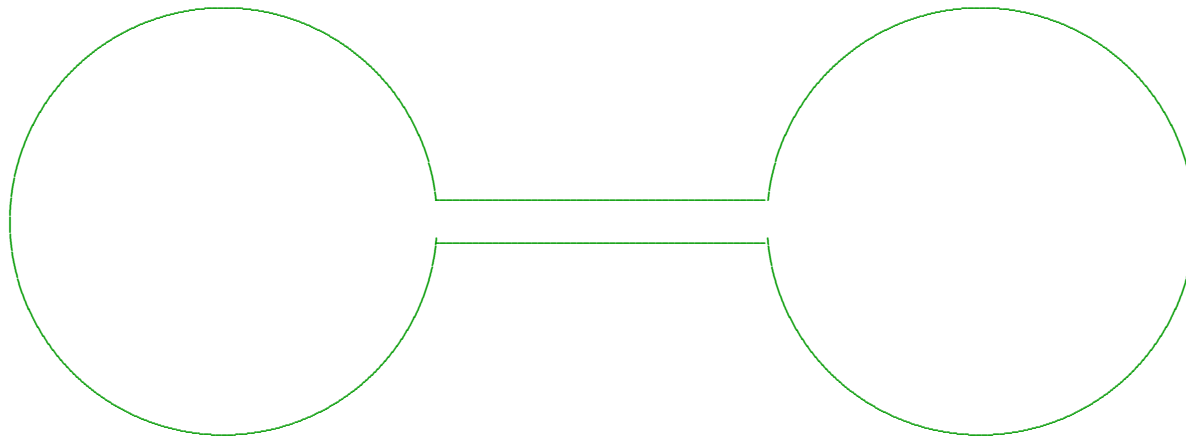
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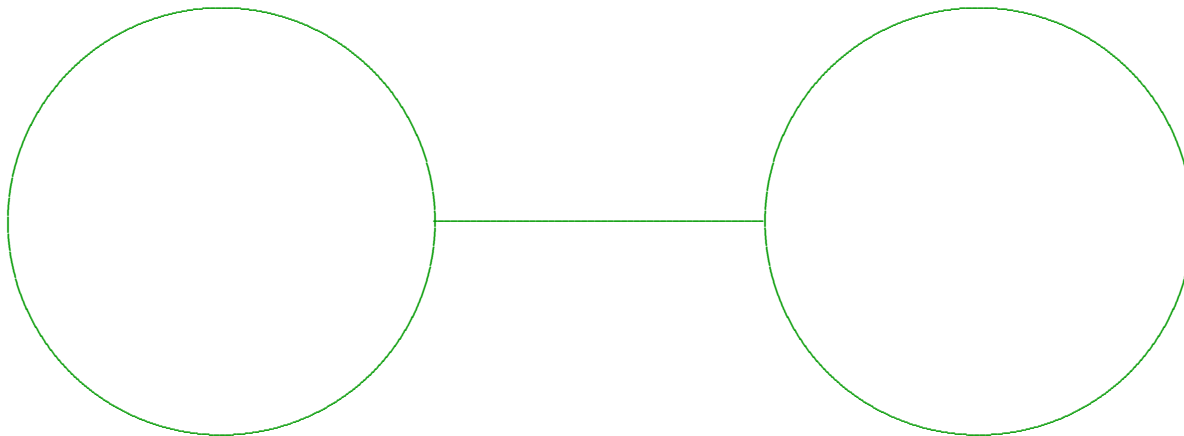
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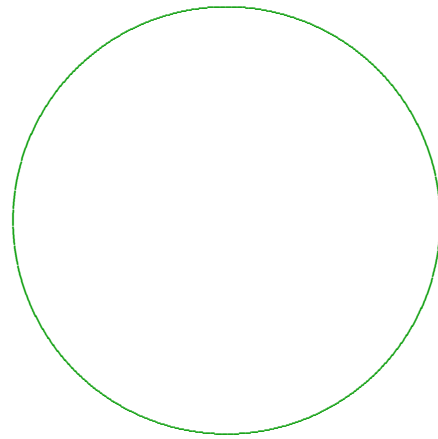
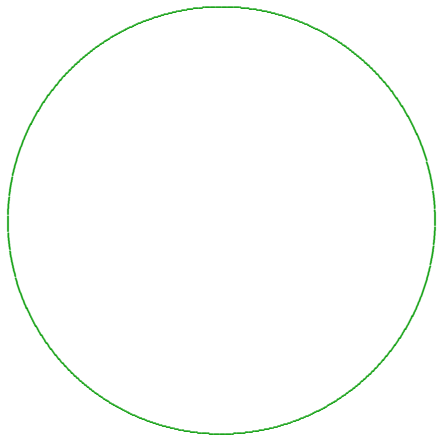
Ahlfors-Bers: Quasi-conformal mappings

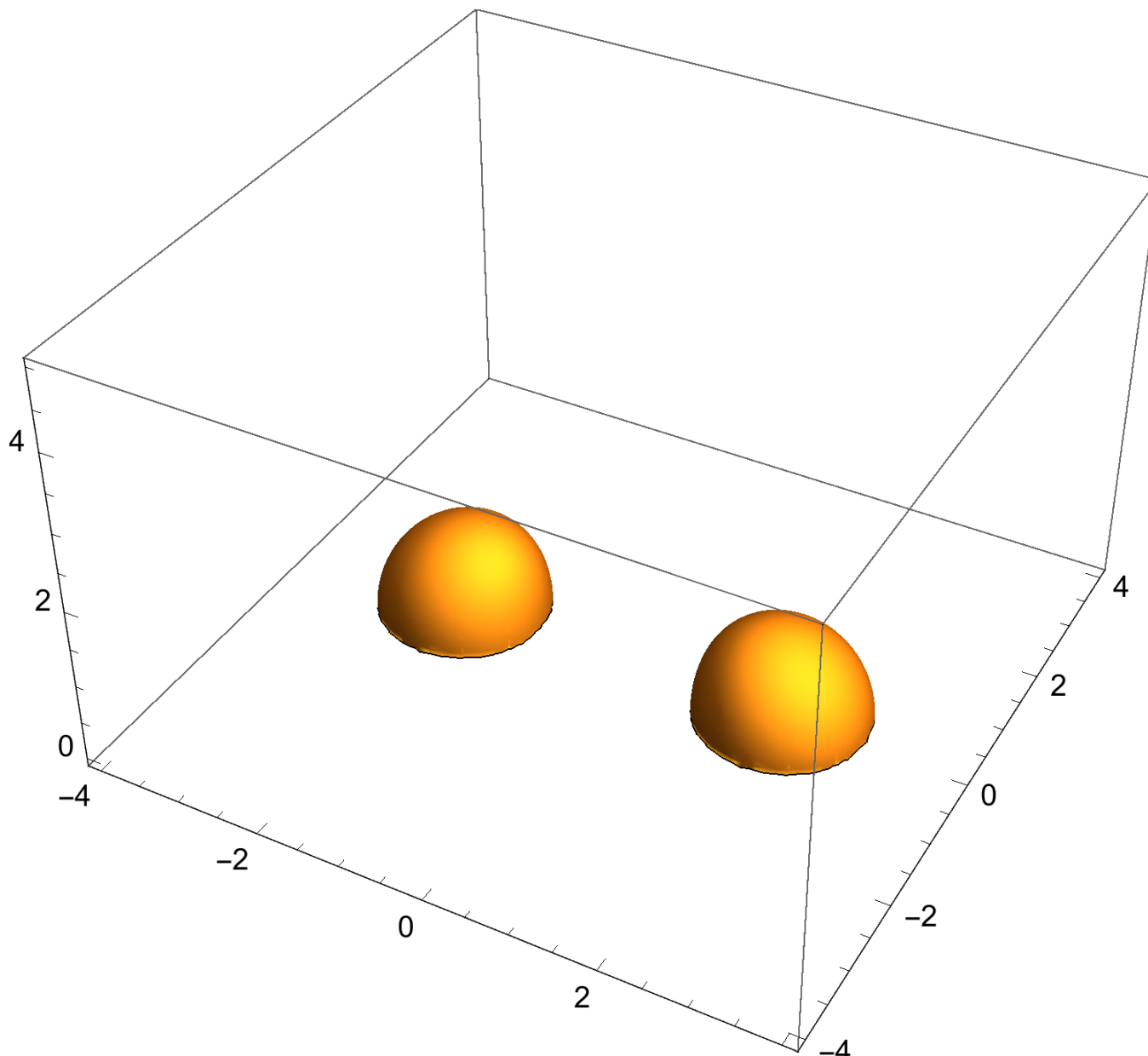


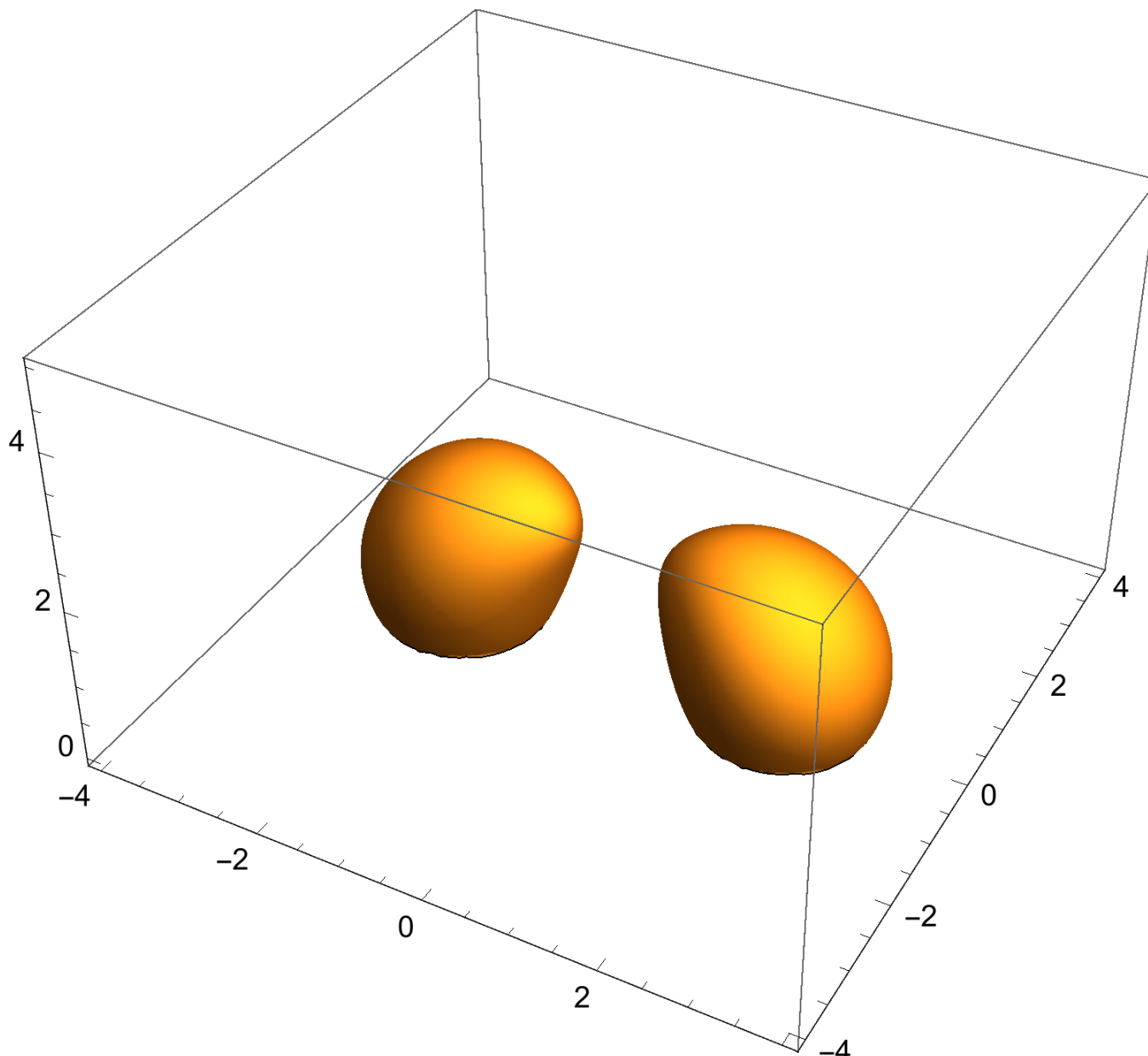


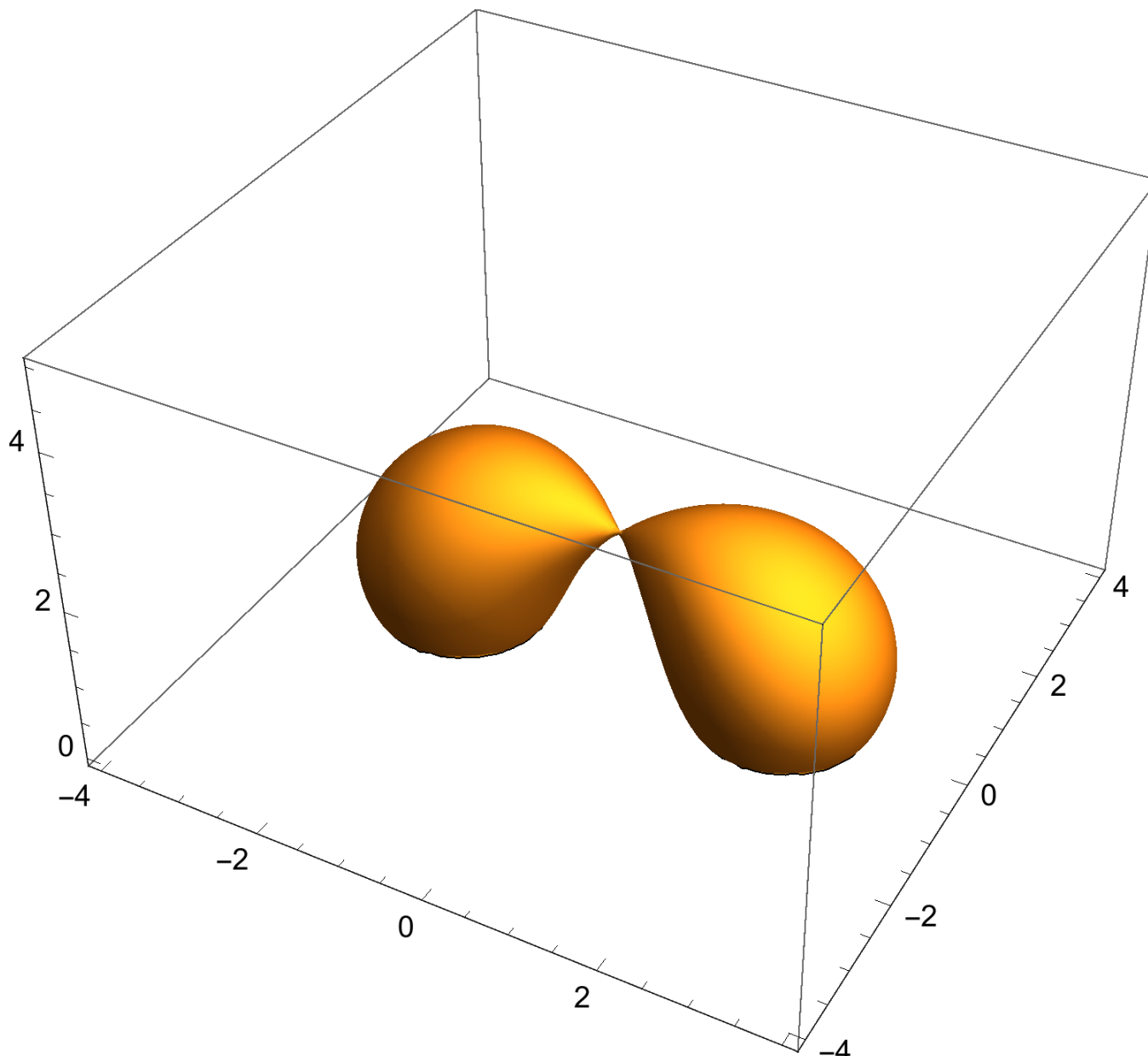


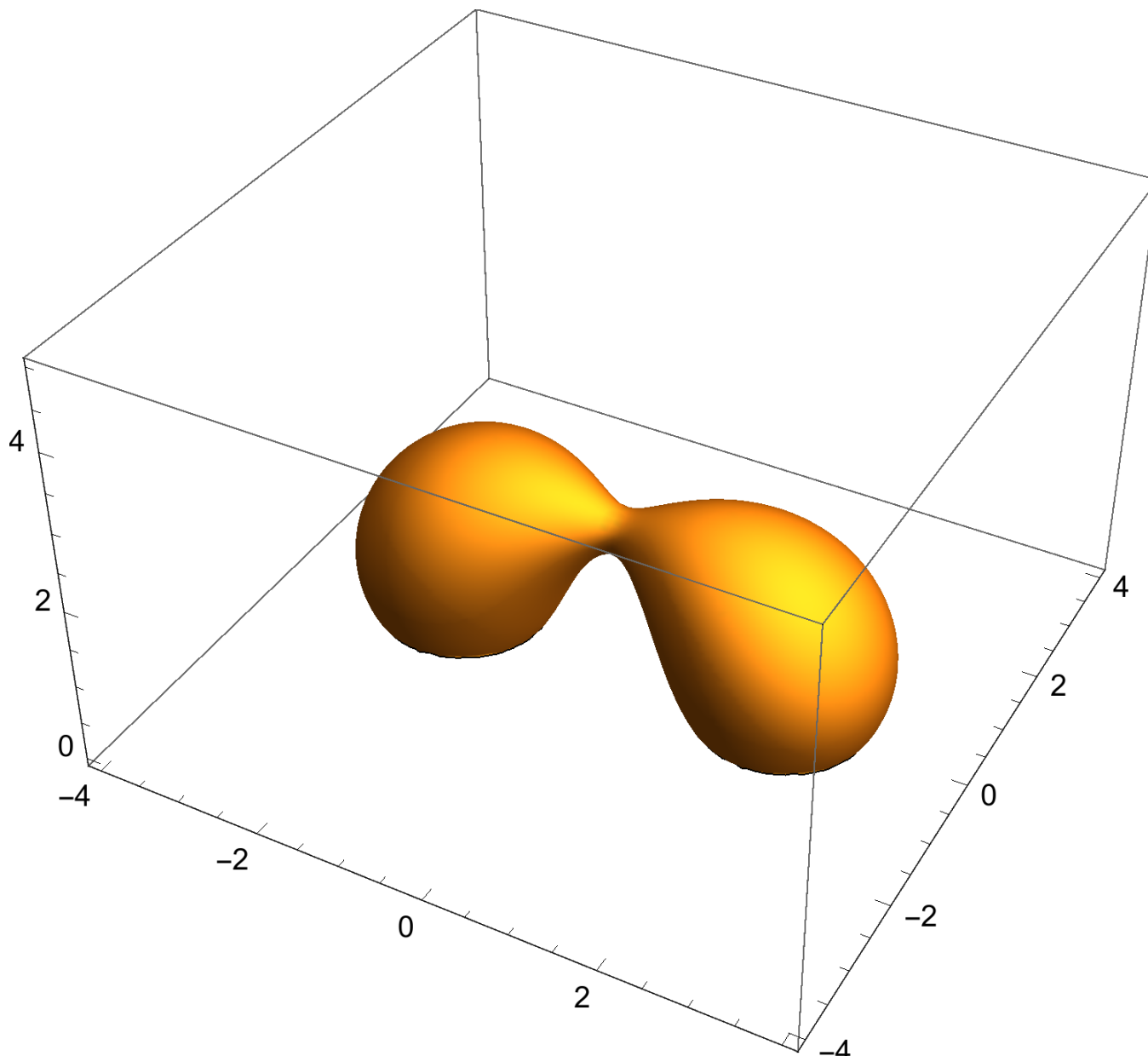


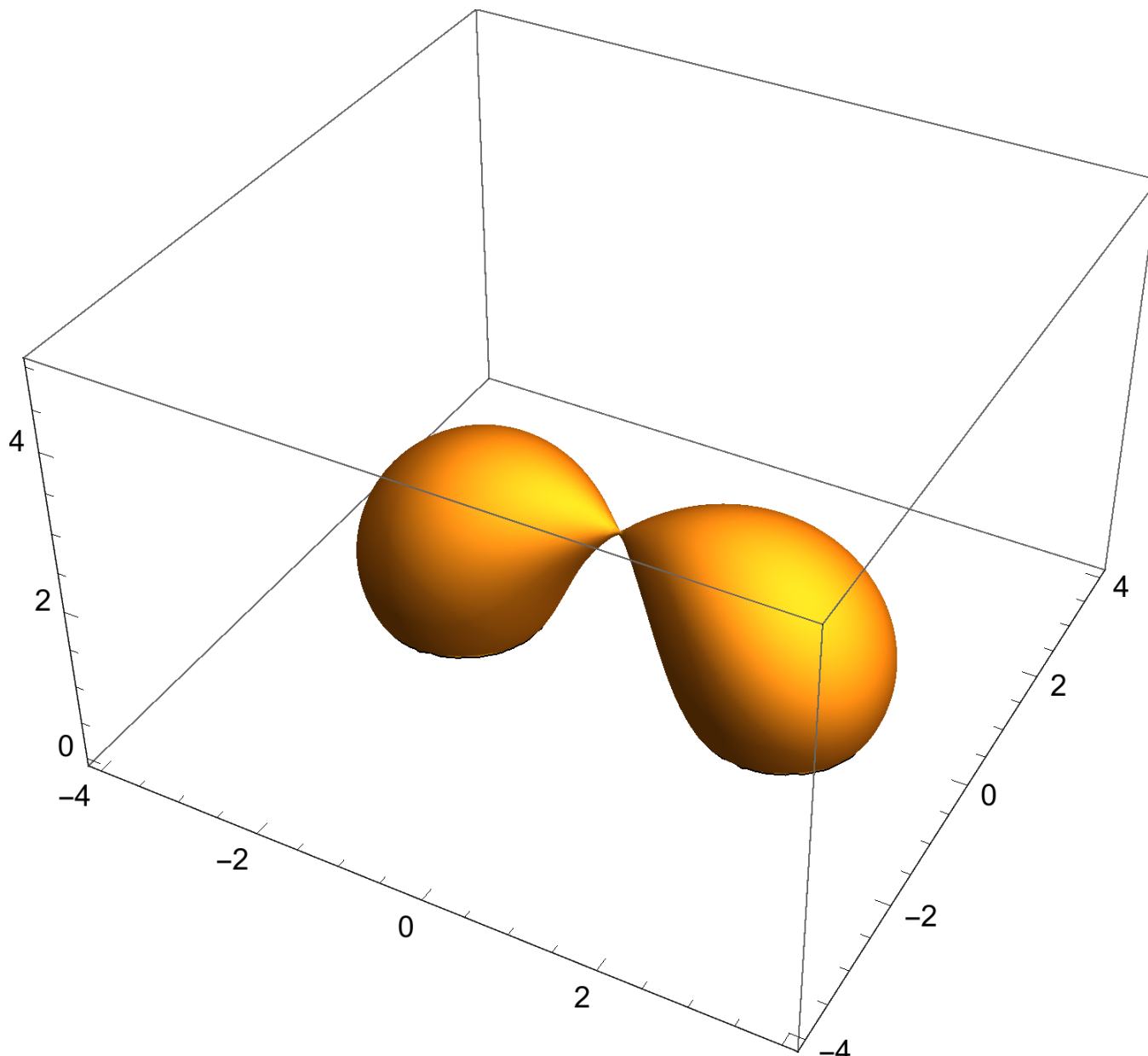


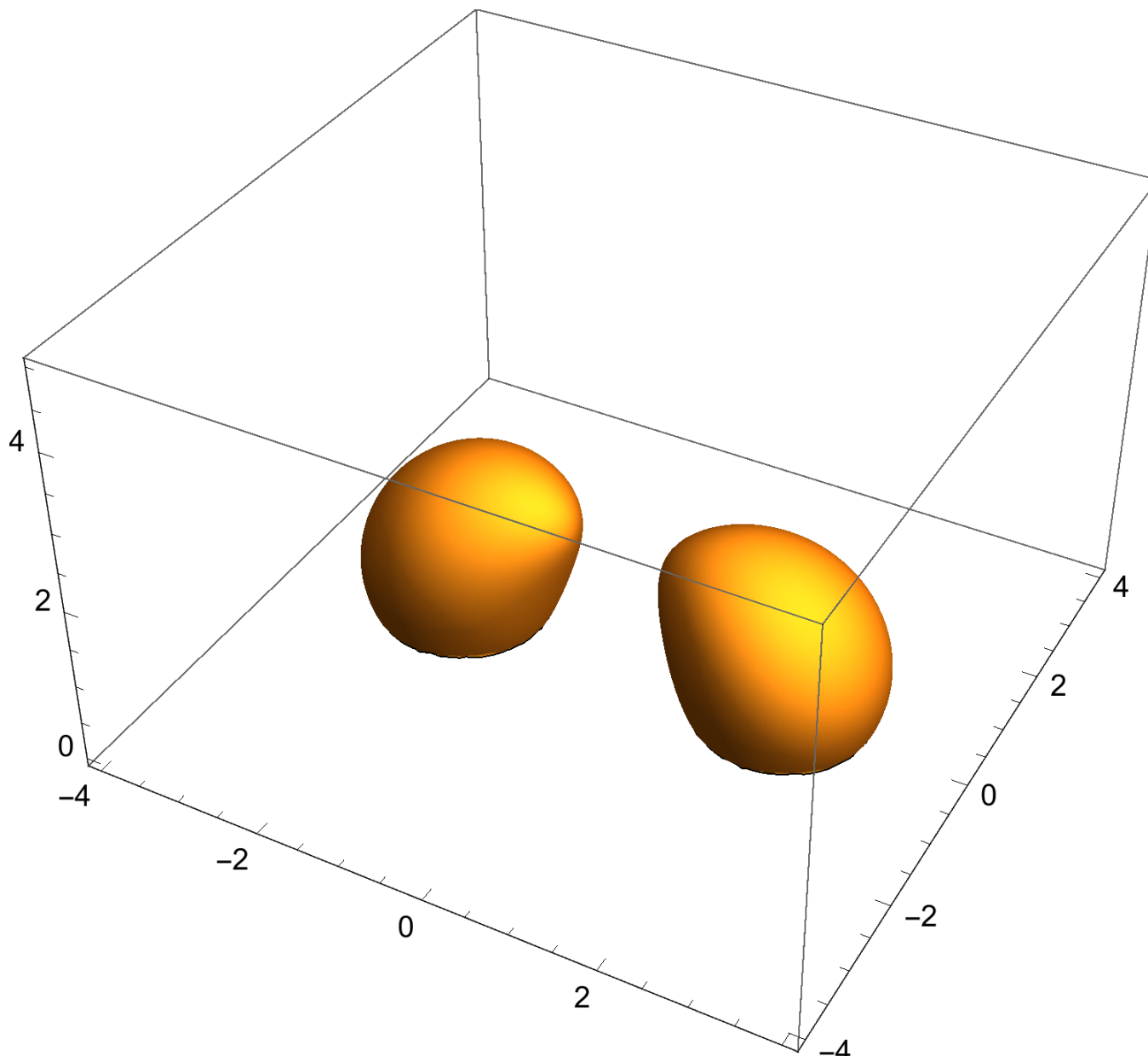


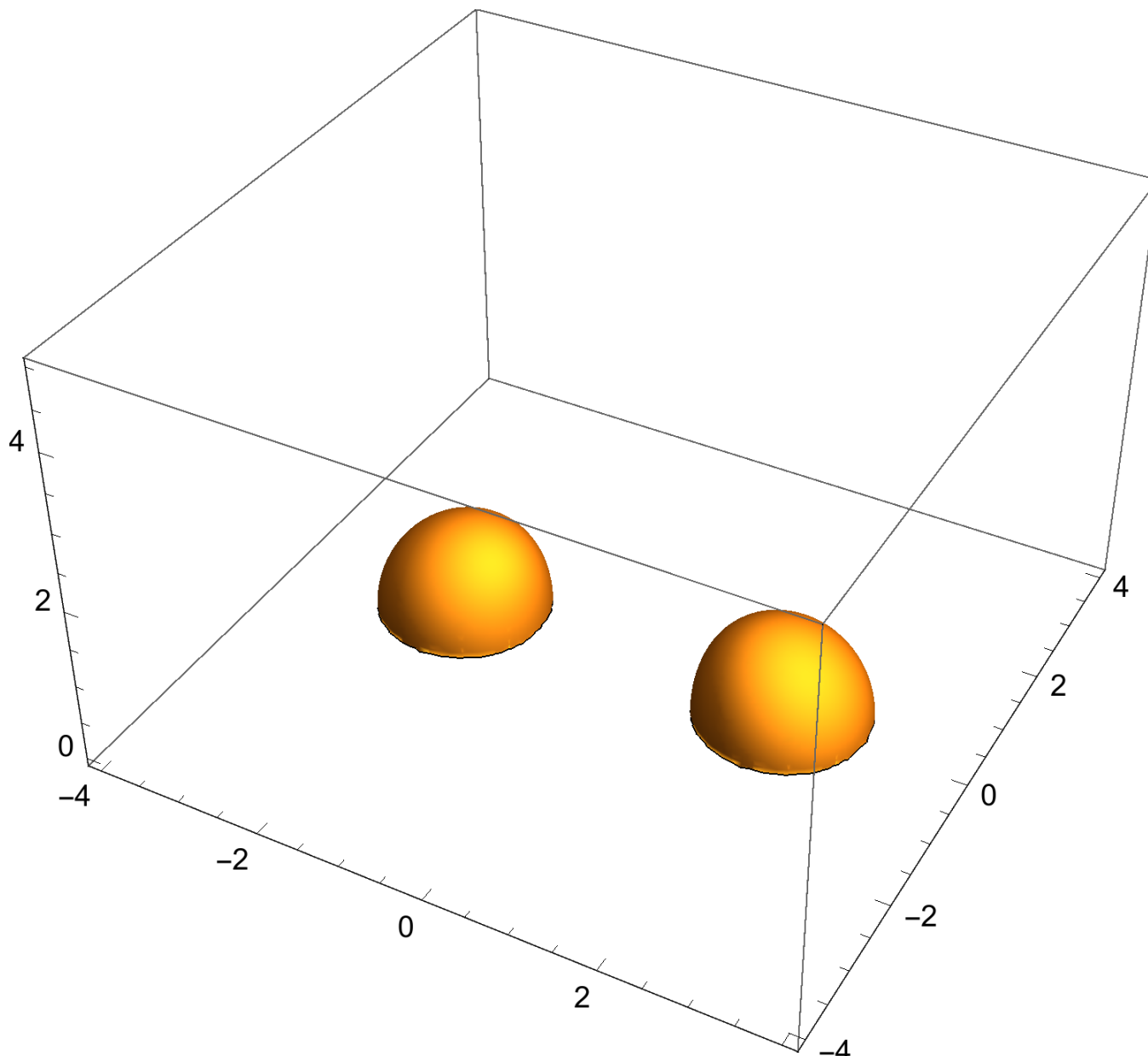


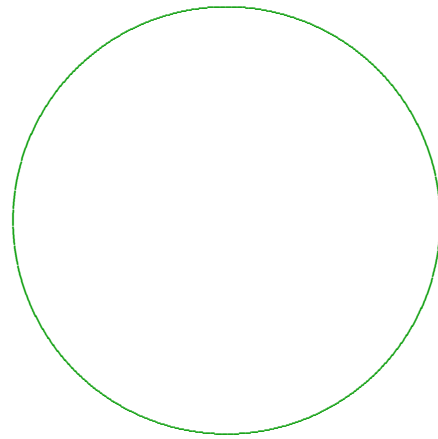
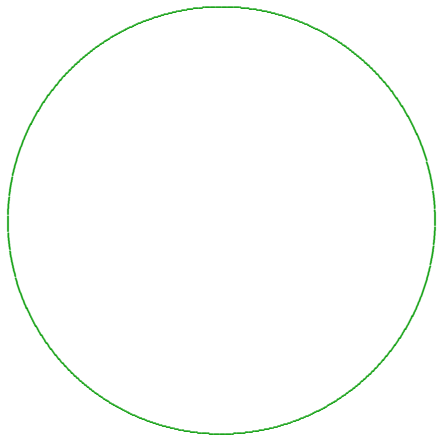


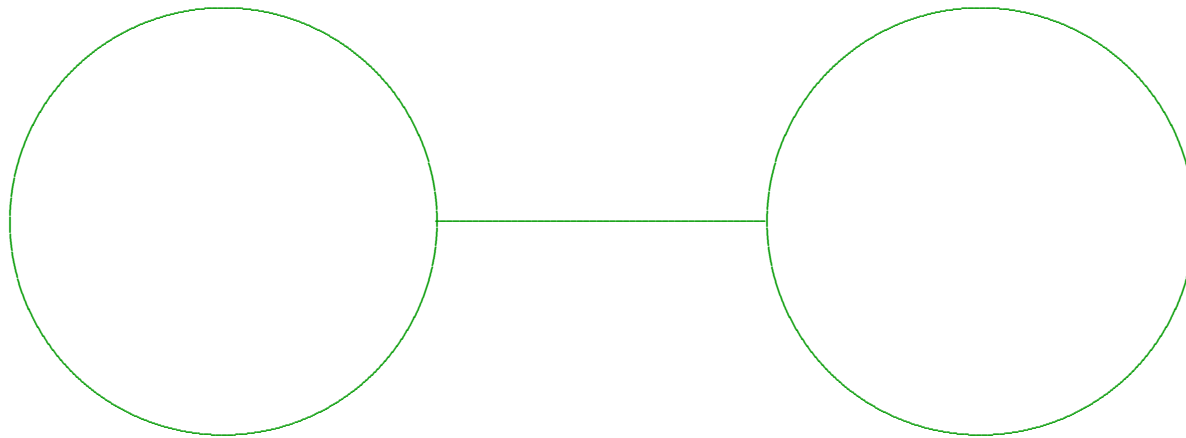


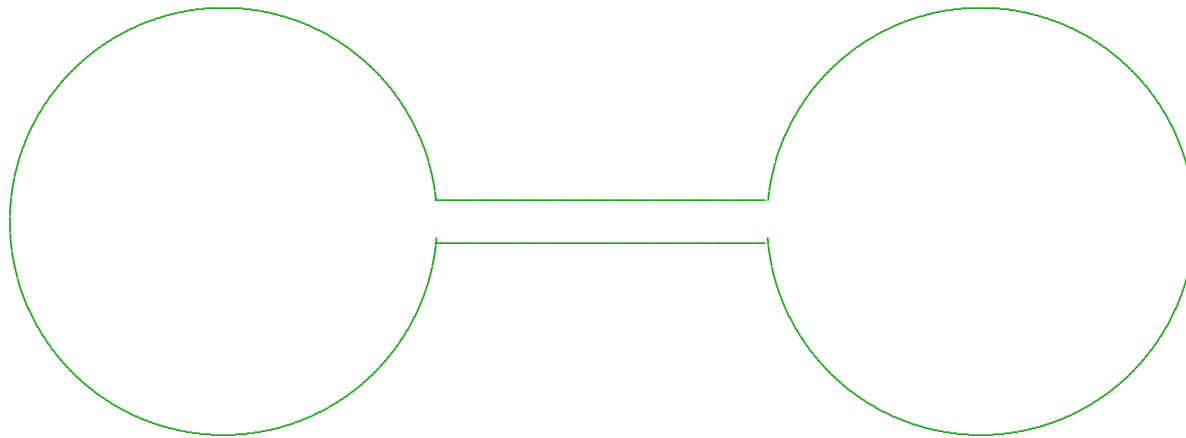


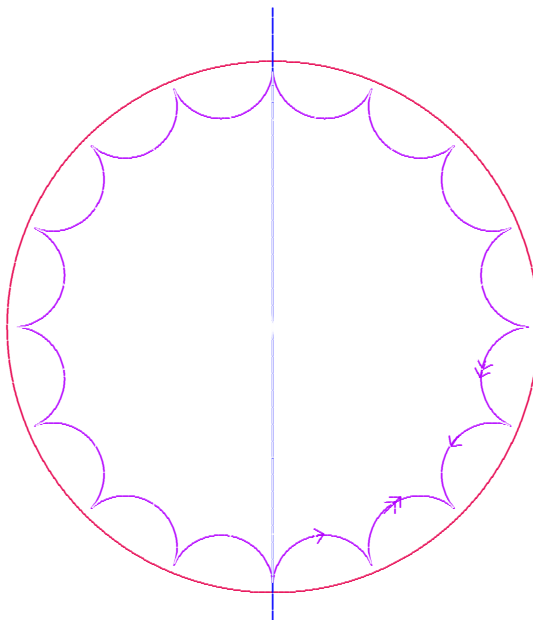
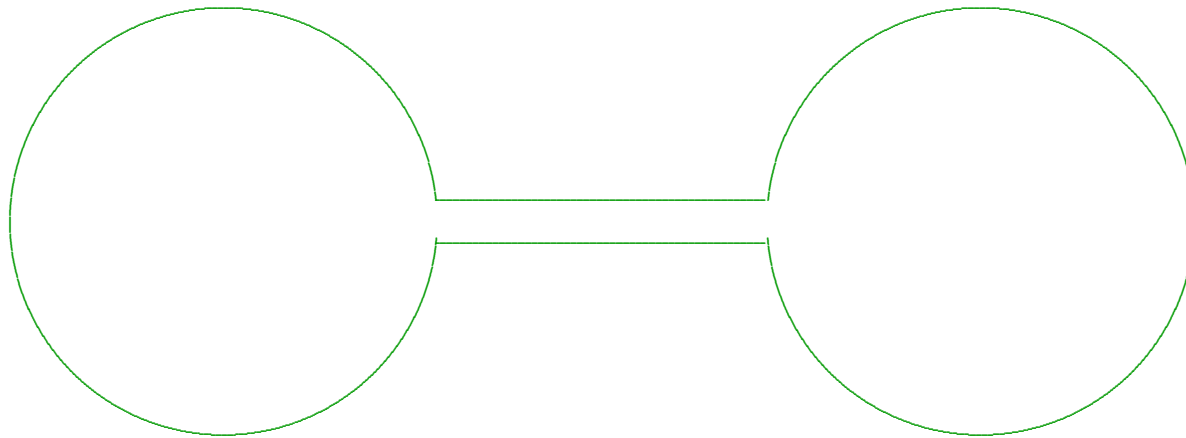


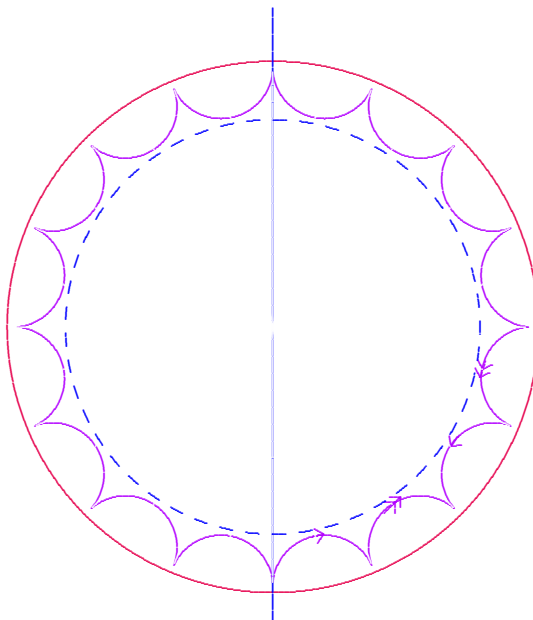
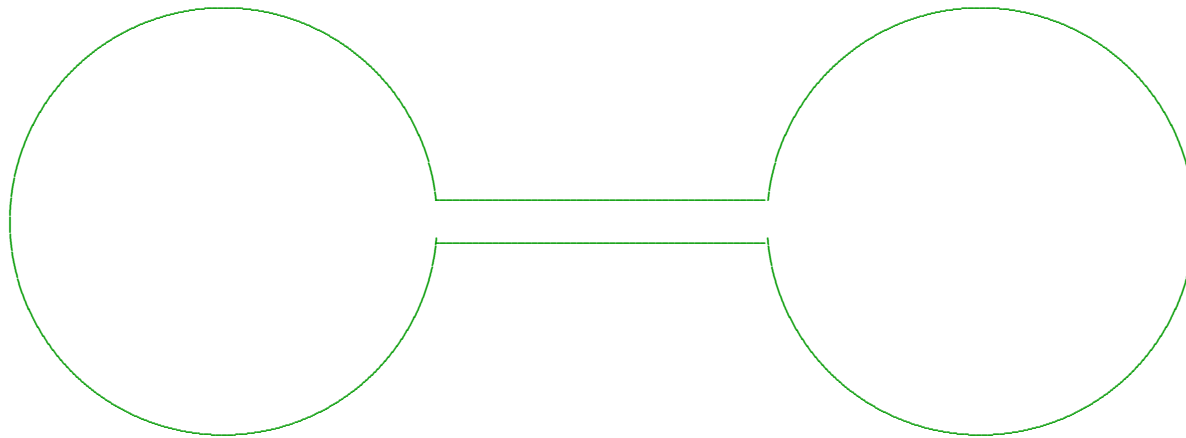


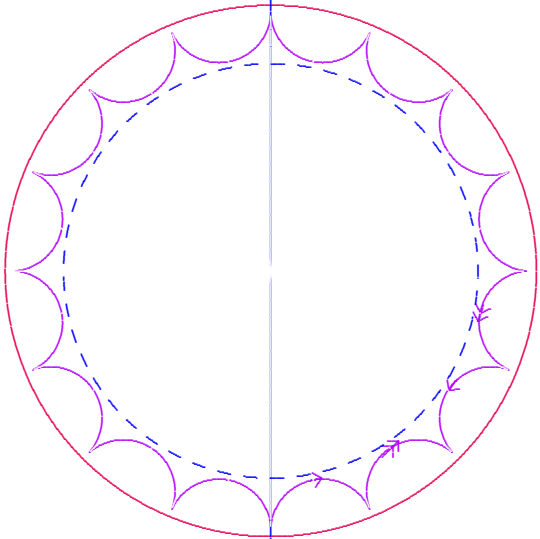
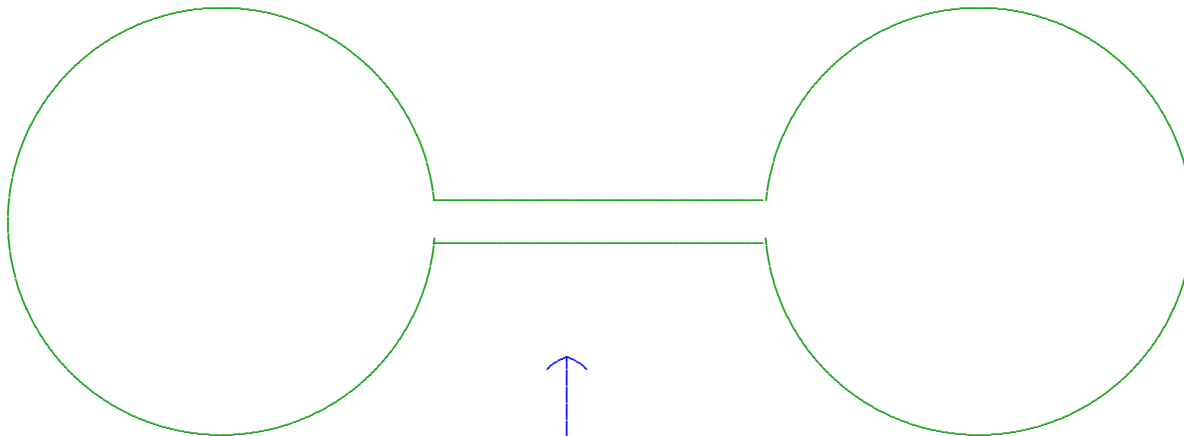


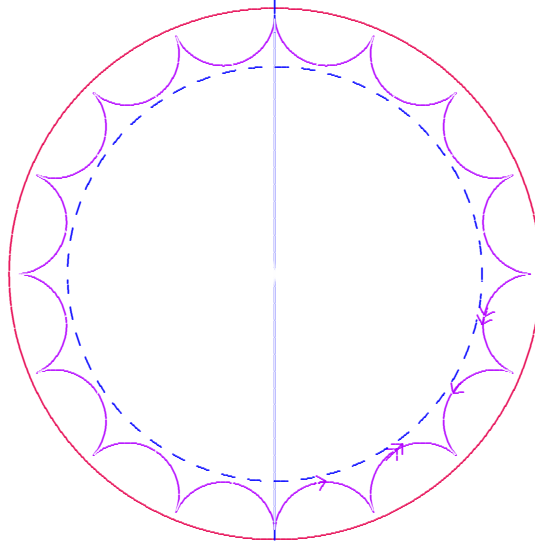
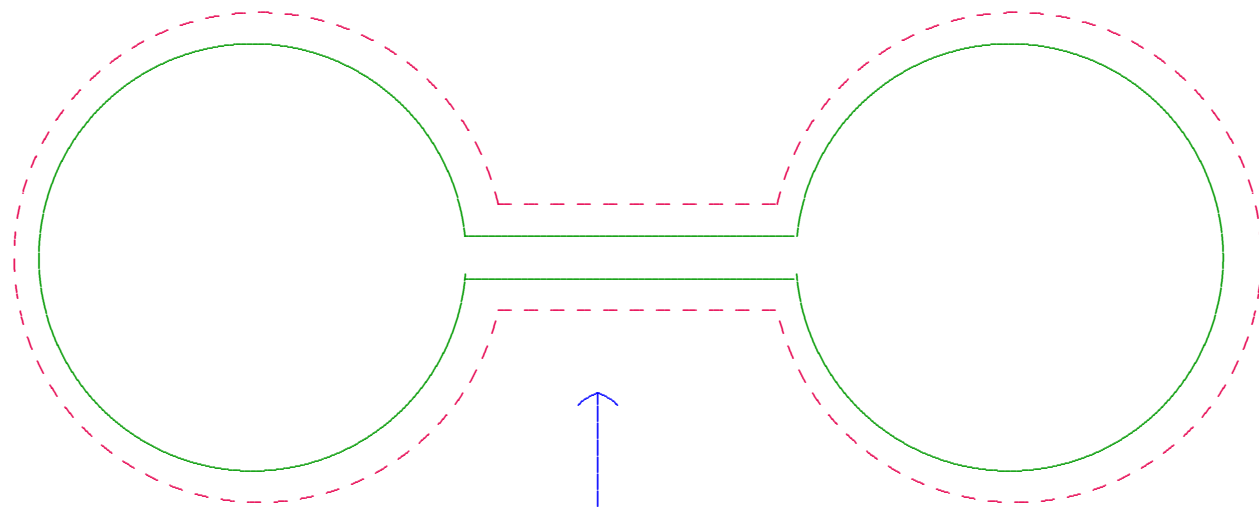


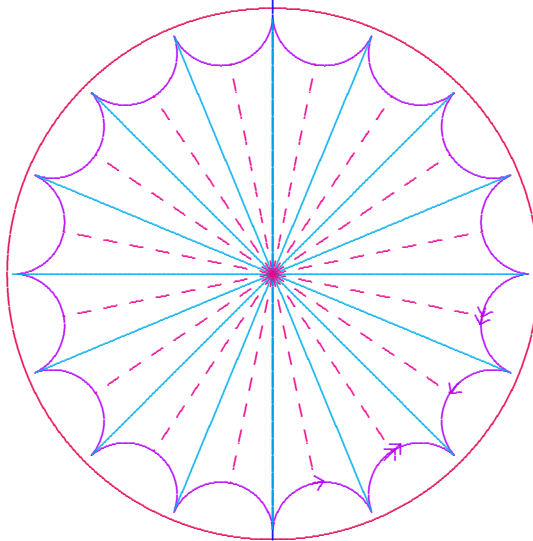
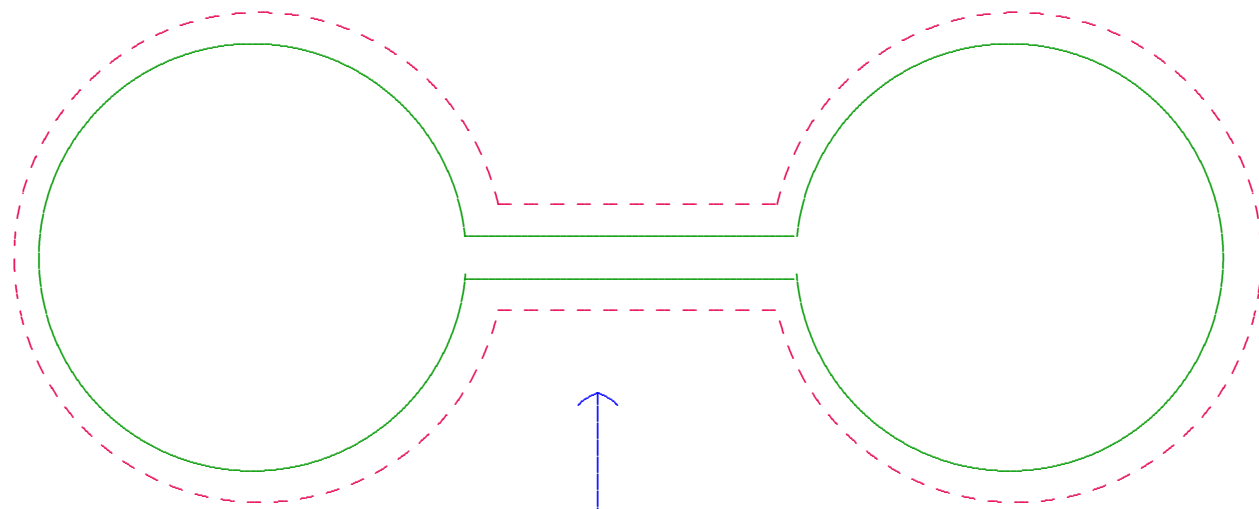


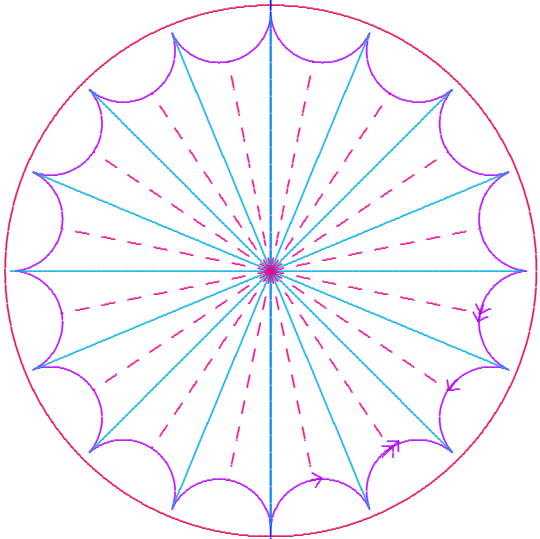
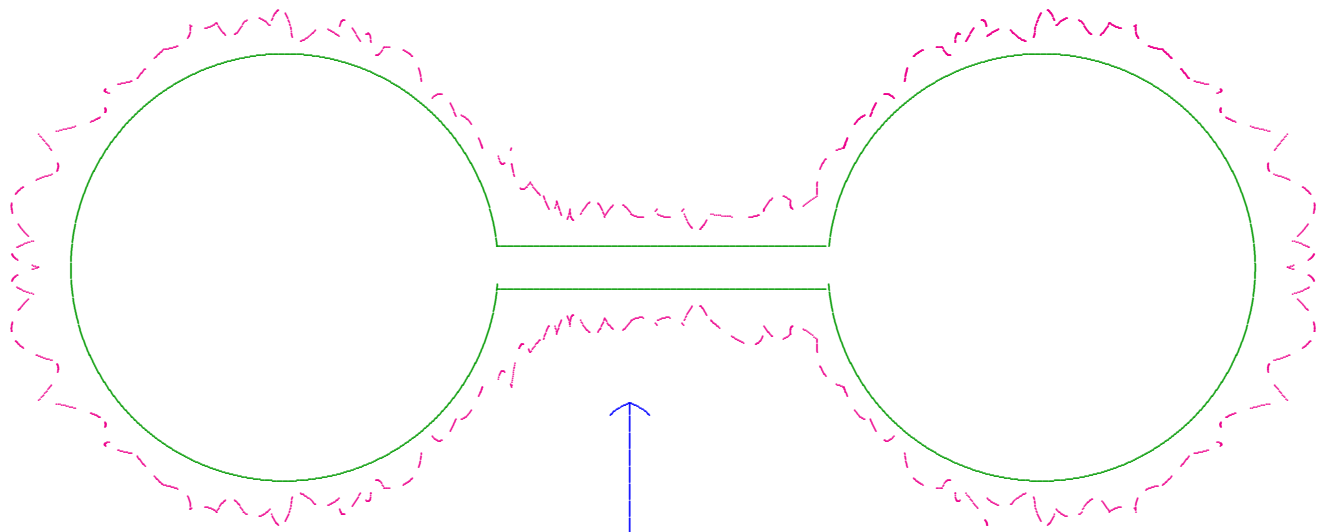












Theorem A. Consider 4-manifolds $M = \Sigma \times S^2$, where Σ compact Riemann surface of genus g .

Then \forall even $g \gg 0$, \exists family $[g_t]$, $t \in [0, 1]$, of locally-conformally-flat classes on M , such that

- \exists scalar-flat Kähler metric $g_0 \in [g_0]$; but
- \nexists almost-Kähler metric $g \in [g_1]$.

Theorem B. Fix an integer $k \geq 2$, and then consider the 4-manifolds $M = (\Sigma \times S^2) \#^k \overline{\mathbb{C}\mathbb{P}}_2$, where Σ compact Riemann surface of genus g .

Then \forall even $g \gg 0$, \exists family $[g_t]$, $t \in [0, 1]$, of anti-self-dual conformal classes on M , such that

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