Mass in

Kähler Geometry

Claude LeBrun
Stony Brook University

31st Annual Geometry Festival
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Joint work with

Joint work with

Hans-Joachim Hein
University of Maryland

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e-print: arXiv:1507.08885 [math.DG]

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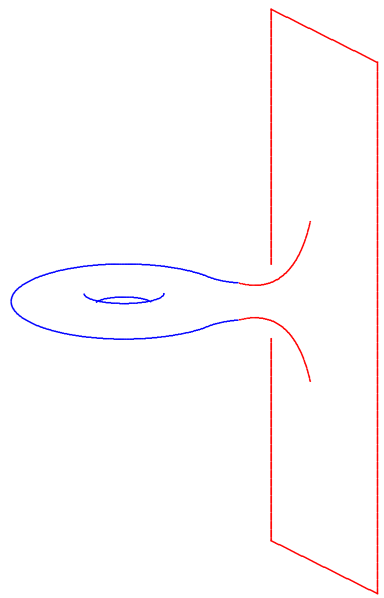
To appear in Comm. Math. Phys.

Definition. *A complete, non-compact Riemannian n-manifold* (M^n, g)



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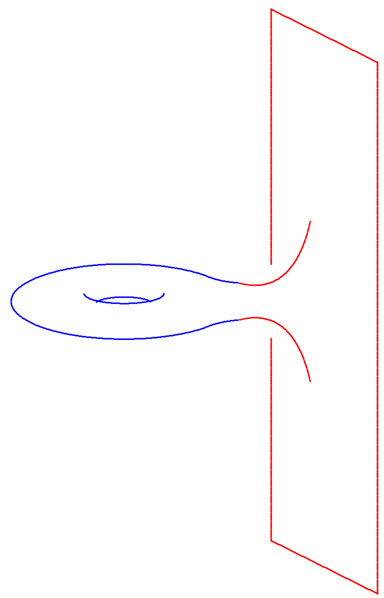

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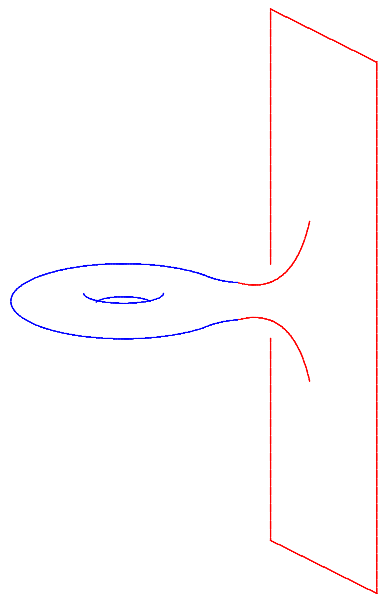

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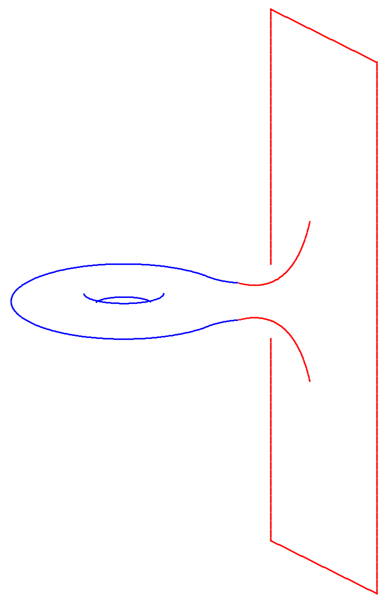

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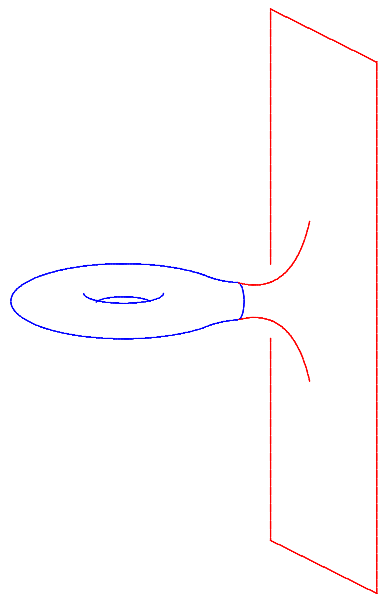

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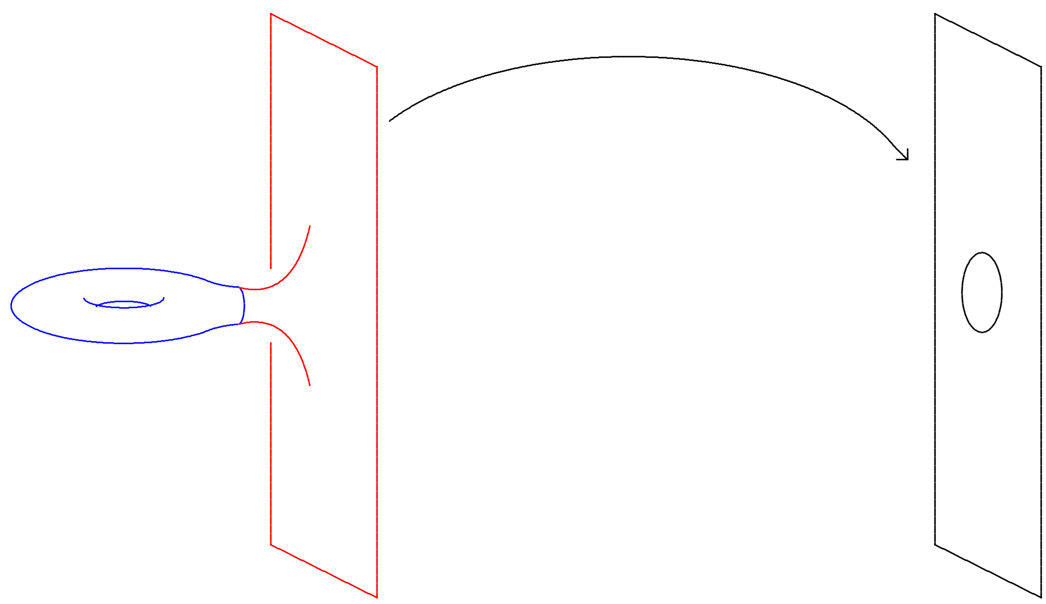

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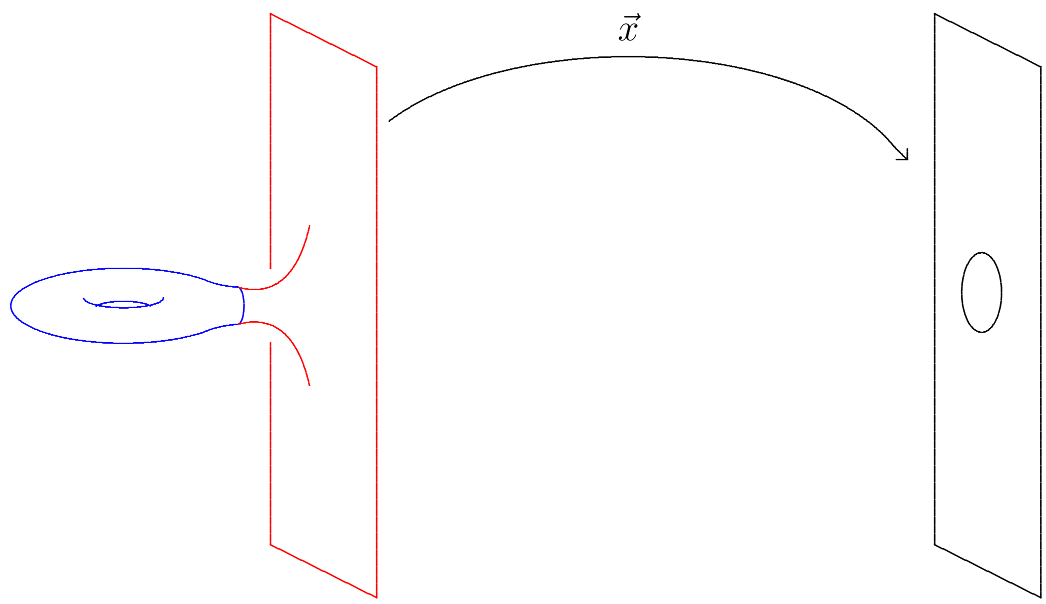

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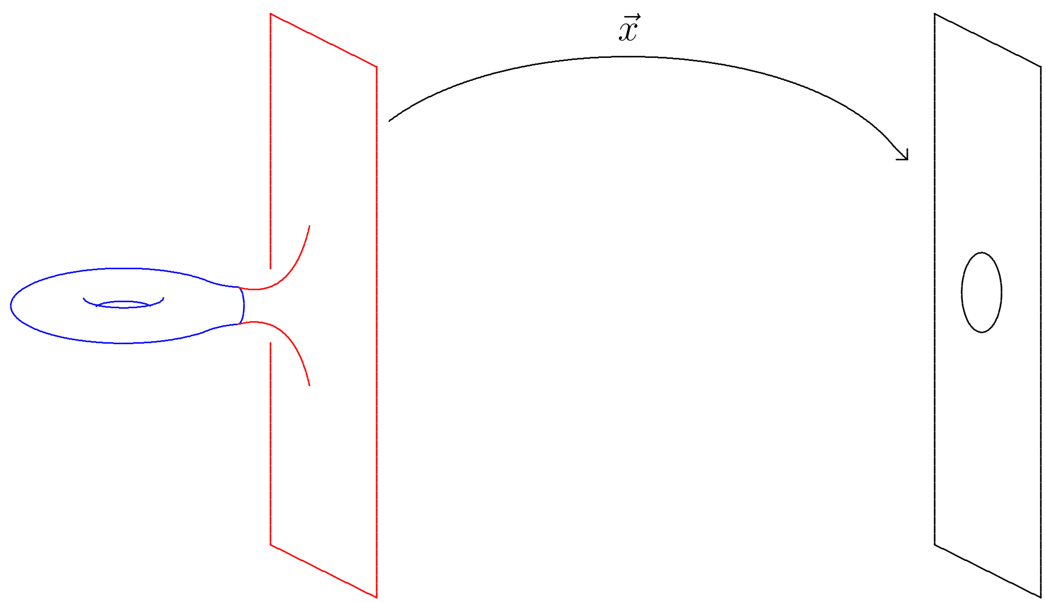

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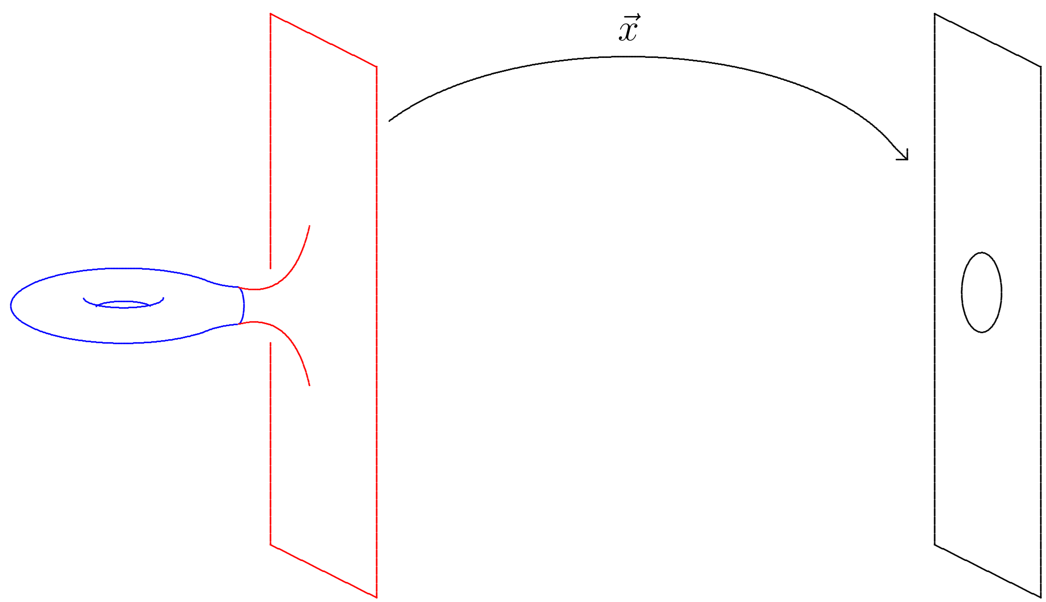

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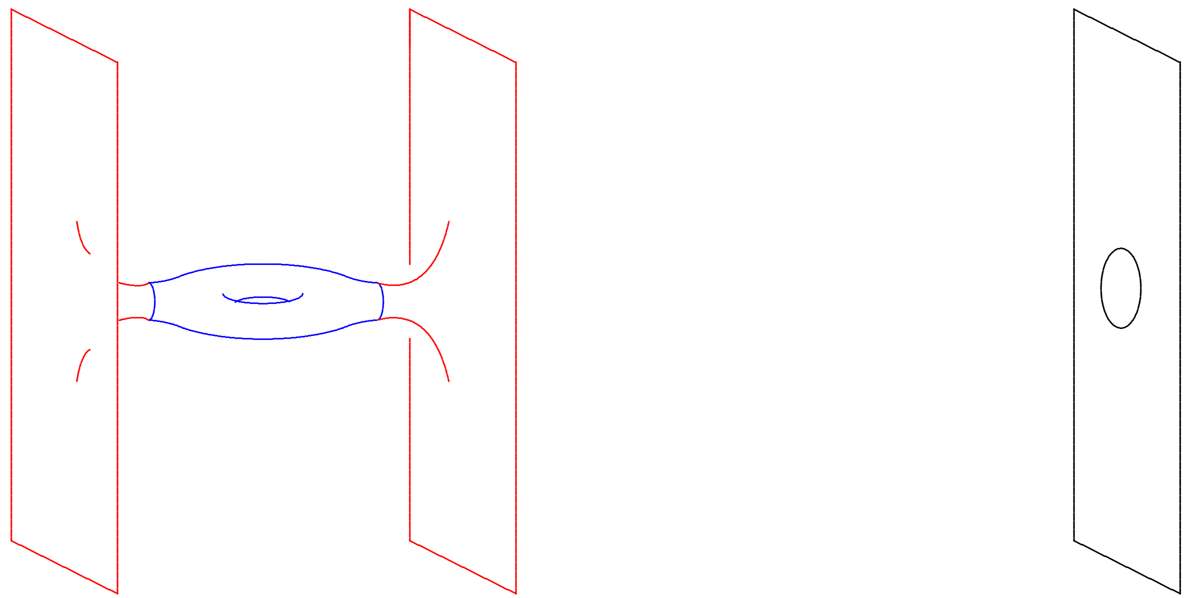

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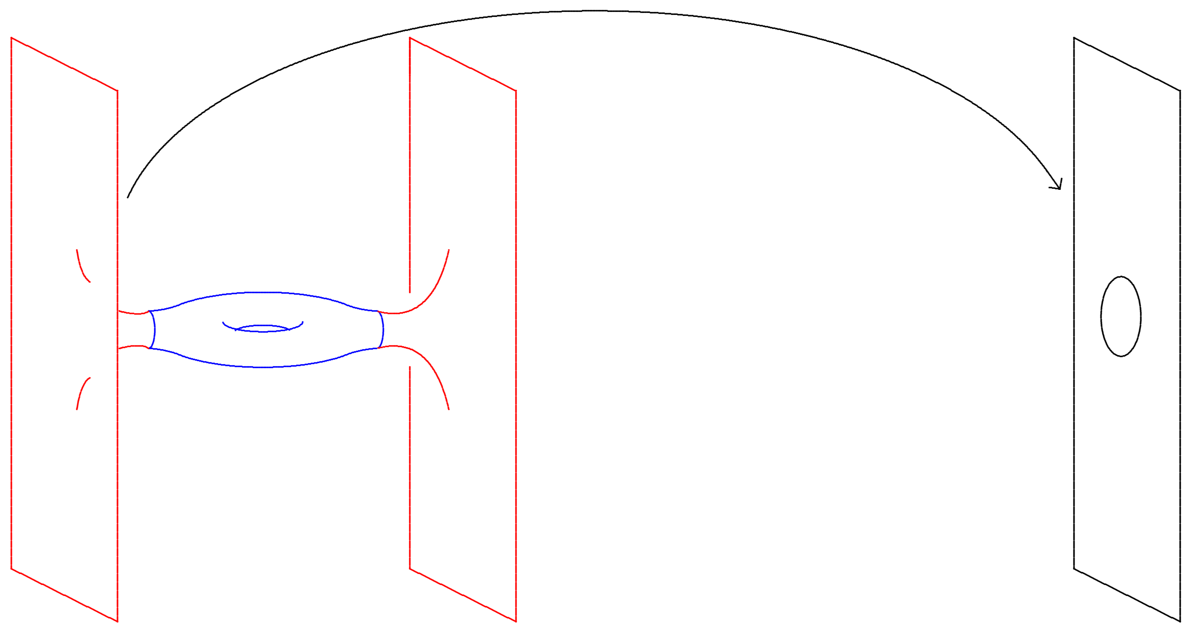

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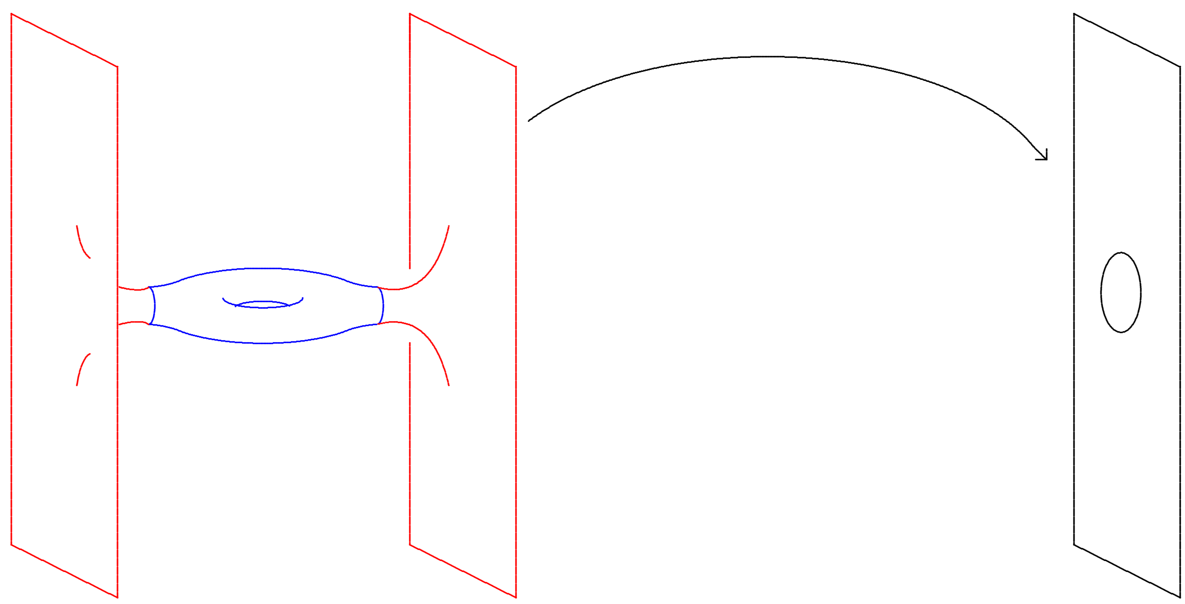

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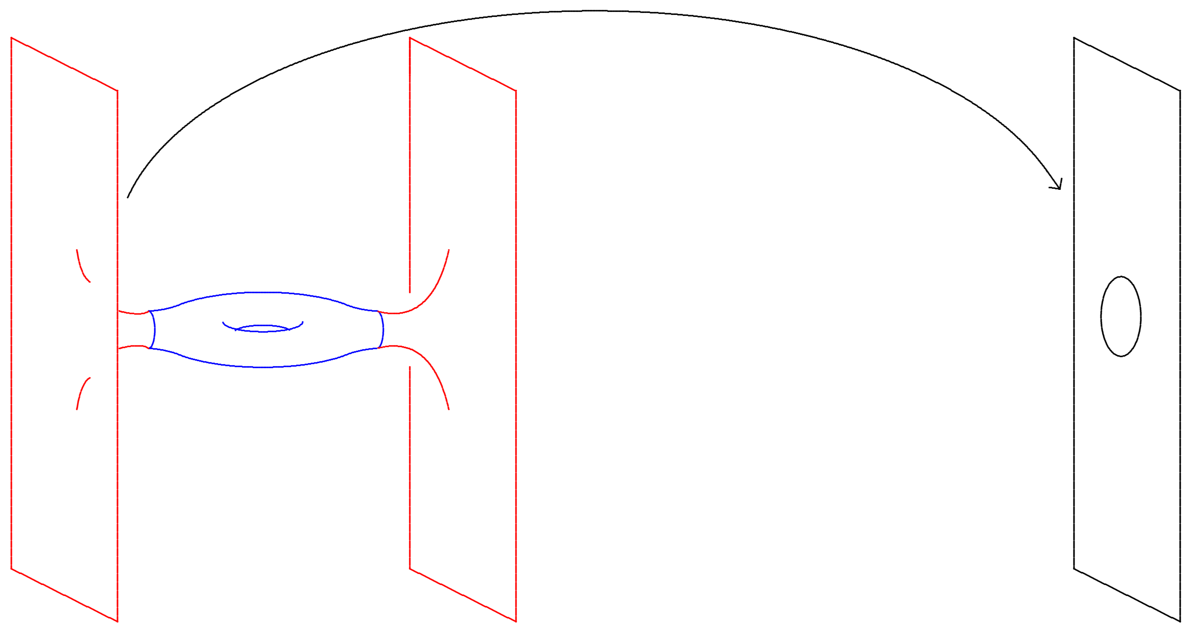

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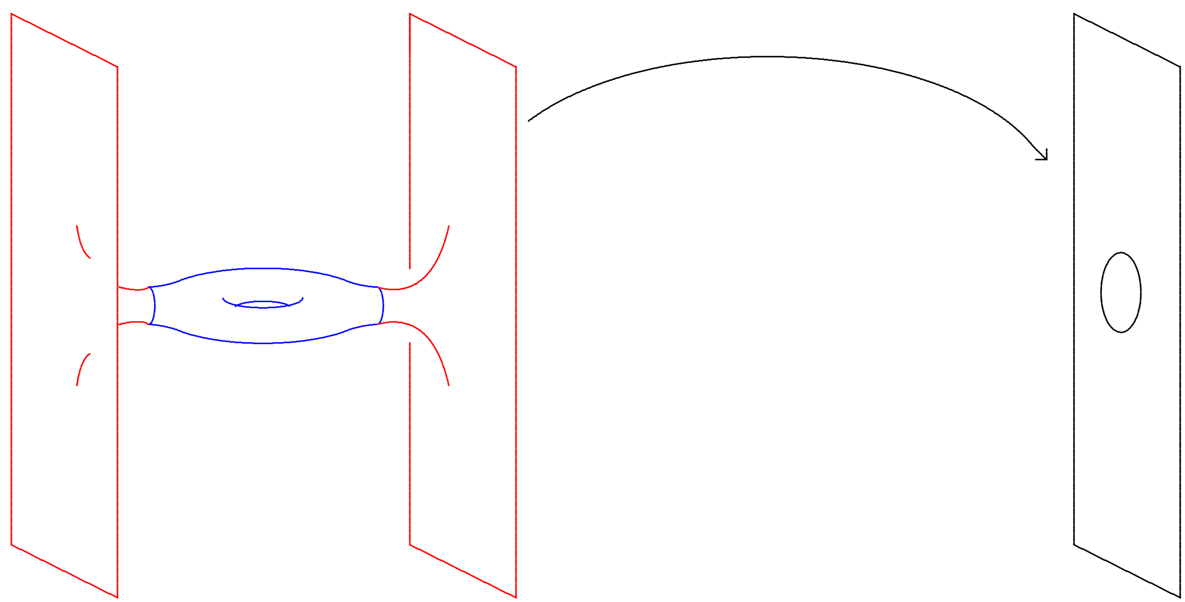

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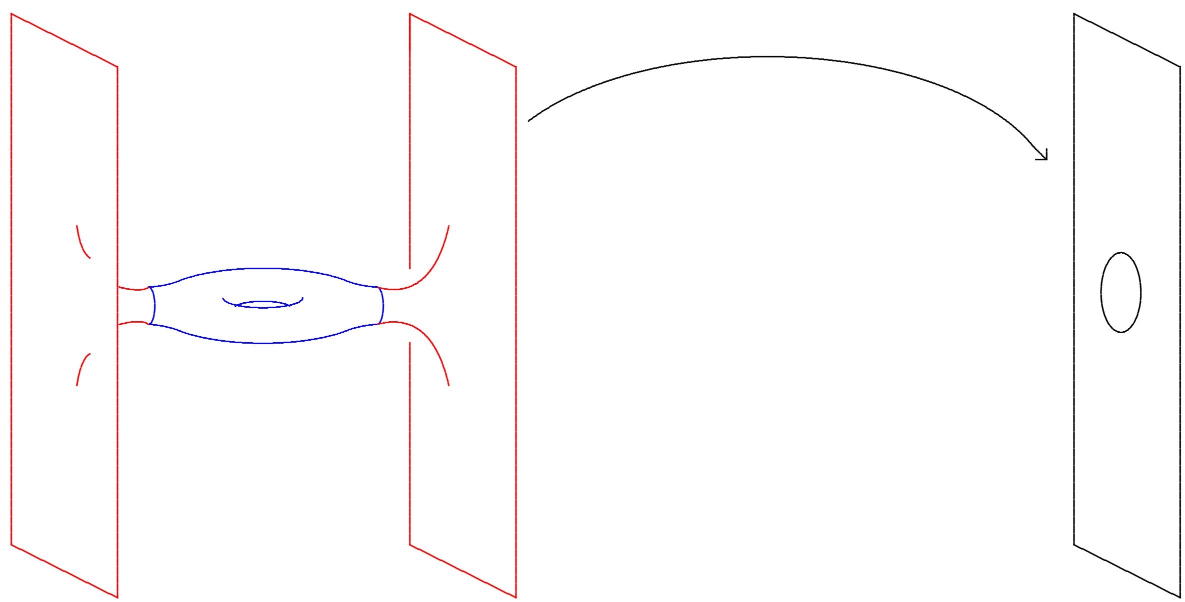

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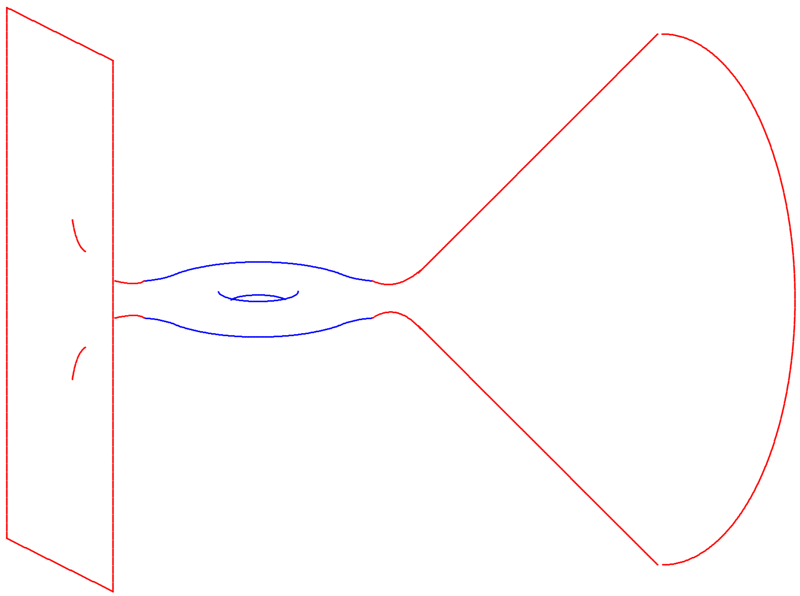

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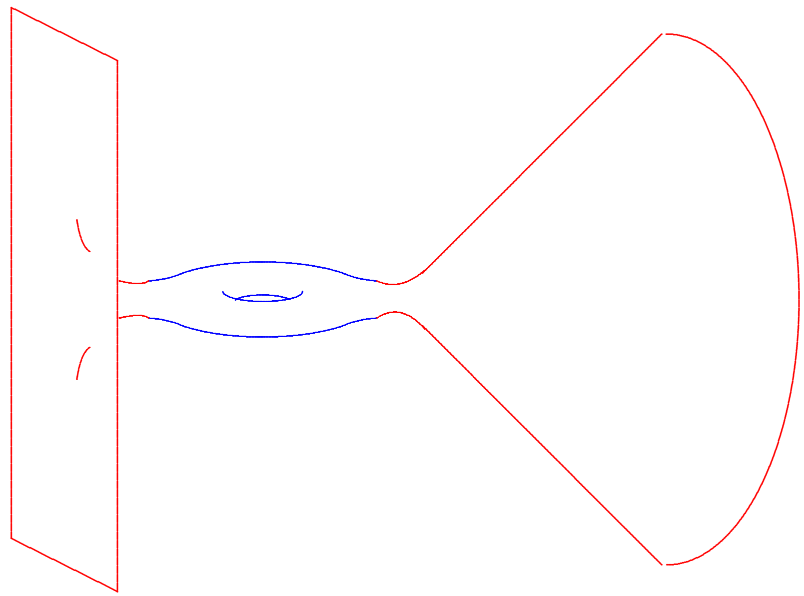

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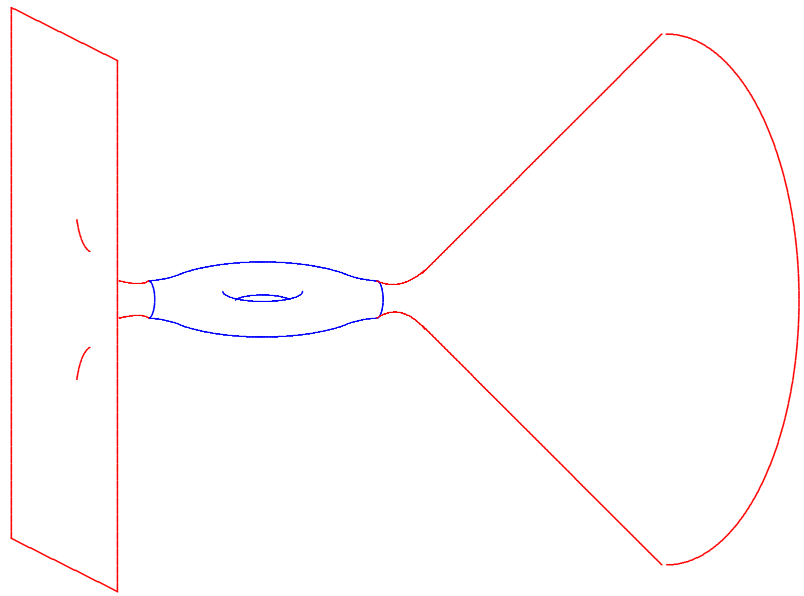

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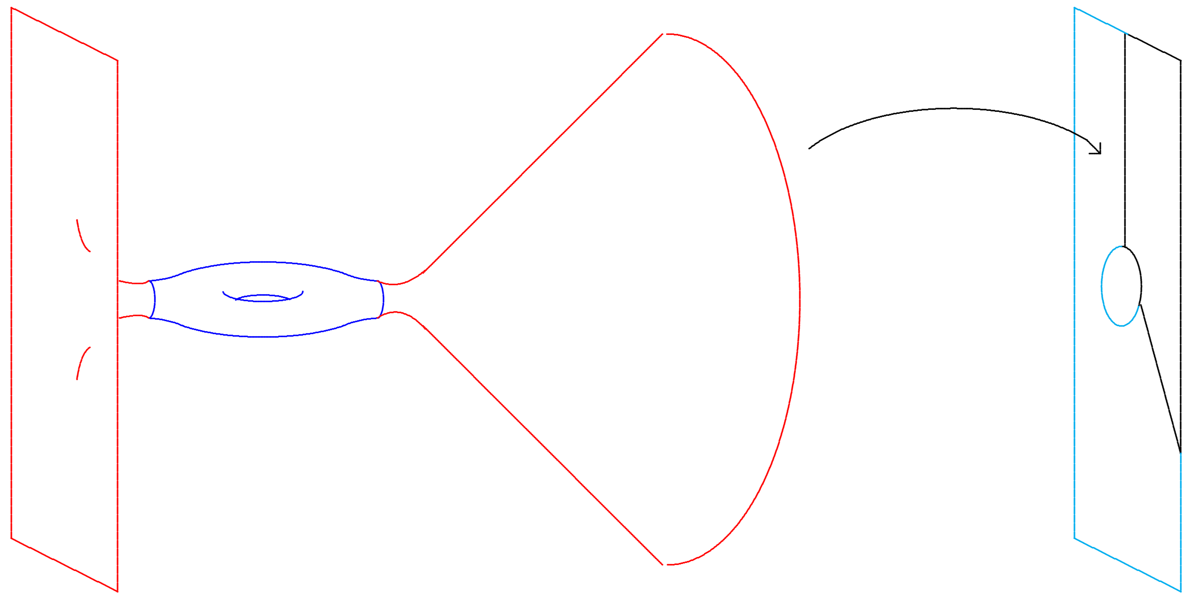

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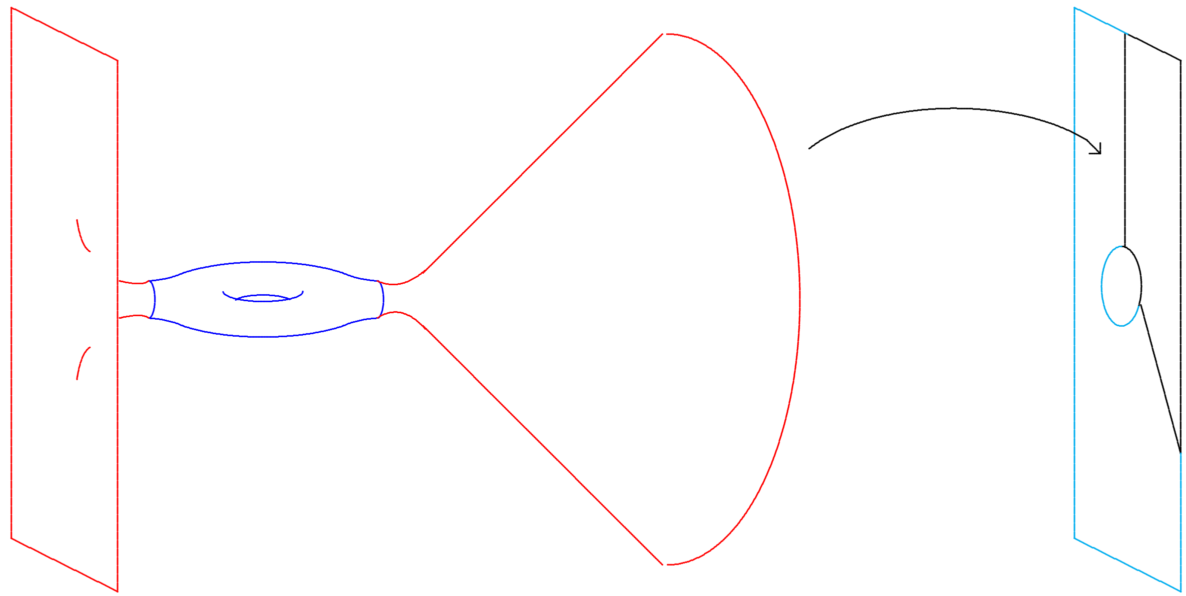

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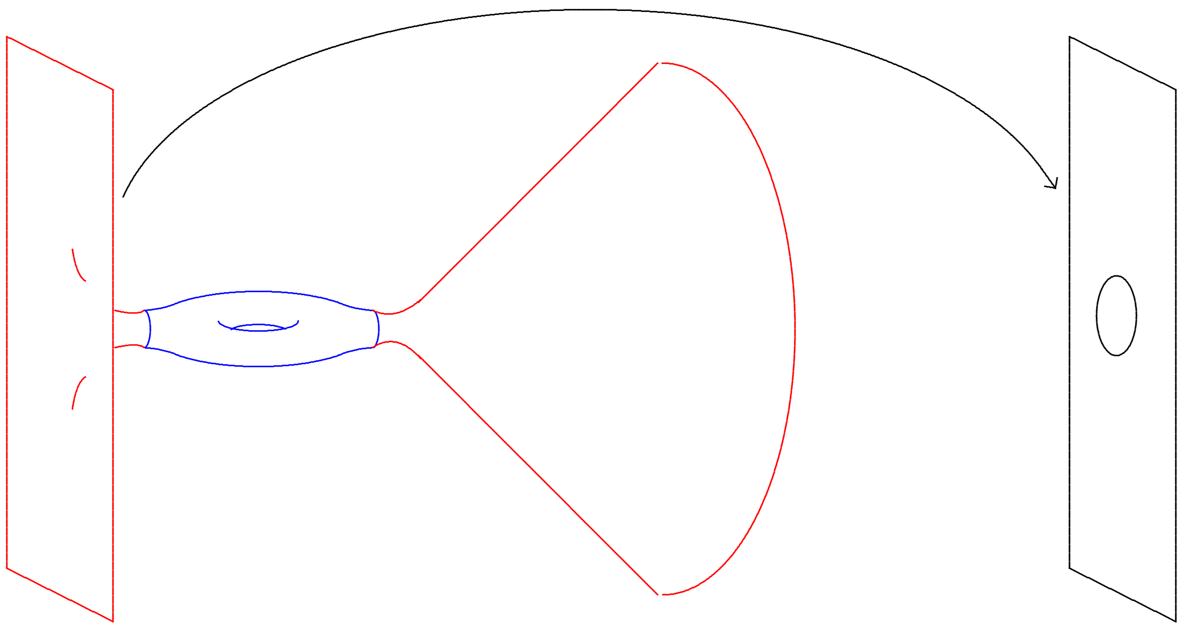

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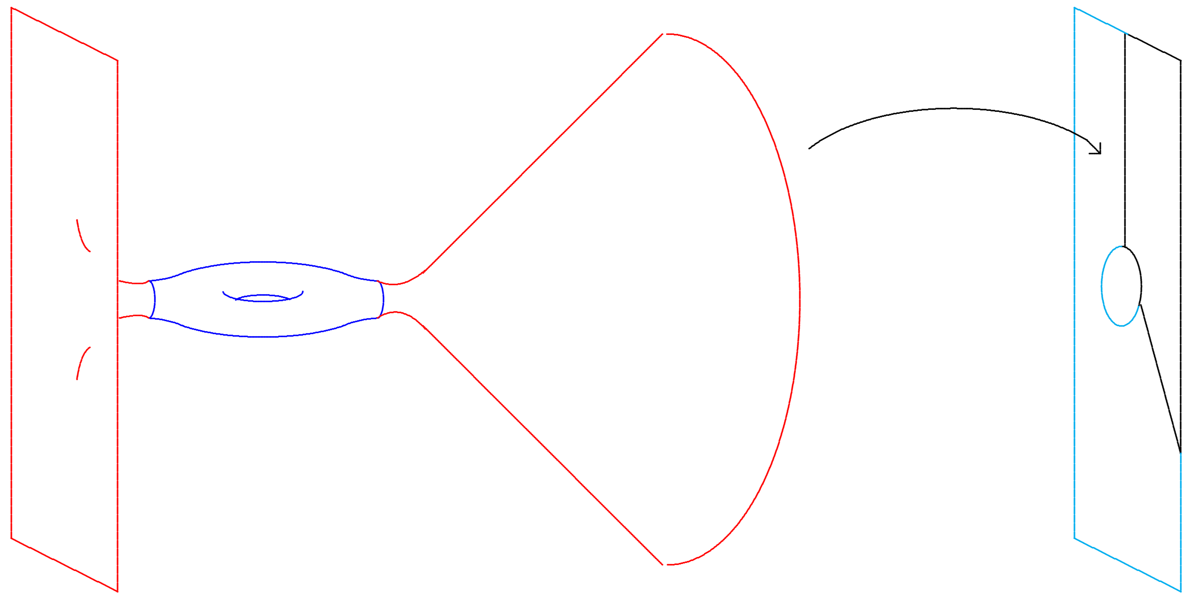

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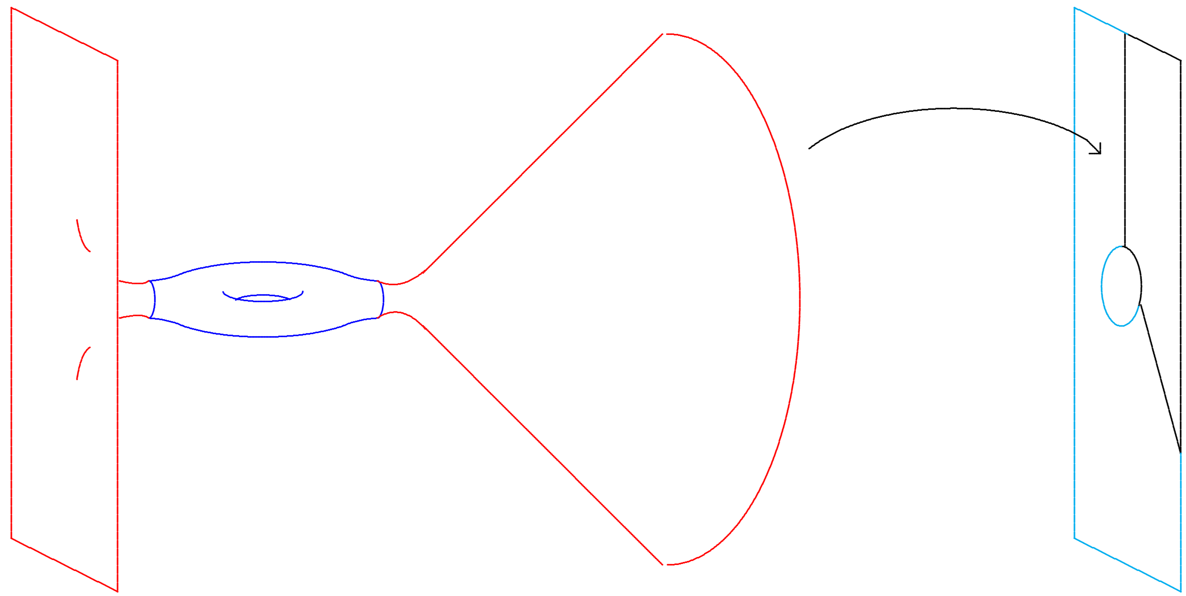

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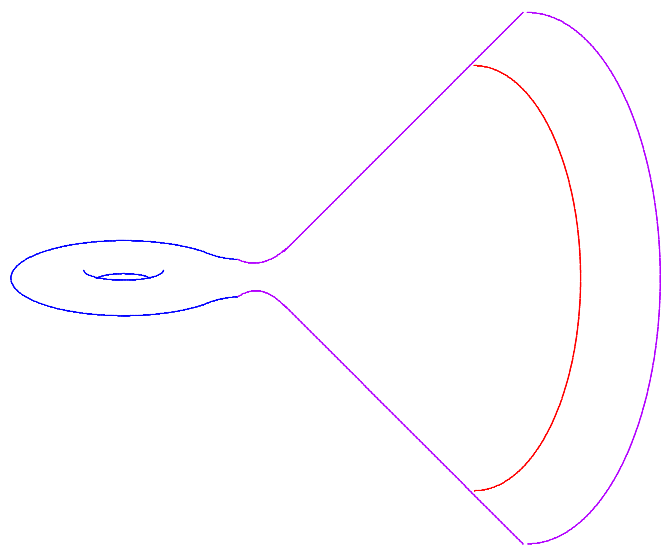

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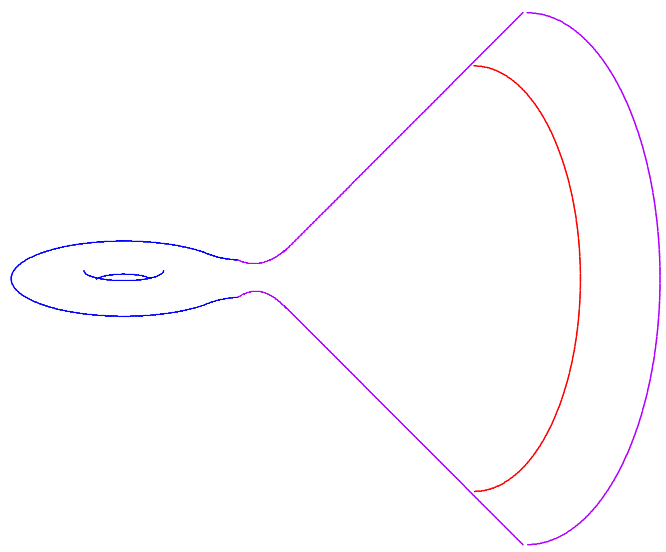

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$$g_{jk} - \delta_{jk} \in C^1_{-\tau}, \quad \tau > \frac{n-2}{2}$$

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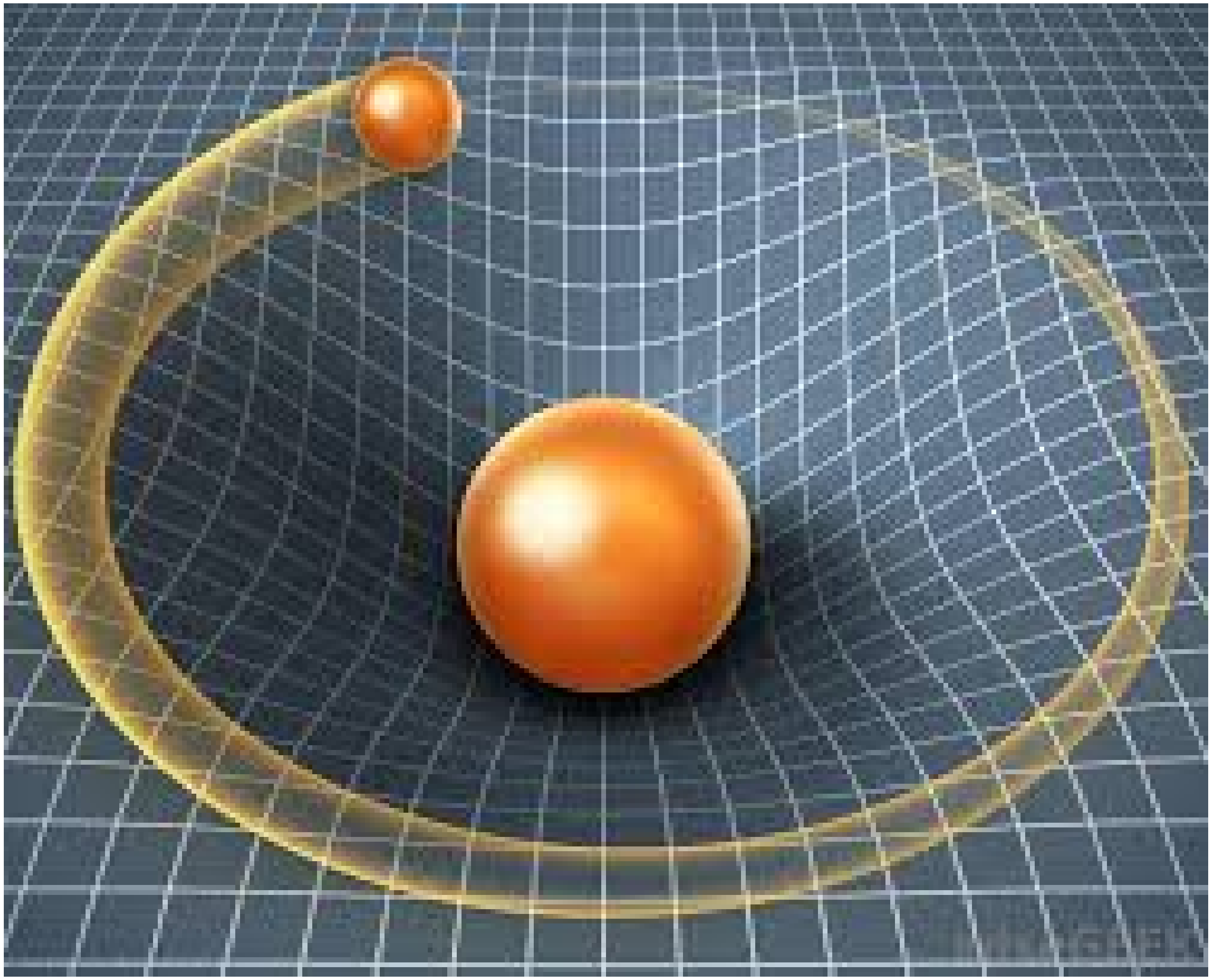

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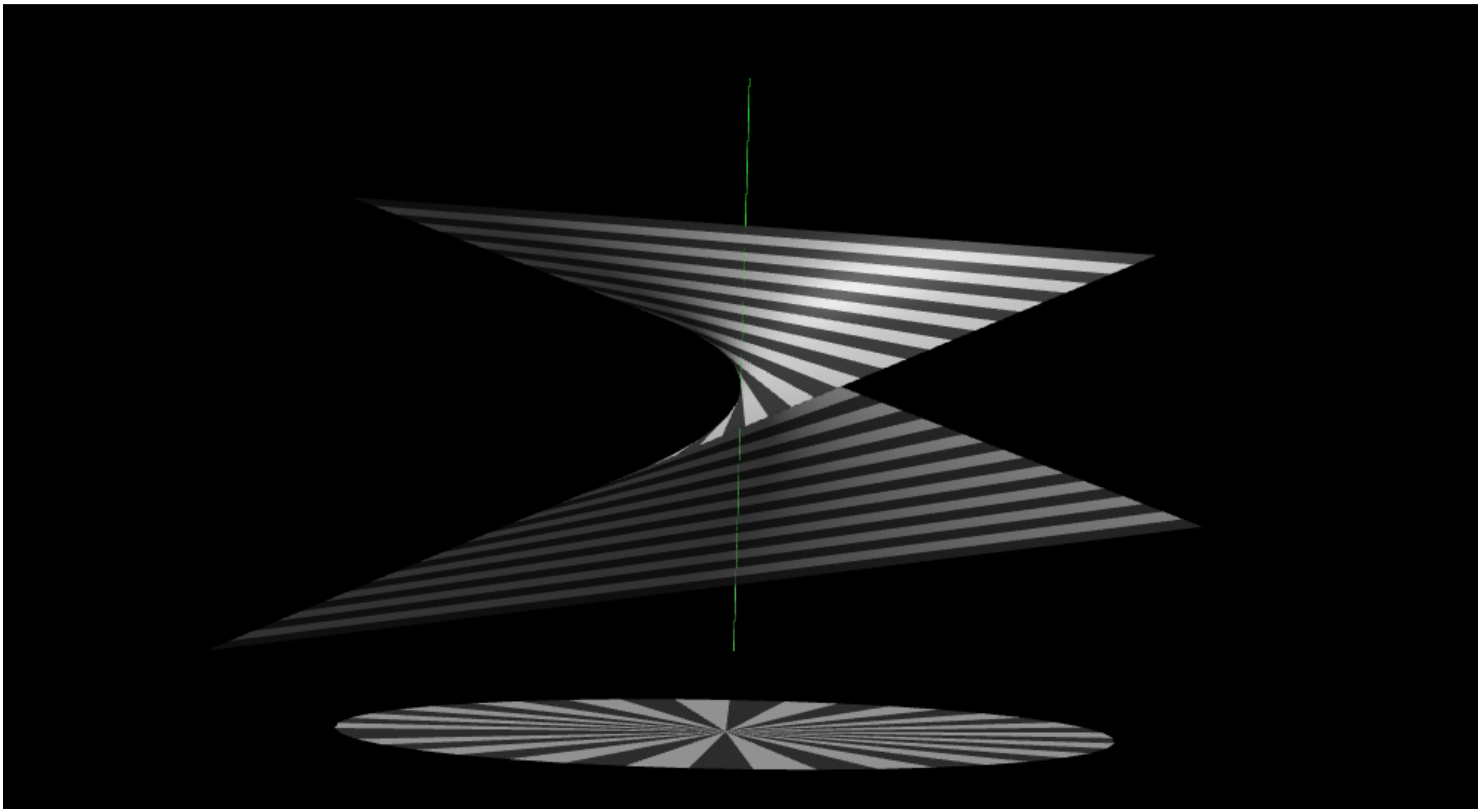

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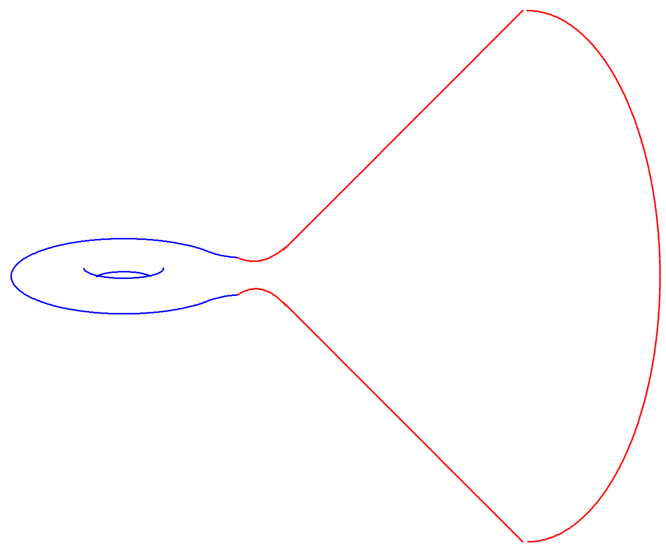

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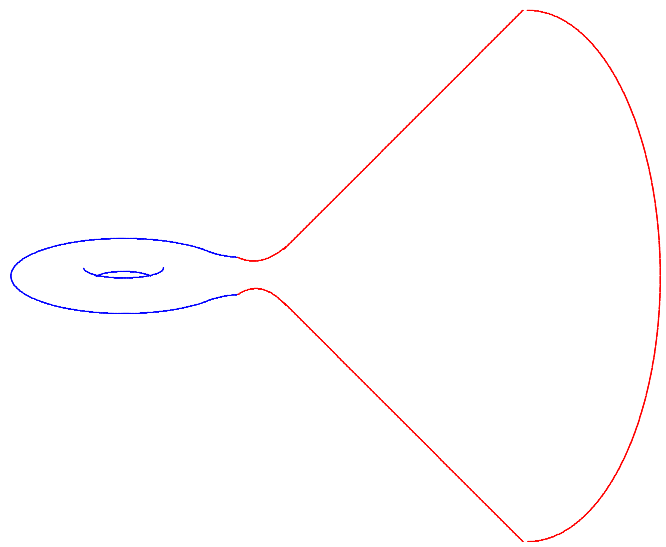

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In fact, we will see that there is an explicit formula for the mass in terms of these data!

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(Discovered independently by Rollin, Singer, & Şuvaina, using different methods.)

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So the mass is a "boundary correction" to the topological formula for the total scalar curvature.

Theorem C. *Any ALE Kähler manifold* (M, g, J) *of complex dimension* m *has mass given by*

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Corollary. *Any ALE scalar-flat Kähler manifold (M, g, J) of complex dimension m has mass given by*

$$\mathfrak{m}(M, g) = -\frac{\langle \clubsuit(c_1), [\omega]^{m-1}\rangle}{(2m-1)\pi^{m-1}}.$$

Corollary. *Any ALE scalar-flat Kähler manifold* (M, g, J) *of complex dimension* m *has mass given by*

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So **Theorem A** is an immediate consequence!

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$$g^{jk}\left(g_{j\ell,k} - g_{jk,\ell}\right)\nu^\ell \alpha_E = -\star d \log\left(\sqrt{\det g}\right) + O(\varrho^{-3-\varepsilon}).$$

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$$d(\theta\wedge\omega) = \rho\wedge\omega = \frac{s}{4}\omega^2 = 0.$$

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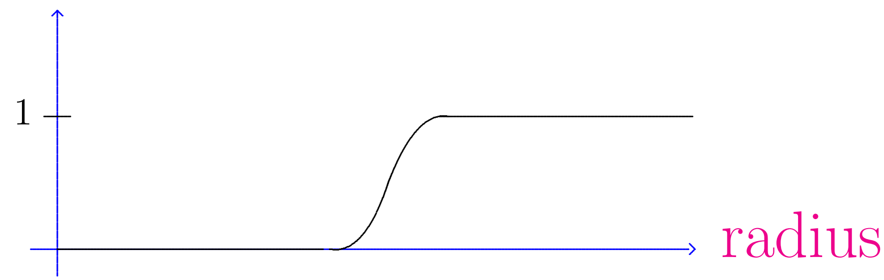

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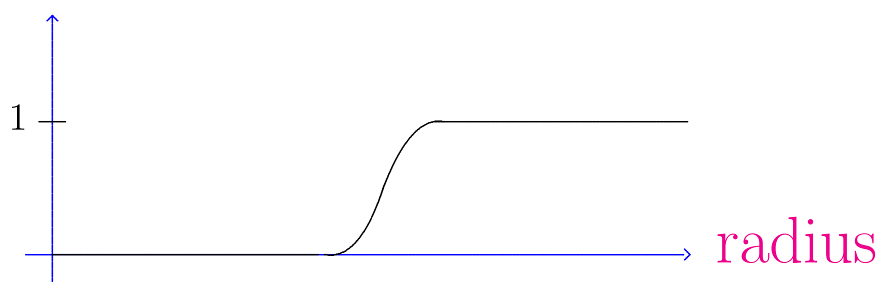

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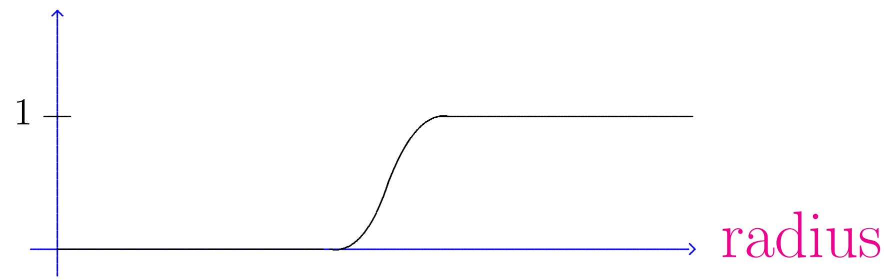

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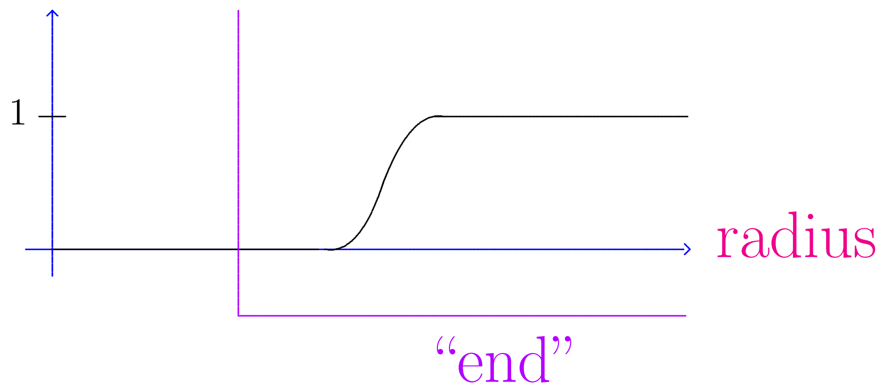

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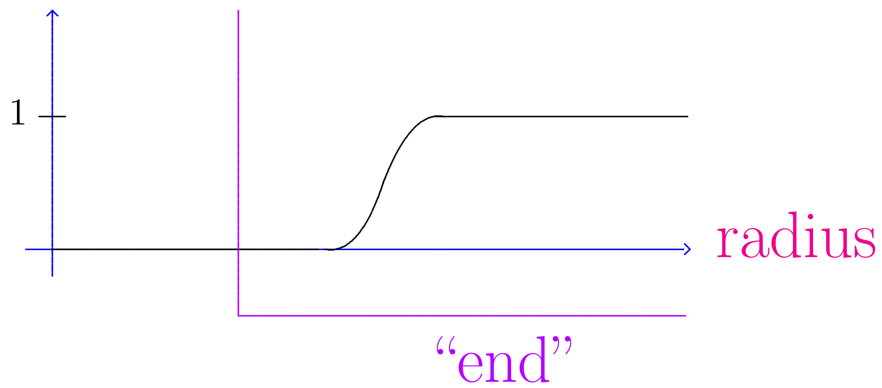

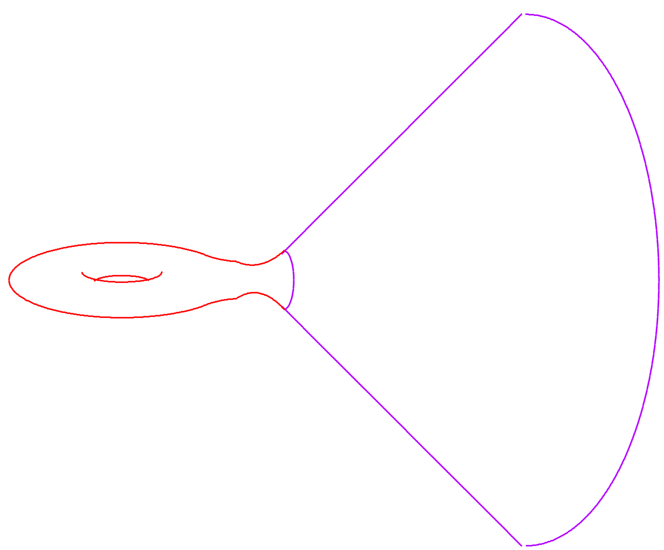

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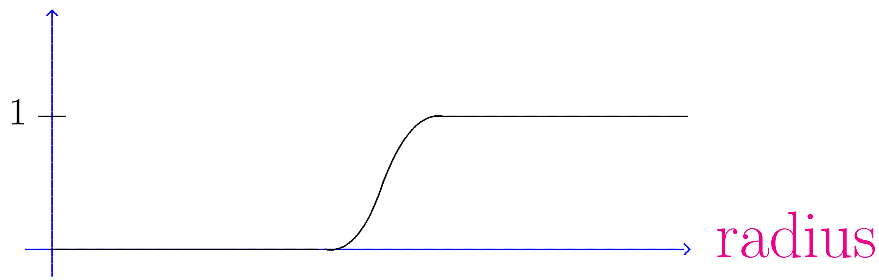

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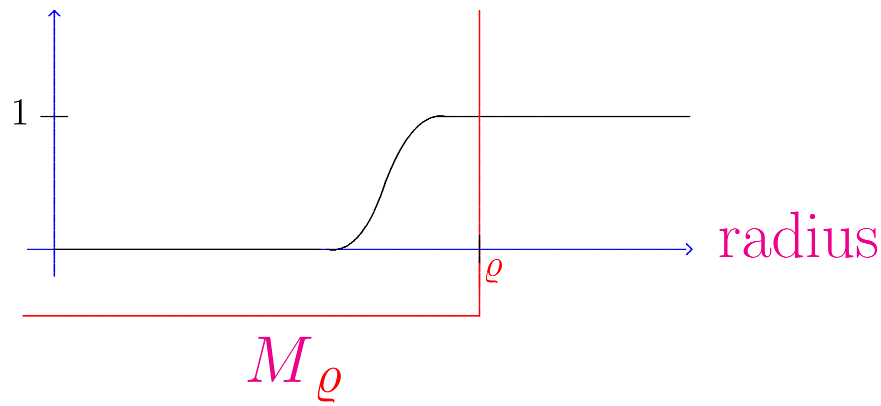

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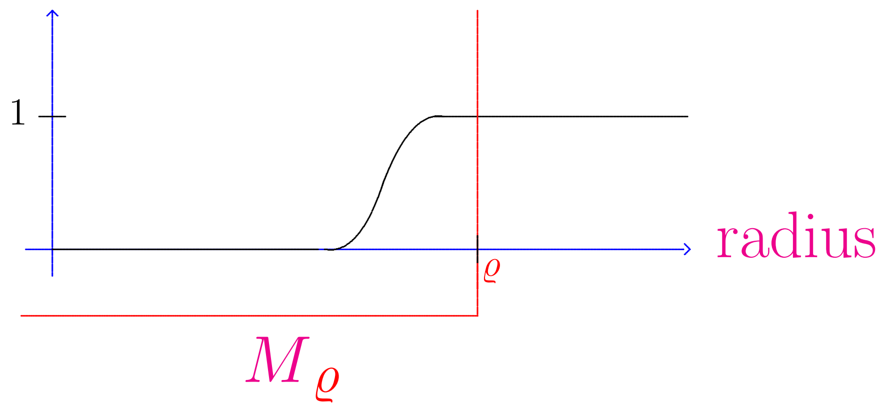

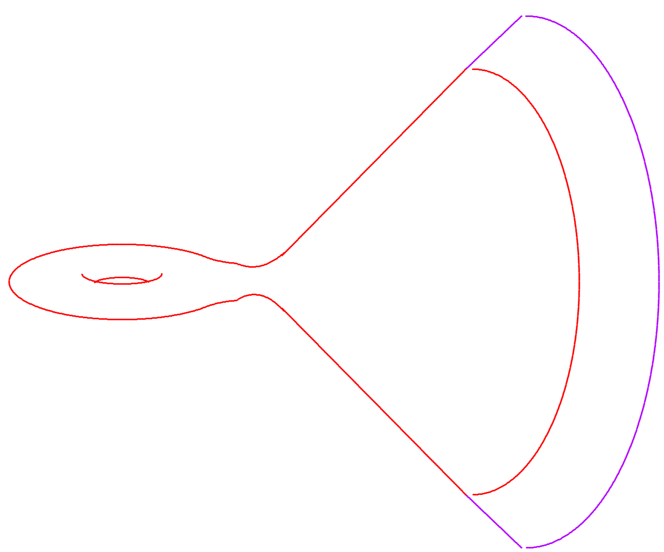

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Compactly supported, because $d\theta = \rho$ near infinity.

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where M_ϱ defined by radius $\leq \varrho$.

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because scalar-flat $\Longrightarrow$ $\rho \wedge \omega = 0.$

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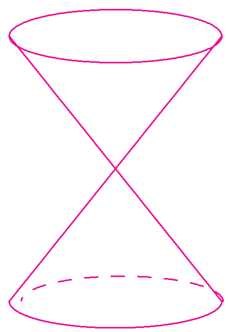

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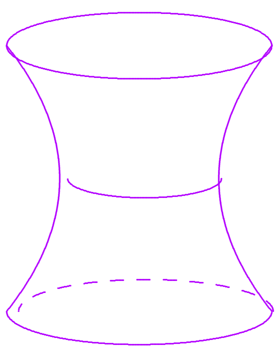

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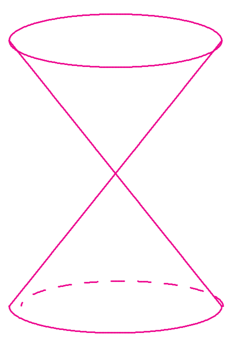

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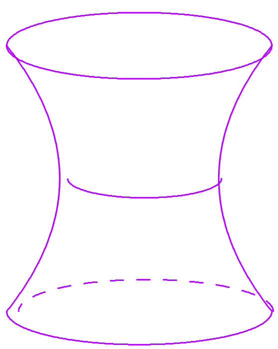

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$$J = J_0 + O(\varrho^{-3}), \qquad \nabla J = O(\varrho^{-4})$$

in suitable asymptotic coordinates adapted to g.

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This has some interesting consequences...

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Proof actually shows something stronger!

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with $= \Longleftrightarrow (M, g, J)$ *is scalar-flat Kähler.*

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so the mass formula implies the claim.

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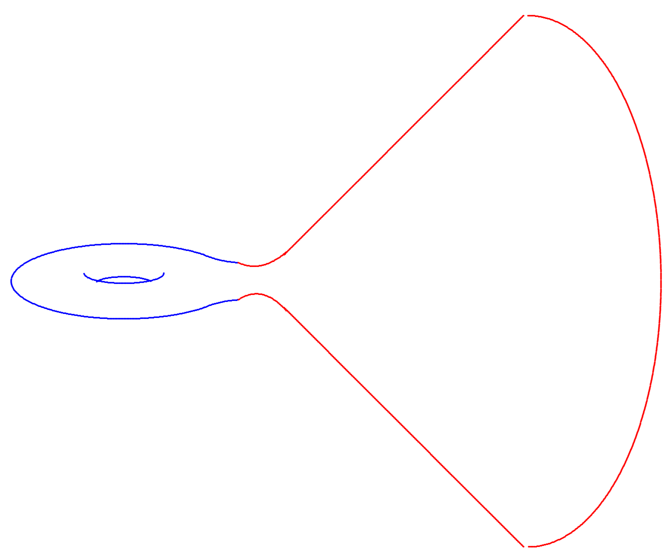

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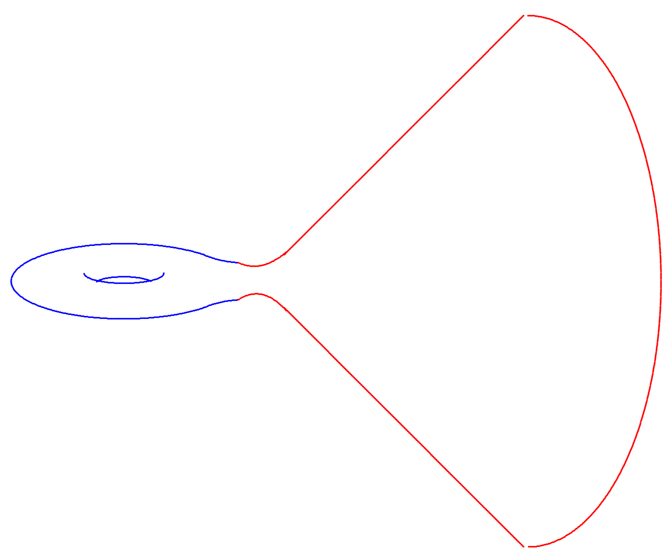

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