The Geometry of 4-Manifolds:

Curvature in the Balance

III

Claude LeBrun
Stony Brook University

Centro di Ricerca Matematica Ennio De Giorgi,
Pisa, Italia. Il 10 giugno 2022.

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Λ^+ self-dual 2-forms.

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However, these are not independent!

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$$\tau(M) = \frac{1}{12\pi^2} \int_M \left(|W_+|^2 - |W_-|^2 \right) d\mu$$

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Today's theme: How do these compare in size,

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Today's theme: How do these compare in size,

for specific classes of metrics on interesting 4-manifolds?

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More general Riemannian metrics?

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Excluded: Round S^4, Fubini-Study $\overline{\mathbb{CP}}_2$.

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(K-E after at worst passing to a double cover.)

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Excluded: Del Pezzo Surfaces (10 diffeotypes)

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$$\Longrightarrow \exists \widehat{g} = u^2 g \quad \text{s.t.} \quad \widehat{\mathfrak{s}} := \widehat{s} - 2\sqrt{6}\widehat{|W_+|} \leq 0.$$

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$$\int_M |W_+|^2 d\mu_g \overset{?}{\geq} \int_M \left(\frac{s^2}{24} - \frac{|\mathring{r}|^2}{2} \right) d\mu_g$$

agrees with previous question in the Einstein case.

Equivalent to

$$\frac{1}{4\pi^2} \int_M |W_+|^2 d\mu_g \overset{?}{\geq} \frac{1}{3}(2\chi + 3\tau)(M).$$

Since

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$$W_- = 0$$

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If g has s of fixed sign, agrees with sign of $Y_{[g]}$.

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Gursky '98 later gave a much simpler proof. . .

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Pursuing this lead will lead to interesting places!

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Natural Generalization:

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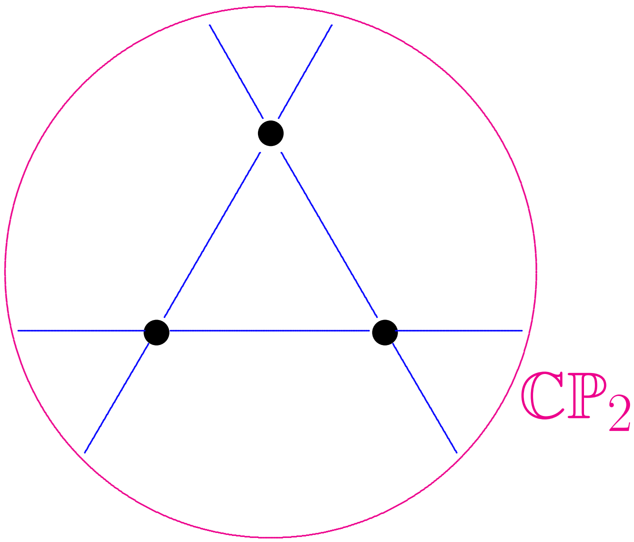
Conjecture. *On any del Pezzo surface* (M^4, J), *the conformally Kähler, Einstein product metric minimizes the Weyl functional* $\mathscr{W}$.

Del Pezzo surfaces:

(M^4, J) for which c_1 is a Kähler class $[\omega]$.
Shorthand: "$c_1 > 0$."

Blow-up of $\mathbb{CP}_2$ at k distinct points, $0 \leq k \leq 8$, in general position, or $\mathbb{CP}_1 \times \mathbb{CP}_1$.



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Theorem. *Each del Pezzo (M^4, J) admits a J-compatible conformally Kähler, Einstein metric, and this metric is geometrically unique.*

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Persuasive partial results.

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But problem still not settled!

Theorem (Gursky '98). *Let M be a smooth compact 4-manifold with $b_+(M) \neq 0$.*

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Applies in much greater generality.

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But says nothing about $Y([g]) < 0$ realm.

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But says nothing about "most" conformal classes.

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Method: Weitzenböck formula

$$0 = \frac{1}{2}\Delta|\omega|^2 + |\nabla\omega|^2 - 2W_+(\omega, \omega) + \frac{s}{3}|\omega|^2$$

for self-dual harmonic 2-form ω.

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$$\Longrightarrow \exists \widehat{g} = u^2 g \quad \text{s.t.} \quad \widehat{\mathfrak{s}} := \widehat{s} - 2\sqrt{6}\widehat{|W_+|} \leq 0.$$

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Open condition in C^2 topology on metrics.

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Open condition in C^2 topology on metrics.

(Harmonic forms depend continuously on metric.)

Theorem (L '15). *Let M be the underlying smooth oriented* 4*-manifold of a del Pezzo surface.*

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This recovers Gursky's inequality — but for a different open set of conformal classes!

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Inequality not limited to the positive Yamabe realm!

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Same method shows conformally Kähler, Einstein metrics on $\mathbb{CP}_2 \# \overline{\mathbb{CP}}_2$ and $\mathbb{CP}_2 \# 2\overline{\mathbb{CP}}_2$ minimize $\int_M |W_+|^2 d\mu$ among toric symplectic-type $[g]$.

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This is apparently not an accident!

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What happens there in the Yamabe-negative realm?

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Another result involving these ideas.

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with equality only when $\nabla\omega = 0$.

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Che piacere, tornare a Pisa!